Intermittent Kac's flights and the generalized telegrapher's equation

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A generalized one-dimensional telegrapher equation associated with an intermittent change of sign in the velocity of a Kac's flight has been proposed. To solve this random differential equation, we used the enlarged master equation approach to obtain the exact differential equation for the evolution of a normalized positive distribution. This distribution is associated with a generalized finite-velocity diffusionlike process. We studied the robustness of the ballistic regime, the cutoff of its domain, and the time-dependent Gaussian convergence. The second moment for the evolution of the profile has been studied as a function of non-Poisson statistics (possibly intermittent) for the time intervals Δ_{ij} in the Kac's flight. Numerical results for the evolution of sharp and wide initial profiles have also been presented. In addition, for comparison with a non-Gaussian process at all times, we have revisited the non-Markov Poisson's flight with exponential pulses. A theory for generalized random flights with intermittent stochastic velocity and in the presence of a force is also presented, and the stationary distribution for two classes of potential has been obtained.

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I. INTRODUCTION

One hundred years ago, the great mathematician Norbert Wiener presented a fundamental work that was the origin of stochastic calculus [1]. In this manner, *the* Wiener's process was the starting point to write the path integral representation for the general description of *the* Brownian motion, and so it was also the inspiration for Feynman's path integral in quantum mechanics [2,3]. We refer to Kac's work for a historical account of the Wiener integral [4] and its physical applications in diffusion processes [5].

A pioneer path integral approach, in the context of hyperbolic diffusion [6–11], was presented by Kac to introduce a simple representation of the solution of the telegrapher's equation (TE) [12]. In fact, Kac was able to find the solution of the TE as a path integral in terms of the Poisson process. That is, the hyperbolic diffusion in one dimension can be represented as the space motion produced by binary changes (sign) in the stochastic velocity. The key ingredient in this description was to characterize statistically the time intervals ($\Delta_{i,j} \equiv \{t_i, t_j\}$) when the velocity is constant in the random flight. Therefore, Kac proved that when the time intervals $\Delta_{i,j}$ are subordinated to the Poisson statistics, the transport turns out to be characterized by the Cattaneo-Fick hyperbolic diffusion. A generalization of Kac's idea has also recently been presented in [11]. An important problem related to Kac's approach is the study of a finite-velocity diffusion in the presence of a force (generalized overdamped Smoluchowski approach). This problem was recently tackled by finding the stationary state [13,14]. The time-dependent solution in the presence of a force (overdamped limit) can be represented as a generalized Kac's path integral [15].

The telegrapher process has also recently been generalized with the aim of considering the motion of particles subject to collisions that produce direction changes [16]. Also the telegraph process has been wrapped onto a non-Euclidean space (a circle) [17], and for studying the coalescence phenomenon, a family of interacting particles, each one governed by a generalized integrated telegraph process, has been proposed [18].

A Poisson-Kac process with intermittent stochastic velocity (noise) has several real-world applications. One example is in the modeling of biological systems, such as the spread of infectious diseases and intermittent locomotion [19]. The Poisson-Kac process is also used to describe the evolution of the disease over time, while the intermittent noise represents random events, such as outbreaks or fluctuations in disease transmission. Another application is in the modeling of intermittent financial systems [20]. The Poisson-Kac process with intermittent stochastic velocity can be used to analyze the behavior of asset prices and capture unpredictable events in markets, such as bursts of high volatility or abrupt changes in trends. Additionally, this type of process is used in solid-state physics to describe phenomena like the transport of charged particles in disordered media, where the intermittent noise represents interaction with random fluctuations in the medium [21,22].

In biology, hyperbolic diffusion has been used to study the propagation of bacteria [11]. In fact, the common E. Coli mimics the run-and-tumble motion, i.e., the transport is composed of an alternating mixing of runs (the agent

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propels itself) and tumbles (changing at random the direction of propagation) [23–26]. The so-called run-and-tumble process, using Levy jump statistics, allows to include the intermittence phenomenon, which has been detected in motion patterns of biological species [27–29]. Transport analysis in terms of different time-tumble statistics is a fundamental problem to be studied (distribution of the running times). The present work can be framed in a run-and-tumble one-dimensional (1D) picture, where the particle changes the velocity sign with non-Poisson *time* statistics allowing intermittence phenomena.

Here we have analyzed a random flight in terms of different statistics for the random time intervals Δ_{ij} characterizing the velocity changes (sign). Exact analytical result is presented. As a by-product, we study how robust the ballistic motion is in terms of the statistic of time intervals Δ_{ij}. In addition, a theoretical approach concerning a particle in the presence of a force and driven by intermittent noise has also been presented. Then, the exact stationary distribution has been found for different types of potential.

The organization of this paper is as follows: In Sec. II we have revisited the random flight model. In the case when the stochastic velocity is a Gaussian white noise, the Wiener process is recovered. If the velocity is a Markovian binary noise (dichotomic process), a TE is recovered for the evolution of the probability distribution function (PDF). While if the velocity is a general binary non-Markovian noise (possibly intermittent), the evolution of the PDF is a partial differential equation of fourth order. All these conclusions are obtained in an exact manner by solving a random differential equation with Markovian coefficients (the enlarged master equation approach [22,30,31]). In addition, and for comparison, we have revisited Poisson's flight to show a non-Gaussian diffusionlike process having a ballistic regime at short times. In Sec. III we have solved the second moment for the evolution of the PDF when the statistic of time intervals Δ_{ij} is non-Poisson, and also for the non-Markovian Poisson's flights (all-time non-Gaussian process). In Sec. IV we have shown numerical results for the evolution of the PDF, of our generalized TE, for different initial conditions, and we have studied its time-dependent Gaussian convergence. In Sec. V a general formulation for a random flight in the presence of a force is presented, and the stationary state is found for stable and unstable potentials. Finally, conclusions are introduced in Sec. VI. Appendixes are dedicated to mathematical details concerning Poisson's flight, the enlarged master equation approach, the intermittent binary noise, the Green-Kubo theorem applied to our random flight process, and Gaussian invariants.

II. RANDOM FLIGHTS

A. The functional approach

The diffusion process can be stated in terms of the fundamental Wiener's differential dW, which is written in terms of a Gaussian white noise $\epsilon(t)$ in the form $dW = \epsilon(t)dt$. This relation can also be written as a stochastic differential equation (SDE) $dW/dt = \epsilon(t)$. This SDE can also be interpreted as a random flight. Therefore, a generalized diffusion process can be expressed in terms of the SDE [2,21,22]:

$$\frac{dx}{dt} = \xi(t),\ t \in (0, \infty),\ x \in (-\infty, +\infty);\ x(0) = x_0, \quad (1)$$

where $x(t)$ is the position of the particle and $\xi(t)$ is a generic stochastic velocity characterized by its functional $G_\xi([k(\bullet)])$; see Appendix A. Therefore, the complete Kolmogorov hierarchy for the set of $n-$ times PDF of the process $x(t)$ can be written by Fourier inversion of the functional [22]. In particular, when the characteristic functional of the noise is Gaussian, $G_\xi([k]) = \exp \frac{-1}{2} \int_0^\infty k(t)^2 dt$, the complete hierarchy of $n-$ times PDF of the process $x(t)$ can be obtained and corresponds to the Wiener process (see Appendix A 1). In the limit $t_n - t_{n-1} \to 0$ the path integral representation is recovered [2,3]. Many noises have well-known functionals; therefore, all the information for a generalized diffusion process can immediately be inferred using the functional approach; for example, the Levy flight [32], the Poisson flight [33], etc. [34]. We note that this theory is very suitable when the generic noise $\xi(t)$ is non-Markovian and the functional $G_\xi([k])$ is known; see Appendix A for a non-Markovian Poisson's flight example. We remark that solving a SDE with generic non-Markovian noise demands an infinite Terwiel series expansion (or functional series approach) for the calculation of the PDF [15].

B. The master equation approach

In the particular situation when the noise $\xi(t)$ in (1) is Markovian, it is possible to work out the mean value of the random differential equation associated with its distribution using the *enlarged master equation* approach (EME) [22,30,31]. In the present paper, we will use this technique to work out the evolution equation of the $1-$ time PDF—in an exact way—for a binary stochastic velocity [the noise $\xi(t)$ in 1]. Note that the "dichotomic" noise does not have a closed expression for its functional [34]; putting a dichotomic noise in (1) corresponds to Kac's flight with Poisson time increments Δ_{ij}; see the next section.

We will also apply the EME approach in Sec. II D to work out an intermittent (non-Markovian) binary stochastic noise in (1), characterized by a biexponential waiting-time for the time increments Δ_{ij}. In Refs. [31,35,36] we have shown that a binary noise with a biexponential renewal waiting-time can be written as a Markov process using an embedded fourth-state process.

C. Kac's flights

Kac solved the problem (1) in a very elegant way [12] considering that the particle moves with random velocity $\pm v$. However, an elementary collision could change the direction of the velocity while keeping the module constant. In other words, the stochastic velocity can be written as $\xi(t) = (-1)^{N(t)} v$, $\forall t \geqslant 0$, where $N(t)$ is a Poisson process. Then, the TE was obtained for the PDF of the variable x.

In this section, we tackle this problem using the EME technique for a Markov binary noise $\xi(t)$ [22,30,31]. By defining velocity-conditional distributions $P_\pm(x,t)$ and using (1), we can write the EME (see Appendix B),

$$\partial_t P_+(x,t) = -\partial_x v P_+(x,t) - aP_+(x,t) + aP_-(x,t), \quad (2)$$

$$\partial_t P_-(x,t) = +\partial_x v P_-(x,t) + aP_+(x,t) - aP_-(x,t). \quad (3)$$

Here we have used the fact that the binary noise $\xi(t)$ has an exponential stationary correlation function: $\langle\xi(t_1)\xi(t_2)\rangle = \exp[-|t_1-t_2|/\tau]$ and $a = 1/2\tau$. Adding and subtracting these equations, we get

$$\partial_t P(x,t) = -v\partial_x Q(x,t), \quad (4)$$

$$\partial_t Q(x,t) = -v\partial_x P(x,t) - Q(x,t)/\tau, \quad (5)$$

where $P(x,t) \equiv P_+(x,t) + P_-(x,t)$ is the PDF [corresponding to averaging over all realizations of the noise $\xi(t)$; see (B5)], and $Q(x,t) = P_+(x,t) - P_-(x,t)$ is proportional to a current. So, by combining (4) and (5), it is simple to obtain the TE for $P(x,t)$:

$$\left[\partial_t^2 + \frac{1}{\tau}\partial_t - v^2\partial_x^2\right]P(x,t) = 0, \ \tau^{-1} = 2a, \quad (6)$$

where τ^{-1} is the rate of energy loss and v is the finite-velocity propagation in a diffusionlike process. This equation can be solved with initial conditions (ICs) $P(x,t)|_{t=0}$ and $\partial_t P(x,t)|_{t=0}$ consistent with the initial conditions for $P_\pm(x,t=0)$. We note that this is the evolution equation obtained by Kac using Poisson time increments Δ_{ij} for the random changes in the velocity sign. The novelty of our work will be to use the EME approach to tackle a non-Markovian binary noise $\xi(t)$ (possibly intermittent) in (1).

