Discord in the voter model for complex networks

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Online social networks have become primary means of communication. As they often exhibit undesirable effects such as hostility, polarization, or echo chambers, it is crucial to develop analytical tools that help us better understand them. In this paper we are interested in the evolution of discord in social networks. Formally, we introduce a method to calculate the probability of discord between any two agents in the multistate voter model with and without zealots. Our work applies to any directed, weighted graph with any finite number of possible opinions, allows for various update rates across agents, and does not imply any approximation. Under certain topological conditions, the opinions are independent and the joint distribution can be decoupled. Otherwise, the evolution of discord probabilities is described by a linear system of ordinary differential equations. We prove the existence of a unique equilibrium solution, which can be computed via an iterative algorithm. The classical definition of active links density is generalized to take into account long-range, weighted interactions. We illustrate our findings on real-life and synthetic networks. In particular, we investigate the impact of clustering on discord and uncover a rich landscape of varied behaviors in polarized networks. This sheds lights on the evolution of discord between, and within, antagonistic communities.

I. INTRODUCTION

There is growing concern over the nefarious effects of online social networks. One phenomenon in particular has attracted more and more attention. Similar-minded users tend to gather and form strongly opinionated communities: echo chambers. This is particularly salient in polarized debates regarding politics, societal issues, or conspiracy theories [1–5]. Echo chambers tend to foster animosity between opposite sides and fuel reinforcement of preexisting beliefs [6,7]— hence their name. To better understand the phenomenon, it is crucial to first examine how *discord*—i.e., disagreement— evolves in social networks. In this paper we contribute insight towards this goal. In the context of the celebrated Voter Model, we demonstrate a formula to calculate discord probabilities on complex networks, defined as

$$\rho_{ij} := P(\sigma_i \neq \sigma_j), \qquad (1)$$

where σ_i is the opinion of agent i (resp. σ_j, agent j).

In the Voter Model, agents within a graph are endowed with individual binary opinions $\{0, 1\}$. They proceed to choose a random neighbor and adopt their opinion, repeating this process multiple times. On the $\mathbb{Z}^d$ lattice ($d \leqslant 2$) as well as on any finite connected network, this dynamic leads to a state of consensus where all agents agree [8,9]. The average discord $\langle \rho \rangle$ over all edges, commonly referred to as *active links density*, is an order parameter of great interest. A low active links density indicates the presence of large clusters wherein all agents agree, and disagreement is found only at their borders. On the other hand, when it is high, there are no such structures and opinions are spread uniformly on the network.

In finite uncorrelated networks, $\langle \rho \rangle$ typically decays exponentially fast, and the convergence time scales linearly with N [10–12]. Similar results have been obtained for small-world networks, while scale-free networks present different behaviors depending on the exponent of the power-law degree distribution [12–15]. However, most of these rely on approximations, and no general solution applicable to any given network has been found. If we wish to study the active links density in real-world networks, this is of primary importance.

In this paper we present a universal formula for discord probabilities that applies to any given directed, weighted network, without any requirement regarding the topology or the degree distribution of the network. Our framework can accomodate any finite number of opinions [9,16], zealots [17,18], and individual update rates [19]. Importantly, we make no approximation and the results are exact. We also propose a way of computing the active links density that uses the exponential of the adjacency matrix to account for long-range, weighted interactions.

The rest of this paper is organized as follows. Section II introduces the model. In Sec. III we derive equations for the evolution of individual opinion distributions. In Sec. IV we present our main results: the formula for discord probabilities and the generalized active links density. Section V is devoted

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to numerical experiments that demonstrate the use cases of our results. Finally, we conclude in Sec. VI.

II. SYSTEM

Consider a group $\mathcal{N} = \{1, \ldots, N\}$ of agents interacting through a social network and holding discrete opinions $\sigma_1, \ldots, \sigma_N$ valued in some set $\mathcal{S} = \{1, \ldots, S\}$. Opinions are bound to evolve due to influence from peers, and we let $w_{ij} \geqslant 0$ denote the weight of the edge $j \to i$. Agents may also be under the influence of zealots, one-sided sources that never change opinions. Possible examples include strongly opinionated characters, such as politicians, journalists, or lobbyists.

