Dynamics at the edge for independent diffusing particles

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We study the dynamics of the outliers for a large number of independent Brownian particles in one dimension. We derive the multitime joint distribution of the position of the rightmost particle, by two different methods. We obtain the two-time joint distribution of the maximum and second maximum positions, and we study the counting statistics at the edge. Finally, we derive the multitime joint distribution of the running maximum, as well as the joint distribution of the arrival time of the first particle at several space points.

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I. INTRODUCTION

The maximum of a large number $N \gg 1$ of identical independent Brownian motions, started from the origin in one dimension, properly rescaled and centered, is distributed according to the Gumbel distribution, one of the three classes of extreme value statistics [1–4]. Recently, there has been renewed interest in the statistics of the particles at the edge of a cloud of Brownian particles diffusing in a common space-time-dependent random environment [5–13]. For a large number of particles, it was shown that on top of the Gumbel fluctuations, there is a random, environment-dependent shift in the position of the rightmost particle. Furthermore, the statistics of this shift was found to be related to those of some solutions of the Kardar-Parisi-Zhang equation. Tracer diffusion experiments, involving colloid or photons, are presently aiming to test that prediction [14].

This prediction is about the position of the maximum at a given time, a one-time observable. It would be interesting to extend it to multitime observables. With this longstanding aim in mind, one can start by asking about a much simpler problem, namely multitime observables for identical independent Brownian motions, in the absence of a background environment. That should be useful, e.g., as a benchmark in the analysis of such experiments.

In the present paper we study the dynamics of the outliers for a large number of independent Brownian particles on the line, all starting at the origin. We derive the multitime joint distribution of the position of the rightmost particle, i.e., of the maximum of all the particle positions, by two elementary methods, which lead to different, though equivalent, formulas. The first method is standard for extreme value statistics, the second uses the diffusion equation. We obtain explicit formula for the cumulants of the fluctuations of the maximum at different times. We then extend these results to the multitime joint distribution of the maximum and of the second maximum particle positions. In parallel, we obtain some predictions about the counting statistics at the edge of the cloud, which describes the outliers. Next, we study some continuum time observables of the rescaled maximum process. We obtain the probability that it remains below some level during some time interval. We study the running maximum, that is the maximum over all the particles and up to some fixed time, and obtain its multitime joint distribution. Finally, we obtain the joint distribution of the arrival times of the first particle (i.e., the first detection times) at different locations in space, and the distribution of the time delays between these detection events.

It must be noted that this class of problems is related to the so-called multivariate extreme value theory and max-stable processes, which has a long history, starting with the Brown-Resnick process [15,16] and the Husler-Reiss (HR) distributions [17,18]. Since these seminal papers there were a number of extensions [19–28], though apparently mainly in the fields of probability and statistics. Our modest aim here is to instead study the problem with simple heuristic methods of statistical physics. In the course of our work we will encounter some known objects, such as the HR distributions, in sometimes different forms, but we will also study more general outlier properties, counting statistics, the running maximum problem, and arrival time statistics. One must also note an independent work in preparation on these topics [29], following upon the recent work [30] which studies counting statistics for stochastic processes with an extended initial condition.

The outline of the paper is as follows. In Sec. II we focus on one-time observables (maximum, order, and counting statistics of outliers) and recall the standard results obtained for these quantities from extreme value statistics of i.i.d. random variables. In Sec. III (and Appendix A) we give a first derivation of the multitime distribution of the maximum. In Sec. IV we give more explicit formula for some marginals, moments and two- and three-time correlations of the maximum. The relevant calculations are detailed in Appendices B and C. In Sec. V we present a second derivation of the multitime distribution of the maximum based on the heat equation. It naturally allows to obtain these distributions recursively. In Sec. VI we study some multitime properties of the outliers, e.g., we obtain the multitime joint distribution for the maximum and second maximum, either directly, in Appendix D, or by studying counting statistics near the edge of the cloud. In Appendix E it is indicated how to extend these results to any rank at number of times. Finally, in Sec. VII and Appendix F we study continuum time observables of the rescaled maximum process, as well as the multitime statistics of the running maximum and of the arrival times of the first particle.

II. OUTLIERS AT A GIVEN TIME

Let us start by recalling standard results of extreme value statistics [1–4], applied to one-time observables for diffusing particles.

A. Maximum at a given time

Let us consider a particle whose position $x(t)$ evolves according to a random process. Let us denote $p_t(x)$ the one-time probability distribution function (PDF) of the position at time t. The example on which we will focus will be a Brownian particle, $x(t) = \sqrt{D}B(t)$, started at $x = 0$ at time $t = 0$. In that case

$$p_t(x) = \frac{1}{\sqrt{2\pi Dt}} e^{-x^2/(2Dt)}, \quad \text{for a Brownian particle (below } D = 1), \tag{1}$$

is the standard diffusion kernel. Below we set $D = 1$ since any value of D can be restored by a rescaling of time.

Let us now consider N identical copies of that particle evolving independently. In our canonical example all the Brownian particles have the same diffusion coefficient and all start from $x = 0$ at time $t = 0$. One now defines the position of the rightmost particle, i.e., the maximum $X(t) = \max_{i=1,\dots,N} x_i(t)$. We are interested in the case where N is large. By definition the cumulative distribution function (CDF) of the maximum is given by

$$\begin{aligned} Q(X) &= \text{Prob}(X(t) < X) = P_{<,t}(X)^N \\ &= e^{N \ln(1 - P_{>,t}(X))} \underset{N \gg 1}{\simeq} e^{-N P_{>,t}(X)}, \end{aligned} \tag{2}$$

where we denote $P_{>,t}(X) = \int_X^{+\infty} dx p_t(x)$ and $P_{<,t}(X) = \int_{-\infty}^X dx p_t(x)$. As is well known, the standard diffusion (1) falls in the Gumbel extremal class. For any process in that class one can simply perform the change of variable from X to z defined by

$$N P_{>,t}(X) = e^{-z}. \tag{3}$$

In the case of diffusion it gives $\frac{N\sqrt{t}}{\sqrt{2\pi}X} e^{-X^2/(2t)} = e^{-z}$ leading to

$$X \simeq \sqrt{2t}\left(\sqrt{\ln N} + \frac{z + c_N}{2\sqrt{\ln N}}\right), \quad c_N = -\frac{1}{2}\ln(4\pi \ln N). \tag{4}$$

In this new variable the CDF is simply the Gumbel distribution, $e^{-e^{-z}}$, i.e., one has

$$Q(X) = \text{Prob}(X(t) < X) \simeq e^{-e^{-z}}. \tag{5}$$

Note that the change of variable (3) [hence, also Eq. (4) to leading order] is such that one has exactly

$$N p_t(X) dX = e^{-z} dz. \tag{6}$$

In the following, we will often (abusively) also consider z as the random variable defined by Eq. (4) with $X = X(t)$, i.e., as the rescaled position of the maximum.

B. Order statistics of outliers

It is well known how to extend this to the k first particles [2–4]. Let us denote $X^{(j)}(t)$ the same set of particles, but ordered by their rank, i.e., $X^{(1)}(t) > X^{(2)}(t) > \dots > X^{(N)}(t)$, so that $X(t) = X^{(1)}(t)$ is the position of the maximum, $X^{(2)}(t)$ of the second maximum and so on.

The joint PDF of the k first outliers is

$$N(N-1)\dots(N-k+1) p_t(X^{(1)}) p_t(X^{(2)}) \dots p_t(X^{(k)}) \theta_{X^{(1)} > \dots > X^{(k)}} P_{<,t}(X^{(k)})^{N-k} dX^{(1)} \dots dX^{(k)}. \tag{7}$$

For large N it becomes

$$\simeq \left(\prod_{j=1}^{k} (N p_t(X^{(j)}) dX^{(j)})\right) \theta_{X^{(1)} > \dots > X^{(k)}} e^{-N P_{>,t}(X^{(k)})}. \tag{8}$$

Thus, the same change of variable

$$N P_{>,t}(X^{(j)}) = e^{-z^{(j)}}, \tag{9}$$

which in the case of diffusion reads

$$X^{(j)} \simeq \sqrt{2t}\left(\sqrt{\ln N} + \frac{z^{(j)} + c_N}{2\sqrt{\ln N}}\right), \tag{10}$$

allows to put the large N asymptotic joint PDF of the position of the k first particles in the well-known form

$$q(z^{(1)}, \dots, z^{(k)}) = \theta_{z^{(1)} > z^{(2)} > \dots > z^{(k)}} e^{-z^{(1)} - z^{(2)} - \dots - z^{(k)}} e^{-e^{-z^{(k)}}}, \tag{11}$$

or equivalently as

$$\begin{aligned} q(z^{(1)}, \dots, z^{(k)}) = \theta_{z^{(1)} > z^{(2)} > \dots > z^{(k)}} \prod_{\ell=1}^{k-1} \ell e^{-\ell(z^{(\ell)} - z^{(\ell+1)})} \\ \times \frac{1}{(k-1)!} e^{-k z^{(k)} - e^{-z^{(k)}}}. \end{aligned} \tag{12}$$

Hence, to generate the k largest points, one first chooses z_k and then the successive gaps as independent exponentially distributed variables, with distinct parameters.