D. The generalized Kac's flights

Consider now a non-Markovian (biexponential) binary stochastic velocity (noise) in (1), i.e., $v\xi(t) = \pm v, \forall t \geqslant 0$ (where v is the amplitude of the noise). Therefore, we characterize the time intervals Δ_{ij} with a biexponential waiting-time function [31,35,36]:

$$\varphi(\Delta_{ij}) = \alpha p e^{-\alpha\Delta_{ij}} + \beta q e^{-\beta\Delta_{ij}}, \quad (7)$$

where $p+q=1$ and the parameters fulfill $\alpha, \beta, p, q \geqslant 0$ (with probability p the rate is fixed to be α, while with probability q the rate is β). In this manner, keeping track of fourth-states, $\{\xi^\pm_{\alpha,\beta}\}$, the embedded process turns out to be Markovian [35]; see Appendix C.

We note that the key ingredient in Kac's approach is a Poisson statistics for the time intervals when the velocity is constant in the random flight. This corresponds to an exponential waiting-time function which is recovered from (7) in the case $\alpha = \beta$.

The current model has the advantage that when $p \gg q$ and $\alpha \gg \beta$, the statistics for time intervals Δ_{ij} will be intermittent, while in the limit $\alpha \to \beta$, the stochastic process $\xi(t)$ goes to the Markovian binary noise (dichotomic process—associated with a Poisson statistics).

Ordering the set of states as $\{\xi^+_\alpha, \xi^+_\beta, \xi^-_\alpha, \xi^-_\beta\}$—our fourth state *Markov* (biexponential) binary noise—we can introduce auxiliary conditional-velocity functions: $\{P^\pm_{\alpha,\beta}\}$. Then using the notation $\tilde{\mathbf{P}} = \{P^+_\alpha, P^+_\beta, P^-_\alpha, P^-_\beta\}$, we can write the EME, see Appendix B, as

$$\partial_t\tilde{\mathbf{P}} = \mathbf{A}\cdot\tilde{\mathbf{P}} + \mathbf{H}\cdot\tilde{\mathbf{P}}, \quad (8)$$

with

$$\mathbf{A} = \begin{pmatrix} -v\partial_x & 0 & 0 & 0 \\ 0 & -v\partial_x & 0 & 0 \\ 0 & 0 & v\partial_x & 0 \\ 0 & 0 & 0 & v\partial_x \end{pmatrix}, \quad (9)$$

$$\mathbf{H} = \begin{pmatrix} -\alpha & 0 & \alpha p & \beta p \\ 0 & -\beta & \alpha q & \beta q \\ \alpha p & \beta p & -\alpha & 0 \\ \alpha q & \beta q & 0 & -\beta \end{pmatrix}, \quad (10)$$

where $\mathbf{H}$ is the master Hamiltonian for the biexponential binary noise [note that now we need four levels to characterize a "Markovian" binary noise $\xi(t)$] [31,35]. We remark that the name "master Hamiltonian" has nothing to do with the classic Hamiltonian matrix. The matrix $\mathbf{H}$ is written in terms of the transition probability matrix, and diagonal terms appear due to the normalization constraint on the probability at all times.

Introducing the basis $P = P^+_\alpha + P^+_\beta + P^-_\alpha + P^-_\beta$, $\mathcal{Q} = P^+_\alpha + P^+_\beta - P^-_\alpha - P^-_\beta$, $\mathcal{M} = P^+_\alpha - P^+_\beta - P^-_\alpha + P^-_\beta$, $\mathcal{N} = P^+_\alpha - P^+_\beta + P^-_\alpha - P^-_\beta$, and after a little bit of algebra, it is possible to rewrite (8) in the form

$$\partial_t P = -v\partial_x\mathcal{Q}, \quad (11)$$

$$\partial_t\mathcal{Q} = -v\partial_x P + (\beta-\alpha)\mathcal{M} - (\alpha+\beta)\mathcal{Q}, \quad (12)$$

$$\partial_t\mathcal{M} = -v\partial_x\mathcal{N} - (p\alpha+q\beta)\mathcal{M} - (p\alpha-q\beta)\mathcal{Q}, \quad (13)$$

$$\partial_t\mathcal{N} = -v\partial_x\mathcal{M} - (q\alpha+p\beta)\mathcal{N} - (q\alpha-p\beta)P. \quad (14)$$

These equations must be solved while providing ICs for the auxiliary functions,

$$P(x,t)|_{t=0},\ \mathcal{Q}(x,t)|_{t=0},\ \mathcal{M}(x,t)|_{t=0},\ \mathcal{N}(x,t)|_{t=0}, \quad (15)$$

consistent with the IC for $P^\pm_{\alpha,\beta}(x,t)|_{t=0}$.

Solving Eqs. (11)–(14) for P we can obtain a differential equation for the evolution of the PDF $P(x,t)$:

$$\begin{aligned} 0 &= \partial_t^4 P + f_{0,3}\partial_t^3 P + f_{0,2}\partial_t^2 P + f_{0,1}\partial_t P \\ &\quad + f_{2,0}\partial_x^2 P + f_{4,0}\partial_x^4 P + f_{2,2}\partial_x^2\partial_t^2 P + f_{2,1}\partial_x^2\partial_t P. \end{aligned} \quad (16)$$

This is a fourth-order partial derivative equation (in time and space), where the coefficients are

$$\begin{aligned} f_{0,3} &= 2(\alpha+\beta), \\ f_{0,2} &= 2(\alpha+\beta) + (\beta^2 p + \alpha^2 q + \alpha\beta p^2 + \alpha\beta q^2 + \alpha^2 pq \\ &\quad + \beta^2 pq + 3\alpha\beta), \\ f_{0,1} &= 2(\alpha\beta^2 p^2 + \alpha^2\beta q^2 + \alpha^2\beta pq + \alpha\beta^2 pq), \\ f_{2,0} &= v^2(\alpha\beta - \alpha\beta p^2 - \alpha\beta q^2 - \alpha^2 pq \\ &\quad - \beta^2 pq - \beta^2 p - \alpha^2 q), \\ f_{4,0} &= v^4, \\ f_{2,2} &= -2v^2, \\ f_{2,1} &= -(\alpha p v^2 + \beta p v^2 + \alpha q v^2 + \beta q v^2 + \alpha v^2 + \beta v^2). \end{aligned} \quad (17)$$

By construction, the generalized TE (16) with coefficients (17) fulfilling the conditions $\{\alpha, \beta, v^2\} > 0, 1 \geqslant p \geqslant 0, q = 1 - p$ admits a normalized positive solution $P(x, t)$ at any time $t > 0$, if it is a PDF for the IC $P(x, t)|_{t=0}$.

In particular, we will be interested in the solution for an initially nonmoving symmetric pulse at the origin, $P(x, t)|_{t=0} = \delta(x)$ [see Sec. IV for a general $P(x, t)|_{t=0} = f(x)$]. Then, in the Fourier representation $\mathcal{F}[\cdots]$,

$$\hat{P}(k, t) \equiv \mathcal{F}[P(x, t)] = \int_{-\infty}^{+\infty} e^{ikx} P(x, t) dx,$$

we obtain the initial conditions

$$\begin{aligned}
\mathcal{F}[P(x,t)]|_{t=0} = 1,\ \ \partial_t \mathcal{F}[P(x,t)]|_{t=0} &= 0,\\
\partial_t \mathcal{F}\left[\partial_x^2 P(x,t)\right]\Big|_{t=0} &= 0,\\
\partial_t^2 \mathcal{F}[P(x,t)]\Big|_{t=0} &= -v^2 k^2,\\
\mathcal{F}\left[\partial_x^2 P(x,t)\right]\Big|_{t=0} &= -k^2,\\
\partial_t^3 \mathcal{F}[P(x,t)]\Big|_{t=0} &= (\alpha+\beta) v^2 k^2.
\end{aligned} \tag{18}$$

The solution for the PDF in the Laplace $\mathcal{L}[\cdots]$ and Fourier representation is denoted as

$$\hat{P}(k, s) = \mathcal{L}[\hat{P}(k, t)] = \int_0^{+\infty} e^{-st} \hat{P}(k, t) dt.$$

To simplify the notation, we use $g(s) \equiv \mathcal{L}[g(t)]$. Then, we write

$$\hat{P}(k, s) = \frac{N_1(k, s)}{D_1(k, s)}, \tag{19}$$

where

$$\begin{aligned}
N_1(k, s) = {} & v^2(\alpha+\beta+s)k^2 + (\beta p + \alpha q + s)(2\alpha\beta(p+q)\\
& + s(\alpha+\beta+\alpha p + \beta q) + s^2)
\end{aligned}$$

and

$$\begin{aligned}
D_1(k, s) = {} & v^4 k^4 + v^2(\alpha^2(p+1)q - 2\alpha\beta q p + \beta^2 p(q+1)\\
& + 2s(\alpha+\beta) + 2s^2)k^2 + s(\beta p + \alpha q + s)(2\alpha\beta\\
& + s(\alpha+\beta+\alpha p + \beta q) + s^2).
\end{aligned} \tag{20}$$

Alternatively, we can use the change of parameters $\alpha = 1/2\tau_1$ and $\beta = 1/2\tau_2$ to rewrite $\hat{P}(k, s)$ in terms of the timescales $\{\tau_1, \tau_2\}$.

1. The Poisson-Kac limit (ordinary TE)

For $\tau_1 = \tau_2 = \tau$ (or equivalently $\beta = \alpha \to 1/2\tau$), the ordinary TE is recovered. That is, in the Fourier-Laplace representation, the solution is obtained as

$$\hat{P}(k, s) = \frac{(s + 1/\tau)}{s(s+1/\tau) + v^2 k^2}. \tag{21}$$

This solution considers both short ($t \ll \tau$) and long ($t \gg \tau$) times for wave and diffusion behaviors. In the solution (19), both timescales (τ_1, τ_2) and probability weight p control the short and long time regimes. The transition between both regimes can be characterized by the dispersion of the PDF, or by studying time-dependent Gaussian invariants. These analyses will be presented in the next sections.

III. THE SECOND MOMENT $\langle x(t)^2\rangle$ USING DIFFERENT STOCHASTIC VELOCITIES $\xi(t)$

The Laplace representation of the second moment for the random flight (1) is calculated as

$$\langle x(s)^2\rangle = -\partial_k^2 \hat{P}(k, s)|_{k=0}, \tag{22}$$

therefore we can calculate the dispersion for the generalized diffusion process $x(t)$ with different models for the stochastic velocity $\xi(t)$.