As in [20,21], we consider all zealots with the same opinion s as one single entity exogenous to the network, which we name the s-zealot. The word *agent* will specifically refer to nonzealous individuals. In this case the notion of zealots has a broader interpretation as exogenous sources of influence, and may, for example, describe some inner bias of the agents or an advertising campaign. We let z_i^s denote the total amount of influence exerted on agent i by the s-zealot. We say that agent i can be influenced by the s-zealot if either $z_i^s > 0$ or there exists an agent j with $z_j^s > 0$ and a path from j to i. For the latter, the strength of influence exerted by the s-zealot on i will depend on the cumulated length of all such paths. We discuss long-range influences in more detail in Sec. IV C.

We call *leaders of i* agents in the set $\mathcal{L}_i = \{j \in \mathcal{N} : w_{ij} > 0\}$. We allow self-loops, i.e., $i \in \mathcal{L}_i$. They can be used to describe (1) an unwillingness of agents to reevaluate their opinions or (2) agents reaffirming their current opinions. The former can effectively adjust the update rates. The latter can be useful when using the Voter Model to describe the dynamics of people retweeting their own previous posts.

For the sake of simplicity and without loss of generality, we consider edge weights to be normalized: for any agent $i \in \mathcal{N}$,

$$\sum_{j \in \mathcal{L}_i} w_{ij} + \sum_{s \in \mathcal{S}} z_i^s = 1. \tag{2}$$

In particular, each node is assumed to either have a leader or to be connected to a zealot. The directed, weighted graph of all agents is denoted by $\mathcal{G}$. We assume $\mathcal{G}$ to be weakly connected—if it is not, one can apply the results to each component of the graph separately. We let $W = (w_{ij})_{1 \leqslant i, j \leqslant N}$ denote the weighted adjacency matrix of the graph.

III. EVOLUTION OF OPINIONS

Each agent is endowed with an exponential clock of parameter 1. Whenever i's clock rings, i updates their opinion by copying a random neighbor or a zealot. Neighbor j is copied with probability w_{ij}, and the s-zealot with probability $z_i^{(s)}$. In other words, with rate w_{ij} agent i copies agent j, and with rate z_i^s agent i adopts opinion s via the s-zealot.

Let x_i^s denote the probability for agent i to hold opinion s. To avoid cumbersome notations, we omit the time parameter. We shall remember that x_i^s is a time-dependent quantity and is implicitly followed by (t). A direct extension of [20, Eq. (3)]

gives

$$\frac{dx_i^s}{dt} = \left(1 - x_i^s\right) \left[\sum_{j \in \mathcal{L}_i} w_{ij} x_j^s + z_i^s \right] \tag{3}$$

$$- x_i^s \left[\sum_{j \in \mathcal{L}_i} w_{ij} \left(1 - x_j^s\right) + \sum_{s' \neq s} z_i^{s'} \right], \tag{4}$$

which reduces to

$$\frac{dx_i^s}{dt} = \sum_{j \in \mathcal{L}_i} w_{ij} x_j^s + z_i^s - x_i^s. \tag{5}$$

Hence, at equilibrium we have

$$x_i^s = \sum_{j \in \mathcal{L}_i} w_{ij} x_j^s + z_i^s. \tag{6}$$

As expressed by [22, Prop. 3.2], x_i^s is the probability that a backward random walk initiated at i reaches the s-zealot before another zealot. Hence, $x_i^s > 0$ if and only if i can be influenced by the s-zealot.

The dynamics correspond to those of a continuous-time Friedkin-Johnsen model [23]. The number of distinct equilibrium states depends on the topology of the agent graph and the influence of zealots. Assuming that any agent can be influenced by at least one zealot, Eq. (2) and [24, Lemma 2.1] imply that the spectral radius of W is strictly less than 1, hence there is a unique equilibrium state; see the Supplemental Material [25, Sec. I]. If $z_i^s = 0$ for all i, s, we uncover the continuous-time French-DeGroot model [26,27]. In that case, consensus is reached if there exists an agent able to reach every other.