C. Counting statistics of outliers

Another standard way to characterize the outliers is the counting statistics. Let us define n_X as the number of particles at a given time t with $x_i(t) > X$. Since the particles are identical and independent the probability that $n_X = n$ follows the binomial distribution

$$P_X(n) = \frac{N!}{n!(N-n)!} P_{<,t}(X)^{N-n} P_{>,t}(X)^n. \tag{13}$$

In the edge regime for large X, i.e., for $NP_{>,t}(X) = O(1)$ this becomes a Poisson distribution

$$P_X(n) \simeq \frac{1}{n!}(NP_{>,t}(X))^n e^{-NP_{>,t}(X)} = \frac{1}{n!} e^{-nz} e^{-e^{-z}}, \tag{14}$$

where in the last equality we have used the change of variable (3). In the case of independent Brownian motions, all starting from the origin, the variables X and z are related through Eq. (4). From it one recovers the Gumbel CDF of the maximum

$$P_X(n=0) = \text{Prob}(X(t) < X) \simeq e^{-e^{-z}}. \tag{15}$$

Note that the other probabilities have also some interpretations in terms of order statistics, i.e., $P_X(1) = \text{Prob}(X(t) > X, X^{(2)}(t) < X)$, where $X(t) = X^{(1)}(t)$ and $X^{(2)}(t)$ are the maximum and second maximum, respectively, and so on. Furthermore, one sees that $X^{(k)} < X$ is equivalent to $n_X \in \{0, 1, \dots, k-1\}$, hence

$$\text{Prob}(X^{(k)} < X) \simeq \left(\sum_{n=0}^{k-1} \frac{1}{n!} e^{-nz}\right) e^{-e^{-z}}. \tag{16}$$

One can check that taking ∂_z of the right-hand side (r.h.s.) one recovers the PDF of $z = z^{(k)}$, i.e., $q(z^{(k)}) = \frac{1}{(k-1)!} e^{-kz^{(k)} - e^{-z^{(k)}}}$.

How does one recover the joint PDF of $X(t)$ and $X^{(2)}(t)$? For that one needs the joint PDF of n_{X_1} and n_{X_2} with $X_1 > X_2$ where we recall that n_X the number of particles with $x_i(t) > X$. To obtain it we split the line into three disjoint intervals (a) $x > X_1$, (b) $X_2 < x < X_1$, and (c) $x < X_2$ and write the product of sums of probabilities of these events

$$1 = [P(x > X_1) + P(X_2 < x < X_1) + P(x < X_2)]^N. \tag{17}$$

Here for convenience we adopt the shorthand notations, e.g., $P(X_2 < x < X_1) = \int_{X_2}^{X_1} dx p_t(x)$ and so on. Expanding Eq. (17), we see that the probability $P_{X_1,X_2}(n_a, n_b, n_c)$ that there are n_a, n_b, n_c particles in each of these intervals, is given by the multinomial distribution

$$\begin{aligned} P_{X_1,X_2}(n_a, n_b, n_c) = {} & \frac{N!}{n_a! n_b! n_c!} \delta_{N, n_a+n_b+n_c} P(x > X_1)^{n_a} \\ & \times P(X_2 < x < X_1)^{n_b} P(x < X_2)^{n_c}. \end{aligned} \tag{18}$$

In the large N limit and choosing X_1 and X_2 near the edge so that typically $n_b, n_c = O(1)$ while $n_a \simeq N$, one obtains by similar manipulations as above that the probability of n_a, n_b is a multiple independent Poisson distribution

$$P_{X_1,X_2}(n_a, n_b) \simeq \frac{1}{n_a! n_b!} (e^{-z_2} - e^{-z_1})^{n_b} e^{-n_a z_1} e^{-e^{-z_2}}, \tag{19}$$

which is correctly normalized to unity. This form applies to any problem of i.i.d. random variables in the Gumbel class through the change of variable (3), and for our purpose here X_1 and z_1 are related through Eq. (4) and so are X_2 and z_2.

Several observables can be obtained from Eq. (19). For instance, the joint PDF of $X(t)$ and $X^{(2)}(t)$ can be retrieved from taking $-\partial_{X_1}\partial_{X_2}$ of the following "CDF":

$$\begin{aligned} & \text{Prob}(X^{(2)}(t) < X_2, X^{(1)}(t) > X_1) \\ & \quad = P_{X_1,X_2}(n_a = 1, n_b = 0) \simeq e^{-z_1} e^{-e^{-z_2}}, \end{aligned} \tag{20}$$

and one can check that it is indeed equal to $\int_{y_1 > z_1} \int_{y_2 < z_2} e^{-y_1 - y_2} e^{-e^{-y_2}}$.

Another interesting observable is the joint PDF of the couple (n_{X_1}, n_{X_2}). Since the intervals $[X_2, +\infty]$ and $[X_1, +\infty]$ overlap, the two variables are correlated. One can write $(n_{X_1}, n_{X_2}) = (n_a, n_a + n_b)$, where n_a, n_b are independent Poisson variables. Hence,

$$\text{Prob}(n_{X_1} = n_1, n_{X_2} = n_2) = \theta_{n_2 \geqslant n_1} \frac{\lambda_a^{n_1}}{n_1!} \frac{\lambda_b^{n_2 - n_1}}{(n_2 - n_1)!} e^{-\lambda_a - \lambda_b}, \tag{21}$$

where $\lambda_a = e^{-z_1}$ and $\lambda_b = e^{-z_2} - e^{-z_1}$ are the mean parameters of the distribution (19).

III. MULTITIME JOINT CDF FOR THE MAXIMUM: FIRST METHOD

Let us now consider the dynamics of the maximum $X(t)$. Its one-time CDF is given by Eqs. (4) and (5). What is the multitime joint CDF of $X(t_1), X(t_2), \dots X(t_n)$?

Let us first determine on what timescale these variables remain correlated. The first simple consideration is as follows. As recalled in Sec. II B, at any fixed time the gap between the maximum $z^{(1)}$ and the second maximum $z^{(2)}$ (in the variable z seen as random variables) is $z^{(1)} - z^{(2)} = O(1)$. Hence, at a given time $t = t_1$, from Eq. (10), the first gap is $X(t) - X^{(2)}(t) = O(\sqrt{t_1/\ln N})$. These two rightmost particles undergo diffusion, hence it takes typically a time $t_2 - t_1 \sim (X(t) - X^{(2)}(t))^2 \sim t_1/\ln N$ for them to meet. This gives the scale of the time difference at which the order of the first few particles at the edge reshuffles, and correlations start decaying. For time differences much larger $t_2 - t_1 \gg t_1/\ln N$ we expect that $X(t_1)$ and $X(t_2)$ become uncorrelated, each described, under the appropriate scaling, by Gumbel distributions.