A. Using Markovian binary noise $\xi(t)$

For the binary Markovian noise, from (21) and (22), using $x_0 = 0$ we get

$$\langle x(s)^2\rangle = \frac{2v^2}{s^2(s+1/\tau)}. \tag{23}$$

Taking the inverse Laplace transform of this equation, we obtain $\langle x(t)^2\rangle = 2\tau v^2(\tau e^{-t/\tau} + t - \tau)$. If $t \ll \tau$, we get $\langle x(t)^2\rangle \sim v^2 t^2$ (ballistic regimen), while in the opposite regime, i.e., $t \gg \tau$, we obtain $\langle x(t)^2\rangle \sim 2\tau v^2 t$ (diffusive regimen). These results correspond to the ordinary TE [8,12,22,31].

B. Using biexponential (non-Markovian) binary noise $\xi(t)$

For the non-Markovian binary noise, using (19) in (22) and taking $x_0 = 0$ we obtain (in terms of parameters τ_1, τ_2, p)

$$\langle x(s)^2\rangle = \frac{2v^2\{\tau_2^2[-p^2 + 2s\tau_1(2s\tau_1+1)+1] + 2\tau_2\tau_1[(p-1)p + s\tau_1] - (p-2)p\tau_1^2\}}{s^2[\tau_2(-p+2s\tau_1+1) + p\tau_1][(p+1)s\tau_2 + s\tau_1(-p+2s\tau_2+2)+1]}. \tag{24}$$

From the Tauberian theorem, if $t \to 0$, we get $\langle x(t)^2\rangle \sim v^2 t^2$ (ballistic behavior), while in the regime $t \to \infty$, we get a diffusive behavior,

$$\begin{aligned}
\langle x(t)^2\rangle &\sim 2v^2 \frac{\left[p(q+1)\tau_1^2 - 2\tau_2\tau_1 pq + (p+1)q\tau_2^2\right]}{(p\tau_1 + q\tau_2)} t && (25)\\
&= v^2 \frac{\left[2p - 4p\alpha\tau_w + (1+p)\alpha^2\tau_w^2\right]}{\alpha^2 q \tau_w} t, && (26)
\end{aligned}$$

where

$$\tau_w = \frac{\alpha}{p} + \frac{\beta}{q}$$

is the mean waiting-time τ_w for the biexponential function (7), which characterizes the sign changes in the stochastic velocity $v\xi(t) = \pm v, \forall t \geqslant 0$ in the SDE (1).

With these results, we conclude that the fourth-order partial differential equation (16) also includes the wave and diffusive regimes as in the TE. This generalized TE contains two timescales, τ_1 and τ_2, characterizing two rates for the absorption of energy. These timescales can lead to an intermittent regime depending on the parameters, as we have mentioned before. In the case $\tau_1 = \tau_2$, we recover the results from the ordinary TE.

C. Poisson's flight: $\dot{x}(t) = \xi(t)$

A Poisson flight with an exponential-shaped pulse is non-Markovian and non-Gaussian at all times. From the cumulants of the Poisson flight (A10) and (A11), we get (for any x_0)

$$\langle x(t)\rangle = x_0 - Q\omega_0\tau_c(1 - e^{-t/\tau_c}),\ \forall t \geqslant 0, \tag{27}$$

$$\langle x(t)^2\rangle - \langle x(t)\rangle^2 = 2!Q\tau_c\omega_0^2\, B(3, 0, 1 - e^{-t/\tau_c}),\ \forall t \geqslant 0. \tag{28}$$

In particular, the second moment follows immediately, using (A12) as

$$\langle x(t)^2\rangle = \left(x_0^2 + 2Q\omega_0^2 t\right) - 2Q\omega_0^2\tau_c\left(1 + \frac{x_0}{\omega_0}\right)y + Q\omega_0^2\tau_c(-1 + Q\tau_c)y^2,\ \forall t \geqslant 0, \tag{29}$$

with

$$y \equiv (1 - e^{-t/\tau_c}).$$

We will readily check $\langle x^2(t = 0)\rangle = x_0^2$ and also the limits,

$$\langle x^2(t \to 0)\rangle \sim Q\omega_0^2\tau_c\left[\left(1 + \frac{x_0}{\omega_0}\right) + (-1 + Q\tau_c)\right]\left(\frac{t}{\tau_c}\right)^2 + O(t^3), \tag{30}$$

that is, a ballistic behavior for short times, while in the long-time regime we obtain

$$\langle x^2(t \to \infty)\rangle \sim \left(x_0^2 + 2Q\omega_0^2 t\right) - 2Q\omega_0^2\tau_c\left(1 + \frac{x_0}{\omega_0}\right) + Q\omega_0^2\tau_c(-1 + Q\tau_c) + \cdots, \tag{31}$$

that is, a time linear behavior.

We note that the second moment of a Poisson flight has similar short (quadratic) and long (linear) time behaviors to those for the random flight (1) with binary noise. Nevertheless, we remark that a Poisson flight, in general, is never Gaussian because $K_n(t \to \infty) \sim t, \forall n \geqslant 2$ (all cumulants are non-null); see Appendix A for the definition of the moments M_n and cumulants K_n. So it is interesting to show here a non-Gaussian process, having the same short and long time asymptotic behaviors.

General remark

After Laplace inversion of (24) the corresponding second moments are shown in Fig. 1. This analysis shows a quadratic behavior for short time ($t \to 0$) and a linear behavior in the long-time limit ($t \to \infty$). We notice that $\langle x(t)^2\rangle$ (for all cases studied) has similar asymptotic behaviors. The most interesting case corresponds to the *intermittent* binary noise $\xi(t)$ limit ($p = 0.98$, $\tau_2 = 20$), where the ballistic regimen is delayed compared to the remaining cases. Additionally, we plot an arbitrary non-Markovian (nonintermittent) case

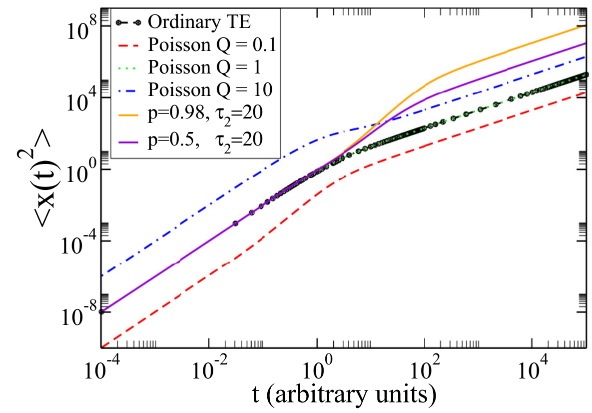

FIG. 1. Second moment $\langle x(t)^2\rangle$ of the PDF $P(x, t)$ as a function of time. From top to bottom (in log-log): Ordinary TE (black circles), Poisson noise: $Q = 0.1$ (dashed lines), $Q = 1$ (dotted lines), $Q = 10$ (dashed and lines), non-Markovian noise with $\tau_2 = 20$ and $p = 0.98$ (intermittent case), and $p = 0.5$ noises $\xi(t)$ in Eq. (1). The remaining parameters are set to unit.

($p = 0.5$, $\tau_2 = 20$) for contrast. As can be seen, the inclusion of generalized non-Poisson statistics is a crucial ingredient to control the crossover between the ballistic and diffusive regime in the transport process.

For the sake of completeness, we mention here that for any noise $\xi(t)$, using the second moment $\langle x(t)^2\rangle$ and the Green-Kubo formula, we can readily calculate (analytically) the stationary velocity autocorrelation function (VAF) of the random flight process (1), and the diffusion coefficient of the process; see Appendix D.

IV. NUMERICAL SOLUTION OF THE GENERALIZED TE (16)

To observe the transition from the ballistic to the diffusive regime, we present several results from the numerical solution of the evolution of the PDF for the generalized random flight (1). We have solved (11)–(14), which is equivalent to solving (16) using two different initial conditions for the PDF $f(x)$, with the consistent auxiliary initial conditions:

$$P(x, t = 0) = f(x) \geqslant 0,\ \mathcal{Q}(x, t = 0) = 0,\ \mathcal{M}(x, t = 0) = 0,\ \mathcal{N}(x, t = 0) = 0.$$

A. Initial conditions

First, we consider $f(x)$ as a narrow Gaussian at $t = 0$ (sharp IC), that is,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{32}$$

and we take $\sigma = 0.06$. In Fig. 2, we show $P(x, t)$ as a function of position x for some values of $t = 0, 2, 4, 6, 8, 10, 12, 14$ for the Markovian case with [$\beta = \alpha$ or $\tau_2 = \tau_1 = 1$ in (16)]. We observe that the ballistic regimen vanishes with increasing time. In Fig. 3, we compare these results with the intermittent case for the non-Markovian binary noise $\xi(t)$, i.e., $p = 0.98$, $\tau_2 = 20$ ($\alpha \gg \beta$ or $\tau_2 \gg \tau_1$), for $t = 0, 4, 8, 12, 16, 20, 24, 28, 32$ (the rest of parameters are

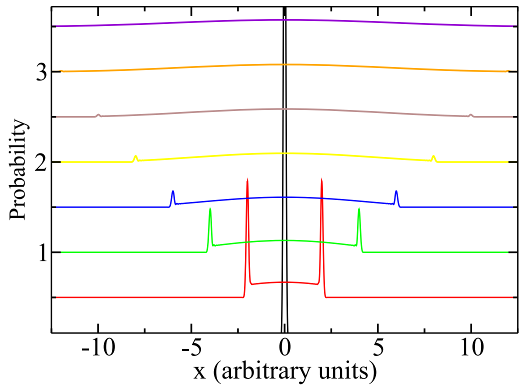

FIG. 2. Evolution of PDF $P(x,t)$ as a function of x for the ordinary TE, $t=0$ (black), 2 (red), 4 (green), 6 (blue), 8 (yellow), 10 (brown), 12 (orange), 14 (violet). The initial condition is sharp: $P(x,t)|_{t=0}=f(x)=e^{-\frac{x^2}{2\sigma^2}}/\sqrt{2\pi}\sigma$, with $\sigma=0.06$. Probabilities at different times have been shifted for better visualization. The shift between consecutive PDFs is 0.5 and uniform.