IV. DISCORD PROBABILITIES

Let ρ_{ij} denote the probability $P(\sigma_i \neq \sigma_j)$ of discord between agents i and j. The quantity $1 - \rho_{ij}$ is called probability of *harmony*. Trivially $\rho_{ii} = 0$ and $\rho_{ij} = \rho_{ji}$. For the sake of conciseness we again omit the time parameters, and denote indistinctively by ij or ji the unordered agent pair $\{i, j\}$. One can then write

$$\rho_{ij} = P(\sigma_i \neq \sigma_j) \tag{7}$$

$$= \sum_{s \in \mathcal{S}} P(\sigma_i = s, \sigma_j \neq s) \tag{8}$$

$$= \sum_{s \in \mathcal{S}} P(\sigma_i = s) P(\sigma_j \neq s) \tag{9}$$

$$= \sum_{s \in \mathcal{S}} x_i^s \left(1 - x_j^s\right). \tag{10}$$

However, the above derivation implicitly assumes that the opinions σ_i and σ_j are independent, which is not guaranteed, e.g., if i and j are neighbors. A toy example is presented in Fig. 1. We focus on the general case, valid for any i, j. Later we characterize precisely the cases where Eq. (10) holds.

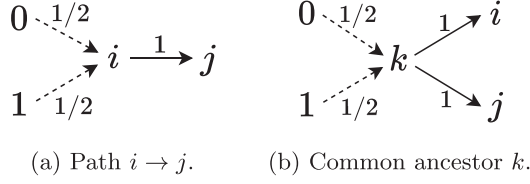(a) Path $i \to j$.   (b) Common ancestor k.

FIG. 1. Dependency between opinions. Nodes 0 and 1 are the 0- and 1-zealot, respectively. Numbers along the arrows denote edge weights. In (a) there is a path from i to j. In (b) there is none, but i and j have a common ancestor k. In both cases, Eq. (15) gives $\rho_{ij} = 1/4$, while Eq. (10) gives $\rho_{ij} = 1/2$.

A. General case

There are two types of events that lead to i adopting an opinion different than j's:

(1) i copies agent $k \neq j$, who holds another opinion than j's. This happens at rate $w_{ik}\rho_{jk}$.

(2) i copies an s-zealot while j holds an opinion different than s. This happens at rate $z_i^s(1 - x_j^s)$.

Hence, i adopts another opinion than j's at rate

$$\sum_{k \in \mathcal{L}_i} w_{ik}\rho_{jk} + \sum_{s \in \mathcal{S}} z_i^s(1 - x_j^s). \qquad (11)$$

The same reasoning gives us the rate at which j adopts another opinion than i's, and we find that the pair ij switches from harmony to discord at rate

$$\Delta_{ij}^- = (1 - \rho_{ij})\left[\sum_{k \in \mathcal{L}_i} w_{ik}\rho_{jk} + \sum_{s \in \mathcal{S}} z_i^s(1 - x_j^s) + \sum_{k \in \mathcal{L}_j} w_{jk}\rho_{ik} \right.$$
$$\left. + \sum_{s \in \mathcal{S}} z_j^s(1 - x_i^s) \right], \qquad (12)$$

and from discord to harmony at rate

$$\Delta_{ij}^+ = \rho_{ij}\left[\sum_{k \in \mathcal{L}_i} w_{ik}(1 - \rho_{jk}) + \sum_{s \in \mathcal{S}} z_i^s x_j^s + \sum_{k \in \mathcal{L}_j} w_{jk}(1 - \rho_{ik}) \right.$$
$$\left. + \sum_{s \in \mathcal{S}} z_j^s x_i^s \right]. \qquad (13)$$

Subtracting Δ_{ij}^- from Δ_{ij}^+, we obtain the master equation

$$\frac{d\rho_{ij}}{dt} = \Delta_{ij}^+ - \Delta_{ij}^- = \sum_{k \in \mathcal{L}_i} w_{ik}\rho_{jk} + \sum_{k \in \mathcal{L}_j} w_{jk}\rho_{ik}$$
$$+ \sum_{s \in \mathcal{S}} z_i^s(1 - x_j^s) + \sum_{s \in \mathcal{S}} z_j^s(1 - x_i^s) - 2\rho_{ij}. \qquad (14)$$

The evolution of discord probabilities is thus governed by a system of $N(N-1)/2$ linear differential equations—although not all will be needed, as stressed below. Setting the left-hand side to zero gives us the equilibrium discord probability for the pair ij,

$$\rho_{ij} = \frac{1}{2}\left[\sum_{k \in \mathcal{L}_i} w_{ik}\rho_{jk} + \sum_{k \in \mathcal{L}_j} w_{jk}\rho_{ik} \right.$$
$$\left. + \sum_{s \in \mathcal{S}} z_i^s(1 - x_j^s) + \sum_{s \in \mathcal{S}} z_j^s(1 - x_i^s) \right]. \qquad (15)$$

The discord probabilities are thus the solution of a linear system of the form

$$\rho = V\rho + y \qquad \text{(equilibrium)}, \qquad (16)$$

$$\frac{d\rho}{dt} = 2(V - I)\rho + 2y \qquad \text{(evolution)}. \qquad (17)$$

The entries of V, y are given by (15). In the absence of zealots, we have $y = 0$, and the speed of convergence depends on the spectrum of V. If consensus is reached (e.g., a strongly connected network), then $\rho \to 0$. Otherwise, the various equilibrium states are given by the leading eigenvectors of V.