Let us give the result for the joint CDF $\text{Prob}(X(t_1) < X_1, \dots, X(t_n) < X_n)$. In view of the previous paragraph we define the dimensionless rescaled time differences τ_i as

$$t_j = t_1\left(1 + \frac{\tau_{j,1}}{\ln N}\right), \tag{22}$$

with the notation $\tau_{i,j} = \tau_i - \tau_j$ and $\tau_1 = 0$. As above one performs the change of variable

$$\begin{aligned} X_j &= \sqrt{2t_j}\sqrt{\ln N}\left(1 + \frac{z_j + c_N}{2\ln N}\right) \\ &\simeq \sqrt{2t_1}\sqrt{\ln N}\left(1 + \frac{z_j + \tau_{j,1} + c_N}{2\ln N}\right). \end{aligned} \tag{23}$$

Then, at large N we obtain (by similar manipulations as in the previous section, see Appendix A) that the joint CDF takes the form

$$\text{Prob}(X(t_1) < X_1, \dots, X(t_n) < X_n) \simeq \exp\left(-\Phi(z_1, \dots, z_n; \tau_{2,1}, \dots, \tau_{n,n-1})\right), \tag{24}$$

$$\Phi(z_1, \dots, z_n; \tau_{2,1}, \dots, \tau_{n,n-1}) = \int_{y_1,\dots,y_n}\left(1 - \prod_{i=1}^{n}\theta_{y_i<z_i}\right)e^{-y_1}G(y_{2,1}, \tau_{2,1})\dots G(y_{n,n-1}, \tau_{n,n-1}), \tag{25}$$

where here and below we often use the notations $y_{i,j} = y_i - y_j$ and the shorthand $\int_y = \int dy = \int_{-\infty}^{+\infty} dy$ as well as $\theta_{y<z} = \theta_{z>y} = \theta(z-y)$, where $\theta(x)$ is the Heaviside function, and

$$G(y, \tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(y+\tau)^2}{4\tau}} \tag{26}$$

is simply the free diffusion kernel, however *with a negative unit drift* (and with $D = 2$). This drift originates from the fact that the position of the maximum increases with time, as can be seen, e.g., in Eq. (23). Hence, the front of the cloud of particles moves to the right and with respect to this front the diffusion of a single particle, which is symmetric, has a negative drift. This is why, as we will see below, the correlations decay exponentially at large time: after a scaled time difference $\tau = O(1)$ the cloud of particles overtakes the particle which was the rightmost at time t_1. This is illustrated in Fig. 1. Finally, the factor e^{-y_1} reminds that the Gumbel distribution is a (nonnormalizable) stationary distribution of the rescaled maximum process in the z variable (see below).

Although exact, the above formula is delicate. Indeed, each of the two terms in $(1 - \prod_{i=1}^{n}\theta_{y_i<z_i})$ is a divergent integral

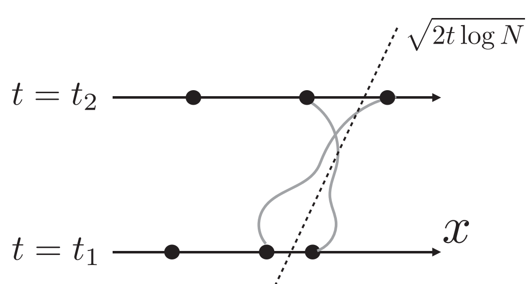

FIG. 1. Schematic view of the positions of the three rightmost particles at two times, with $t_2 - t_1 = t_1(1 + \frac{\tau}{\ln N})$. The dotted line is the average position of the edge, which moves to the right with unit velocity in the scaled space-time variables z, τ. The scaled maximum process is stationary in the frame moving with the edge. However, the particle of maximal position at t_1 undergoes symmetric diffusion, and as τ increases is overtaken by the other particles in a time $\tau = O(1)$, leading to exponential decay of correlations.

(for $y_j \to y_j + y$ with $y \to -\infty$), and only the combination is finite and the terms cannot be separated.

Let us give an equivalent formula for $n = 2$. One recombine as

$$\begin{aligned} 1 - \theta_{y_1<z_1}\theta_{y_2<z_2} &= 1 - (1 - \theta_{z_1<y_1})(1 - \theta_{z_2<y_2}) \\ &= \theta_{z_1<y_1} + \theta_{z_2<y_2} - \theta_{z_1<y_1}\theta_{z_2<y_2}. \end{aligned} \tag{27}$$

We use the important property that $e^{-z_1}dz_1$ is a (nonnormalizable) stationary measure of the free diffusion with a negative unit drift, i.e., it satisfies

$$\int dz_1 e^{-z_1}G(z_{2,1}, \tau_{2,1}) = e^{-z_2} \tag{28}$$

for any real z_1, z_2 and $\tau_{2,1} > 0$, where we here and below use the notation $z_{2,1} = z_2 - z_1$. The other important property is the normalization condition

$$\int dy\, G(y, \tau) = 1. \tag{29}$$

Using these two properties, inserting Eq. (27) into Eq. (25) one finds

$$\Phi(z_1, z_2; \tau_{2,1}) = e^{-z_1} + e^{-z_2} - \int_{z_1<y_1, z_2<y_2} e^{-y_1}G(y_{2,1}, \tau_{2,1}). \tag{30}$$

This function admits an explicit expression in terms of error functions, it is given below in Eqs. (40) and (41) and in Eq. (B10) in the Appendix.

Let us now generalize this formula to arbitrary n. Starting from $n = 3$ one must also use a third property, the convolution identity

$$\int dy\, G(z-y, \tau')G(y-x, \tau) = G(z-x, \tau+\tau'). \tag{31}$$

It is then easy to see, using these three properties, that one has

$$\Phi(z_1, z_2; \tau_{2,1}) = e^{-z_1} + e^{-z_2} - g(z_1, z_2; \tau_{2,1}), \tag{32}$$

$$\begin{aligned} &\Phi(z_1, z_2, z_3; \tau_{2,1}, \tau_{3,2}) \\ &\quad = e^{-z_1} + e^{-z_2} + e^{-z_3} - g(z_1, z_2; \tau_{2,1}) - g(z_2, z_3; \tau_{3,2}) \\ &\qquad - g(z_1, z_3; \tau_{31}) + g_3(z_1, z_2, z_3; \tau_{2,1}, \tau_{3,2}), \end{aligned} \tag{33}$$

and so on, where we have defined

$$g(z_1, z_2; \tau_{2,1}) = \int_{z_1 < y_1, z_2 < y_2} e^{-y_1} G(y_{2,1}, \tau_{2,1})$$
$$= e^{-z_1} \int_{0<y_1, 0<y_2} e^{-y_1} G(y_{2,1} + z_{2,1}, \tau_{2,1}), \quad (34)$$
$$g_3(z_1, z_2, z_3; \tau_{2,1}, \tau_{3,2})$$
$$= \int_{z_1<y_1, z_2<y_2, z_3<y_3} e^{-y_1} G(y_{2,1}, \tau_{2,1}) G(y_{3,2}, \tau_{3,2}), \quad (35)$$
$$= e^{-z_1} \int_{0<y_1, 0<y_2, 0<y_3} e^{-y_1} G(y_{2,1} + z_{2,1}, \tau_{2,1}) \times G(y_{3,2} + z_{3,2}, \tau_{3,2}), \quad (36)$$

and so on, with more generally, for $a_1 < \cdots < a_k$, $k \geqslant 1$

$$g_k(z_{a_1}, \ldots, z_{a_k}; \tau_{a_2,a_1}, \ldots, \tau_{a_k,a_{k-1}}) = \int_{z_{a_1}<y_{a_1}, \ldots, z_{a_k}<y_{a_k}} e^{-y_{a_1}} \times G(y_{a_2,a_1}, \tau_{a_2,a_1}) \ldots G(y_{a_k,a_{k-1}}, \tau_{a_k,a_{k-1}}), \quad (37)$$

with $g_2 = g$ and $g_1(z) = e^{-z}$. It is easy to see from the above properties (31) and (28) that if any argument z_{a_j} for some j in g_k is set to $z_{a_j} = -\infty$ then $g_k \to g_{k-1}$ and z_{a_j} and τ_j are dropped from the list of arguments. Also, any g_k obviously vanishes when any z_{a_j} for some j argument is taken to $+\infty$.

One can see in Eqs. (34) and (36) that one can always factor out the term e^{-z_1}, the rest being only function of differences of z_j's. This important factorization property is true for any n, i.e., one can always write

$$\Phi(z_1, \ldots, z_n; \tau_{2,1}, \ldots, \tau_{n,n-1}) = e^{-z_1} \phi_{\tau_{2,1},\ldots,\tau_{n,n-1}}(z_{2,1}, \ldots, z_{n,n-1}). \quad (38)$$

The functions ϕ will be further studied below.

The above result provides explicit, although lengthy, expressions for the multitime joint CDF of the maximum. As we will see below, using a second method based on the heat equation, the functions Φ can also be computed recursively with n. This leads to more compact expressions. Before explaining that, let us give some explicit results for marginals, cumulants and correlation functions for $n = 2$ and $n = 3$.