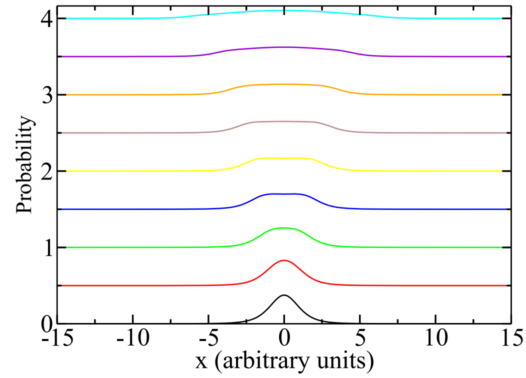

FIG. 4. Evolution of $P(x,t)$ as a function of x for the ordinary TE, for $t=0$ (black), 0.5 (red), 1 (green), 1.5 (blue), 2 (yellow), 2.5 (brown), 3 (orange), 4 (violet), 6 (celeste). The initial condition is wide: $P(x,t)|_{t=0}=f(x)=\nu^{\frac{\nu+1}{2}}(\nu+x^2)^{-\frac{\nu+1}{2}}/\sqrt{\nu}B(\frac{\nu}{2},\frac{1}{2})$, with $\nu=4$. Probabilities at different times have been shifted for better visualization. The shift between consecutive PDFs is 0.5 and uniform.

set to unit). We note that, for a process $x(t)$ with a narrow Gaussian IC, the evolution of the pulse has a longer ballistic regime than for the ordinary TE. Note that even after $t=30$ the evolution of the pulse is ballistic.

As a second IC, we consider a Student T-Distribution for $f(x)$ at $t=0$, i.e.,

$$f(x)=\frac{\left(\frac{\nu}{\nu+x^2}\right)^{\frac{\nu+1}{2}}}{\sqrt{\nu}B\left(\frac{\nu}{2},\frac{1}{2}\right)}. \tag{33}$$

This is a power-law distribution for the IC. We take the parameter $\nu=4$ with $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 dz z^{a-1}(1-z)^{b-1}$ as the Euler beta function [37]. In Fig. 4 we plot $P(x,t)$ as a function of x for a Markovian noise [$\beta=\alpha$ or $\tau_2=\tau_1=1$ in (16)] for $t=0, 0.5, 1, 1.5, 2, 2.5, 3, 4, 6$. The IC (33) is wide in $t=0$, and $P(x,t)$ hardly develops two peaks with increasing time (in comparison with the previous IC). Instead of developing two peaks, the profile is almost flattened, so the profile quickly reaches the diffusive regime. In contrast, in Fig. 5, for the non-Markovian (intermittent) binary noise with $p=0.98$, $\tau_2=20$ (the rest of parameters are set to unit), a wide initial pulse (33) develops two wide peaks moving in opposite directions, one with respect to the other. The peak of the right is moving with velocity v (telegrapher's velocity); in this case we do not observe any abrupt cutoff in the probability, as it is for the case of sharp IC (a similar behavior is obtained for the left peak).

Therefore, we reach the following conclusions: the cutoff is due to the sharp IC and can be softened with a wider IC. The parameters of the noise are crucial to preserve the ballistic regime, but the peak moves (with the velocity of the TE) independently of the noise parameters. Nevertheless, intermittent

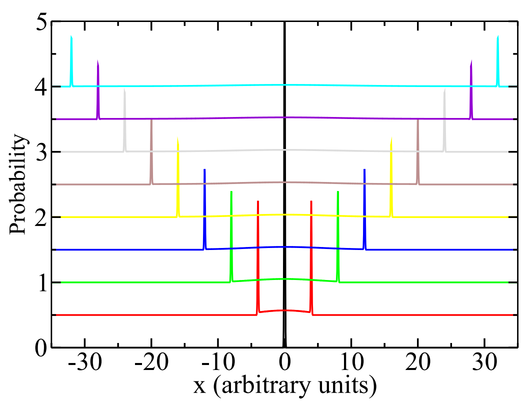

FIG. 3. PDF $P(x,t)$ as a function of x for the intermittent noise case with $p=0.98$, $\tau_2=20$, $\tau_1=1$ [$\alpha\gg\beta$ in (8)] for $t=0$ (black), 4 (red), 8 (green), 12 (blue), 16 (yellow), 20 (brown), 24 (gray), 28 (violet), 32 (celeste). The initial condition is sharp: $P(x,t)|_{t=0}=f(x)=e^{-\frac{x^2}{2\sigma^2}}/\sqrt{2\pi}\sigma$, with $\sigma=0.06$. Probabilities at different times have been shifted for better visualization. The shift between consecutive PDFs is 0.5 and uniform.

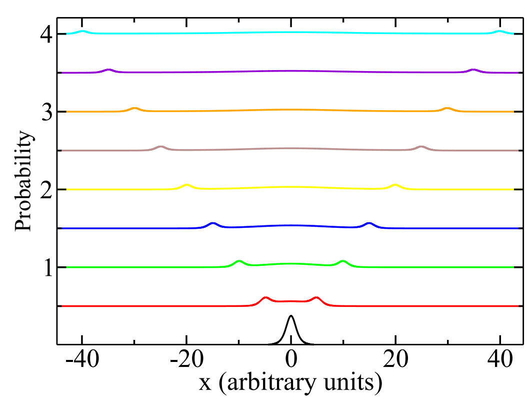

FIG. 5. $P(x,t)$ as a function of x for an intermittent noise with $p=0.98$, $\tau_2=20$, $\tau_1=1$ [$\alpha\gg\beta$ in (8)] for $t=0$ (black), 5 (red), 10 (green), 15 (blue), 20 (yellow), 25 (brown), 30 (orange), 35 (violet), 40 (celeste). The initial condition is $P(x,t)|_{t=0}=f(x)=\nu^{\frac{\nu+1}{2}}(\nu+x^2)^{-\frac{\nu+1}{2}}/\sqrt{\nu}B(\frac{\nu}{2},\frac{1}{2})$, with $\nu=4$. Probabilities at different times have been shifted for better visualization. The shift between consecutive PDFs is 0.5 and uniform.

binary stochastic velocity delays the ballistic regime in the time evolution.

B. Softness in the support of the PDF

It is known that the solution of the ordinary TE presents a cutoff in its support if the IC is a δ peak. This fact can readily be seen from the presence of the step function in its solution; that is, there is a time-dependent cutoff characterized by the presence of the multiplicative factor $[\Theta(x+vt)-\Theta(x-vt)]$ in the solution of the TE, if the IC is $P(x,t)|_{t=0}=\delta(x)$ and $\partial_t P(x,t)|_{t=0}=0$ [6,9].

When the IC is not sharp, the cutoff is not abrupt because there is *mass* in all the domain of the initial distribution [10]. Therefore, it is interesting to present a figure showing how this cutoff is softened by a wide IC. In addition to this issue, in the same figure, we are going to show that the cutoff is an intrinsic phenomenon due to the finite-velocity increments in Eq. (1) (the binary noise). Also, we will show that this cutoff only depends on the magnitude of the velocity v and it is independent of noise parameters: $\{\tau_1, \tau_2, p\}$.

We have studied half of the area of the PDF as a function of x and for different noise parameters. That is, we define the function

$$\mathcal{A}(x,t)=\int_0^x P(x',t)dx'. \tag{34}$$

In Fig. 6, we show $\mathcal{A}(x,t)$ as a function of x for different times for the ordinary TE and the generalized TE. We study this function considering two ICs for the initial pulse. Then, as expected, it can be seen that the plots corresponding to a sharp IC are always above the ones corresponding to a wide IC. Also, by using the wide IC (33), it can be seen that the cutoff is softened (before and after $x=vt$), while if the IC is sharp (32), the cutoff is abrupt and occurs at the time-dependent position: $x=vt$, even for an intermittent stochastic velocity in the random flight.

C. Gaussian convergence of the PDF

It is known that the solution of the ordinary TE (6) converges to the Gaussian distribution as time goes on. The same happens with the solution of the generalized TE (16). Here, the point that we want to emphasize is to show how this convergence happens, and the way to control this time-dependent convergence by changing the noise parameters $\{\tau_1, \tau_2, p\}$. To show this issue, we calculate some Gaussian invariants like the Kurtosis for each particular case using Markovian, non-Markovian, and intermittent noises $\xi(t)$ in the random flight (1). If the process were Gaussian, the Kurtosis would be $\mathcal{K}=M_4/M_2^2=3$; see Appendix E. For a generic distribution, another important measure that uses higher cumulants is the invariant $\mathcal{H}$, which we define as

$$\mathcal{H}=\frac{M_6}{M_2^3}=15+15C_4+C_6. \tag{35}$$

Here, C_j is calculated in terms of cumulants K_j and moments M_j as

$$C_4=K_4/M_2^2,\ C_6=K_6/M_2^3. \tag{36}$$

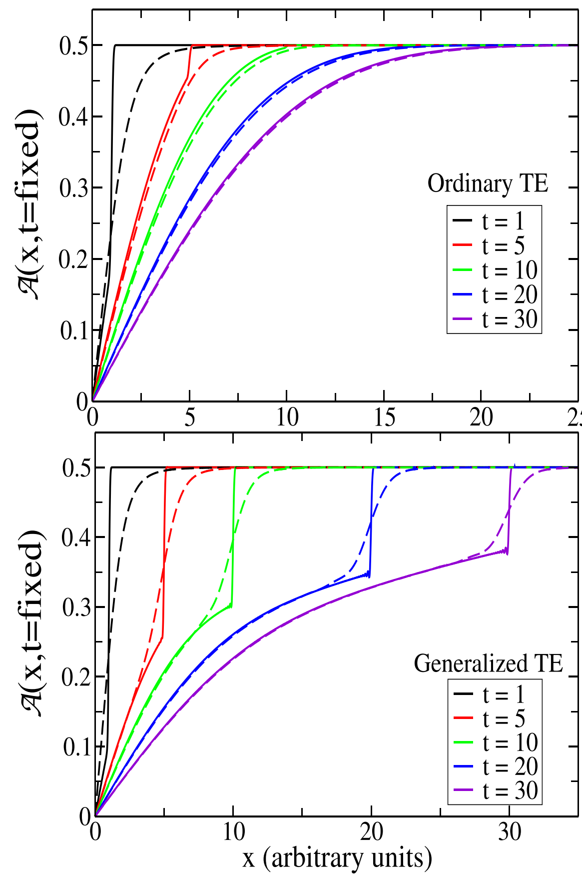

FIG. 6. Half of the area of the PDF $P(x,t)$. The measure $\mathcal{A}(x,t)$ as a function of space x for times $t=1,5,10,20,30$. Top figure: Markovian noise $\xi(t)$ (with $\tau=1$), and bottom figure: non-Markovian noise $\xi(t)$ (with $p=0.98,\ \tau_2=20$). Dashed lines for a wide T-student IC, and continuous lines for a sharp IC. In the Markovian case—at short times $t=1,5$—the cutoff can clearly be seen, while for later times it is harder to see because the peak disappears in the time-evolution of the PDF. In the non-Markovian case (intermittent) the cutoff can readily be seen at all times. The remaining parameters are set to the unit.

Therefore, from (35) we see that for a Gaussian distribution, $\mathcal{H}=15$.

