

Truncated stochastically switching processesPaul C. Bressloff *Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom*

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There are a large variety of hybrid stochastic systems that couple a continuous process with some form of stochastic switching mechanism. In many cases the system switches between different discrete internal states according to a finite-state Markov chain, and the continuous dynamics depends on the current internal state. The resulting hybrid stochastic differential equation (hSDE) could describe the evolution of a neuron's membrane potential, the concentration of proteins synthesized by a gene network, or the position of an active particle. Another major class of switching system is a search process with stochastic resetting, where the position of a diffusing or active particle is reset to a fixed position at a random sequence of times. In this case the system switches between a search phase and a reset phase, where the latter may be instantaneous. In this paper, we investigate how the behavior of a stochastically switching system is modified when the maximum number of switching (or reset) events in a given time interval is fixed. This is motivated by the idea that each time the system switches there is an additive energy cost. We first show that in the case of an hSDE, restricting the number of switching events is equivalent to truncating a Volterra series expansion of the particle propagator. Such a truncation significantly modifies the moments of the resulting renormalized propagator. We then investigate how restricting the number of reset events affects the diffusive search for an absorbing target. In particular, truncating a Volterra series expansion of the survival probability, we calculate the splitting probabilities and conditional MFPTs for the particle to be absorbed by the target or exceed a given number of resets, respectively.

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There are a wide range of stochastic processes in cell biology that involve the coupling between continuous and discrete random variables (stochastic hybrid systems) [1,2]. The continuous process could represent the concentration of proteins synthesized by a gene [3–8], the membrane voltage of a neuron [9–17], the position of a swimming bacterium [18–21], or a molecular motor [22–26]. The corresponding discrete process could represent the activation state of the gene, the conformational state of an ion channel, or the velocity state of an active particle. Let $(\mathbf{X}(t), N(t))$ denote the state of the system at time t with $\mathbf{X}(t) \in \mathbb{R}^d$ and $N(t) \in \Gamma$, where Γ is a discrete set. Assuming that $N(t) = n$, the continuous variables typically evolve according to a hybrid stochastic differential equation (hSDE) of the form $d\mathbf{X} = \mathbf{A}_n(\mathbf{X})dt + \sqrt{2D}d\mathbf{W}$, where \mathbf{W} is a vector of independent Wiener processes and \mathbf{A}_n is an n -dependent drift term. (The diffusivity could also depend on n .) The discrete variable switches between the different discrete states according to a continuous time Markov chain whose matrix generator could itself depend on $\mathbf{X}(t)$. In the limit $D \rightarrow 0$, the dynamics reduces to a so-called piecewise deterministic Markov process [27].

In many applications of hSDEs, there is a separation of time scales, whereby the switching between discrete states of the Markov chain is fast compared to the relaxation dynamics of the continuous process. Suppose that τ is the characteristic time-scale of the relaxation dynamics and $\epsilon\tau$ is the characteristic time-scale of the Markov chain for some small positive parameter ϵ . Taking the limit $\epsilon \rightarrow 0$ then leads to an effective continuous dynamical system that is obtained by averaging the piecewise dynamics with respect to the corresponding unique stationary measure of the Markov chain (assuming the latter exists). In the weak-noise regime $0 < \epsilon \ll 1$, various approaches have been used to study noise-induced transitions between metastable states of the averaged system. These include large deviation theory [28–31], WKB approximations and matched asymptotics [6,7,12,15,16], and stochastic hybrid path integrals [32–34].

Another important example of a randomly switching process is a search process with stochastic resetting. (See the review in Ref. [35] and references therein.) The simplest version of a resetting protocol is to instantaneously reset the position of a diffusing particle to some fixed point \mathbf{x}_r at a constant rate r [36–38]. One of the characteristic properties of a search process with stochastic resetting is that the mean time for a Brownian particle to find a hidden target in an unbounded domain is finite, and has an optimal value as a function of the resetting rate r . This is a consequence of the fact that the mean first passage time (MFPT) to find the target diverges in the limits $r \rightarrow 0$ and $r \rightarrow \infty$. Analogous behavior has been observed in other search processes with resetting, including diffusion with time-dependent resetting

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[39], nondiffusive search processes such as Levy flights [40], active run and tumble particles [41,42] and directed velocity jump processes [43,44], diffusion in potential landscapes [45] or switching environments [46–48], resetting followed by a refractory period [49,50], resetting with finite return times [51–56], and encounter-based models of absorbing targets [57–59].

In this paper we consider a different aspect of stochastically switching systems, namely, conditioning the process on the maximum number of switching events that can occur. That is, if $\mathcal{M}(t)$ denotes the number of switching events in the interval $[0, t]$, then we impose the condition $\mathcal{M}(t) \leq \mu < \infty$ for all t . One motivation for such a construction is that state transitions in an hSDE tend to cost energy, so that the maximum number of such transitions could be limited. Alternatively, conditioning on the number of transitions provides another type of statistic that could be measured experimentally. For example, in the case of gene networks, transitions from the inactive to active state often results on some form of bursting. In the case of search processes, the cost of stochastic resetting has been explored in a recent paper [60], which assumes that the cost is additive, and the contribution of each reset is a function of the distance a particle must travel to the reset position \mathbf{x}_r . These authors focus on the mean cost accrued by a search process that is terminated when the target is found. In contrast, we take the cost to be equal to the number of reset events, and terminate the search process as soon as one or other of the following occurs: the particle finds the target or the number of reset events crosses some threshold.

The structure of the paper is as follows. In Sec. II we give a general definition of an hSDE and write down the evolution equation for the associated propagator. In Sec. III we construct an integral equation for the propagator, which is expanded as a Volterra series, whose individual terms correspond to fixing the number of state transitions. Truncating the Volterra series is then equivalent to restricting the maximum number of allowed state transitions. We use this to define a renormalized propagator and its associated moments. The theory is illustrated in Sec. IV using the example of an OU process with random drift, which has previously been used to model the motion of an RTP with diffusion in a harmonic potential [61,62] and protein synthesis in a two-state gene network [4,5]. We use the corresponding diagrammatic expansion to calculate moments of the hSDE that are conditioned on the maximum number of switching events. In Sec. V, we develop the analogous theory for a diffusive search process with stochastic resetting. In this case, we expand the standard last renewal equation for the survival probability as a Volterra series in the number of resetting events. Truncating the series now corresponds to restricting the maximum number of resets. We use this to calculate the splitting probabilities and conditional MFPTs for the particle to be absorbed by the target or exceed a given number of resets, respectively.

II. HYBRID SDE IN \mathbb{R}^d

Consider a system whose states are described by a pair of stochastic variables $(\mathbf{X}(t), N(t)) \in \mathbb{R}^d \times \{0, \dots, K-1\}$. When the discrete state is $N(t) = n$, the system evolves

according to the SDE

$$d\mathbf{X}(t) = \mathbf{A}_n(\mathbf{X}(t))dt + \sqrt{2D}d\mathbf{W}(t), \quad (2.1)$$

where \mathbf{W} is a vector of d independent Wiener processes. The discrete stochastic variable $N(t)$ evolves according to a K -state continuous-time Markov chain with a $K \times K$ matrix generator \mathbf{Q} , which is taken to be independent of $\mathbf{X}(t)$. That is, the probability distribution $P_n(t) = \mathbb{P}[N(t) = n]$ evolves according to the $\mathbf{X}(t)$ -independent master equation

$$\frac{dP_n(t)}{dt} = \sum_{m=0}^{K-1} Q_{nm}P_m(t), \quad (2.2)$$

with

$$Q_{nm} = W_{nm} - \delta_{n,m}\Gamma_m, \quad \Gamma_m = \sum_{k=0}^{K-1} W_{km}, \quad (2.3)$$

where W_{nm} is the rate of the transition $n \rightarrow m$. Given the definition of Γ_m , we can introduce the decomposition $W_{nm} = P_{nm}\Gamma_m$ with $\sum_n P_{nm} = 1$. The positive quantity Γ_m is the rate at which a transition from the state m occurs and P_{nm} is the probability that such a transition is to the state n . We assume that the generator is irreducible so that there exists a stationary density ρ for which $\sum_m Q_{nm}\rho_m = 0$. In the case of a two-state hSDE ($n = 0, 1$), the matrix generator takes the form

$$\mathbf{Q} = \begin{pmatrix} -\beta & \alpha \\ \beta & -\alpha \end{pmatrix}, \quad (2.4)$$

and

$$\rho_0 = \frac{\alpha}{\alpha + \beta}, \quad \rho_1 = 1 - \rho_0 = \frac{\beta}{\alpha + \beta}. \quad (2.5)$$

Note that α is the rate of the transition $0 \rightarrow 1$ and β is the rate of the transition $1 \rightarrow 0$.

Given the initial conditions $\mathbf{X}(0) = \mathbf{x}_0, N(0) = n_0$, we introduce the propagator $G_{nn_0}(\mathbf{x}, t | \mathbf{x}_0, 0)$ with

$$G_{nn_0}(\mathbf{x}, t | \mathbf{x}_0, 0)d\mathbf{x} = \mathbb{P}[\mathbf{X}(t) \in (\mathbf{x}, \mathbf{x}+d\mathbf{x}), N(t) = n | \mathbf{x}_0, n_0], \quad (2.6)$$

and $G_{nn_0}(\mathbf{x}, 0 | \mathbf{x}_0, 0) = \delta_{n,n_0}\delta(\mathbf{x} - \mathbf{x}_0)$. It can be shown that G_{nn_0} evolves according to the forward differential Chapman-Kolmogorov (CK) equation [2]

$$\frac{\partial G_{nn_0}}{\partial t} = -\nabla \cdot [\mathbf{A}_n(\mathbf{x})G_{nn_0}] + D\nabla^2 G_{nn_0} + \sum_{m=0}^{K-1} Q_{nm}G_{mn_0}. \quad (2.7)$$

The first two terms on the right-hand side represent the probability flow associated with the SDE for a given n , whereas the third term represents jumps into or out of the discrete state n . Integrating Eq. (2.7) with respect to \mathbf{x} and setting $\bar{G}_{nn_0} = \int_{\mathbb{R}^d} G_{nn_0}(\mathbf{x}, t | \mathbf{x}_0, 0)d\mathbf{x}$ shows that \bar{G}_{nn_0} satisfies the master equation (2.2). In addition, since $\sum_n Q_{nm} = 0$ for all m , it follows that $d \sum_n \bar{G}_{nn_0} / dt = 0$. We thus have

$$\sum_{n=0}^{K-1} \int_{\mathbb{R}^d} G_{nn_0}(\mathbf{x}, t | \mathbf{x}_0, 0)d\mathbf{x} = 1, \quad (2.8)$$

which expresses conservation of probability. In the absence of switching with n fixed, the system reduces to a single SDE

whose corresponding Fokker-Planck (FP) equation takes the form

$$\frac{\partial p_n}{\partial t} = -\nabla \cdot [\mathbf{A}_n(\mathbf{x})p_n] + D\nabla^2 p_n, \quad (2.9)$$

and $p_n(\mathbf{x}, 0|\mathbf{x}_0, 0) = \delta(\mathbf{x} - \mathbf{x}_0)$. We will refer to p_n as the bare (no switching) propagator.