In the presence of zealots, Lemma 2.1 from [24] implies that the spectral radius of W is strictly less than 1, assuming every agent can be influenced by a zealot; see the Supplemental Material [25, Sec. I] for details. Hence, the system has a unique solution. It can be efficiently computed by iterating

$$\rho^{(k)} = V\rho^{(k-1)} + y \qquad (18)$$

for any initialization $\rho^{(0)}$ with values in $]0,1[$, and the convergence rate depends on the spectral radius of V. The proof of that statement can be found in [28, Thm. 4].

Finally, note that all the equations we derived can be adapted to account for various update rates $r_1, \ldots, r_N$ across agents. It suffices to scale the rates w_{ij} by r_i, to replace $2\rho_{ij}$ by $(r_i + r_j)\rho_{ij}$ in Eq. (14) and $1/2$ by $1/(r_i + r_j)$ in Eq. (15). Equations (5) and (6) are unchanged. Here we stick to the traditional setting $r_i = 1$ for all i.

B. Independent pairs

There are some cases where the opinions σ_i and σ_j are independent, and one can use Eq. (10) to calculate ρ_{ij} without having to solve a possibly large linear system. Independence holds if one of the following is verified:

(1) σ_i or σ_j is constant or

(2) There is no path from i to j nor from j to i, and i and j have no common ancestor.

A formal proof can be found in the Supplemental Material [25, Sec. II]. The second condition assures us that i and j do not influence each other, and that they are exposed to strictly different sources: their opinions evolve in total independence. As illustrated in Fig. 1, if one of these assumptions is violated, then Eq. (10) gives an incorrect result.

When i and j have the same opinion distribution x, Eq. (10) is akin to the entropy of x. Hence the more uniform x is, the higher the discord. Equation (10) is also exactly 1 minus the cosine similarity between x_i and x_j. While for dependent agent pairs the discord probabilities are given by Eqs. (14) and (15), Eq. (10) may still be used for the purpose of measuring dissimilarity of opinion distributions.

C. Generalized active links density

The active links density is the average discord between all neighboring agents. While convenient for regular, unweighted graphs, this definition suffers from two shortcomings when it comes to general networks. First, not all edges are created equal: if $w_{ij} = 0.80$ and $w_{ik} = 0.01$, then j holds a strong power of influence over i, while k barely has any at all. Discord ρ_{ij} between i and j will thus often be much more relevant to the analysis than ρ_{ik}, a difference not accounted for when taking a simple unweighted average.

Second, two agents may be closer than they appear: if $w_{ij} = 0$ but $w_{ik} = w_{kj} = 0.9$, agent j exerts non-negligible influence on i via k, despite their not being directly connected by an edge. The opinion of agent j reaches i in two steps with rate

$$\sum_{k \in \mathcal{L}_i} w_{ik} w_{kj}. \tag{19}$$

Indeed, i copies leader k with rate w_{ik}, and each time it happens there is a probability w_{kj} that k's opinion was copied directly from j.

From these considerations stems a metric that is better suited for complex networks: the *generalized active links density*, defined over all agent pairs by

$$\langle \rho \rangle = \frac{\sum_{i<j}(w_{ij}^{\infty} + w_{ji}^{\infty})\rho_{ij}}{\sum_{i<j}(w_{ij}^{\infty} + w_{ji}^{\infty})}. \tag{20}$$

The w_{ij}^{∞} can be any measure of influence of j on i. Inspired by [29], here w_{ij}^{∞} is the (i, j)-th component of the matrix exponential

$$e^W = \sum_{k=1}^{\infty} \frac{1}{k!} W^k. \tag{21}$$

Summing all powers of W generalizes the reasoning made above (19), and lets us quantify the multistep influence that agents may have on one another. Scaling by the inverse of the path length factorial attributes a rapidly decaying importance to longer paths. The sum $w_{ij}^{\infty} + w_{ji}^{\infty}$ is then a measure of long-range, weighted influence between i and j.