For a generic symmetric distribution, all cumulants adopt a very simple expression [22]:

$$K_4=M_4-3M_2^2,\ K_6=M_6-15M_4M_2+30M_2^3. \tag{37}$$

Then, using the solution (19) of our generalized TE (16), we can calculate all moments $M_j\equiv\langle x(s)^j\rangle$, and therefore after Laplace inversion, the invariants $\mathcal{H}$ and $\mathcal{K}$ as a function of time and for different values of noise parameters are $\{\tau_1=1/2\alpha,\ \tau_2=1/2\beta,\ p=1-q\}$.

In Fig. 7, we show $\mathcal{H}$ and $\mathcal{K}$ as functions of time for several values of $\{\tau_1,\tau_2,p\}$ for the generalized TE. These functions exhibit nonmonotonic behavior, showing a maximum departure from the Gaussian invariant values. The invariants $\mathcal{H}(t)$ and $\mathcal{K}(t)$ converge from above to the Gaussian values indicating that the solution of the generalized TE is different from that of the ordinary TE. In the long-time regime, the convergence to the Gaussian values follows a power-law: $\mathcal{H}(t)\sim$

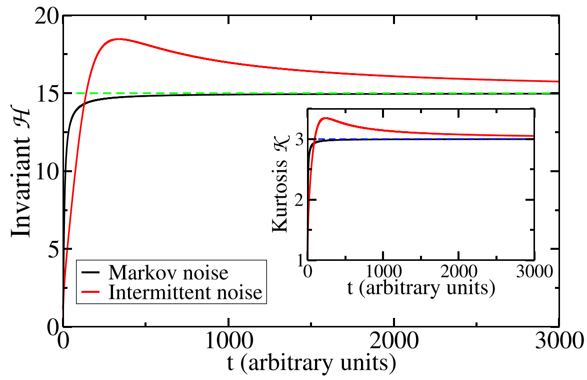

FIG. 7. Invariant $\mathcal{H}(t)$ and Kurtosis $\mathcal{K}(t)$ (in the inset) both as a function of time and for two models of noises $\xi(t)$ in Eq. (1): Markovian ($\tau = 1$) and non-Markovian ($p = 0.98$, $\tau_2 = 20$) in the intermittent case. The rest of the parameters are set to the unit. In both figures, the IC is sharp (32). In both figures, plots below the constant values 15 and 3 correspond to the convergence from the solution of the ordinary TE (black lines).

$15 + \tau_H/t$ and $\mathcal{K}(t) \sim 3 + \tau_K/t$. While these functions have similar long-time behaviors, $\mathcal{H}$ magnifies the non-Gaussian relaxation. In Appendix E, we explicitly give the timescale τ_H as a function of the noise parameters.

V. INTERMITTENT HYPERBOLIC DIFFUSION IN THE PRESENCE OF A FORCE

Using Terwiel's diagrams, the evolution equation for the PDF for a random flight in the presence of a force (in the overdamped limit) and driven by Markovian binary velocity increments was presented in [14]. The time-dependent solution was introduced as a generalized Kac's path integral, see Appendix B.1 in [15]. In Refs. [13,14], the stationary PDF for the generalized Smoluchoswki-like dynamics has been solved for several stable and unstable potentials. Notably, the support of the PDF is characterized by a finite domain and there is a multimodal transition as a result of anharmonic potentials and the competition between the two transport regimens: ballistic and diffusive. In all cases, the shape of the asymptotic PDF depends on the flatness of the potential and the values of TE parameters, i.e., finite velocity v and relaxation time $\tau = 1/2a$. Thus, it is important to generalize this analysis considering an intermittent binary stochastic velocity, as well as to study how the stationary state of the PDF is modified by noise's parameters $\{\tau_1, \tau_2, p\}$.

Using the present EME approach we can also solve the evolution equation for the PDF for a random flight in the presence of a force and driven by non-Markovian (intermittent) binary stochastic velocity. To work out this problem, we follow the method presented in Sec. II D.

Consider a Smoluchowski-like process with finite-velocity diffusion, described by the SDE,

$$\frac{dx}{dt} = h(x) + \xi(t),\ t \in (0, \infty),\ x(0) = x_0. \tag{38}$$

With $\xi(t)$ chosen to be our four-state Markov binary noise of Sec. II D, the continuity equation in the presence of a force $h(x)$ is now

$$\begin{aligned} \partial_t \rho(x,t) &= -\partial_x\{[h(x) + \xi(t)]\rho(x,t)\}, \\ \rho(x,t_0) &= \delta(x - x_0),\ t \geqslant t_0. \end{aligned} \tag{39}$$

Using the auxiliary conditional functions $\{P^{\pm}_{\alpha,\beta}\}$ and the notation $\tilde{\mathbf{P}} = \{P^+_\alpha, P^+_\beta, P^-_\alpha, P^-_\beta\}$, we can write the EME as (see Appendix B)

$$\partial_t \tilde{\mathbf{P}} = \mathbf{B} \cdot \tilde{\mathbf{P}} + \mathbf{H} \cdot \tilde{\mathbf{P}}, \tag{40}$$

with

$$\mathbf{B} = \begin{pmatrix} -\partial_x[h(x)+v] & 0 & 0 & 0 \\ 0 & -\partial_x[h(x)+v] & 0 & 0 \\ 0 & 0 & -\partial_x[h(x)-v] & 0 \\ 0 & 0 & 0 & -\partial_x[h(x)-v] \end{pmatrix}. \tag{41}$$

The master Hamiltonian $\mathbf{H}$ in (40) is the same as that given in (10).

The procedure to get the mean value of the density, $\langle\rho(x,t)\rangle = P^+_\alpha + P^+_\beta + P^-_\alpha + P^-_\beta$, follows as in Sec. II D [see (B5)]. Here the analysis of the stationary state will be presented. From (40) and in the limit $p \to 1$ or $q \to 1$, or $\tau_2 \to \tau_1$ ($\beta \to \alpha$), the stochastic process $\xi(t)$ goes to the Markovian binary noise. Then, using Terwiel operators, the time evolution equation for $\langle\rho\rangle \equiv P$ can be written as a hyperbolic Smoluchoswki-like differential equation [15].

Another way to calculate the average of (39) is by using (40) and our previous basis: $P = P^+_\alpha + P^+_\beta + P^-_\alpha + P^-_\beta$, $\mathcal{Q} = P^+_\alpha + P^+_\beta - P^-_\alpha - P^-_\beta$, $\mathcal{M} = P^+_\alpha - P^+_\beta - P^-_\alpha + P^-_\beta$, $\mathcal{N} = P^+_\alpha - P^+_\beta + P^-_\alpha - P^-_\beta$; see Sec. II D. Therefore, we obtain equations similar to (11)–(14), plus new terms depending on $h(x)$:

$$\partial_t P = -v\partial_x \mathcal{Q} - \partial_x h(x) P, \tag{42}$$

$$\partial_t \mathcal{Q} = -v\partial_x P - \partial_x h(x)\mathcal{Q} + (\beta - \alpha)\mathcal{M} - (\alpha + \beta)\mathcal{Q}, \tag{43}$$

$$\partial_t \mathcal{M} = -v\partial_x \mathcal{N} - \partial_x h(x)\mathcal{M} - (p\alpha + q\beta)\mathcal{M} - (p\alpha - q\beta)\mathcal{Q}, \tag{44}$$

$$\partial_t \mathcal{N} = -v\partial_x \mathcal{M} - \partial_x h(x)\mathcal{N} - (q\alpha + p\beta)\mathcal{N} - (q\alpha - p\beta)P. \tag{45}$$

Here we focus on the stationary PDF, i.e., $\partial_t P = 0$ and $\partial_t \mathcal{Q} = 0$, $\partial_t \mathcal{M} = 0$, $\partial_t \mathcal{N} = 0$ for $t \to \infty$. For the Markovian noise ($p \to 1$ or $q \to 1$, or $\beta \to \alpha$), we obtain a simple differential equation for the PDF $P_{\rm st}$ as in [14]:

$$\frac{2h(x)[\alpha \ + \ h'(x)]P_{\rm st}(x)}{v} - \left(v - \frac{h(x)^2}{v}\right)P'_{\rm st}(x) = 0. \tag{46}$$

Solving this equation, we obtain an analytical solution (where $\mathcal{C}$ is a normalization constant):

$$P_{\rm st}(x) = \mathcal{C} \exp\left[\int_0^x \frac{2h(y)[\alpha \ + h'(y)]}{v^2 - h(y)^2} dy\right]. \tag{47}$$

We note that we can write the solution as a function of a deterministic potential $U(x)$ as in [14],

$$h(x) = -U'(x). \tag{48}$$

In the general case [non-Markovian, possibly intermittent stochastic velocity $\xi(t)$], we obtain [using (42)–(45)] an analytical differential equation for $P_{st}(x)$ as follows:

$$\begin{aligned} &- [h'(x) + \beta p + \alpha q]G(y) + [v^2 - h(x)^2]\{h(x)P'(x)[\alpha + \beta + 5h'(x) + \alpha p + \beta q] + [h(x) - v][h(x) + v]P''(x)\} \\ &- P(x)[R(x) + 2h(x)^2\{h'(x)[\alpha + \beta + 2h'(x) + \alpha p + \beta q] + \alpha^2 p + \beta^2 q\}] = 0, \end{aligned} \tag{49}$$

where

$$G(x) = \int_0^x \{A(y) + B(y)\}dy, \tag{50}$$

with

$$\begin{aligned} A(y) = {}& [P''(y)]^3 + \{[p\alpha + \alpha + q\beta + \beta + 5h'(y)]P'(y) \\ & + 2P(y)h''(y)\}h(y)^2, \end{aligned} \tag{51}$$

$$\begin{aligned} B(y) = {}& [2P(y)\{p\alpha^2 + q\beta^2 + h'(y)[p\alpha + \alpha + q\beta + \beta \\ & + 2h'(y)]\} - v^2P''(y)]h(y) \\ & - v^2[p\alpha + q\beta + h'(y)]P'(y), \end{aligned} \tag{52}$$

and

$$\begin{aligned} R(x) = {}& -2v^2h(x)[h''(x)]^3[h''(x)]^2\{(\alpha - \beta)(\alpha q - \beta p) \\ & - h'(x)[\alpha + \beta + 2h'(x)]\}. \end{aligned} \tag{53}$$

By construction, the solution $P(x,t)$ of this set of equations must be positive for all values of the noise parameters $\{\alpha, \beta, p + q = 1\}$ if $P(x,t)|_{t=0} > 0, \forall x \in \mathcal{D}_x$. We note that it is easier to solve numerically the coupled differential system of Eqs. (42)–(45) than the integral-differential equations (49)–(53). Therefore, we obtain $P_{\rm st}(x)$ numerically from (42)–(45) using $\partial_t P = 0,\ \partial_t \mathcal{Q} = 0,\ \partial_t \mathcal{M} = 0,\ \partial_t \mathcal{N} = 0$ (for $t \to \infty$), and $P_{\rm st}(0)/\text{const} = 1$, with $\mathcal{Q}(0) = 0,\ \mathcal{M}(0) = 0,\ \mathcal{N}(0) = 0$ for $t \to \infty$.