III. INTEGRAL EQUATION AND VOLTERRA SERIES EXPANSION

The propagator G_{nm} satisfies an integral equation of the form

$$G_{nm}(\mathbf{x}, t|\mathbf{x}_0, 0) = \delta_{n,m}e^{-\Gamma_m t} p_m(\mathbf{x}, t|\mathbf{x}_0, 0) + \sum_l W_{nl} \int_0^t d\tau \int d\mathbf{y} e^{-\Gamma_n(t-\tau)} p_n(\mathbf{x}, t|\mathbf{y}, \tau) G_{lm}(\mathbf{y}, \tau|\mathbf{x}_0, 0). \quad (3.1)$$

The first term on the right-hand side is the contribution from all paths that never switch in the interval $[0, t]$, which only occurs if $n = m$. The probability of no switching from the state m is $e^{-\Gamma_m t}$. The second term on the right-hand side represents the sum over all trajectories that switch at least once, with the final transition occurring at the time τ . Iterating the integral equation (3.1) generates a Volterra series representation of the propagator:

$$\begin{aligned} G_{nm}(\mathbf{x}, t|\mathbf{x}_0, 0) &= \delta_{n,m}e^{-\Gamma_m t} p_m(\mathbf{x}, t|\mathbf{x}_0, 0) + W_{nm} \int_0^t d\tau \int d\mathbf{y} e^{-\Gamma_n(t-\tau)} p_n(\mathbf{x}, t|\mathbf{y}, \tau) e^{-\Gamma_m \tau} p_m(\mathbf{y}, \tau|\mathbf{x}_0, 0) \\ &+ \sum_l W_{nl} W_{lm} \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \int d\mathbf{y}_2 \int d\mathbf{y}_1 e^{-\Gamma_n(t-\tau_2)} p_n(\mathbf{x}, t|\mathbf{y}_2, \tau_2) e^{-\Gamma_l(\tau_2-\tau_1)} p_l(\mathbf{y}_2, \tau_2|\mathbf{y}_1, \tau_1) e^{-\Gamma_m \tau_1} p_m(\mathbf{y}_1, \tau_1|\mathbf{x}_0, 0) \\ &+ \dots \end{aligned} \quad (3.2)$$

The j th term in the series expansion, $j \geq 0$, has the following interpretation: it specifies the contribution to the propagator from paths that undergo exactly j switching events. For example, if $n \neq m$, then the $j = 1$ term has a factor $P_{nm}\Gamma_m e^{-\Gamma_n(t-\tau)} e^{-\Gamma_m \tau}$, after setting $W_{nm} = P_{nm}\Gamma_m$. The probability that the first transition occurs in the time interval $[\tau, \tau + d\tau]$ is $\Gamma_m e^{-\Gamma_m \tau} d\tau$, the probability that $m \rightarrow n$ is P_{nm} , and the probability that there are no transitions from the state n is $e^{-\Gamma_n(t-\tau)}$. Hence, the total probability that there is a single transition $m \rightarrow n$ in the time interval $[0, t]$ is

$$P_{nm}^{(1)}(t) = W_{nm} e^{-\Gamma_n t} \int_0^t e^{-[\Gamma_m - \Gamma_n]\tau} d\tau = \frac{W_{nm}}{\Gamma_m - \Gamma_n} [e^{-\Gamma_n t} - e^{-\Gamma_m t}]. \quad (3.3)$$

Similarly, the probability that there are two transitions in the interval $[0, t]$ is

$$P_{nm}^{(2)}(t) = \sum_l W_{nl} W_{lm} \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 e^{-\Gamma_n(t-\tau_2)} e^{-\Gamma_l(\tau_2-\tau_1)} e^{-\Gamma_m \tau_1} = \sum_l \frac{W_{nl} W_{lm}}{\Gamma_m - \Gamma_l} \left[\frac{e^{-\Gamma_n t} - e^{-\Gamma_l t}}{\Gamma_l - \Gamma_n} - \frac{e^{-\Gamma_n t} - e^{-\Gamma_m t}}{\Gamma_m - \Gamma_n} \right]. \quad (3.4)$$

In addition, integrating Eq. (3.2) with respect to \mathbf{x} , summing over n , and then using the unit normalization of the propagator, see Eq. (2.8), shows that

$$1 = e^{-\Gamma_m t} + \sum_{j \geq 1} \sum_n P_{nm}^{(j)}(t). \quad (3.5)$$

Again this is an expression of conservation of probability.

For the sake of illustration, consider a two-state hSDE with matrix generator (2.4). Suppose that the system starts in the state $n_0 = 0$. Then

$$\begin{aligned} G_{00}(\mathbf{x}, t|\mathbf{x}_0, 0) &= e^{-\beta t} p_0(\mathbf{x}, t|\mathbf{x}_0, 0) \\ &+ \alpha \int_0^\infty d\tau \int d\mathbf{y} e^{-\beta(t-\tau)} p_0(\mathbf{x}, t|\mathbf{y}, \tau) G_{10}(\mathbf{y}, \tau|\mathbf{x}_0, 0), \end{aligned} \quad (3.6a)$$

$$\begin{aligned} G_{10}(\mathbf{x}, t|\mathbf{x}_0, 0) &= \beta \int_0^\infty d\tau \int d\mathbf{y} e^{-\alpha(t-\tau)} p_1(\mathbf{x}, t|\mathbf{y}, \tau) G_{00}(\mathbf{y}, \tau|\mathbf{x}_0, 0). \end{aligned} \quad (3.6b)$$

Similarly, if $n_0 = 1$, then

$$\begin{aligned} G_{01}(\mathbf{x}, t|\mathbf{x}_0, 0) &= \alpha \int_0^\infty d\tau \int d\mathbf{y} e^{-\beta(t-\tau)} p_0(\mathbf{x}, t|\mathbf{y}, \tau) G_{11}(\mathbf{y}, \tau|\mathbf{x}_0, 0) \end{aligned} \quad (3.6c)$$

$$\begin{aligned} G_{11}(\mathbf{x}, t|\mathbf{x}_0, 0) &= e^{-\alpha t} p_1(\mathbf{x}, t|\mathbf{x}_0, 0) \\ &+ \beta \int_0^\infty d\tau \int d\mathbf{y} e^{-\alpha(t-\tau)} p_1(\mathbf{x}, t|\mathbf{y}, \tau) G_{01}(\mathbf{y}, \tau|\mathbf{x}_0, 0). \end{aligned} \quad (3.6d)$$

The first term on the right-hand side of Eq. (3.6a) represents the contribution from all trajectories that never switch to the state $n = 1$. The latter occurs with probability $e^{-\beta t}$. However, the integral term sums over all trajectories that switch at least once, with the last switch $1 \rightarrow 0$ occurring at a rate β at a time τ , $0 < \tau < t$. Similar interpretations apply to

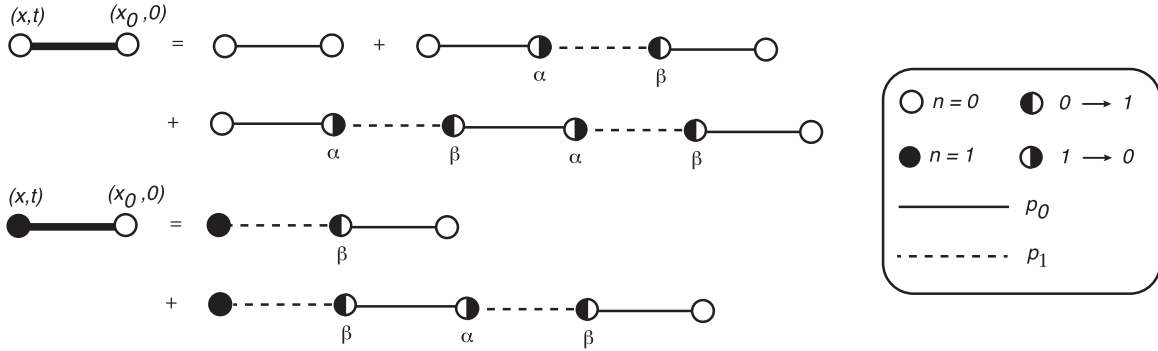


FIG. 1. First few terms in the diagrammatic expansions of the full propagators $G_{00}(\mathbf{x}, t|\mathbf{x}_0, 0)$ and $G_{10}(\mathbf{x}, t|\mathbf{x}_0, 0)$ in terms of the bare propagators $p_n(\mathbf{x}, t|\mathbf{x}_0, t_0)$ for the two-state Markov chain. α and β are the transition rates for $1 \rightarrow 0$ and $0 \rightarrow 1$, respectively. Time flows from right to left.