IV. MULTITIME MARGINALS, MOMENTS, AND CORRELATIONS FOR THE MAXIMUM

Let us analyze in more details the joint PDF obtained in the previous section. We will derive very explicit results for $n = 2$, and give some general formula for $n = 3$.

A. Two-time correlations

Let us start with $n = 2$.

1. Two-time joint CDF

One recalls that at large N with $t_2 - t_1 = t_1 \frac{\tau_{2,1}}{\ln N}$, with $\tau_{2,1} = \tau = O(1)$, the joint CDF for the position of the maximum takes the form

$$\text{Prob}(X(t_1) < X_1, X(t_2) < X_2) \simeq Q_{<,<}(z_1, z_2) = \exp(-\Phi(z_1, z_2; \tau)), \quad (39)$$

where we denote $Q_{<<}(z_1, z_2)$ the two-time CDF of the process in the variables z_j (implicitely at times t_j), and $\Phi(z_1, z_2; \tau)$ was defined in Eq. (30). Through the change of variable $y_j \to y_j + z_j$ it can also be written as

$$\Phi(z_1, z_2; \tau) = e^{-z_1} \phi_\tau(z_{2,1}),$$
$$\phi_\tau(z) = 1 + e^{-z} - \int_{y_1>0, y_2>0} e^{-y_1} G(y_{2,1} + z, \tau). \quad (40)$$

The integral can be computed, see Appendix B. This leads to the explicit form

$$\phi_\tau(z) = \frac{1}{2}\left[\text{erf}\left(\frac{\tau + z}{2\sqrt{\tau}}\right) + e^{-z}\text{erfc}\left(\frac{z - \tau}{2\sqrt{\tau}}\right) + 1\right], \quad (41)$$

which obeys the important symmetry (see Appendix B)

$$\phi_\tau(z) e^z = \phi_\tau(-z). \quad (42)$$

This symmetry is equivalent to the fact that $\Phi(z_1, z_2; \tau)$ is symmetric in z_1, z_2 as is visible on its explicit form (B10) in the Appendix.

In summary, the two-time CDF of the maximum, in the rescaled variables, has the form

$$Q_{<<}(z_1, z_2) \simeq e^{-e^{-z_1}\phi_\tau(z_{2,1})}, \quad (43)$$

where the function $\phi_\tau(z)$ is given in Eq. (41). As mentioned in the Introduction, this distribution appeared before [17,18] and is known as the bivariate HR distribution.

The function $\phi_\tau(z)$ has the following asymptotic behaviors for large argument

$$\phi_\tau(z) \simeq e^{-z} + \psi_\tau(z), \quad z \to -\infty, \quad (44)$$
$$\phi_\tau(z) \simeq 1 + \psi_\tau(z), \quad z \to +\infty, \quad (45)$$
$$\psi_\tau(z) = e^{-\frac{(z+\tau)^2}{4\tau}} \frac{2\tau^{3/2}}{z^2\sqrt{\pi}}\left[1 + \frac{\tau(\tau - 6)}{z^2} + \frac{\tau^2(60 + \tau(\tau - 20))}{z^4} + O(z^{-6})\right], \quad |z| \to +\infty. \quad (46)$$

Note that $\psi_\tau(-z) = e^z \psi_\tau(z)$. These asymptotics guarantee that one recovers the one-time Gumbel CDF for $z_1 \to +\infty$ or $z_2 \to +\infty$, i.e., one has

$$Q_{<<}(z, +\infty) = Q_{<<}(+\infty, z) = e^{-e^{-z}}. \quad (47)$$

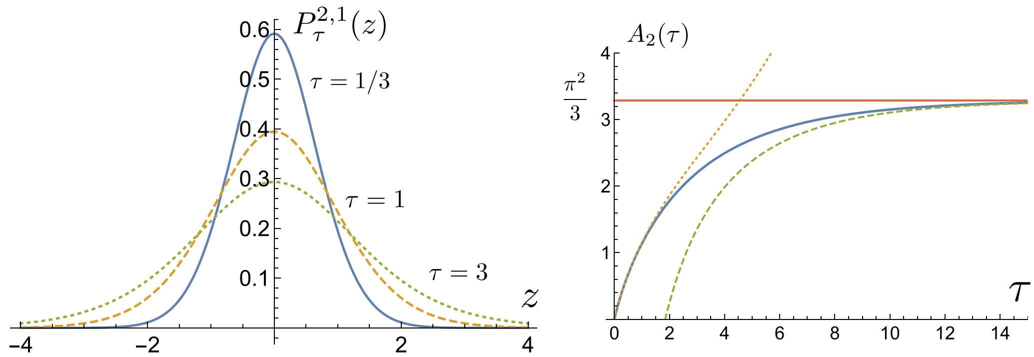

FIG. 2. Left: marginal PDF of $z = z_{2,1} = z_2 - z_1$, the scaled distance traveled by the maximum, with $X(t_2) - X(t_1) \simeq \sqrt{\frac{t_1}{2\ln N}}(\tau + z)$. It is plotted for $\tau = 1/3$ (plain), $\tau = 1$ (dashed), $\tau = 3$ (dotted). Right: its second moment $\langle z_{2,1}^2 \rangle = A_2(\tau)$, plotted versus τ (blue, thick). The first three terms in the small τ asymptotics (65) (dotted), and the first two terms in the large τ asymptotics (63) are also plotted (dashed), together with the limiting value $\pi^2/3$ (horizontal line). Recall that $\pi^2/3 - A_2(\tau)$, i.e., the curve reflected versus $\pi^2/3$ describes the two-time covariance of the maximum, see Eq. (72).

In addition, at large τ, since the free diffusion kernel with drift $G(y_{2,1} + z, \tau) \to 0$, one sees from Eq. (40) that

$$\phi_\tau(z) \simeq 1 + e^{-z}, \quad \Phi(z_1, z_2; \tau) \simeq e^{-z_1} + e^{-z_2}, \quad \tau \to +\infty, \tag{48}$$

which corresponds to two uncorrelated Gumbel variables.

2. Exponential moments of z_1 and z_2

Through Eq. (23) with $X_j = X(t_j)$ one can (abusively) also think of z_1 and z_2 as random variables. Furthermore, from now on it is useful to consider z_1 and $z_{2,1} = z_2 - z_1$ as the two random variables of interest.

A first result, see Appendix C, is an explicit integral expression for the joint moment generating function, which reads

$$\langle e^{-sz_1 - bz_{2,1}} \rangle = \Gamma(1+s) \int dz e^{-bz} \partial_z \left[\left(1 + \frac{\phi_\tau'(z)}{\phi_\tau(z)} \right) \frac{1}{\phi_\tau(z)^s} \right], \tag{49}$$

where here and below $\langle \dots \rangle$ denotes an expectation value. A simpler expression also exists for Eq. (49), see Eq. (C13), but it is less convenient for the analysis of the moments. For $b = 0$ the integral in Eq. (49) can be performed and the boundary term is 1 at $z = +\infty$ and vanish at $z = -\infty$ (see Appendix C), recovering the generating function of the one-time Gumbel distribution

$$\langle e^{-sz_1} \rangle = \Gamma(1+s). \tag{50}$$

One also recovers the same result for z_2, by setting $b = s$, although the algebra is slightly more involved, see Appendix C.