In Fig. 8, we study the anharmonic potential $U(x) = \frac{1}{3}x^3$ for Markovian and non-Markovian stochastic binary velocity $\xi(t)$. The parameters are $\beta = \alpha/40$, $p = 0.98$, $v = 1$, and $\alpha = 1, 1.5, 2.5, 27$ (all corresponding to intermittent cases). We notice that for α close to 1, the intermittence produces a stationary state that is similar to the Markovian (dichotomic) case presented in Ref. [14], while for $\alpha \gg 1$ the intermittent noise prevents the (generic) transition from infinite to zero at the border of $\mathcal{D}_x$; that is, the transition between $\cup$ to $\cap$ shapes.

In Fig. 9, we study the harmonic potential $U(x) = \frac{1}{2}x^2$ for both Markovian and non-Markovian $\xi(t)$. The parameters are $\beta = \alpha/40$, $p = 0.98$, $v = 1$, and $\alpha = 0.9, 1.1, 5, 27$ [all corresponding to intermittent stochastic velocities or noise $\xi(t)$].

As before, the intermittence produces a stationary state that is similar to the Markovian (dichotomic) case [14]. Nevertheless, for $\alpha = 1.1$ and 5 both cases prevent the transition from $\cup$ to $\cap$ shapes. In addition, for $\alpha \gg 1$ it can be seen that the stationary distribution approaches the Gaussian distribution better.

In all figures, it is possible to see that the domain $\mathcal{D}_x$ where the stationary $P_{\rm st}(x)$ is defined remains similar to the case when the noise is Markovian [13,14]. It is interesting to point out that, in the intermittent regime, new scenarios for the support of the stationary distribution may appear as a function of the noise parameters. These results must be studied as a function of all phase-space parameters $\{v, \tau_1, \tau_2, p\}$ and compared with stochastic simulations for the interpretation of the physical model.

Another possibility is to consider the joint dynamics in the space and velocity [11,15]. In this case, instead of (38)

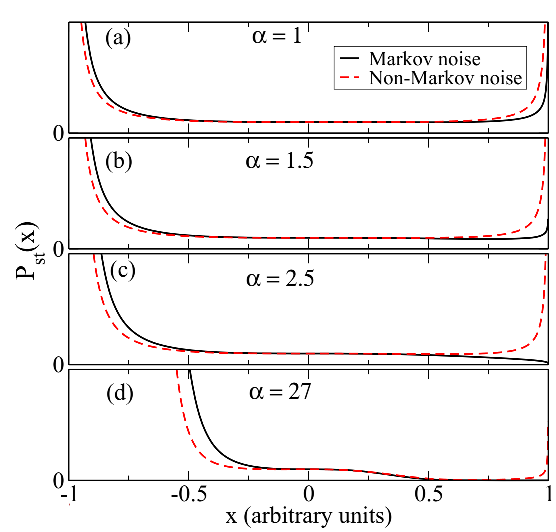

FIG. 8. PDF $P_{\rm st}(x)$ as a function of position x, for Markovian (full black lines) and intermittent noise (red dashed lines). The parameters are $\beta = \alpha/40$, $p = 0.98$, $v = 1$ and $\alpha = 1,\ 1.5,\ 2.5,\ 27$ (for both cases). For $U(x) = \frac{1}{3}x^3$.

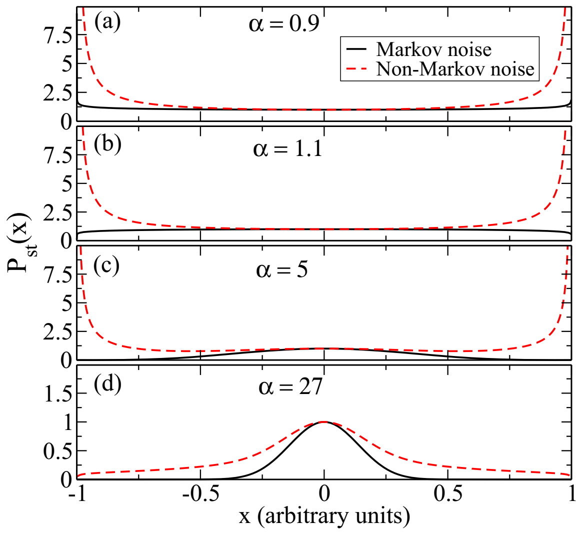

FIG. 9. PDF $P_{\rm st}(x)$ as a function of position x, for Markovian (full black lines) and intermittent noise (red dashed lines). The parameters are $\beta = \alpha/40$, $p = 0.98$, $v = 1$ and $\alpha = 0.9,\ 1.1,\ 5,\ 27$ (for both cases). For $U(x) = \frac{1}{2}x^2$.

we have two equations: $\dot{x} = v(t)$, $\dot{v} = -\gamma v + h(x) + \xi(t)$, where $\xi(t)$ is the noise and γ is a dissipative parameter. Therefore, the continuity equation is now for the density $\rho(x, v, t)$, and the matrix system corresponding to the EME will be of dimension $\mathcal{R}_e^{8\times 8}$.

VI. CONCLUSIONS

It is well known that Wiener's process can be written as a random flight when the stochastic velocity is a Gaussian white noise. If the stochastic velocity is a binary Markovian noise (dichotomic noise for the changes in the direction of the 1D velocity [12]), Kac proved, many years ago, that the diffusion turns out to be hyperbolic (that is, the PDF is governed by the TE). In the present work, based on Kac's random flight idea, we have used different statistics for the time intervals Δ_{ij}. Then we have analyzed the robustness of the ballistic regime as well as the Gaussian convergence of the profile.

We have presented a random flight in which the stochastic velocity is a binary non-Markovian noise with the possibility of intermittence. This situation is quite different from the usual Poisson-Kac flight [12] and provides more control over the ballistic-diffusion transition. We have found that the evolution equation for this wave-diffusion dynamics is a fourth-order partial differential equation that preserves positivity and normalization for a generalized diffusion-like process, and we have found the exact solution in the Fourier-Laplace representation. We have also revisited a non-Markovian Poisson flight (always non-Gaussian) in order to compare this process with the one associated with a random flight with a binary stochastic velocity. Then, the characteristic functional of the Poisson noise with exponential pulses has been used to present analytical results of the Poisson flights.

Using the Fourier-Laplace representation, we have calculated the exact solution of our generalized TE (random flights having a non-Poisson time statistics interval Δ_{ij}). We have then calculated the second moment to study the transition from the ballistic to diffusive regime. We have presented asymptotic expressions to study the robustness of the ballistic behavior as a function of noise parameters.

The cutoff of the support and Gaussian invariants have been studied to show the time-dependent convergence of the PDF to the diffusive regime. That is, $\mathcal{H}(t\to\infty) \sim 15 + \tau_H/t$ and $\mathcal{K}(t\to\infty) \sim 3 + \tau_K/t$ with characteristic timescales.

We have found that the PDF for our generalized hyperbolic diffusion is more malleable in terms of timescales $\{\tau_1, \tau_2\}$ than the ordinary telegrapher's profile. We have conducted a numerical analysis for the time-evolution of the PDF, using two different ICs, to study the support of the distribution.

We conclude by noting that the EME approach has also been used to solve the PDF for a random flight in the presence of a force and driven by intermittent binary velocity increments. We have explicitly calculated the stationary PDF $P_{\rm st}(x)$ for a random flight in the presence of two different potentials (stable and unstable), and we have presented its behavior to be compared with the Markovian dichotomic noise case. The interesting analysis of the support in the stationary state, as a function of noise's parameters, is currently under investigation. Extended numerical analysis concerning these subjects will be presented elsewhere.

The wave characteristics for the solution of the ordinary TE can readily be demonstrated by calculating the localized gap and/or studying the penetration of a wave [39,40]. This localized gap is described by a critical Fourier number k_c, which can also be calculated from the solution (19) of the present generalized TE (16). Works in these directions will be presented in a future contribution.

The analysis of the survival probability for a run-and-tumble particle with and without potential [41], and in the presence of intermittence in the stochastic velocity, is an important issue that is outside the scope of the present work and will be considered in the future.

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ACKNOWLEDGMENTS

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APPENDIX A: REVISITING A POISSON FLIGHT

An *integrated* Poisson process $x(t)$ is defined by the SDE (1) when $\xi(t)$ is a Poisson's noise. That is, considering the stationary zero mean-value Poisson noise:

$$\xi(t') = \sum_{j=1}^{n(t')} \omega_j h(t' - t_j) - Q\omega_0 \text{ with } t' \in (0, \infty),$$

$$\xi(0) = -Q\omega_0,\ h(u) = 0,\ \forall u < 0. \tag{A1}$$

The positive random variables ω_j are statistically independent and equally distributed with the distribution $\phi(\omega) = e^{-\omega/\omega_0}/\omega_0$ (so its mean value is $\overline{\omega} = \omega_0$), and $h(t)$ is an arbitrary positive normalized function characterizing the *shape* of each pulse [22,33,38]. In (A1) the number $n(t')$ states that it is a Poisson counting process of rate Q, with events occurring at random times t_j. If $h(t)$ is not a δ function, the process (A1) is called *colored* Poisson noise. Using the characteristic functional of a noise $\xi(t)$ [the average $\langle\cdots\rangle$ is over all stochastic realizations of $\xi(t)$, and $k(t)$ is a test function],

$$G_\xi([k(\bullet)]) = \left\langle \exp i\int_0^\infty k(t)\xi(t)dt\right\rangle \text{ with } \int_0^\infty k(t)dt < \infty, \tag{A2}$$

we write for the Poisson noise,

$$\ln G_\xi([k(\bullet)]) = -iQ\omega_0\int_0^\infty dt_n\, k(t_n) + Q\int_0^\infty dt_n\left(\frac{i\omega_0\int_0^\infty dt' k(t')\,h(t'-t_n)}{1-i\omega_0\int_0^\infty dt' k(t')\,h(t'-t_n)}\right). \tag{A3}$$

With these prescriptions, any Poisson noise $\xi(t)$ bounded from below can be worked out. That is, at any time t, the value of the noise fulfills $\xi\in(-Q\omega_0,\infty)$. The Poisson noise and integrated Poisson noise (Poisson flights) with square, exponential, and power-law shape pulses have been studied in [33].