Eqs. (3.6b)–(3.6d). Iterating Eq. (3.6a) gives

$$\begin{aligned}
 G_{00}(\mathbf{x}, t|\mathbf{x}_0, 0) &= e^{-\beta t} p_0(\mathbf{x}, t|\mathbf{x}_0, 0) + \alpha\beta \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \int d\mathbf{y}_2 \int d\mathbf{y}_1 \\
 &\times e^{-\beta(t-\tau_1)} p_0(\mathbf{x}, t|\mathbf{y}_2, \tau_2) e^{-\alpha(\tau_2-\tau_1)} p_1(\mathbf{y}_2, \tau_2|\mathbf{y}_1, \tau_1) \\
 &\times e^{-\beta\tau_2} p_0(\mathbf{y}_1, \tau_2|\mathbf{x}_0, 0) + \dots
 \end{aligned} \quad (3.7)$$

Since the initial and final discrete states are the same, the number of switches has to be even. Using similar arguments, we obtain analogous series expansions of G_{11} , G_{01} , and G_{10} . For example, contributions to G_{11} involve sequences of transitions of the form $1 \rightarrow 0 \rightarrow 1$, whereas contributions to G_{10} involves the transition $1 \rightarrow 0$ followed by additional transitions of the form $0 \rightarrow 1 \rightarrow 0$. The first few terms in the diagrammatic expansions of G_{00} and G_{10} are shown in Fig. 1.

A few comments are in order. First, as we show in Sec. IV, the series expansion (3.7) is not uniformly convergent due to the presence of secular terms involving powers of αt and βt . Thus one cannot interpret Eq. (3.7) as a perturbation expansion in the slow switching limit $\alpha, \beta \rightarrow 0$. However, as we have already highlighted, the terms in Eq. (3.7) have a natural probabilistic interpretation based on the number of state transitions. In particular, truncating the series is equivalent to conditioning the propagator with respect to the maximum number of transitions. For a general hSDE, let $G_{nm}^{(\mu)}(\mathbf{x}, t|\mathbf{x}_0, 0)$ denote the contribution to the propagator from paths that have a maximum of μ transitions, which is given by the first $\mu + 1$ terms in the corresponding diagrammatic expansion. Taking the random variable $\mathcal{M}(t)$ to denote the number of transitions over the interval $[0, t]$, it follows that

$$\begin{aligned}
 \mathbb{P}[\mathcal{M}(t) \leq \mu] &= \int_{\mathbb{R}^d} G_{nm}^{(\mu)}(\mathbf{x}, t|\mathbf{x}_0, 0) d\mathbf{x} \\
 &= \delta_{n,m} e^{-\Gamma_n t} + \sum_{j=1}^{\mu} P_{nm}^{(j)}(t).
 \end{aligned} \quad (3.8)$$

We then introduce a renormalized propagator that is conditioned to undergo a maximum of μ transitions:

$$\mathcal{G}_{nm}^{(\mu)}(\mathbf{x}, t|\mathbf{x}_0, 0) = \frac{G_{nm}^{(\mu)}(\mathbf{x}, t|\mathbf{x}_0, 0)}{\sum_{l=0}^{K-1} \int_{-\infty}^{\infty} G_{lm}^{(\mu)}(\mathbf{x}, t|\mathbf{x}_0, 0) d\mathbf{x}}. \quad (3.9)$$

IV. OU PROCESS WITH RANDOM DRIFT

In this section we illustrate the theory by considering the particular example of an OU process with random drift. This has previously been used to model an RTP with diffusion in a harmonic potential [61,62] and protein synthesis in a gene network [4,5]. In the former case, $X(t) \in \mathbb{R}$ represents the position of the RTP at time t , whereas $N(t) = n \in \{0, 1\}$ specifies the current velocity state v_n of the particle. If $v_0 = v$ and $v_1 = -v$, then the motion becomes unbiased when the mean time spent in each velocity state is the same ($\alpha = \beta$). However, in the case of the gene network, $X(t)$ represents the concentration of synthesized protein and $N(t)$ specifies whether the gene is active or inactive. That is, v_n is the rate of synthesis with $v_0 > v_1 \geq 0$. In both examples, the variable $X(t)$ evolves according to the piecewise SDE

$$dX(t) = [-\kappa_0 X(t) + v_n] dt + \sqrt{2D} dW(t), \quad N(t) = n, \quad (4.1)$$

where κ_0 represents an effective “spring constant” for an RTP in a harmonic potential, whereas it corresponds to a protein degradation rate in the case of a gene network. Comparison with Eq. (2.1) implies that $A_n(x) = -\kappa_0 x + v_n$. One major difference between an RTP and a gene network is that the continuous variable $X(t)$ has to be positive in the latter case. However, we will assume that the effective “harmonic potential” for $v_0 > v_1 \geq 0$ restricts $X(t)$ to positive values with high probability so that we do not have to impose the condition $X(t) \geq 0$ explicitly. (If $D = 0$, then $X(t) \in \Sigma = [v_0/\kappa_0, v_1/\kappa_0]$ and the CK equation can be restricted to the finite interval Σ with reflecting boundary conditions at the ends. In this case, the steady-state CK equation can be solved explicitly [3–5].)

A. Bare propagator

First suppose that there is no switching ($\alpha = \beta = 0$). The FP equation for the bare propagator p_n is

$$\frac{\partial p_n}{\partial t} = \frac{\partial(\kappa_0 x - v_n)p_n}{\partial x} + D \frac{\partial^2 p_n}{\partial x^2}. \quad (4.2)$$

One way to determine the propagator $p_n(x, t|x_0, 0)$ is to use the fact that we have a Gaussian process so we only need

to determine the first and second moments of p_n . The k th moment is defined according to

$$m_n^{(k)}(x_0, t) = \langle X(t)^k \rangle \equiv \int_{-\infty}^{\infty} x^k p_n(x, t|x_0, 0) dx. \quad (4.3)$$

Taking expectations of both sides of Eq. (4.1) and using $\langle dW(t) \rangle = 0$ yields the deterministic differential equation

$$\frac{d\langle X \rangle}{dt} = -\kappa_0 \langle X \rangle + v_n.$$

This has the solution

$$\langle X(t) \rangle = x_0 e^{-\kappa_0 t} + \frac{v_n}{\kappa_0} (1 - e^{-\kappa_0 t}). \quad (4.4)$$

Similarly, using $\langle dX(t)dW(t) \rangle = 0$ and $dW(t)^2 = dt$, we have

$$\begin{aligned} \langle X(t+dt)^2 \rangle &= \langle [X(t) + dX(t)]^2 \rangle \\ &= \langle [(1 - \kappa_0 dt)X(t) + v_n dt + \sqrt{2D}dW(t)]^2 \rangle \\ &= (1 - \kappa_0 dt)^2 \langle X(t)X(t) \rangle \\ &\quad + v_n dt^2 + 2v_n(1 - \kappa_0 dt)X(t)dt + 2Ddt. \end{aligned}$$

Subtracting $\langle X(t)X(t) \rangle$ from both sides, dividing through by dt and taking the limit $dt \rightarrow 0$ leads to the second-order moment equation

$$\frac{d\langle X^2 \rangle}{dt} = -2\kappa_0 \langle X^2 \rangle + 2v_n \langle X \rangle + 2D, \quad (4.5)$$

which has the solution

$$\begin{aligned} \langle X(t)^2 \rangle &= e^{-2\kappa_0 t} x_0^2 + \frac{D}{\kappa_0} (1 - e^{-2\kappa_0 t}) \\ &\quad + \frac{2v_n x_0}{\kappa_0} e^{-\kappa_0 t} (1 - e^{-\kappa_0 t}) + \left(\frac{v_n}{\kappa_0}\right)^2 [1 - e^{-\kappa_0 t}]^2. \end{aligned} \quad (4.6)$$

It immediately follows that

$$\text{Var}[X(t)] = \frac{D}{\kappa_0} (1 - e^{-2\kappa_0 t}). \quad (4.7)$$

Hence, the bare propagator p_n has the explicit solution

$$\begin{aligned} p_n(x, t|x_0, 0) &= \frac{1}{\sqrt{2\pi\Sigma(t)}} \exp\left(-\frac{[x - x_0 e^{-\kappa_0 t} - v_n(1 - e^{-\kappa_0 t})/\kappa_0]^2}{2\Sigma(t)}\right), \end{aligned} \quad (4.8)$$

where

$$\Sigma(t) = \frac{D}{\kappa_0} (1 - e^{-2\kappa_0 t}). \quad (4.9)$$

B. Conditional moments

When switching is included, the moments of the hSDE are determined by the full propagator G_{nn_0} , which satisfies the equation

$$\frac{\partial G_{nn_0}}{\partial t} = \frac{\partial(\kappa_0 x - v_n)G_{nn_0}}{\partial x} + D \frac{\partial^2 G_{nn_0}}{\partial x^2} + \sum_{m=0,1} Q_{nm} G_{mno}, \quad (4.10)$$

where \mathbf{Q} is given by Eq. (2.4). The corresponding moments are

$$M_{nn_0}^{(k)}(x_0, t) = \int_{-\infty}^{\infty} dx x^k G_{nn_0}(x, t|x_0, 0). \quad (4.11)$$

Although calculating the full propagator is nontrivial, exact expressions for the moments $M_{nn_0}^{(k)}(x_0, t)$ can be obtained using statistical field theory [62]. However, as we now illustrate, the truncated and conditional moments can be evaluated in terms of the bare propagators p_n . The truncated moments are defined according to

$$M_{nn_0}^{(k,\mu)}(x_0, t) = \int_{-\infty}^{\infty} dx x^k G_{nn_0}^{(\mu)}(x, t|x_0, 0), \quad (4.12)$$

and the conditional moments are given by

$$\begin{aligned} \mathcal{M}_{nn_0}^{(k,\mu)}(x_0, t) &= \int_{-\infty}^{\infty} dx x^k \mathcal{G}_{nn_0}^{(\mu)}(x, t|x_0, 0) \\ &= \frac{M_{nn_0}^{(k,\mu)}(x_0, t)}{\sum_{l=0,1} M_{ln_0}^{(0,\mu)}(x_0, t)}, \end{aligned} \quad (4.13)$$

with $\mathcal{G}^{(\mu)}$ defined in Eq. (3.9).