3. PDF of $z_2 - z_1$

It is possible to obtain explicitly the PDF of $X(t_2) - X(t_1)$ the distance over which the maximum has moved. One has from Eq. (23) that $X(t_2) - X(t_1) = \sqrt{\frac{t_1}{2\ln N}}(\tau + z_{2,1})$, where we consider $z_{2,1} = z_2 - z_1$ as a random variable. Its distribution is easily obtained. Indeed, Eq. (49) for $s = 0$ implies the following expression for the generating function of the moments of $z_{2,1}$,

$$\langle e^{-bz_{2,1}} \rangle = \int dz e^{-bz} \partial_z \left(1 + \frac{\phi_\tau'(z)}{\phi_\tau(z)} \right). \tag{51}$$

This, in turn, implies the following expressions for the PDF [denoted $P_\tau^{(2,1)}(z)$] and the CDF of the variable $z_{2,1}$,

$$P_\tau^{(2,1)}(z) = \partial_z \frac{\phi_\tau'(z)}{\phi_\tau(z)} = \partial_z^2 \ln \phi_\tau(z),$$
$$\text{Prob}(z_{2,1} < z) = 1 + \partial_z \ln \phi_\tau(z), \tag{52}$$

where $\phi_\tau(z)$ is given explicitly in Eq. (41). Thanks to the symmetry (42) we see that the PDF is an even function of z,

$$P_\tau^{2,1}(-z) = P_\tau^{2,1}(z), \tag{53}$$

hence all odd moments of $z_{2,1}$ vanish. The PDF $P_\tau^{(2,1)}(z)$ is plotted in Fig. 2. Its behavior for large $|z|$ at fixed τ is

$$P_\tau^{2,1}(z) \simeq \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(|z|+\tau)^2}{4\tau}} \left(1 + \frac{2\tau}{|z|} + \frac{2\tau^2}{z^2} + O(|z|^{-3} \right), \tag{54}$$

hence for large argument it becomes equal to the free diffusion kernel with unit drift. Around $z = 0$ it is analytic and behaves as

$$P_\tau^{2,1}(z) = \frac{1}{4} + \frac{e^{-\tau/4}}{\sqrt{\pi}\sqrt{\tau}\left(2\text{erf}\left(\frac{\sqrt{\tau}}{2}\right) + 2\right)} - \frac{z^2}{16} \left(1 + \frac{e^{-\tau/2}\left(\sqrt{\pi} e^{\tau/4}(5\tau + 2)\left(\text{erf}\left(\frac{\sqrt{\tau}}{2}\right) + 1\right) + 6\sqrt{\tau}\right)}{\pi\tau^{3/2}\left(\text{erf}\left(\frac{\sqrt{\tau}}{2}\right) + 1\right)^2} \right) + O(z^4). \tag{55}$$

In the limit of large τ and fixed z it converges to a finite limit

$$P_\tau^{2,1}(z) = \frac{1}{4\cosh^2\left(\frac{z}{2}\right)} + O(e^{-\tau/4}\tau^{-1/2}), \tag{56}$$

which is simply the PDF of the difference of two uncorrelated Gumbel variables, as it should. Indeed, Eq. (56) implies

$$\lim_{\tau\to+\infty}\langle e^{-bz_{2,1}}\rangle = \frac{\pi b}{\sin(\pi b)} = \Gamma(1+b)\Gamma(1-b). \tag{57}$$

One can study how that limit is reached. One has, at large $\tau \gg 1$,

$$\phi_\tau(z) = 1 + e^{-z} - e^{-\frac{(z+\tau)^2}{4\tau}}\chi_\tau(z), \quad \chi_\tau(z) = \frac{2}{\sqrt{\pi\tau}}\left(1 - \frac{2}{\tau} + \frac{12+z^2}{\tau^2} + O(1/\tau^3)\right). \tag{58}$$

Inserting into Eq. (52), expanding the logarithm to first order in $\chi_\tau(z)$, writing the exponential as $e^{-\frac{\tau}{4}-\frac{z}{2}} \times e^{-\frac{z^2}{4\tau}}$ and expanding the last factor in powers of $1/\tau$, one obtains the large τ expansion

$$P_\tau^{2,1}(z) = \frac{1}{4\cosh^2(\frac{z}{2})} + \frac{2e^{-\frac{\tau}{4}}}{\sqrt{\pi\tau}}\left(\frac{1}{16}\frac{3-\cosh(z)}{\cosh^3\left(\frac{z}{2}\right)} + \frac{-3z^2+(z^2+16)\cosh(z)-8z\sinh(z)-16}{64\cosh^3\left(\frac{z}{2}\right)\tau} + O(\tau^{-2})\right). \tag{59}$$

4. Moments of $z_2 - z_1$

We recall that the odd moments of $z_{2,1} = z_2 - z_1$ vanish. One can further obtain formulas for the the even moments of $z_{2,1}$ by integration by part (boundary terms vanish), as, with $n \geqslant 1$,

$$\left\langle z_{2,1}^{2n}\right\rangle = A_{2n}(\tau) = 2\int_0^{+\infty} dz z^{2n}\partial_z^2 \ln\phi_\tau(z) = 4n(2n-1)\int_0^{+\infty} dz z^{2n-2}\ln\phi_\tau(z), \tag{60}$$

which are easy to evaluate numerically. The function $A_2(\tau)$ is plotted in Fig. 2. Using Eq. (57) one finds that at large time the moments tend to the moments of the difference of two uncorrelated Gumbel variables, namely

$$\lim_{\tau\to+\infty}\left\langle z_{2,1}^{2n}\right\rangle = A_{2n}(+\infty) = (-1)^{n+1}(2^{2n}-2)\pi^{2n}B_{2n}, \tag{61}$$

where the B_k are the Bernoulli numbers. The details asymptotics of $A_{2n}(\tau)$ at large τ is obtained as follows. From Eq. (58) one has

$$\ln\phi_\tau(z) = \ln(1+e^{-z}) - e^{-\frac{(z+\tau)^2}{4\tau}}\frac{\chi_\tau(z)}{1+e^{-z}} + O(e^{-2\frac{(z+\tau)^2}{4\tau}}), \tag{62}$$

and one performs similar manipulations as above. Using the last identity in Eq. (60), the integral on z converges term by term in the expansion, leading to

$$\begin{aligned}A_2(\tau) &= \frac{\pi^2}{3} - \frac{4\sqrt{\pi}e^{-\tau/4}}{\sqrt{\tau}}\left(1 - \frac{8+\pi^2}{\tau} + \frac{12+\frac{3\pi^2}{2}+\frac{5\pi^4}{32}}{\tau^2} - \frac{120+15\pi^2+\frac{25\pi^4}{16}+\frac{61\pi^6}{384}}{\tau^3} + O(\tau^{-4})\right),\\ A_4(\tau) &= \frac{7\pi^4}{15} - \frac{24\pi^{5/2}e^{-\tau/4}}{\sqrt{\tau}}\left(1 - \frac{2+\frac{5\pi^2}{4}}{\tau} + \frac{12+\frac{15\pi^2}{2}+\frac{61\pi^4}{32}}{\tau^2} - \frac{120+75\pi^2+\frac{305\pi^4}{16}+\frac{1385\pi^6}{384}}{\tau^3} + O(\tau^{-4})\right).\end{aligned} \tag{63}$$

At short time difference $\tau \ll 1$ the variable z_{21} is of order $\sqrt{\tau}$. Defining the $O(1)$ random variable w such that $z_{21} = w\sqrt{\tau}$ one finds that its PDF $p_\tau(w)$ admits a small τ expansion

$$p_\tau(w) = \frac{e^{-\frac{w^2}{4}}}{2\sqrt{\pi}} - \frac{\sqrt{\tau}\left(\sqrt{\pi}e^{-\frac{w^2}{4}}w\,\mathrm{erf}\left(\frac{w}{2}\right) + \pi\left(\mathrm{erf}\left(\frac{w}{2}\right)^2 - 1\right) + 2e^{-\frac{w^2}{2}}\right)}{4\pi} + O(\tau). \tag{64}$$

So not surprisingly $z_{2,1}$ undergoes free diffusion at short time, but there are corrections as τ increases. The short time expansion of the lowest moments and cumulants, as well as of the kurtosis, reads

$$\begin{aligned}&\left\langle z_{2,1}^2\right\rangle = \left\langle z_{2,1}^2\right\rangle^c = A_2(\tau) = 2\tau - \frac{4}{3}\sqrt{\frac{2}{\pi}}\tau^{3/2} + 0.217996\tau^2 - 0.0129398\tau^{5/2} + O(\tau^3),\\ &\left\langle z_{2,1}^4\right\rangle = A_4(\tau) = 12\tau^2 - \frac{72}{5}\sqrt{\frac{2}{\pi}}\tau^{5/2} + O(\tau^3),\ \left\langle z_{2,1}^4\right\rangle^c = \frac{8}{5}\sqrt{\frac{2}{\pi}}\tau^{5/2} + O(\tau), \quad \mathrm{Ku} = \frac{2}{5}\sqrt{\frac{2}{\pi}}\sqrt{\tau} + O(\tau),\\ &\left\langle z_{2,1}^6\right\rangle = A_6(\tau) = 120\tau^3 - \frac{1380}{7}\sqrt{\frac{2}{\pi}}\tau^{7/2} + O(\tau^4).\end{aligned} \tag{65}$$

The small and large τ asymptotics are compared with the numerical calculation of $A_2(\tau)$ in Fig. 2.