1. The $n-$times joint PDF of the noise $\xi(t)$

For any stochastic process, it is true that the $n-$times (joint) PDF $P(\xi(t_1)\cdots\xi(t_n))$ can be calculated from the Fourier inversion of the $n-$times characteristic function $G_\xi(k_1,t_1;\cdots;k_n,t_n)$ [22]. The function $G_\xi(k_1,t_1;\cdots;k_n,t_n)$ follows from the characteristic functional $G_\xi([k(\bullet)])$ evaluated at the test function $k(t)=k_1\ \delta(t-t_1)+\cdots+k_n\ \delta(t-t_n)$. Thus, the $1-$time characteristic function of Poisson noise is $G_\xi(k_1,t_1)=G_\xi([k_1\ \delta(t-t_1)])$. From this formula, moments $M_m(t_1)\equiv\langle\xi(t_1)^m\rangle=\int\xi^m P(\xi,t_1)d\xi$, and cumulants $K_m(t_1)\equiv\langle\langle\xi(t_1)^m\rangle\rangle$, follow as

$$M_m(t_1) = \frac{d^m}{d(ik_1)^m}G_\xi(k_1,t_1)|_{k_1=0},\ \forall m\geqslant 1,$$
$$K_m(t_1) = \frac{d^m}{d(ik_1)^m}\ln G_\xi(k_1,t_1)|_{k_1=0},\ \forall m\geqslant 1,$$

and so we obtain the $1-$time cumulants:

$$K_1(t_1) = -Q\omega_0 + Q\overline{\omega}\int_{-\infty}^{t_1}h(u)du, \tag{A4}$$

$$K_m(t_1) = Q\,\overline{\omega^m}\int_{-\infty}^{t_1}h(u)^m du,\ \forall m\geqslant 2. \tag{A5}$$

Noting that $h(u)=0$ for $u<0$ and $\int_0^\infty h(u)du=1$, we see that in the stationary state $K_1(t\to\infty)\to 0$. The stationary PDF $P_{\rm st}(\xi)$ can also be analytically calculated [33].

2. Poisson's flights

The complete Kolmogorov hierarchy for the set of $n-$times PDF of the process $\dot{x}(t)=\xi(t)$ can be written as the Fourier inversion of the functional:

$$G_x([z(\bullet)]) = e^{iz_0x_0}G_\xi\left(\left[\int_t^\infty z(u)du\right]\right);\ z_0=\int_0^\infty z(u)du, \tag{A6}$$

evaluated at the test function: $z(t)=z_n\delta(t-t_n)+\cdots+z_1\delta(t-t_1)$ in the form [22]

$$P_n(x_n,t_n;\cdots;x_1,t_1) = \left(\frac{1}{2\pi}\right)^n\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Pi_{j=1}^n\,dz_j\,e^{-i(z_nx_n+\cdots z_1x_1)} \times G_x([z(\bullet)]). \tag{A7}$$

Thus, we can call the integral of the Poisson noise the "*Poisson flight*." In the case when $\xi(t)$ is a shot-noise (the pulse is a δ-function), the process $x(t)$ can be studied using the Feller–Van Kampen formula to obtain the $1-$time PDF [38]. But when the shape of the pulse is not a δ-function, the increments of Poisson's process are not independent, thus the process $x(t)$ is non-Markovian and the situation is more complex to analyze. Using our approach, this situation can be overcome.

In general for any stochastic process $\xi(t)$ in (1), the characteristic functional of the process $x(t),\ t\in(0,\infty)$ follows as (A6). Then using $G_\xi([k(\bullet)])$ for the Poisson noise, we obtain

$$\ln G_x([z(\bullet)]) = iz_0x_0 + Q\int_0^\infty dt_n\left\{\overline{\exp i\int_0^\infty \omega\int_{t'}^\infty z(u)du\,h(t'-t_n)dt'} - i\omega_0\int_{t_n}^\infty z(u)du - 1\right\}. \tag{A8}$$

This formula can be used for any intensity distribution $\phi(\omega)$ [the mean-value is represented by $\overline{(\cdots)}$] and shape pulse by $h(t)$. In particular, here we will exemplify Poisson's flight with exponential shape pulses $h(t)$ and exponentially distributed intensities ω_j; see (A3). This readily allows taking the limit to the Wiener process; other shapes can also be analyzed in a similar way [33].

a. *Poisson's flights with exponential pulses and exponentially distributed intensities*

The $1-$time characteristic function follows from (A8) as $G_x(k_1,t_1)=\ G_x([z(t)=k_1\ \delta(t-t_1)])$. Like Wiener's process, Poisson's flights are nonstationary. Using an exponential shape $h(u)=\Theta(u)e^{-u/\tau_c}/\tau_c$, we get

$$\ln G_x(k_1,t_1) = ik_1x_0 - ik_1Q\omega_0\tau_c(1-e^{-t_1/\tau_c}) + Q\tau_c\sum_{n=2}^\infty(i\omega_0k_1)^nB(n+1,0,1-e^{-t_1/\tau_c}). \tag{A9}$$

Here $B(\nu,\mu,y)$ is the Beta function [37]. From this result, we can calculate any moment or cumulant of Poisson's flight $x(t)$. In particular, the first and second cumulants are

$$K_1(t) = x_0 - Q\omega_0\tau_c(1-e^{-t/\tau_c}), \tag{A10}$$

$$K_2(t) = 2!Q\tau_c\omega_0^2\,B(3,0,1-e^{-t/\tau_c}). \tag{A11}$$

Using the formula

$$B(n+1,0,y) = -\ln(1-y) - y - \frac{y^2}{2} - \cdots - \frac{y^n}{n}, \tag{A12}$$

it is readily seen that any cumulant is given by

$$K_n(t) = n!Q\omega_0^n\left(t - \tau_c(1 - e^{-t/\tau_c}) - \frac{\tau_c}{2}(1 - e^{-t/\tau_c})^2 - \cdots - \frac{\tau_c}{n}(1 - e^{-t/\tau_c})^n\right), \quad \forall n \geqslant 2. \tag{A13}$$

Thus, in the long-time limit $K_1(t \to \infty) \to x_0 - Q\omega_0\tau_c$ and $K_n(t \to \infty) \sim t, \forall n \geqslant 2$, showing a clear departure from diffusive behavior. The Wiener limit can be recovered by taking $\tau_c \to 0$ and $Q \to \infty$, $\omega_0 \to 0$ such that $Q\omega_0^2 \to D$.

Moments $M_n(t) \equiv \langle x(t)^n \rangle$ and correlations $\langle\langle x(t_1)x(t_2)\cdots\rangle\rangle$ follow in a similar way. For example, the $2-$times correlation function can be calculated from the knowledge of the $2-$times characteristic function $G_x(k_1, t_1; k_2, t_2) = G_x([z(t) = k_1 \ \delta(t - t_1) + k_2 \ \delta(t - t_2)])$, which can be computed from the functional $G_x([z(\bullet)])$ given in (A8). In addition, using (A8) in (A7) and taking the limit $t_i - t_{i-1} \to 0$, formula (A7) gives the path integral representation for a general non-Markovian Poisson's flight.

APPENDIX B: ENLARGED MASTER EQUATION APPROACH

In general, for any nonwhite noise $\xi(t)$ the first-order differential equation (1) is associated with a non-Markov process $x(t)$, which for a given realization of the noise is characterized by the Liouville flux:

$$\partial_t \rho(x,t) = -\partial_x[\xi(t)\rho(x,t)], \ \ \rho(x,t_0) = \delta(x - x_0), \ \ t \geqslant t_0. \tag{B1}$$

If the stochastic process $\xi(t)$ is Markovian, there exists a semigroup associated with its conditional probability $\Pi_{t,t_0}(\xi|\xi')$. Therefore, there is a master Hamiltonian $\mathbf{H}$ such that $\frac{d}{dt}\Pi_{t,t_0}(\xi|\xi') = \sum_{\xi''}\mathbf{H}_{\xi\xi''}\Pi_{t,t_0}(\xi''|\xi')$ [if $\xi(t)$ is continuous, $\mathbf{H}$ is the Fokker-Planck operator]. Although the process $x(t)$ defined by (1) is itself not Markovian, it can be considered as a projection of an enlarged Markovian process: $\{x(t), \xi(t)\}$. Therefore, using the fact that the noise $\xi(t)$ is Markovian and discrete, we can write an *enlarged master equation* for the joint probability $\mathcal{P}(\rho, \xi, t)$:

$$\partial_t \mathcal{P}(\rho, \xi, t) = -\partial_\rho[(-\partial_x \xi \rho)\mathcal{P}(\rho, \xi, t)] + \sum_{\xi'} \mathbf{H}_{\xi\xi'}\mathcal{P}(\rho, \xi', t), \quad t \geqslant t_0. \tag{B2}$$

The initial condition in (B2) is taken in the stationary ensemble of process $\xi(t)$; that is, $\mathcal{P}(\rho, \xi, t_0) = \Pi_{t_0}(\xi) \ \delta(\rho - \rho(x, t_0))$.

In general, for a scalar process $x(t)$, the Markovian character of the $(1+1)$-dimensional process, $\{x(t), \xi(t)\}$, is due to the Markovian character of process $\xi(t)$ and to the fact that for a given realization the solution of (B1) depends on the values of $\xi(\tau)$ only in the time interval $t_0 \leqslant \tau \leqslant t$. Therefore, as the enlarged process $\{x(t), \xi(t)\}$ is Markovian, one can write the master equation (B2) for the joint probability $\mathcal{P}(\rho, \xi, t)$, which varies in time due to the flow in ρ-space and jumps of ξ in an enlarged Liouville's phase-space [22,30].

If only the mean value of ρ is wanted, we can go one step further and calculate the conditional average:

$$P_\xi(x,t) \equiv \int d\rho \ \mathcal{P}(\rho, \xi, t)\rho. \tag{B3}$$

Thus, if Eq. (B2) is multiplied by $\rho(x,t)$ and integrated over $d\rho$, we get

$$\partial_t P_\xi(x,t) = -\partial_x \xi P_\xi(x,t) + \sum_{\xi'} \mathbf{H}_{\xi\xi'}P_{\xi'}(x,t). \tag{B4}$$

Now considering the set of discrete values $\xi = \pm v$ for the dichotomous process $\xi(t)$, we get the EME for the (velocity) conditional PDF $\tilde{\mathbf{P}} = \{P_+, P_-\}$, that is, Eqs. (2) and (3).

In the case of considering the biexponential binary noise $\xi(t)$ of Sec. II D, we do an embedding to consider a "Markovian" binary noise with four states. In this manner, the set of values of the process $\xi(t)$ is now $\{\xi_\alpha^+, \xi_\beta^+, \xi_\alpha^-, \xi_\beta^-\}$ and the set of conditional averages is $\tilde{\mathbf{P}} = \{P_\alpha^+, P_\beta^+, P_\alpha^-, P_\beta^-\}$. Thus from (B4), we get the EME (8). Higher-order statistical objects can also be considered using the EME approach; see Appendix A in [31].