For the sake of illustration, consider the first moment for $n = n_0 = 0$ and $\mu = 2$:

$$\mathcal{M}_{00}^{(1,2)}(x_0, t) = \int_{-\infty}^{\infty} dx x^k \mathcal{G}_{00}^{(2)}(x, t|x_0, 0), \quad (4.14)$$

$$= \frac{M_{00}^{(1,2)}(x_0, t)}{M_{00}^{(0,2)}(x_0, t) + M_{10}^{(0,2)}(x_0, t)}, \quad (4.15)$$

Using Eq. (3.7) we have

$$\begin{aligned} M_{00}^{(0,2)}(x_0, t) &= e^{-\beta t} + P_{00}^{(2)}(t) \\ &= e^{-\beta t} + \alpha\beta \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 e^{-\beta(t-\tau_2)} e^{-\alpha(\tau_2-\tau_1)} e^{-\beta\tau_1} \\ &= e^{-\beta t} + \frac{\alpha\beta}{\beta-\alpha} e^{-\beta t} \left[\frac{e^{(\beta-\alpha)t} - 1}{\beta-\alpha} - t \right]. \end{aligned} \quad (4.16)$$

In the limit $\beta \rightarrow \alpha$ this reduces to

$$M_{00}^{(0,2)}(x_0, t) = e^{-\alpha t} \left[1 + \frac{\alpha^2 t^2}{2} \right]. \quad (4.17)$$

Similarly,

$$\begin{aligned} M_{10}^{(0,2)}(x_0, t) &= P_{10}^{(1)}(t) = \frac{\beta}{\alpha-\beta} [1 - e^{-(\alpha-\beta)t}] \\ &\rightarrow \alpha t e^{-\alpha t} \quad \text{as } \beta \rightarrow \alpha. \end{aligned} \quad (4.18)$$

The corresponding expressions for the full zeroth moments are [62].

$$M_{00}^{(0)}(x_0, t) = e^{-\alpha t} \cosh(\alpha t), \quad M_{10}^{(0)}(x_0, t) = e^{-\alpha t} \sinh(\alpha t). \quad (4.19)$$

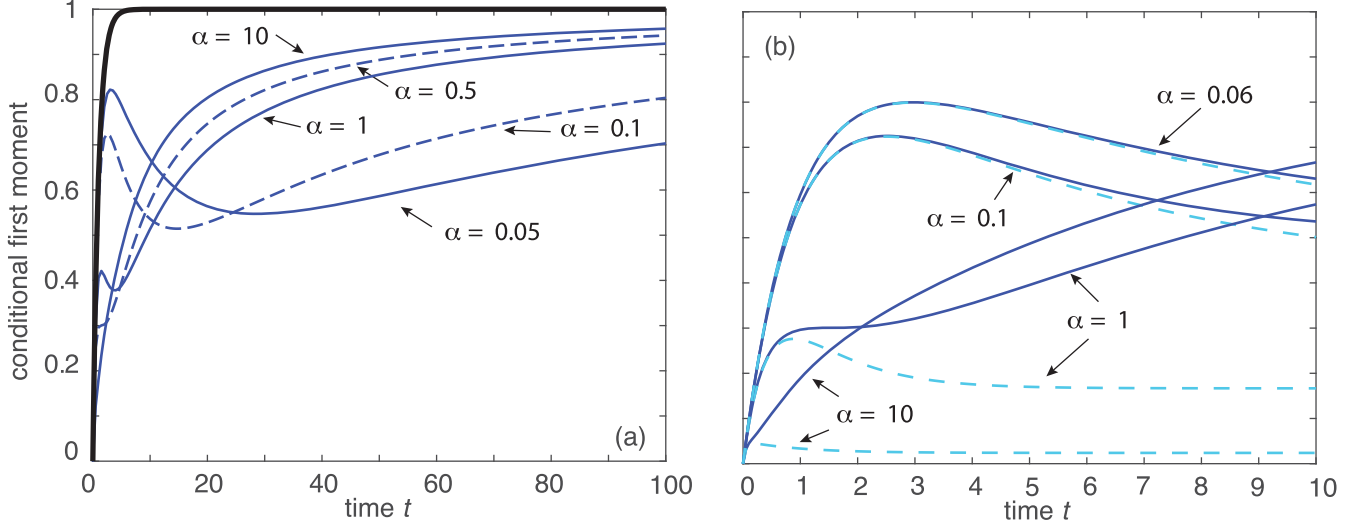


FIG. 2. (a) Plot of conditional first moment $\mathcal{M}_{00}^{(1,2)}(x_0, t)$ given by Eq. (4.22) as a function of time for different values of α . The thick curve represents the first moment $m_0^{(1)}(x_0, t)$ of the bare propagator ($\alpha = 0$). Other parameter values are $v_0 = -v_1 = 1$, $\kappa_0 = 1$ and $x_0 = 0$. The curves approach unity as $t \rightarrow \infty$. (b) Comparison of $\mathcal{M}_{00}^{(1,2)}(x_0, t)$ (solid curves) with the unconditional first moment $M_{00}^{(1)}(x_0, t)$ (dashed curves) given by Eq. (4.21).

Turning to the numerator in Eq. (4.15) we find that for $\alpha = \beta$ (see the Appendix)

$$\begin{aligned} M_{00}^{(1,2)}(x_0, t) = & x_0 \left(1 + \frac{\alpha^2 t^2}{2} \right) e^{-[\kappa_0 + \alpha]t} \\ & + \frac{v_0}{\kappa_0} \left(1 + \frac{\alpha^2 t^2}{2} \right) (1 - e^{-\kappa_0 t}) e^{-\alpha t} \\ & + \frac{2\alpha^2 (v_0 - v_1)}{\kappa_0^3} (1 - e^{-\kappa_0 t}) e^{-\alpha t} \\ & - \frac{\alpha^2 t (v_0 - v_1)}{\kappa_0^2} (1 + e^{-\kappa_0 t}) e^{-\alpha t}. \end{aligned} \quad (4.20)$$

The basic steps in the derivation of Eq. (4.20) are as follows. First, we substitute the 1D version of the series expansion (3.7) into the definition of $M_{00}^{(1,2)}(x_0, t)$, see Eq. (4.11). This reduces the calculation of the various terms to the evaluation of moments of the bare propagator p_0 , which can be carried out explicitly. If $v_0 = -v_1 = v$, then our result is consistent with Taylor expanding the exact expression [62], which can be written in the form

$$\begin{aligned} M_{00}^{(1)}(x_0, t) = & x_0 e^{-\kappa_0 t} \cosh(\alpha t) \\ & + \frac{v\kappa_0 (1 - e^{-\kappa_0 t}) \cosh(\alpha t) e^{-\alpha t}}{\kappa_0^2 - 4\alpha^2} \\ & - \frac{2v\alpha (1 + e^{-\kappa_0 t}) \sinh(\alpha t) e^{-\alpha t}}{\kappa_0^2 - 4\alpha^2}. \end{aligned} \quad (4.21)$$

(All α -dependent terms are expanded except for the exponential factors $e^{-\alpha t}$.) Note that Eqs. (4.17), (4.18), and (4.20) involve the secular terms αt and $(\alpha t)^2$. Hence, $M_{00}^{(1,2)}$ does not yield a good approximation of $M_{00}^{(1)}$ unless $t \ll 1/\alpha$. Similarly for the zeroth moments.

Finally, substituting Eqs. (4.17), (4.18), and (4.20) into Eq. (4.15) yields the following expression for the conditional

first moment given a maximum of two transitions:

$$\begin{aligned} \mathcal{M}_{00}^{(1,2)}(x_0, t) = & \frac{1 + \alpha^2 t/2}{1 + \alpha t + \alpha^2 t^2/2} \left[x_0 e^{-\kappa_0 t} + \frac{v_0}{\kappa_0} (1 - e^{-\kappa_0 t}) \right] \\ & + \frac{\alpha^2}{1 + \alpha t + \alpha^2 t^2/2} \frac{2(v_0 - v_1)}{\kappa_0^3} (1 - e^{-\kappa_0 t}) \\ & - \frac{\alpha^2 t}{1 + \alpha t + \alpha^2 t^2/2} \frac{(v_0 - v_1)}{\kappa_0^2} (1 + e^{-\kappa_0 t}). \end{aligned} \quad (4.22)$$

Note that,

$$\lim_{t \rightarrow \infty} \mathcal{M}_{00}^{(1,2)}(x_0, t) = \frac{v_0}{\kappa_0} = \lim_{t \rightarrow \infty} m_0^{(1)}(x_0, t). \quad (4.23)$$

The fact that this limit is independent of the leftward velocity v_1 reflects the fact that restricting the dynamics to two switching events means that the fraction of time spent in the right-moving state approaches unity in the limit $t \rightarrow \infty$. Note, however, that the behavior of the conditional moment $\mathcal{M}_{00}^{(1,2)}(x_0, t)$ differs significantly from the bare moment $m_0^{(1)}(x_0, t)$ for finite t . In particular, the conditional moment takes much longer to approach the steady-state v_0/κ_0 , and is a nonmonotonic function of t . This is illustrated in Fig. 2(a) for the initial position $x_0 = 0$. The nonmonotonicity reflects the fact that all paths that switch twice include a finite time interval over which the particle is moving leftward, and thus appears to be returning towards the initial position $x_0 = 0$. In Fig. 2(b), we compare the conditional moment $\mathcal{M}_{00}^{(1,2)}(t)$ with the unconditional moment $M_{00}^{(1)}$ given by Eq. (4.21) for $v_0 = -v_1 = v$. The latter has the asymptotic limit

$$\lim_{t \rightarrow \infty} M_{00}^{(1)}(x_0, t) = \frac{v}{\kappa_0 + 2\alpha}. \quad (4.24)$$

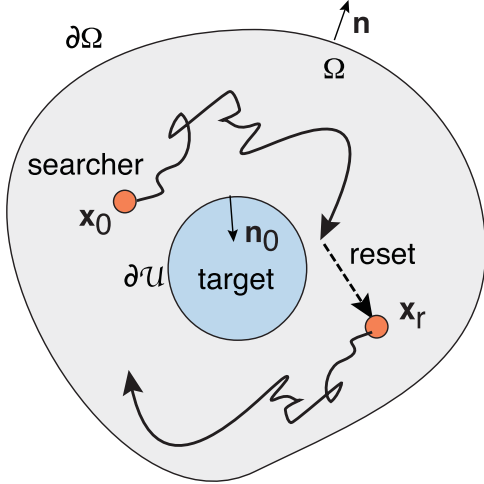


FIG. 3. Domain $\Omega \subset \mathbb{R}^d$ containing a single target \mathcal{U} with a totally absorbing surface $\partial\mathcal{U}$. Particle starts at \mathbf{x}_0 and resets to the point \mathbf{x}_r at a constant rate r .