Note that with this definition, zero entries in the ith row w_i^{∞} correspond to users unable to reach i, and nonzero entries correspond to ancestors of i. The cosine similarity

$$\cos\left(w_i^{\infty}, w_j^{\infty}\right) = \frac{w_i^{\infty} \cdot w_j^{\infty}}{\|w_i^{\infty}\| \|w_j^{\infty}\|} \tag{22}$$

informs us about the similarity of i and j's ancestry, meaning the extent to which they are exposed to the same channels of influence.

V. NUMERICAL EXPERIMENTS

A. Dependency between opinions

One may be interested only in certain values of ρ_{ij}, wishing to avoid the burden of computing them all. It seems natural to use

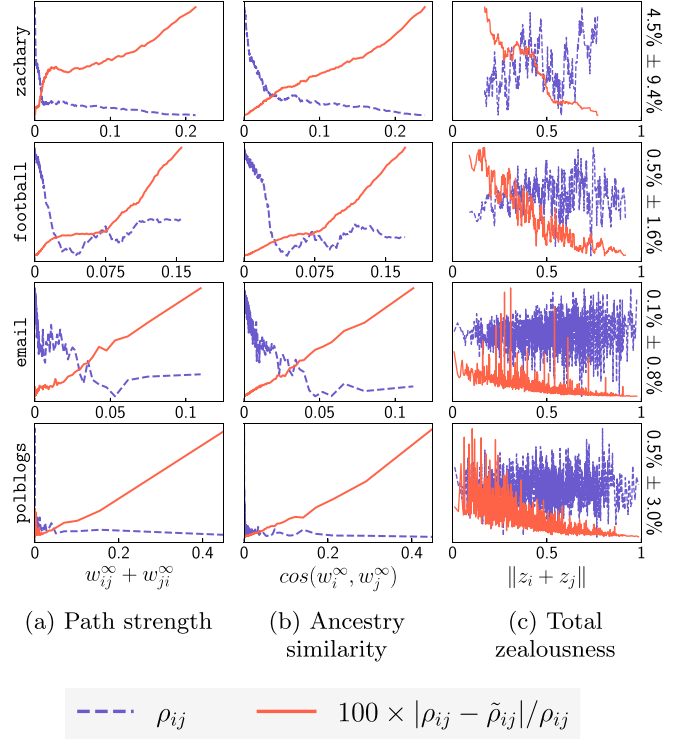$$\tilde{\rho}_{ij} = \sum_{s \in \mathcal{S}} x_i^s (1 - x_j^s), \tag{23}$$

FIG. 2. Moving average for ρ_{ij} (dotted blue line) and error percentage of $\tilde{\rho}_{ij}$ (orange line) for all dependent agent pairs, function of (a) path strength, (b) ancestry similarity, and (c) total zealousness. All curves differ in their range of values, which we don't make precise on the y axes for the sake of clarity. On the right are shown average errors and standard deviation for the distribution of errors.

as given by Eq. (10), even if σ_i and σ_j are not independent. While sometimes effective, this approximation does not always fare well; see Fig. 1. As the dependency of i's and j's opinions relies on the strength of paths joining them and the similarity of their ancestry, we expect ρ_{ij} to decrease with these and the error made by $\tilde{\rho}_{ij}$ to increase. We verify this at equilibrium on four data sets: zachary [30], football [31], email [32], and polblogs [33]. More details on the data and the preprocessing are available in the Supplemental Material [25, Sec. III]. To quantify path strength and ancestry similarity we use, respectively, $w_{ij}^{\infty} + w_{ji}^{\infty}$ and $\cos(w_i^{\infty}, w_j^{\infty})$. We also look at the effect of the total zealousness $\|z_i + z_j\|$ of i and j.

Results are shown in Fig. 2 and confirm our hypotheses: ρ_{ij} decreases with the strength of paths joining (i, j) and the similarity of their ancestry, while the error made by $\tilde{\rho}_{ij}$ increases. Moreover, the error decreases with the total zealousness. This is not surprising, as higher values mean lighter weights on interagent edges and thus less influence from peers. The errors are quite low on average, but can peak very high for certain agent pairs (maximum error ranges from 15% for football to 187% for polblogs). The least accurate $\tilde{\rho}_{ij}$ is on zachary, which could be because of the higher path strengths due to the graph's smaller size.