In general, the mean value of the density $\rho(x,t)$ is therefore reduced to the addition of $P_\xi(x,t)$ for the different set of values ξ, that is,

$$\langle \rho(x,t) \rangle = \sum_\xi P_\xi(x,t). \tag{B5}$$

APPENDIX C: ON THE BINARY NOISE

1. Markov binary noise

A Markov binary process [dichotomic noise $\xi(t)$] can be built by considering a sequence of constant values ± 1 alternating at random times t_j, where these times are in correspondence with a stationary Markov renewal process. Therefore, a realization of the process, $\xi(t) = \pm 1, \forall t \geqslant 0$ corresponds to being exponentially correlated; that is, $\langle \xi(t)\xi(t')\rangle = e^{-|t-t'|/T}$. A realization of this symmetric binary noise $\xi(t)$ can be generated by

$$\xi(t) = \sum_{j=1}^{n} (-1)^j \ W(t_j, t_{j+1}|t), \quad t \geqslant 0, \quad \xi(0) = 1, \tag{C1}$$

where $W(t_j, t_{j+1}|t)$ is the window function: $W(t_j, t_{j+1}|t) = \Theta(t - t_j) - \Theta(t - t_{j+1})$, with $\Theta(u)$ a step function, the number of dots n in (C1) is Poisson-distributed, and the random location of independent times t_j is uniformly distributed in $[0, \infty]$ with density a. That is, we generate statistically independent time-increments $\Delta_{i,j} \equiv t_j - t_{j-1}$ with an exponential waiting-time: $\varphi(\Delta_{ij}) = a\exp(-a\Delta_{ij})$, where $a = 1/2T$ in (C1).

2. Intermittent (non-Markov) binary noise

A symmetric intermittent binary noise, $\xi(t) = \pm 1, \forall t \geqslant 0$, can be represented with a nonexponential correlation function having two characteristic timescales. The stationary distribution $\Pi_{\mathrm{St}}(\xi) = [\delta_{\xi,1} + \delta_{\xi,-1}]/2$ remains the same as for the Markovian dichotomic case. The only difference in the intermittent case is the waiting-time function that generates the statistically independent time increments $\Delta_{i,j} \equiv t_j - t_{j-1}$. It can be proved that intermittent binary noise can be obtained by introducing a biexponential waiting-time [see (7)] for the random time increments Δ_{ij} in (C1). We note that the important parameters to characterize intermittence are $\alpha \gg \beta$ (different timescales) and $p \gg q$, very different statistical weight for each timescale [35]. If $\alpha = \beta$, we recover the Markovian case.

3. The master equation approach

It is well known that the conditional probability $\mathbf{P}(\xi, t|\xi', t')$ for the dichotomous process $\xi(t) = \pm 1, \forall t \geqslant 0$ with exponential correlation (the Markov case) can be obtained by solving the master equation: $\partial_t \mathbf{P}(\xi, t|\xi_0, t_0) = \sum_{\xi'=1}^{2} \mathbf{H}_{\xi\xi'} \mathbf{P}(\xi', t|\xi_0, t_0)$, where the "master Hamiltonian" is given by $\mathbf{H} = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$.

Notably, for the intermittent case, and due to the fact that the waiting-time (7) is biexponential, it is possible to solve the conditional probability $\mathbf{P}(\xi, t|\xi', t')$ for this non-Markov binary intermittent noise, as the embedding of a four-state Markov process $\xi_{\{\alpha,\beta\}}(t) = \pm 1, \forall t \geqslant 0$, governed by the four-state master equation: $\partial_t \mathbf{P}(\xi, t|\xi'', t'') = \sum_{\xi'=1}^{4} \mathbf{H}_{\xi\xi'} \mathbf{P}(\xi', t|\xi'', t'')$, where the "master Hamiltonian" is given by (10); see [35].

APPENDIX D: VELOCITY AUTOCORRELATION FUNCTION OF THE INTEGRATED PROCESS

Using the second moment $\langle x(t)^2 \rangle$ of the random flight (1), we can write the VAF for the process $x(t)$ [22]. Thus, for the case $x_0 = 0$ using the Green-Kubo theorem, we obtain

$$\frac{s^2}{2} \langle x(s)^2 \rangle = D_{\mathrm{eff}}(s), \tag{D1}$$

where $D_{\mathrm{eff}}(s)$ is the Laplace transform of the stationary VAF:

$$D_{\mathrm{eff}}(s) \equiv \int_0^\infty \langle \xi(0)\xi(t) \rangle e^{-st} dt. \tag{D2}$$

In addition, the function $D_{\mathrm{eff}}(s)$ allows for the calculation of the diffusion coefficient $D = D_{\mathrm{eff}}(s = 0)$ for the integrated process $x(t)$, which is nothing more than Einstein's formula: $D = \int_0^\infty \langle \xi(0)\xi(t) \rangle dt$.

On the other hand, in the real-time representation, we can write

$$\langle \xi(0)\xi(t) \rangle = \frac{1}{2} \frac{d^2}{dt^2} \langle x(t)^2 \rangle. \tag{D3}$$

1. The Markovian binary noise case

Using (23), (D1) and (D2), we get

$$\langle \xi(0)\xi(s) \rangle = \frac{v^2}{(s + 1/\tau)}. \tag{D4}$$

Taking the inverse Laplace transform, we get $\langle \xi(0)\xi(t) \rangle = v^2 e^{-t/\tau}$, in accordance with Rice's method applied to the SDE (1).

2. The non-Markovian binary noise case

From (24), (D1), and (D2), we obtain

$$\langle \xi(0)\xi(s) \rangle = \frac{v^2 \{ \tau_2^2 [-p^2 + 2s\tau_1(2s\tau_1 + 1) + 1] + 2\tau_2\tau_1[(p-1)p + s\tau_1] - (p-2)p\tau_1^2 \}}{[\tau_2(-p + 2s\tau_1 + 1) + p\tau_1][(p+1)s\tau_2 + s\tau_1(-p + 2s\tau_2 + 2) + 1]}, \tag{D5}$$

which is the Laplace representation of the stationary correlation function of the biexponential binary noise characterized by the waiting-time function (7) [35].

3. Poisson's flight

The exact second moment for Poisson's flight is given in (29). Thus, applying the Green-Kubo formula (D3) to the integrated process, we get

$$\langle \xi(0)\xi(t) \rangle = \frac{Q\omega_0^2}{\tau_c} [(2 - Q\tau_c) + (-2 + 2Q\tau_c) e^{-t/\tau_c}] e^{-t/\tau_c}, \quad \forall t \geqslant 0, \tag{D6}$$

which is the stationary correlation function of the Poisson noise (A1), with exponential-shaped pulses.

APPENDIX E: GAUSSIAN INVARIANTS

A Gaussian distribution has many interesting characteristics, among the most important being its invariants. These values can be constructed as functions of moments M_j (or cumulants K_j) [22]. For example, the kurtosis is defined as the fourth centered moment divided by the square of the second centered moment; that is, $\mathcal{K} = \langle (x - \langle x \rangle)^4 \rangle / \langle (x - \langle x \rangle)^2 \rangle^2 = 3$. Nevertheless, here we are interested in symmetric distributions, thus $\mathcal{K} = M_4/M_2^2$. Another invariant that is built up with higher-order moments is $\mathcal{H} = M_6/M_2^3 = 15$. The important point is that, for any distribution, the deviation from *Gaussianity* can be computed by calculating these invariants.

For example, the convergence to the diffusion process can be studied by calculating the time-dependent behavior of $\mathcal{K}$ and/or $\mathcal{H}$. From (19) we can calculate any moment, therefore we get asymptotically a power-law convergence,

$$\mathcal{H} = 15 + \tau_H/t + \cdots. \tag{E1}$$

Here the timescale τ_H can be readily calculated as a function of the noise parameters $\{\alpha, \beta, p = 1 - q\}$ (or $\{\tau_1 \equiv 1/2\alpha, \tau_2 \equiv 1/2\beta\}$):

$$\tau_H = \frac{45}{\alpha\beta} \sum_{n=0}^{6} j_n p^n \bigg/ \sum_{n=0}^{5} l_n p^n, \tag{E2}$$

with

$$
\begin{aligned}
j_0 &= \alpha^6,\\
j_1 &= 2\alpha^5\beta + 10\alpha^4\beta^2 - 20\alpha^3\beta^3 + 8\alpha^2\beta^4,\\
j_2 &= 3\alpha^6 - 2\alpha^5\beta - 17\alpha^4\beta^2 + 20\alpha^3\beta^3 + 4\alpha^2\beta^4 - 8\alpha\beta^5,\\
j_3 &= -8\alpha^5\beta + 12\alpha^4\beta^2 + 20\alpha^3\beta^3 - 52\alpha^2\beta^4 + 36\alpha\beta^5 - 8\beta^6,\\
j_4 &= -3\alpha^6 + 8\alpha^5\beta + 10\alpha^4\beta^2 - 60\alpha^3\beta^3 + 85\alpha^2\beta^4 - 52\alpha\beta^5 + 12\beta^6,\\
j_5 &= 6\alpha^5\beta - 30\alpha^4\beta^2 + 60\alpha^3\beta^3 - 60\alpha^2\beta^4 + 30\alpha\beta^5 - 6\beta^6,\\
j_6 &= \alpha^6 - 6\alpha^5\beta + 15\alpha^4\beta^2 - 20\alpha^3\beta^3 + 15\alpha^2\beta^4 - 6\alpha\beta^5 + \beta^6,\\
l_0 &= -\alpha^5,\\
l_1 &= \alpha^5 + 3\alpha^4\beta - 4\alpha^3\beta^2,\\
l_2 &= 2\alpha^5 - 8\alpha^4\beta + 6\alpha^3\beta^2 + 4\alpha^2\beta^3 - 4\alpha\beta^4,\\
l_3 &= -2\alpha^5 + 2\alpha^4\beta + 10\alpha^3\beta^2 - 22\alpha^2\beta^3 + 16\alpha\beta^4 - 4\beta^5,\\
l_4 &= -\alpha^5 + 8\alpha^4\beta - 22\alpha^3\beta^2 + 28\alpha^2\beta^3 - 17\alpha\beta^4 + 4\beta^5,\\
l_5 &= \alpha^5 - 5\alpha^4\beta + 10\alpha^3\beta^2 - 10\alpha^2\beta^3 + 5\alpha\beta^4 - \beta^5.
\end{aligned} \tag{E3}
$$

The intermittent case corresponds to taking $p \gg q$ and $\alpha \gg \beta$, as we have commented before. From (E2), it is possible to recover the time-dependent Gaussian convergence from the solution of the ordinary TE. That is, taking $\alpha = \beta = 1/2\tau$, we get $\mathcal{H} = 15 - 90\tau/t + \cdots$.

A similar analytical formula can also be obtained for the kurtosis: $\mathcal{K} = 3 + \tau_K/t + \cdots$, with a characteristic timescale τ_K. We note that for the solution of the ordinary TE, this is $\mathcal{K} = 3 - 6\tau/t + \cdots$.

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