As expected, the difference between the two moments increases with α .

V. TRUNCATED SEARCH PROCESS WITH STOCHASTIC RESETTING

We now turn to another example of a randomly switching process, namely, a search process with stochastic resetting [35]. Consider a particle (searcher) subject to Brownian motion in $\Omega \subseteq \mathbb{R}^d$, and resetting to a fixed point \mathbf{x}_r at a constant rate r . Suppose that there exists some target $\mathcal{U} \subset \Omega$ whose boundary $\partial\mathcal{U}$ is totally absorbing and $\mathbf{x}_r \notin \mathcal{U}$, see Fig. 3. The probability density $p_r(\mathbf{x}, t|\mathbf{x}_0)$ for the particle to be at position \mathbf{x} at time t given the initial position \mathbf{x}_0 evolves according to the master equation

$$\frac{\partial p_r(\mathbf{x}, t|\mathbf{x}_0)}{\partial t} = \nabla^2 p_r(\mathbf{x}, t|\mathbf{x}_0) - r p_r(\mathbf{x}, t|\mathbf{x}_0) + r \delta(\mathbf{x} - \mathbf{x}_r). \quad (5.1)$$

This is supplemented by the absorbing boundary condition $p_r(\mathbf{x}, t|\mathbf{x}_0) = 0$ for all $\mathbf{x} \in \partial\mathcal{U}$ and the reflecting boundary condition $J(\mathbf{x}, t|\mathbf{x}_0) = 0$ for all $\mathbf{x} \in \partial\Omega$. Here $J(\mathbf{x}, t|\mathbf{x}_0) = -\nabla p(\mathbf{x}, t|\mathbf{x}_0) \cdot \mathbf{n}$ with \mathbf{n} the outward normal on $\partial\Omega$. Let $Q_r(\mathbf{x}_0, t)$ be the survival probability of the particle that started at \mathbf{x}_0 :

$$Q_r(\mathbf{x}_0, t) = \int_{\Omega \setminus \mathcal{U}} p_r(\mathbf{x}, t|\mathbf{x}_0) d\mathbf{x}. \quad (5.2)$$

Integrating Eq. (5.1) with respect to $\mathbf{x} \in \Omega \setminus \mathcal{U}$ and using the divergence theorem shows that

$$\begin{aligned} \frac{\partial Q_r(\mathbf{x}_0, t)}{\partial t} &= \int_{\partial\mathcal{U}} \nabla p_r(\mathbf{x}, t|\mathbf{x}_0) \cdot \mathbf{n}_0 d\mathbf{x} - r Q_r(\mathbf{x}_0, t) \\ &\quad + r Q_r(\mathbf{x}_r, t), \end{aligned} \quad (5.3)$$

with \mathbf{n}_0 the normal into \mathcal{U} , see Fig. 3. Let \mathcal{T} denote the FPT for absorption at $\partial\mathcal{U}$. The MFPT can be expressed in terms of Q_r according to

$$\begin{aligned} T_r(\mathbf{x}_0) &\equiv \mathbb{E}[\mathcal{T}] = -\int_0^\infty t \frac{dQ_r(\mathbf{x}_0, t)}{dt} dt \\ &= \int_0^\infty Q_r(\mathbf{x}_0, t) dt. \end{aligned} \quad (5.4)$$

We have used the fact that the FPT density $f_r(\mathbf{x}_0, t)$ is related to the survival probability according to $f_r(\mathbf{x}_0, t) = -dQ_r(\mathbf{x}_0, t)/dt$.

It is well known that Q_r is related to the survival probability without resetting, Q_0 , according to a last renewal equation [36–39]:

$$\begin{aligned} Q_r(\mathbf{x}_0, t) &= e^{-rt} Q_0(\mathbf{x}_0, t) \\ &\quad + r \int_0^t e^{-r(t-\tau)} Q_0(\mathbf{x}_r, t-\tau) Q_r(\mathbf{x}_0, \tau) d\tau. \end{aligned} \quad (5.5)$$

The first term on the right-hand side represents trajectories with no resets. The integrand in the second term is the contribution from trajectories that last reset at time $\tau \in (0, t)$, and consists of the product of the survival probability starting from \mathbf{x}_0 with resetting up to time $t - \tau$ and the survival probability starting from \mathbf{x}_r without any resetting for the time interval τ . The standard method for solving the renewal equation (5.5) is to introduce the Laplace transform

$$\tilde{Q}_r(\mathbf{x}_0, s) = \int_0^\infty Q_r(\mathbf{x}_0, t) e^{-st} dt, \quad (5.6)$$

and use the convolution theorem. Thus, Laplace transforming Eq. (5.5) and rearranging shows that

$$\tilde{Q}_r(\mathbf{x}_0, s) = \frac{\tilde{Q}_0(\mathbf{x}_0, r+s)}{1 - r\tilde{Q}_0(\mathbf{x}_r, r+s)}. \quad (5.7)$$

The MFPT to reach the target is then given by

$$T_r(\mathbf{x}_0) = \tilde{Q}_r(\mathbf{x}_0, 0) = \frac{\tilde{Q}_0(\mathbf{x}_0, r)}{1 - r\tilde{Q}_0(\mathbf{x}_r, r)}. \quad (5.8)$$

A. Splitting probabilities and conditional MFPTs

Following our analysis of hybrid SDEs, we now consider a truncated version of the search process, in which the maximum number of resets is fixed. This is equivalent to truncating the Volterra series expansion of the renewal equation, which in the time domain takes the form

$$\begin{aligned} Q_r(\mathbf{x}_0, t) &= e^{-rt} Q_0(\mathbf{x}_0, t) + r e^{-rt} \int_0^t Q_0(\mathbf{x}_0, \tau) Q_0(\mathbf{x}_r, t-\tau) d\tau \\ &\quad + r^2 e^{-rt} \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 Q_0(\mathbf{x}_0, \tau_1) Q_0(\mathbf{x}_r, \tau_2) Q_0(\mathbf{x}_r, t-\tau_1-\tau_2) + \dots \end{aligned} \quad (5.9)$$

The corresponding expansion in Laplace space is a geometric series in powers of $r\tilde{Q}_0(\mathbf{x}_r, r+s)$. The ℓ th term in the series (5.9), $\ell \geq 0$, is the joint probability $Q_{r,\ell}(\mathbf{x}_0, t)$ that the particle hasn't been absorbed and has reset exactly ℓ times:

$$Q_{r,\ell}(\mathbf{x}_0, t) = r^\ell e^{-rt} [Q_0(\mathbf{x}_0, \cdot) \otimes Q_0(\mathbf{x}_r, \cdot) \otimes \dots \otimes Q_0(\mathbf{x}_r, \cdot)](t), \quad (5.10)$$

where $Q_0 \otimes Q_0$ denotes the ℓ th order convolution:

$$\begin{aligned} & Q_0 \otimes Q_0(\mathbf{x}_r, t) \\ &= \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \dots \int_0^{t-\sum_{j=1}^{\ell-1} \tau_j} d\tau_{\ell-1} \\ & \quad \times Q_0(\mathbf{x}_r, \tau_1) Q_0(\mathbf{x}_r, \tau_2) \dots Q_0\left(\mathbf{x}_r, t - \sum_{j=1}^{\ell} \tau_j\right). \end{aligned} \quad (5.11)$$

The probability that there are ℓ reset events in the interval $[0, t]$ is given by the Poisson distribution

$$P_\ell(t) = \frac{(rt)^\ell e^{-rt}}{\ell!}. \quad (5.12)$$

Hence, $Q_{r,\ell}(\mathbf{x}_0, t)/P_\ell(t)$ is the survival probability conditioned on exactly ℓ reset events in $[0, t]$.

In Ref. [60] the joint probability distribution for the number of resets, the time of absorption, and a general cost was calculated. One result from that analysis was an expression for the probability distribution $P(N|\mathbf{x}_0)$ given N resets up to the time of absorption with $\mathbf{x}_r = \mathbf{x}_0$. In our notation,

$$\begin{aligned} & P(N|\mathbf{x}_0) \\ &= \int_0^\infty \left[\int_0^t e^{-r(t-\tau)} f_0(t-\tau, \mathbf{x}_0) r Q_{r,N-1}(\tau, \mathbf{x}_0) d\tau \right] dt, \end{aligned} \quad (5.13)$$

where $f_0(t, \mathbf{x}_0)$ is the reset-free first passage time density for absorption. This equation can be interpreted as follows. First, we suppose that the N th reset occurs in the time interval $[\tau, \tau + d\tau]$ and the particle has not yet been absorbed, which is given by the probability $r Q_{r,N-1}(\tau, \mathbf{x}_0) d\tau$. The probability density that there are no more resets and the particle is absorbed at time t is then $e^{-r(t-\tau)} f_0(t-\tau, \mathbf{x}_0)$. Integrating with respect to τ and t then yields $P(N|\mathbf{x}_0)$. We can rewrite the right-hand side of Eq. (5.13) using Laplace transforms so that

$$\begin{aligned} & P(N|\mathbf{x}_0) = r \tilde{f}_0(r, \mathbf{x}_0) \tilde{Q}_{r,N-1}(0, \mathbf{x}_0) \\ &= \tilde{f}_0(r, \mathbf{x}_0) [r \tilde{Q}_0(r, \mathbf{x}_0)]^N, \end{aligned} \quad (5.14)$$

which recovers the result obtained in Ref. [60].