B. Discord and clustering

We investigate the relationship between discord and clustering. To do so, we compute the generalized active links

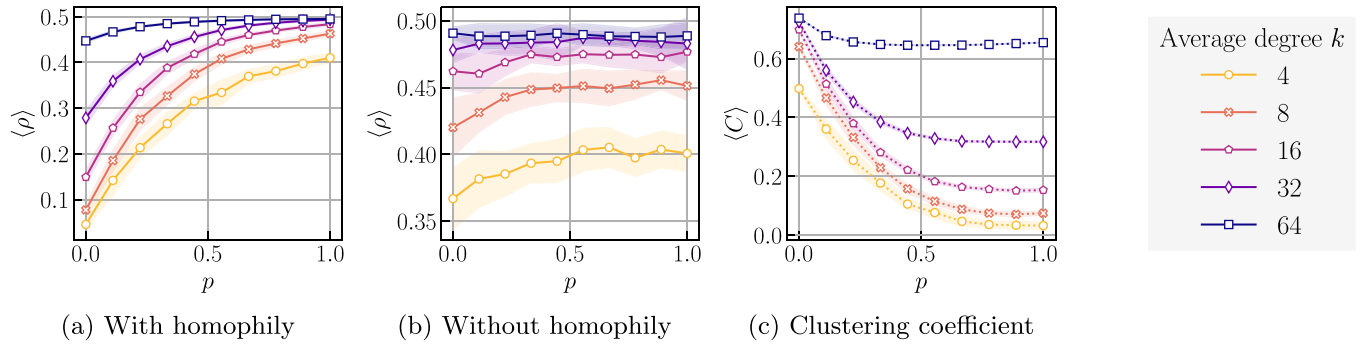(a) With homophily　　　　(b) Without homophily　　　　(c) Clustering coefficient

FIG. 3. Impact of clustering coefficient on $\langle \rho \rangle$. For each set of parameters (k, p) we generate $m = 30$ Watts-Strogatz networks of size $N = 100$, average degree k, and rewiring probability p. We average $\langle \rho \rangle$ over all realizations. Shaded areas cover ± 1 standard deviation. Half of the agents support opinion 0 ($z^{(0)} > 0$), the other half opinion 1 ($z^{(1)} > 0$). (a) With homophily: the majority of edges are between agents supporting the same opinion. (b) Without homophily: no correlation between edges and opinion support. (c) Average clustering coefficient (independent from homophily). Note the different scales on the y axes.

density $\langle \rho \rangle$ on a collection of Watts-Strogatz networks [34], with various average degrees and rewiring probabilities. There are $N = 100$ agents arranged in a circle and two possible opinions $\{0, 1\}$. Each node is connected to its k nearest neighbors, and each edge is rewired at random with probability p. As p increases, the average local clustering coefficient $\langle C \rangle$—defined at node i as the proportion of edges that exist among neighbors of i—decreases. Thus, there are fewer and fewer triangles in the graph.

We assume that half of the agents "support" opinion 0: they have $z^{(0)} > 0$ with a value chosen uniformly at random uniform in $[0,1]$, and $z^{(1)} = 0$. The other half analogously supports opinion 1. We consider two cases:

With homophily: Agents 1 to 50 support opinion 0, and agents 51 to 100 support opinion 1. Thus, most edges connect similar-minded agents.

Without homophily: Supporters of each opinion are placed at random in the graph. Hence, we expect about 50% of edges to connect similar-minded agents, and 50% to connect opposite-minded ones.

Results are presented in Fig. 3. We also plot the evolution of the average local clustering coefficient $\langle C \rangle$. As expected, $\langle C \rangle$ decreases with p for all values of the average degree k. We also observe that $\langle \rho \rangle$ always increases with k. This might be due to the fact that, as degrees increase, each neighbor of a node i is less often copied by i, meaning fewer opportunities for them to agree.

With homophily, discord increases significantly with p. Agents are initially mostly connected with peers who share the same beliefs, but as we rewire more and more edges, two things happen. First, connections with opposite-minded agents increase as those with similar-minded ones decrease. Second, the clustering coefficient, and thus the number of triangles, falls. These two effects consequently increase exposure to a higher diversity of opinions, which entails higher discord overall.