5. Covariance of $X(t_1)$ and $X(t_2)$

Let us translate some of these results in the original variables. Let us recall the relations

$$X(t_1) = X_1 \simeq \sqrt{2t_1}\left(\sqrt{\ln N} + \frac{z_1}{2\sqrt{\ln N}}\right), \quad (66)$$

$$X(t_2) = X_2 \simeq \sqrt{2t_1}\left(\sqrt{\ln N} + \frac{\tau + z_2}{2\sqrt{\ln N}}\right). \quad (67)$$

Hence, one has

$$X_2 - X_1 \simeq \sqrt{\frac{t_1}{2\ln N}}(\tau + z_{2,1}), \quad \tau = \frac{t_2 - t_1}{t_1}\ln N = O(1). \quad (68)$$

The results of the previous subsection thus imply for $\tau = O(1)$ and $p \geqslant 1$

$$\langle X_2 - X_1\rangle \simeq \sqrt{\frac{t_1}{2\ln N}}\,\tau, , \quad \langle (X_2 - X_1 - \langle X_2 - X_1\rangle)^{2p+1}\rangle = o\left(\left(\frac{t_1}{2\ln N}\right)^{2p+1}\right), \quad (69)$$

$$\mathrm{Var}X_1 = \mathrm{Var}X_2 \simeq \frac{t_1}{2\ln N}\frac{\pi^2}{6}, \quad \mathrm{Var}(X_2 - X_1) \simeq \frac{t_1}{2\ln N}A_2(\tau), \quad (70)$$

$$\mathrm{Cov}(X_1, X_2) = \tfrac{1}{2}(\mathrm{Var}(X_1) + \mathrm{Var}(X_2) - \mathrm{Var}(X_2 - X_1)), \quad (71)$$

$$= \frac{t_1}{4\ln N}\left(\frac{\pi^2}{3} - A_2(\tau)\right), \quad A_2(\tau) = 4\int_0^{+\infty} dz \ln\phi_\tau(z), \quad (72)$$

where we have used the values for the first two moments of the Gumbel distribution, $\langle z_1^2\rangle = \langle z_2^2\rangle = \frac{\pi^2}{6} + \gamma_E^2$ and $\langle z_1\rangle = \langle z_2\rangle = \gamma_E$. The covariance is illustrated in Fig. 2. Using the asymptotics (63), we see that at large time difference $\tau \gg 1$ the covariance of the maximum at two different times decay as

$$\mathrm{Cov}(X(t_1), X(t_2)) \simeq \frac{t_1}{4\ln N}\frac{4\sqrt{\pi}e^{-\tau/4}}{\sqrt{\tau}}. \quad (73)$$

As discussed in the previous section, the large time decay is exponential since the particle which is the rightmost at time t_1 undergoes symmetric diffusion, while the front of the other particles advances, with an effective unit drift, on the timescales $\tau = O(1)$.

6. Correlations between $X(t_1)$ and $X(t_2) - X(t_1)$

There are correlations between the variable z_1 and $z_{2,1}$. For instance, from Eq. (49) one obtains

$$\langle z_1 e^{-bz_{2,1}}\rangle - \langle z_1\rangle\langle e^{-bz_{2,1}}\rangle = -\int dz e^{-bz}\partial_z\left(\left(1 + \frac{\phi_\tau'(z)}{\phi_\tau(z)}\right)\ln\phi_\tau(z)\right). \quad (74)$$

The lowest cross moment is trivial

$$\langle z_1 z_{2,1}\rangle = -\tfrac{1}{2}\langle z_{2,1}^2\rangle. \quad (75)$$

The first nontrivial cross cumulant is

$$\langle z_1 z_{2,1}^2\rangle^c = \langle z_1 z_{2,1}^2\rangle - \langle z_1\rangle\langle z_{2,1}^2\rangle = -\int dz z^2\partial_z\left(\left(1 + \frac{\phi_\tau'(z)}{\phi_\tau(z)}\right)\ln\phi_\tau(z)\right), \quad (76)$$

which upon rescaling yields the corresponding cross cumulant for the maximum $X(t_1)$ and its variation $X(t_2) - X(t_1)$.

7. Conditional probability of z_1 given $z_2 - z_1$

It is also interesting to compute the PDF of z_1 conditioned on a given value of $z_{2,1} = z$, which turns out to have a simple form. One finds

$$q_\tau(z_1|z_{2,1} = z) = \frac{\partial_{z_1}\partial_{z_2}e^{-e^{-z_1}\phi_\tau(z_{2,1})}|_{z_{2,1}=z}}{\partial_z^2\ln\phi_\tau(z)} = \phi_\tau(z)(B_\tau(z)e^{-z_1} + \phi_\tau(z)(1 - B_\tau(z))e^{-2z_1})) \times e^{-e^{-z_1}\phi_\tau(z)}, \quad (77)$$

with $B_\tau = \phi_\tau(\phi_\tau' + \phi_\tau'')/(\phi_\tau\phi_\tau'' - (\phi_\tau')^2)$. Upon the deterministic shift

$$z_1 = \tilde{z}_1 + \ln\phi_\tau(z), \quad (78)$$

we see that $\tilde{z}_1$ has the same distribution that the one of a random variable which (i) with probability $B_\tau(z)$ is Gumbel (ii) with probability $1 - B_\tau(z)$ has the PDF of the second maximum $e^{-2\tilde{z}_1}e^{-e^{-\tilde{z}_1}}$. Note that $0 < B_\tau(z) < 1$ is an even function of z which reaches its strictly positive τ-dependent minimum at $z = 0$, and with $B_\tau(\pm\infty) = 1$.

B. Three-time correlations

For three times we will again consider z_1, $z_{2,1} = z_2 - z_1$ and $z_{3,2} = z_3 - z_2$ as the random variables of interest. Note that we use the notation z_{21} and z_{32} for the corresponding arguments of functions, or for real integration variables. We use the same notation for z_1, z_2, z_3 as random variables and function arguments.

Anticipating a bit, in the next section, Sec. V, we will obtain the simplest form for the three-time function Φ. As noted in Eq. (38), once again the dependence in z_1 can be singled out as

$$\Phi(z_1, z_2, z_3; \tau_{3,2}, \tau_{2,1}) = e^{-z_1}\phi_{\tau_{2,1},\tau_{3,2}}(z_{2,1}, z_{3,2}), \quad (79)$$

a property which extends to any number of time n. For $n = 3$ one has the tedious but explicit formula

$$\phi_{\tau_{2,1},\tau_{3,2}}(z_{21}, z_{32}) = \int dy\, G(z_{32} + z_{21} - y, \tau_{3,2})\phi_{\tau_{2,1}}(\min(z_{21}, y)) = \frac{1}{2}\int \frac{dy}{\sqrt{4\pi\tau_{32}}}e^{-\frac{(z_{32}+z_{21}-y+\tau_{3,2})^2}{4\tau_{3,2}}}\left(\mathrm{erf}\left(\frac{\tau_{2,1} + \min(z_{21}, y)}{2\sqrt{\tau_{2,1}}}\right) + e^{-\min(z_{21},y)}\mathrm{erfc}\left(\frac{\min(z_{21}, y) - \tau_{2,1}}{2\sqrt{\tau_{2,1}}}\right) + 1\right). \quad (80)$$

Although we could not perform this integral in closed form, it can be easily evaluated numerically. Let us now put this result to use.