In contrast to Ref. [60], we assume that Brownian motion is killed when either (a) the particle reaches $\partial\mathcal{U}$ or (b) it resets for the $(\mu + 1)$ th times. The unconditional FPT density is then

$$f_r^{(\mu)}(\mathbf{x}_0, t) = -\frac{dQ_r^{(\mu)}(\mathbf{x}_0, t)}{dt}, \quad (5.15)$$

where $Q_r^{(\mu)}(\mathbf{x}_0, t)$ is the corresponding survival probability:

$$Q_r^{(\mu)}(\mathbf{x}_0, t) = \sum_{\ell=0}^{\mu} Q_{r,\ell}(\mathbf{x}_0, t). \quad (5.16)$$

Since $Q_r^{(\mu)}(\mathbf{x}_r, 0) = 1$ and $\lim_{t \rightarrow \infty} Q_r^{(\mu)}(\mathbf{x}_r, t) = 0$, the FPT density has unit normalization. Using similar arguments to previous examples, the unconditional MFPT is

$$\begin{aligned} T_r^{(\mu)}(\mathbf{x}_0) &= \tilde{Q}_r^{(\mu)}(\mathbf{x}_0, 0) = \tilde{Q}_0(\mathbf{x}_0, r) \sum_{\ell=0}^{\mu} (r \tilde{Q}_0(\mathbf{x}_r, r))^\ell \\ &= \tilde{Q}_0(\mathbf{x}_0, r) \frac{1 - (r \tilde{Q}_0(\mathbf{x}_r, r))^{\mu+1}}{1 - r \tilde{Q}_0(\mathbf{x}_r, r)}. \end{aligned} \quad (5.17)$$

If we wish to distinguish between the two types of killing events, then we need to determine the splitting probabilities and conditional MFPTs. Let $p_{r,\ell}(\mathbf{x}, t|\mathbf{x}_0)$ denote the joint probability density for particle position at time t and the number ℓ of resets in the interval $[0, t]$. The forward equation for $p_{r,\ell}$ is

$$\begin{aligned} \frac{\partial p_{r,\ell}(\mathbf{x}, t|\mathbf{x}_0)}{\partial t} &= \nabla^2 p_{r,\ell}(\mathbf{x}, t|\mathbf{x}_0) - r p_{r,\ell}(\mathbf{x}, t|\mathbf{x}_0) \\ & \quad + r \delta(\mathbf{x} - \mathbf{x}_r) Q_{r,\ell-1}(\mathbf{x}_0, t). \end{aligned} \quad (5.18)$$

Integrating with respect to $\mathbf{x} \in \Omega \setminus \mathcal{U}$ implies that

$$\begin{aligned} \frac{\partial Q_{r,\ell}(\mathbf{x}_0, t)}{\partial t} &= \int_{\partial\mathcal{U}} \nabla p_{r,\ell}(\mathbf{x}, t|\mathbf{x}_0) \cdot \mathbf{n}_0 d\mathbf{x} - r Q_{r,\ell}(\mathbf{x}_0, t) \\ & \quad + r Q_{r,\ell-1}(\mathbf{x}_0, t) \\ &= -J_{a,\ell}(\mathbf{x}_0, t) - J_{b,\ell}(\mathbf{x}_0, t), \end{aligned} \quad (5.19)$$

with $Q_{r,-1} \equiv 0$. Here $J_{a,\ell}(\mathbf{x}_0, t)$ is the probability flux into the surface $\partial\mathcal{U}$,

$$J_{a,\ell}(\mathbf{x}_0, t) = - \int_{\partial\mathcal{U}} \nabla p_{r,\ell}(\mathbf{x}, t|\mathbf{x}_0) \cdot \mathbf{n}_0 d\mathbf{x}, \quad (5.20)$$

whereas $J_{b,\ell}(\mathbf{x}_0, t)$ is the probability flux associated with resetting,

$$J_{b,\ell}(\mathbf{x}_0, t) = r Q_{r,\ell}(\mathbf{x}_0, t) - r Q_{r,\ell-1}(\mathbf{x}_0, t). \quad (5.21)$$

Let $\pi_a^{(\mu)}(\mathbf{x}_0)$ and $\pi_b^{(\mu)}(\mathbf{x}_0)$ denote, respectively, the splitting probabilities for absorption at \mathcal{U} and resetting for the $(\mu + 1)$ th time. Then

$$\begin{aligned} \pi_a^{(\mu)}(\mathbf{x}_0) &= \int_0^\infty \left[\sum_{\ell=0}^{\mu} J_{a,\ell}(\mathbf{x}_0, t) \right] dt \\ &= \lim_{s \rightarrow 0} \sum_{\ell=0}^{\mu} \tilde{J}_{a,\ell}(\mathbf{x}_0, s), \end{aligned} \quad (5.22)$$

and

$$\begin{aligned} \pi_b^{(\mu)}(\mathbf{x}_0) &= \int_0^\infty \left[\sum_{\ell=0}^{\mu} J_{b,\ell}(\mathbf{x}_0, t) \right] dt = r \int_0^\infty Q_{r,\mu}(\mathbf{x}_0, t) dt \\ &= r \lim_{s \rightarrow 0} \tilde{Q}_{r,\mu}(\mathbf{x}_0, s). \end{aligned} \quad (5.23)$$

To determine the Laplace transformed flux $\tilde{J}_a(\mathbf{x}_0, s)$, we Laplace transform Eq. (5.18) under the initial condition $p_{r,\ell}(\mathbf{x}, 0|\mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \delta_{\ell,0}$. This yields the equation

$$\begin{aligned} & \nabla^2 \tilde{p}_{r,\ell}(\mathbf{x}, s|\mathbf{x}_0) - (r+s) \tilde{p}_{r,\ell}(\mathbf{x}, s|\mathbf{x}_r) \\ &= -\delta(\mathbf{x} - \mathbf{x}_0) \delta_{\ell,0} - \delta(\mathbf{x} - \mathbf{x}_r) r \tilde{Q}_{r,\ell-1}(\mathbf{x}_0, s). \end{aligned} \quad (5.24)$$

Introduce the Green's function $G(\mathbf{x}, s|\mathbf{y})$ with

$$\nabla^2 G(\mathbf{x}, s|\mathbf{y}) - sG(\mathbf{x}, s|\mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}), \quad (5.25)$$

together with the boundary conditions $\nabla G \cdot \mathbf{n} = 0$ for all $\mathbf{x} \in \partial\Omega$ and $G(\mathbf{x}, s|\mathbf{y}) = 0$ for all $\mathbf{x} \in \partial\mathcal{U}$. We can then write the solution for $\tilde{p}_{r,\ell}$ as

$$\begin{aligned} \tilde{p}_{r,\ell}(\mathbf{x}, s|\mathbf{x}_r) &= G(\mathbf{x}, s+r|\mathbf{x}_0)\delta_{\ell,0} \\ &+ rG(\mathbf{x}, s+r|\mathbf{x}_r)\tilde{Q}_{r,\ell-1}(\mathbf{x}_0, s). \end{aligned} \quad (5.26)$$

Combining with the Laplace transform of Eq. (5.20), we have

$$\begin{aligned} \tilde{J}_{a,\ell}(\mathbf{x}_0, s) &= -\delta_{\ell,0} \int_{\partial\mathcal{U}} \nabla G(\mathbf{x}, s+r|\mathbf{x}_0) \cdot \mathbf{n}_0 d\mathbf{x} \\ &- r\tilde{Q}_{r,\ell-1}(\mathbf{x}_0, s) \int_{\partial\mathcal{U}} \nabla G(\mathbf{x}, s+r|\mathbf{x}_r) \cdot \mathbf{n}_0 d\mathbf{x}. \end{aligned} \quad (5.27)$$

Note that $-\int_{\partial\mathcal{U}} \nabla G(\mathbf{x}, s|\mathbf{x}_0) \cdot \mathbf{n}_0 d\mathbf{x}$ can be identified with the Laplace transform of the probability flux into the target in the absence of resetting, which we denote by $\tilde{J}_0(\mathbf{x}_0, s)$. Finally, substituting this solution into Eq. (5.22) gives

$$\begin{aligned} \pi_a^{(\mu)}(\mathbf{x}_0) &= \lim_{s \rightarrow 0} \tilde{J}_0(\mathbf{x}_0, r+s) \\ &+ r \lim_{s \rightarrow 0} \left[\tilde{J}_0(\mathbf{x}_r, r+s) \sum_{\ell=0}^{\mu-1} \tilde{Q}_{r,\ell}(\mathbf{x}_0, s) \right]. \end{aligned} \quad (5.28)$$

Let $\mathcal{T}_a^{(\mu)}(\mathbf{x}_0)$ be the FPT that the particle is absorbed at $\partial\mathcal{U}$ having started at \mathbf{x}_0 . Since there is a nonzero probability that the particle never exits at a point on $\partial\mathcal{U}$ due to resetting for the $(\mu+1)$ th time prior to absorption, it follows that the unconditional MFPT $\mathbb{E}[\mathcal{T}_a^{(\mu)}(y)] = \infty$. This motivates the introduction of the conditional MFPT,

$$T_a^{(\mu)}(\mathbf{x}_0) = \mathbb{E}[\mathcal{T}_a^{(\mu)}(\mathbf{x}_0) | \mathcal{T}_a^{(\mu)}(\mathbf{x}_0) < \infty]. \quad (5.29)$$

The conditional FPT density for absorption is

$$f_a^{(\mu)}(\mathbf{x}_0, t) = \frac{\sum_{\ell=0}^{\mu} J_{a,\ell}(\mathbf{x}_0, t)}{\pi_a^{(\mu)}(\mathbf{x}_0)}, \quad (5.30)$$

so that

$$\begin{aligned} T_a^{(\mu)}(\mathbf{x}_0) &= \pi_a^{(\mu)}(\mathbf{x}_0)^{-1} \sum_{\ell=0}^{\mu} \int_0^{\infty} t J_{a,\ell}(\mathbf{x}_0, t) dt \\ &= -\pi_a^{(\mu)}(\mathbf{x}_0)^{-1} \sum_{\ell=0}^{\mu} \lim_{s \rightarrow 0} \partial_s \tilde{J}_{a,\ell}(\mathbf{x}_0, s). \end{aligned} \quad (5.31)$$