Without homophily, $\langle \rho \rangle$ also increases with p, but much less so. Thus, when there is no homophily in connections, the amount of triangles in the network has a smaller impact on discord. As the community memberships are initially random, rewiring edges does not have much impact on the diversity of opinions that agents are exposed to. We also observe a higher

variance in $\langle \rho \rangle$, which might be due to the higher entropy in the topological distribution of $(z^{(0)}, z^{(1)})$. Finally note that, as p approaches 1, the difference between Figs. 3(a) and 3(b) vanishes.

C. Discord and communities

An immediate application of discord lies in the analysis of polarized networks. If two groups support different ideas, how fiercely do they disagree? To what extent does it depend on the connections between them, and on the influence of zealots in each camp? To study these questions we generate networks partitioned into two communities $\mathcal{C}_0$ and $\mathcal{C}_1$ via the Stochastic Block Model, and calculate $\langle \rho \rangle$ at equilibrium for various values of the model parameters. The rest of this section is dedicated to the discussion of the results, illustrated in Fig. 4.

When reinforcing connections between different-minded communities, the novel paths of influence create more diverse information flows, and thus a higher exposition to contradicting beliefs. There are two main consequences one could expect from this. Agents may revise their viewpoint to incorporate adverse ideas, leading the system towards a more consensual state of lower discord. Alternatively, they may fiercely cling onto their preexisting opinions, thus creating more tension and reinforcing discord—the so-called *backfire effect* [35].

When community $\mathcal{C}_0$ is much less zealous than $\mathcal{C}_1$, for $z = (0.1, 0.5)$ and $z = (0.1, 0.9)$, outgroup edges can surprisingly reduce the discord within $\mathcal{C}_0$. While adding too few of them will introduce more discord in the community, once they reach a critical mass we observe a decrease in $\langle \rho \rangle$. This stems from the fact that $\mathcal{C}_1$ is more zealous and thus has fiercer partisans, meaning opinion 1 gets more and more prevalent in the network as connectivity between the two communities goes up; see the support for opinion 0 dropping in the bottom plots. Thus, even within $\mathcal{C}_0$, holding opinion 1 guarantees more agreement with peers.

It is striking that contrary to $\langle \rho \rangle$, the average difference in opinion $\langle \Delta x \rangle$ [Figs. 4(b) and 4(c)] always decreases within communities, given enough outgroup edges. The opinion difference between i and j is defined by the distance $\Delta x_{ij} = \|x_i - x_j\|$. Hence, despite a surge of discord amongst
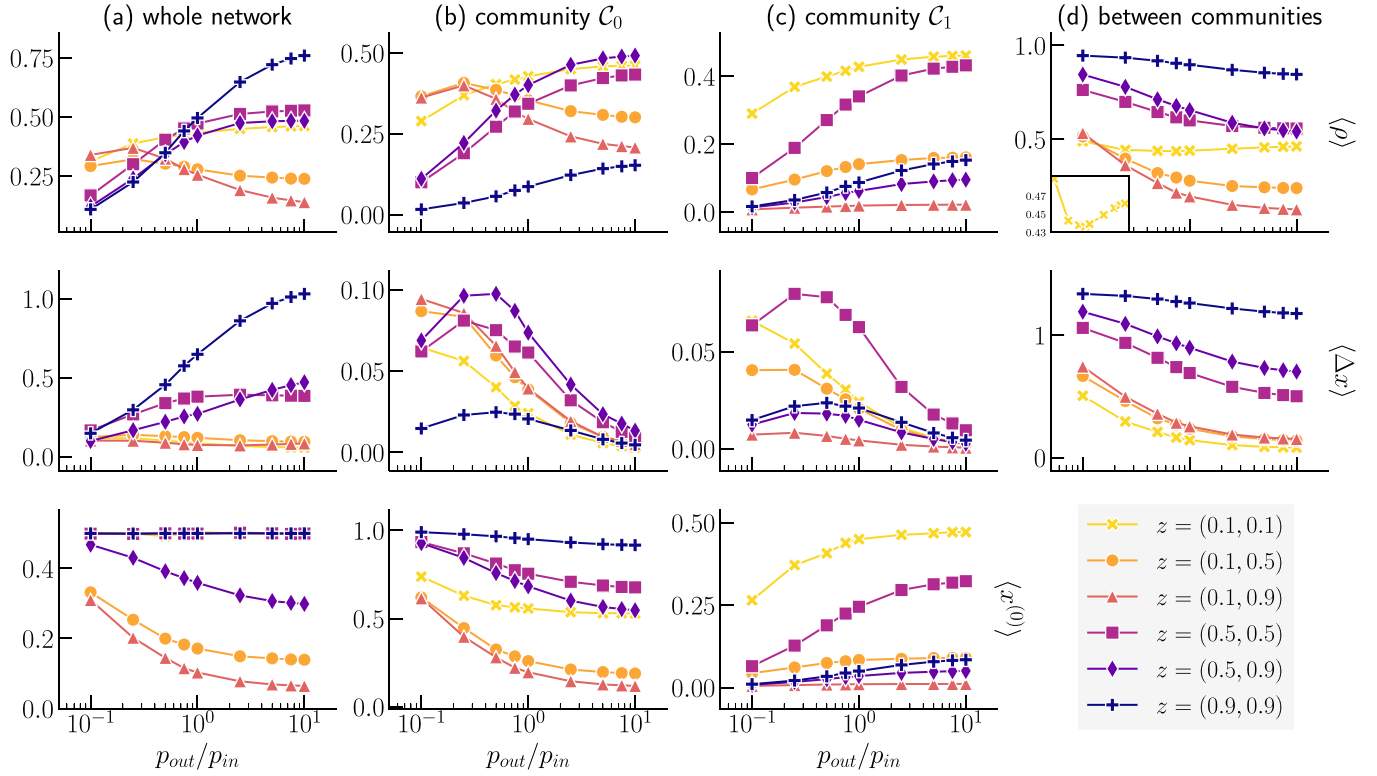
FIG. 4. Network of $N = 100$ agents with two communities $\mathcal{C}_0$ and $\mathcal{C}_1$. The s-zealot exerts a total influence z^s on each agent in $\mathcal{C}_s$ and none on others. In-group edge probability is fixed at $p_{in} = 0.1$, while the out-group edge probability p_{out} and zealousness $z = (z^0, z^1)$ vary. Results are averaged over 20 agent graphs generated under the Stochastic Block Model, for the whole network (a), within each community (b, c), and between them (d). Top: Generalized active links density; middle: opinion difference; bottom: support for opinion 0. The plots have different scales for clarity, but the purpose is to focus on the qualitative dynamics rather than exact values.