One can generalize the manipulations in the previous section (see Appendix C) and obtain again the exponential moments as

$$\begin{aligned}&\langle e^{-sz_1-bz_{2,1}-cz_{3,2}}\rangle\\&=\Gamma(1+s)\int dz_{32}dz_{21}e^{-sz_1-bz_{21}-cz_{32}}\partial_{z_{32}}(\partial_{z_{21}}-\partial_{z_{32}})\\&\quad\times\left(\left(1+\frac{\partial_{z_{21}}\phi_{\tau_{2,1},\tau_{3,2}}(z_{21},z_{32})}{\phi_{\tau_{21},\tau_{32}}(z_{21},z_{32})}\right)\phi_{\tau_{2,1},\tau_{3,2}}(z_{21},z_{32})^{-s}\right),\end{aligned} \tag{81}$$

where now z_{21} and z_{32} denote two real integration variables. Hence, the joint PDF $P_{\tau_{2,1},\tau_{3,2}}(z_{21},z_{32})$ of the random variables $z_{2,1}=z_2-z_1$ and $z_{3,2}=z_3-z_2$ is obtained as

$$P_{\tau_{2,1},\tau_{3,2}}(z_{21},z_{32})=\partial_{z_{32}}(\partial_{z_{21}}-\partial_{z_{32}})\partial_{z_{21}}\ln\phi_{\tau_{2,1},\tau_{3,2}}(z_{21},z_{32}), \tag{82}$$

where $\phi_{\tau_{2,1},\tau_{3,2}}(z_{21},z_{32})$ is given explicitly in Eq. (80). Upon rescaling and shifting, Eq. (82) gives the joint PDF of the variations $X(t_3)-X(t_2)$ and $X(t_2)-X(t_1)$.

From Eq. (82) one can evaluate the corresponding joint moments through a double integral. The lowest nontrivial such moments are of third order, i.e., $\langle z_{2,1}^2z_{3,2}\rangle$ and $\langle z_{2,1}z_{3,2}^2\rangle$. Indeed, the order two correlation is simply related to two-time moments,

$$\langle z_{21}z_{32}\rangle=\langle z_{21}z_{32}\rangle_c=\tfrac{1}{2}(A(\tau_{31})-A(\tau_{21})-A(\tau_{32})). \tag{83}$$

V. MULTITIME JOINT CDF FOR THE MAXIMUM, FROM THE HEAT EQUATION

We now give another calculation of the multitime joint CDF for the maximum, using the diffusion equation. It is slightly less controlled technically, but quite intuitive physically.

Let us start with the one-time CDF. For a single particle $x(t)$ undergoing free diffusion, the CDF, which we denote here for convenience $P_<(x,t)=\text{Prob}(x(t)<x)$, satisfies

$$\partial_tP_<=\tfrac{1}{2}\partial_x^2P_<, \tag{84}$$

with $P_<(-\infty,t)=0$ and $P_<(+\infty,t)=1$. Here the initial condition $P_<(x,t=0)=\theta(x)$, although one can consider more general ones. Writing $P_<(x,t)=e^{-f(x,t)}$, the field $f(x,t)$ satisfies

$$\partial_tf=\tfrac{1}{2}\partial_x^2f-\tfrac{1}{2}(\partial_xf)^2. \tag{85}$$

Considering now N identical copies, the CDF of the maximum is given by $\text{Prob}(X(t)<x)=P_<(x,t)^N=e^{-Nf(x,t)}=e^{-F(x,t)}$ where we defined $F(x,t)=Nf(x,t)$. Hence, the field $F(x,t)$ satisfies

$$\partial_tF=\frac{1}{2}\partial_x^2F-\frac{1}{2N}(\partial_xF)^2, \tag{86}$$

with $F(-\infty,t)=+\infty$ and $F(+\infty,t)=0$.

Let us now use diffusive scaling, i.e., we define $F(x,t)=\tilde{F}(y,t)$ with $y=x/\sqrt{2t}$. This leads to

$$2t\,\partial_t\tilde{F}=y\partial_y\tilde{F}+\frac{1}{2}\partial_y^2\tilde{F}-\frac{1}{2N}(\partial_y\tilde{F})^2. \tag{87}$$

To anticipate the known result for N large, we now make the further change of variable

$$\tilde{F}(y,t)=\hat{F}(z,t),\ \ y=\sqrt{\ln N}\left(1+\frac{z+c_N}{2\ln N}\right),\ \ dy=\frac{dz}{2\sqrt{\ln N}}. \tag{88}$$

It gives

$$\frac{t}{\ln N}\partial_t\hat{F}=\partial_z\hat{F}+\partial_z^2\hat{F}+\frac{z+c_N}{2\ln N}\partial_z\hat{F}-\frac{1}{N}(\partial_z\hat{F})^2, \tag{89}$$

which, until now, is exact for any N.

Let us now consider time $t=O(1)$ and large $N\gg 1$. If we are looking for typical events, i.e., $\hat{F}=O(1)$, then the equation formally becomes stationary,

$$\partial_z\hat{F}+\partial_z^2\hat{F}\simeq 0. \tag{90}$$

Hence, one sees that the Gumbel distribution, which corresponds to $\hat{F}(z)=e^{-z}$ is indeed a stationary distribution. Note that studying the finite N corrections, and convergence to stationarity [31], would require to examine the various regimes in z and their matching (89), but we do not need it here.

It turns out that it is relatively simple to obtain also the multitime joint CDF from Eq. (89), i.e., the dynamics of the maximum. For this one needs to rescale the time, more precisely rescale the relative time from a fixed reference time. To this aim we fix some time $t=t_1$ and consider times very close to t_1, $t-t_1=t_1\frac{\tau}{\ln N}$. Denoting for convenience $\hat{F}[z,t_1(1+\frac{\tau}{\ln N})]\to\hat{F}(z,\tau)$, the left-hand side (l.h.s) of Eq. (89) becomes

$$\frac{t}{\ln N}\partial_t\hat{F}\simeq\frac{t_1}{\ln N}\partial_t\hat{F}=\partial_\tau\hat{F}. \tag{91}$$

Thus, dropping the terms which are subdominant at large N in the region $z=O(1)$, $\hat{F}=O(1)$, we obtain

$$\partial_\tau\hat{F}=\partial_z\hat{F}+\partial_z^2\hat{F}, \tag{92}$$

which is precisely the diffusion with negative unit drift encountered in the previous sections, of associated Green's function $G(z,\tau)$ in Eq. (26).

Let us now apply this method to solve the two-time problem $n=2$. The main point is that one can use the same representation (with $t_2>t_1$),

$$\begin{aligned}&\text{Prob}(X(t_1)<x_1,X(t_2)<x_2)\\&\quad=\text{Prob}(x(t_1)<x_1,x(t_2)<x_2)^N=e^{-F(x_2,t_2;x_1,t_1)}.\end{aligned} \tag{93}$$

The single particle joint CDF, $Q_{t_1,t_2}(x_1,x_2)=\text{Prob}(x(t_1)<x_1,x(t_2)<x_2)$, satisfies the same heat equation (84) as a function of x_2 and t_2, but with "initial" condition $Q_{t_1,t_1}(x_1,x_2)=P_<[\min(x_1,x_2),t_1]$. Hence, $F(x_2,t_2;x_1,t_1)$ satisfies the same equation (86) as a function of x_2 and t_2, but with initial condition $F(x_2,t_1;x_1,t_1)=F[\min(x_1,x_2),t_1]$, and where $F(x,t)$ is the function studied above. In the large N limit and in the variables z_1,z_2,τ one thus finds that $F(x_2,t_2;x_1,t_1)=$

$\hat{F}(z_2,\tau;z_1)$ satisfies the negative unit drift diffusion equation (92) with initial condition

$$\hat{F}(z_2,\tau=0;z_1)=e^{-\min(z_1,z_2)}. \tag{94}$$

This implies that

$$\begin{aligned}\hat{F}(z_2,\tau;z_1) &= \int_{-\infty}^{z_1} dy_2 e^{-y_2} G(z_2-y_2,\tau) + e^{-z_1}\int_{z_1}^{+\infty} dy_2 G(z_2-y_2,\tau)\\ &= e^{-z_1}\left(\int_{-\infty}^{0} dy\, e^{-y} G(z_{2,1}-y,\tau) + \int_0^{+\infty} dy\, G(z_{2,1}-y,\tau)\right).\end{aligned} \tag{95}$$

Remarkably, although the calculations are quite different, this gives exactly the same result as the other method, i.e., one finds by explicit calculation of the integrals in Eq. (95),

$$\begin{aligned}\hat{F}(z_2,\tau;z_1) &= \Phi(z_1,z_2;\tau)=e^{-z_1}\phi_\tau(z_{2,1}),\\ \phi_\tau(z) &= \frac{1}{2}\left[\mathrm{erf}\left(\frac{\tau+z}{2\sqrt{\tau}}\right)+e^{-z}\mathrm{erfc}\left(\frac{z-\tau}{2\sqrt{\tau}}\right)+1\right].\end{aligned} \tag{96}$$