Similarly, the conditional MFPT for the $(\mu+1)$ th reset is

$$\begin{aligned} T_b^{(\mu)}(\mathbf{x}_0) &= \mathbb{E}[\mathcal{T}_b^{(\mu)}(\mathbf{x}_0) | \mathcal{T}_b^{(\mu)}(\mathbf{x}_0) < \infty] \\ &= \pi_b^{(\mu)}(\mathbf{x}_0)^{-1} \sum_{\ell=0}^{\mu} \int_0^{\infty} t J_{b,\ell}(\mathbf{x}_0, t) dt \\ &= -\pi_b^{(\mu)}(\mathbf{x}_0)^{-1} \sum_{\ell=0}^{\mu} \lim_{s \rightarrow 0} \partial_s \tilde{J}_{b,\ell}(\mathbf{x}_0, s) \\ &= -\pi_b^{(\mu)}(\mathbf{x}_0)^{-1} r \lim_{s \rightarrow 0} \partial_s \tilde{Q}_{r,\mu}(\mathbf{x}_0, s). \end{aligned} \quad (5.32)$$

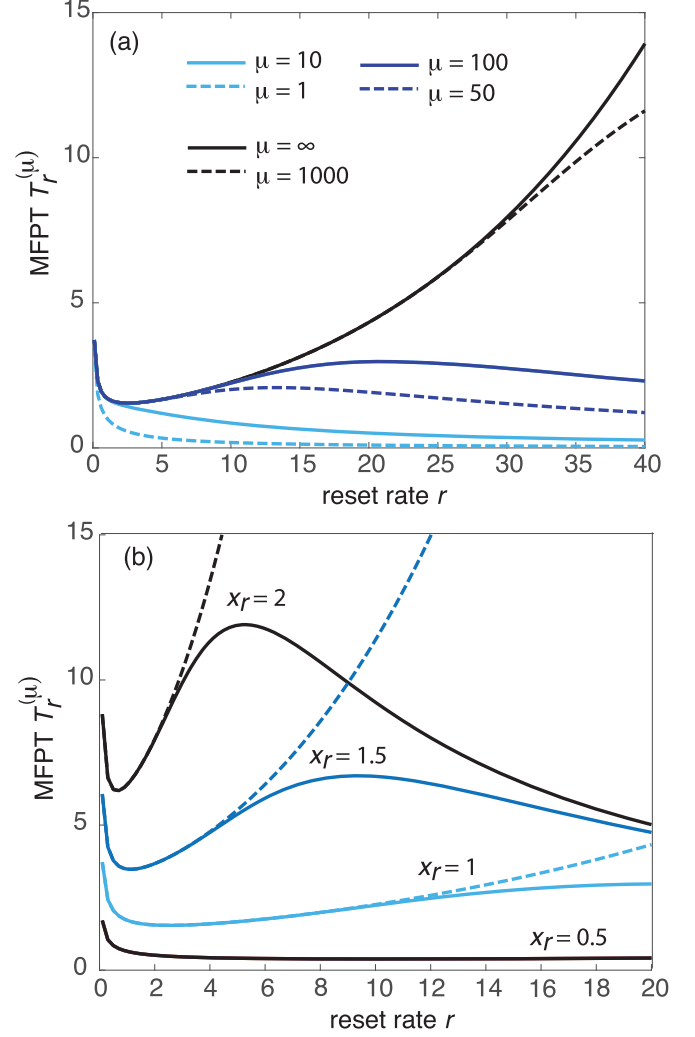


FIG. 4. Plot of the MFPT $T_r^{(\mu)}(x_r)$ as a function of r for a Brownian particle on the half-line that is killed either by reaching the boundary $x = 0$ or resetting for the $(\mu+1)$ th time. (a) Various μ for $x_r = 1$. (b) Various x_r for $\mu = 100$ (solid curves) and $\mu = \infty$ (dashed curves).

B. Diffusion on the half-line

Consider a diffusing particle on the half-line $[0, \infty)$ with an absorbing target at $x = 0$. For simplicity, we set $x_r = x_0$. In the absence of resetting the Laplace transformed survival probability $\tilde{Q}_0(x, s)$ satisfies the equation

$$D \frac{d^2 \tilde{Q}_0}{dx^2} - s \tilde{Q}_0 = -1, \quad x \in (0, \infty), \quad (5.33)$$

together with the boundary condition

$$\tilde{Q}_0(0, s) = 0. \quad (5.34)$$

The solution takes the form [36,37]

$$\tilde{Q}_0(x_r, s) = \frac{1 - e^{-\sqrt{s/D}|x_r|}}{s}, \quad (5.35)$$

which can be inverted to give the error function

$$Q_0(x_r, t) = \text{erf}(x_r/2\sqrt{Dt}). \quad (5.36)$$

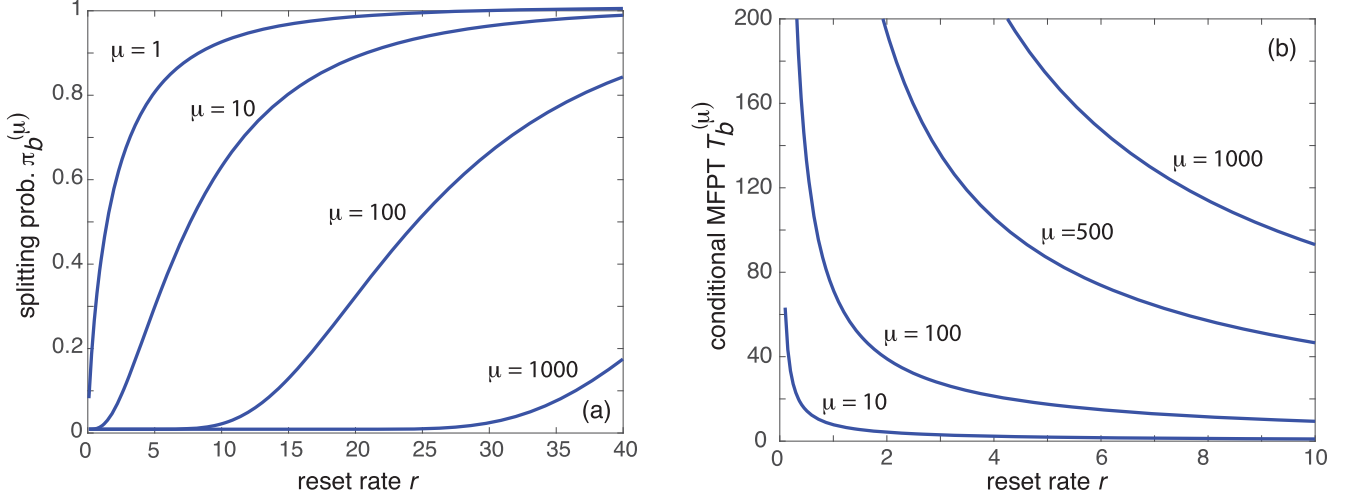


FIG. 5. (a) Plot of the splitting probability $\pi_b^{(\mu)}$ as a function of r for the particle to reset for the $(\mu + 1)$ th time before being absorbed at $\partial\mathcal{U}$. (b) Corresponding plots of the conditional MFPT $T_b^{(\mu)}$. We set $x_r = 1$ and $D = 1$.

Equation (5.8) then implies that

$$T_r(x_r) = \frac{1}{r}(e^{\sqrt{r/D}x_r} - 1). \quad (5.37)$$

Note that in the limit $r \rightarrow 0$, the MFPT diverges as $T_r \sim 1/\sqrt{r}$, which recovers the result that the MFPT of a Brownian particle without resetting to return to the origin is infinite. One also finds that T_r diverges in the limit $r \rightarrow \infty$, since the particle resets to X_r so often that it never has the chance to reach the origin. Finally, the MFPT has a finite and unique minimum at an intermediate value of the resetting rate r [37,38].

The situation is very different if we restrict the maximum number of resets to be μ . For sufficiently small r , the probability that the particle reaches the maximum μ becomes negligible so that the major contribution to the MFPT is absorption at $x = 0$. Assuming that r_{opt} occurs within this regime, we expect the MFPT to have a local minimum as in the absence of truncation. However, as r increases beyond r_{opt} ,

the main contribution to the unconditional MFPT arises from the particle exceeding μ resets. Moreover, the expected time to reset $\mu + 1$ times is a monotonically decreasing function of r . Hence, for a wide range of parameter values we expect the MFPT to have a local minimum for small r and to be monotonically decreasing function of r as $r \rightarrow \infty$. This is indeed found to be the case. In particular, Eq. (5.17) implies that

$$T_r^{(\mu)}(x_r) = \frac{1}{r}(e^{\sqrt{r/D}x_r} - 1)[1 - (1 - e^{-\sqrt{r/D}|x_r|})^{\mu+1}]. \quad (5.38)$$

In Fig. 4(a) we plot $T_r^{(\mu)}(x_r)$ as a function of the resetting rate r for various values of μ and fixed x_r . Corresponding plots for various reset positions x_r and fixed μ are shown in Fig. 4(b). It can also be seen that the value of r where truncation starts to have a noticeable effect decreases as x_r increases.