like-minded agents, the distributions of their equilibrium opinions converge. This might be due to the fact that as edges are added, the network gets closer to a complete one. Individual node particularities thus fade as agents become more similar to one another. But as the distributions of opinions approach the uniform distribution, the probabilities of drawing two different values increase. This highlights the importance of distinguishing between dissimilarity of opinion distributions and discord probabilities.

Between the communities, we observe that discord always diminishes with more outgroup edges. However, as shown in the inset [Fig. 4(d)], for equally low values of zealousness $z = (0.1, 0.1)$, first $\langle \rho \rangle$ decreases, but it quickly goes back up as more outgroup edges are added. Thus, when agents are not too zealous, relations with opposite-minded others become more confrontational as connections between them increase. For high values of z, discord does decrease with the number of outgroup edges, but stays important compared with low-zealousness cases.

Finally, Fig. 4(a) showcases the importance of using other measures than simply average opinions, as also stressed by [11]. When the zealousness is the same on both sides, the average opinion $\langle x^{(0)} \rangle$ over the whole network does not change with more edges, but we observe various behaviors in the evolution of $\langle \rho \rangle$ and $\langle \Delta x \rangle$.

VI. CONCLUSION

We developed a method to compute discord probabilities in the multistate voter model on complex networks, with and without zealots. Our results do not rely on any approximation; they are exact and applicable to any directed, weighted network of agents with any finite number of opinions and individual update rates. This led us to propose a generalized definition of the order parameter $\langle \rho \rangle$ traditionally known as active links density, that quantifies the average discord over all edges. Our definition accounts for long-range, weighted interactions between agents.

We illustrated our findings on real-life and synthetic networks. We uncovered correlations between discord, edge weights, and similarity of ancestors. Through numerical experiments on Watts-Strogatz networks, we observed that clustering had a significant impact on discord in the presence of homophily, but less so when that feature was absent. We also found that increasing the number of connections between antagonistic communities could potentially increase discord, depending on the zealousness of agents and density of edges. To gain a deeper understanding of how discord evolves in diverse scenarios, future research shall experiment with various network topologies. Additionally, the theoretical advancements presented here could assist empirical studies examining opinion dynamics in real-world data sets.

ACKNOWLEDGMENTS

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