Turning to the splitting probabilities and conditional MFPTs, we use the identities $1 = \pi_a^{(\mu)}(x_r) + \pi_b^{(\mu)}(x_r)$ and

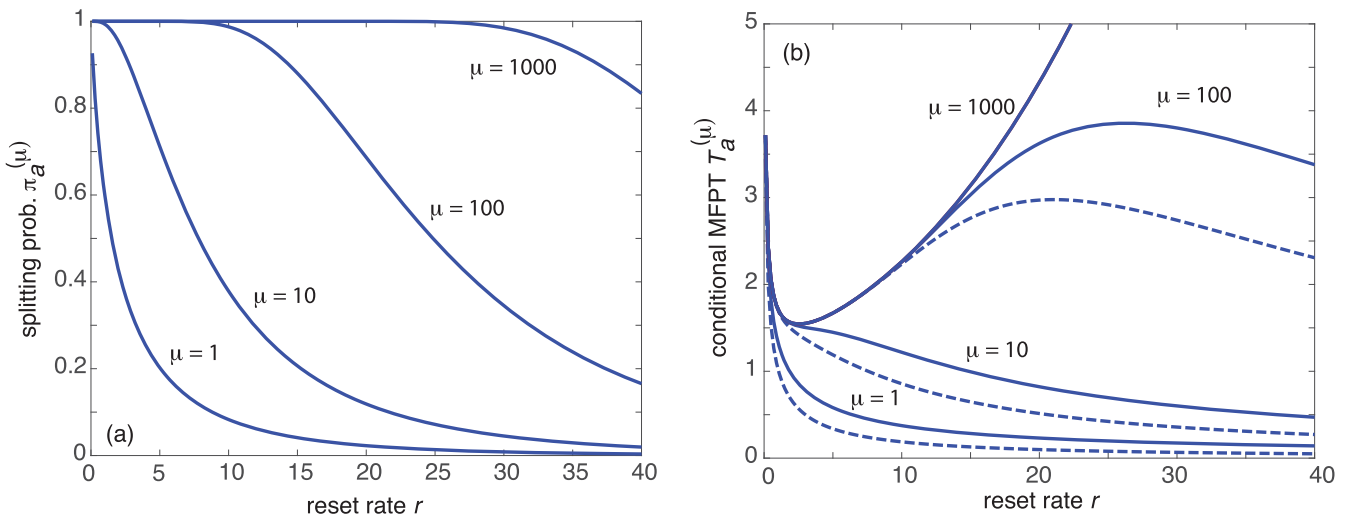


FIG. 6. (a) Plot of the splitting probability $\pi_a^{(\mu)}$ as a function of r for the particle to be absorbed at $\partial\mathcal{U}$ before resetting for the $(\mu + 1)$ th time. (b) Corresponding plots of the conditional MFPT $T_a^{(\mu)}$ (solid curves) and the unconditional MFPT $T_r^{(\mu)}$ (dashed curves). For $\mu = 1000$ the two curves coincide over the given range of r . We set $x_r = 1$ and $D = 1$.

$T_r^{(\mu)} = \pi_a^{(\mu)} T_a^{(\mu)} + \pi_b^{(\mu)} T_b^{(\mu)}$. This means that, given $T_r^{(\mu)}$, we only need to calculate $\pi_b^{(\mu)}$ and $T_b^{(\mu)}$. First, using Eq. (5.23) we have

$$\begin{aligned} \pi_b^{(\mu)}(x_r) &= r \tilde{Q}_{r,\mu}(x_r, 0) = (r \tilde{Q}_0(x_r, r))^{\mu+1} \\ &= (1 - e^{-\sqrt{r/D}x_r})^{\mu+1}. \end{aligned} \quad (5.39)$$

Second, Eq. (5.32) becomes

$$\begin{aligned} \pi_b^{(\mu)}(x_r) T_b^{(\mu)}(x_r) &= -r(\mu+1)(r \tilde{Q}_0(x_r, r))^\mu \partial_s \tilde{Q}_0(x_r, r) \\ &= \frac{\mu+1}{r} (1 - e^{-\sqrt{r/D}x_r})^\mu \\ &\quad \times \left[1 - \frac{(2 + x_r \sqrt{r/D}) e^{-\sqrt{r/D}x_r}}{2} \right]. \end{aligned} \quad (5.40)$$

Example plots of $\pi_b^{(\mu)}$ and $T_b^{(\mu)}$ are shown in Fig. 5. As expected, the probability $\pi_b^{(\mu)}$ that the particle resets for the $(\mu+1)$ th time before being absorbed decreases as the maximum reset threshold μ is increased. However, it is an increasing function of r . The conditional MFPT for exceeding the reset threshold μ is a monotonically decreasing function of r and a monotonically increasing function of μ . This is consistent with the idea that, all other things being equal, a faster reset rate reduces the time to reach $\mu+1$. In Fig. 6 we show corresponding plots of $\pi_b^{(\mu)}$ and $T_b^{(\mu)}$ with

$$\pi_a^{(\mu)} = 1 - \pi_b^{(\mu)}, \quad T_a^{(\mu)} = \frac{T_r^{(\mu)} - \pi_b^{(\mu)} T_b^{(\mu)}}{\pi_a^{(\mu)}}. \quad (5.41)$$

VI. DISCUSSION

In this paper we explored the effects of restricting the maximum number of switching events in a stochastic hybrid system, under the assumption that switching costs energy. We considered two distinct classes of switching dynamics; (i) an hSDE and (ii) diffusion with stochastic resetting. In the former case, we truncated a Volterra series expansion of the particle propagator, and used this to define a renormalized propagator in which the maximum number of switching events is fixed. We illustrated the theory by calculating the renormalized moments of an OU process with random drift. In case (ii), we truncated a Volterra series expansion of the survival probability of a Brownian particle searching for an absorbing target. This led to a modified FPT problem in which the search is terminated when either the particle finds the

target or the number of resets exceeds a fixed threshold. We calculated the splitting probabilities and conditional MFPTs for these mutually exclusive events.

There are a number of natural extensions of the current work. The first is to calculate renormalized propagators for hSDEs beyond the example of a one-dimensional OU process with random drift. One of the challenges is that there are few examples where the bare propagators p_n are known exactly. Moreover, in many cases, the matrix generator \mathbf{Q} depends on the continuous state $\mathbf{X}(t)$ at time t . One notable example is a gene network that is regulated by its own protein product [3]. Suppose that the promoter has a single operator site OS_1 for binding protein X . The gene is assumed to be OFF when X is bound to the promoter and ON otherwise. If O_0 and O_1 denote the unbound and bound promoter states, then the corresponding state transitions are $O_0 \xrightarrow{\beta x} O_1$ and $O_1 \xrightarrow{\alpha} O_0$, where x is the concentration of X . Equation (4.1) still holds but the matrix generator becomes

$$\mathbf{Q}(x) = \begin{pmatrix} -\beta x & \alpha \\ \beta x & -\alpha \end{pmatrix}. \quad (6.1)$$

A second example is protein concentration gradient formation during a particular stages of cell polarization in *Caenorhabditis elegans* zygotes. Experimentally, it found that the underlying mechanism relies on space-dependent switching between fast and slow diffusion [63], see also the theoretical studies of Refs. [64,65]. Another future direction would be to consider other examples of truncated search processes with stochastic resetting. This could involve modifying the underlying stochastic search dynamics (e.g., active particles, Lèvy flights, etc.) or introducing delays such as refractory periods and finite return times. Finally, it would be interesting to modify the additive rule for energy cost along the lines of Ref. [60] by taking the cost of each reset to depend on the distance the particle has to travel to the reset point. This would imply that the threshold μ for the number of resets before the search process is killed is itself a random variable that depends on the history of previous resets.

APPENDIX: CALCULATION OF TRUNCATED FIRST MOMENT $M_{00}^{(1,2)}$

Consider the truncated first moment of the full propagator $G_{00}(x, t|x_0, 0)$, which is defined according to

$$M_{00}^{(1,2)}(x_0, t) = \int_{-\infty}^{\infty} dx x G_{00}^{(2)}(x, t|x_0, 0). \quad (A1)$$

Using Eq. (3.7) and the expression for the first moment $m_0^{(1)}(x_0, t)$ of the propagator p_0 gives

$$\begin{aligned} M_{00}^{(1,2)}(x_0, t) &= e^{-\beta t} m_0^{(1)}(x_0, t) + \alpha \beta \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\beta(t-\tau_1)} m_0^{(1)}(x_1, t-\tau_1) \\ &\quad \times e^{-\alpha(\tau_1-\tau_2)} p_1(x_1, \tau_1|x_2, \tau_2) e^{-\beta\tau_2} p_0(x_2, \tau_2|x_0, 0) \\ &= e^{-\beta t} m_0^{(1)}(x_0, t) + \alpha \beta \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\beta(t-\tau_1)} e^{-\alpha(\tau_1-\tau_2)} \\ &\quad \times \left\{ \frac{v_0}{\kappa_0} (1 - e^{-\kappa_0[t-\tau_1]}) + x_1 e^{-\kappa_0[t-\tau_1]} \right\} e^{-\beta\tau_2} p_1(x_1, \tau_1|x_2, \tau_2) p_0(x_2, \tau_2|x_0, 0), \end{aligned} \quad (A2)$$

after using Eq. (4.4). Performing the integration with respect to x_1 then gives

$$M_{00}^{(1,2)}(x_0, t) = e^{-\beta t} m_0^{(1)}(x_0, t) + \alpha\beta \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-\infty}^{\infty} dx_2 e^{-\beta(t-\tau_1)} e^{-\alpha(\tau_1-\tau_2)} e^{-\beta\tau_2} \\ \times \left[\frac{v_0}{\kappa_0} (1 - e^{-\kappa_0[t-\tau_1]}) + \left\{ \frac{v_1}{\kappa_0} (1 - e^{-\kappa_0[\tau_1-\tau_2]}) + x_2 e^{-\kappa_0[\tau_1-\tau_2]} \right\} e^{-\kappa_0[t-\tau_1]} \right] p_0(x_2, \tau_2 | x_0, 0). \quad (\text{A3})$$

Integrating with respect to x_2 we have

$$M_{00}^{(1,2)}(x_0, t) = e^{-\beta t} m_0^{(1)}(x_0, t) + \alpha\beta \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-\beta(t-\tau_1)} e^{-\alpha(\tau_1-\tau_2)} e^{-\beta\tau_2} \\ \times \left\{ \frac{v_0}{\kappa_0} (1 - e^{-\kappa_0[t-\tau_1]}) + \frac{v_1}{\kappa_0} (1 - e^{-\kappa_0[\tau_1-\tau_2]}) e^{-\kappa_0[t-\tau_1]} + \left[\frac{v_0}{\kappa_0} (1 - e^{-\kappa_0\tau_2}) + x_0 e^{-\kappa_0\tau_2} \right] e^{-\kappa_0[t-\tau_2]} \right\}. \quad (\text{A4})$$

Finally, setting $\beta = \alpha$ and computing the time integrals yields Eq. (4.20).

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