This method can be iterated to obtain the result for $n=3$. One must again solve the negative unit drift diffusion equation (92) in the variables z_3 and $\tau_{3,2}$ with the "initial" condition

$$\hat{F}(z_3,\tau_{3,2};z_2,\tau_{2,1}|z_1)|_{\tau_{3,2}=0}=\hat{F}[\min(z_2,z_3),\tau_{2,1};z_1]. \tag{97}$$

This gives

$$\begin{aligned}&\hat{F}(z_3,\tau_{3,2};z_2,\tau_{2,1};z_1)\\ &\quad=\int dy_3 G(z_3-y_3,\tau_{3,2})\hat{F}[\min(z_2,y_3),\tau_{2,1};z_1].\end{aligned} \tag{98}$$

Expanding into various integrals we obtain an expression which, remarkably again, although being apparently quite different, numerically coincides with the one obtained by the first method (the integrals cannot be evaluated in closed form). Indeed, we have tested numerically the consistency of the conjectured identity

$$\begin{aligned}\hat{F}(z_3,\tau_{3,2};z_2,\tau_{2,1};z_1) &= \Phi(z_1,z_2,z_3;\tau_{2,1},\tau_{3,2})\\ &= e^{-z_1}\phi_{\tau_{21},\tau_{32}}(z_{2,1},z_{3,2})\end{aligned} \tag{99}$$

for random values of the arguments. Here Φ and ϕ are defined in Eqs. (33) and (38).

Hence, this second method gives a recursive way to construct the multitime functions Φ. Specifically, we obtain the recursion, for any $n\geqslant 2$,

$$\Phi(z_1,\dots,z_n;\tau_{2,1},\dots,\tau_{n,n-1})=\int dy\, G(z_n-y,\tau_{n,n-1})\Phi(z_1,\dots,z_{n-2},\min(z_{n-1},y);\tau_{2,1},\dots,\tau_{n-1,n-2}). \tag{100}$$

Equivalently, for the functions ϕ defined in Eq. (38) the recursion reads

$$\phi_{\tau_{2,1},\dots,\tau_{n,n-1}}(z_{2,1},\dots,z_{n,n-1})=\int dy\, G(z_{n,1}-y,\tau_{n,n-1})\phi_{\tau_{2,1},\dots,\tau_{n-1,n-2}}[z_{2,1},\dots,z_{n-1,n-2},\min(z_{n,n-1},y)], \tag{101}$$

where $z_{n,1}=z_{2,1}+z_{3,1}+\dots+z_{n,n-1}$. This recursive construction was used above in Sec. IV B.

VI. MULTITIME OBSERVABLES FOR OUTLIERS

In this section we study some multitime observables for the outliers in the case of N independent Brownian motions all starting from the origin. In the text we focus on the maximum $X^{(1)}(t)$ and second maximum $X^{(2)}(t)$ and obtain their joint two-time distribution. This is achieved by studying the two-time counting statistics with several intervals. In Appendix E it is indicated how to extend these results to a secondary maxima of any rank, and to any number of times.

A. Two-time distribution of maximum and second maximum

In this section for convenience we will adopt slightly different notations from the rest of the paper, so we denote t and t' the two different times, use 1 and 2 for first and second maximum, and prime quantities denote the quantities at time $t'>t$. To study large N we perform again the change of variable

$$NP_{>,t}(X_i)=e^{-z_i},\quad NP_{>,t'}(X_i')=e^{-z_i'},\quad t'-t=t\frac{\tau}{\ln N}, \tag{102}$$

and for the problem at hand this leads to the change of variable (which we use interchangeably for the random process as well as for the real variables)

$$X^{(1)}(t)\equiv X_1\simeq\sqrt{2t}\left(\sqrt{\ln N}+\frac{z_1+c_N}{2\sqrt{\ln N}}\right), \tag{103}$$

$$X^{(2)}(t)\equiv X_2\simeq\sqrt{2t}\left(\sqrt{\ln N}+\frac{z_2+c_N}{2\sqrt{\ln N}}\right), \tag{104}$$

$$X^{(1)}(t')\equiv X_1'\simeq\sqrt{2t'}\left(\sqrt{\ln N}+\frac{z_1'+c_N}{2\sqrt{\ln N}}\right), \tag{105}$$

$$X^{(2)}(t')\equiv X_2'\simeq\sqrt{2t'}\left(\sqrt{\ln N}+\frac{z_2'+c_N}{2\sqrt{\ln N}}\right). \tag{106}$$

In these variables our main result is that for $\tau=O(1)$ the joint PDF $q(z_1,z_2,z_1',z_2';\tau)$ of the random variables z_1,z_2,z_1',z_2' reads, for $z_1>z_2$, $z_1'>z_2'$,

$$\begin{aligned}q(z_1,z_2,z_1',z_2';\tau) &= \partial_{z_1}\partial_{z_1'}\partial_{z_2}\partial_{z_2'}[(\Phi(z_1,z_2';\tau)\Phi(z_2,z_1';\tau)\\ &\quad-\Phi(z_1,z_1';\tau))e^{-\Phi(z_2,z_2';\tau)}] \end{aligned}\tag{107}$$

$$\begin{aligned}&= \partial_{z_1}\partial_{z_1'}\partial_{z_2}\partial_{z_2'}[(e^{-z_1-z_2}\phi_\tau(z_2'-z_1)\\ &\quad\times\phi_\tau(z_1'-z_2)\\ &\quad-e^{-z_1}\phi_\tau(z_1'-z_1))e^{-e^{-z_2}\phi_\tau(z_2'-z_2)}]\end{aligned} \tag{108}$$

and is zero otherwise. This formula is completely explicit if one uses the expression of $\phi_\tau(z)$ given in Eq. (41).

We obtain this result by two different methods. The first one is direct but tedious and given in the Appendix D. The second one uses the counting statistics, which is interesting in its own sake and which we now describe. Note that this method also yields the two-time joint CDF of the second maximum, see Eq. (139).

B. Two-time counting statistics

Let us generalize the considerations about counting statistics of Sec. II C. Given X_1, X_2, X_1', X_2' with $X_1 > X_2$ and $X_1' > X_2'$, we again split the line into three disjoint intervals (a) $x > X_1$, (b) $X_2 < x < X_1$, and (c) $x < X_2$ at time t, and into three disjoint intervals (a′) $x' > X_1'$, (b′) $X_2' < x' < X_1'$, and (c′) $x' < X_2'$ at time t'. Each particle has some probability of being in one of these intervals at t and another one at t'. We denote these probabilities as follows, e.g.,

$$\begin{aligned} P_{aa'} &= P(x > X_1, x' > X_1') = \langle \theta_{x>X_1}\theta_{x'>X_1'}\rangle, \\ P_{ab'} &= P(x > X_1, X_2' < x' < X_1'), \end{aligned} \tag{109}$$

and so on, where we recall that $\langle \dots \rangle$ denote expectation values. Since the events are mutually exclusive one has, for each particle,

$$P_{aa'}+P_{ab'}+P_{ac'}+P_{ba'}+P_{bb'}+P_{bc'}+P_{ca'}+P_{cb'}+P_{cc'} = 1. \tag{110}$$

Raising to the power N and expanding we can read off the joint probabilities that there are $\{n_{i,i'}\} = \{n_{aa'}, n_{ab'}, \dots, n_{cc'}\}$ particles which are, respectively, in the interval $i = a, b, c$ at t, and in the interval $i' = a', b', c'$ at t'. It is simply the multinomial distribution

$$P(\{n_{ii'}\}) = \frac{N!}{\prod_{i=a,b,c}\prod_{i'=a',b',c'} n_{ii'}!}\prod_{i=a,b,c}\prod_{i'=a',b',c'} P_{ii'}^{n_{ii'}}. \tag{111}$$

In the large N limit and at the edge, all the occupation numbers $n_{ii'} = O(1)$, except $n_{cc'}$ which is a macroscopic number, $n_{cc'} \simeq N$, and one has the asymptotics, e.g., from Eq. (24) for $n = 2$,

$$P_{cc'}^{n_{cc'}} \simeq P(x < X_1, x' < X_1')^N \simeq e^{-\Phi(z_1, z_1';\tau)}, \tag{112}$$

where Φ is defined in Eq. (30), and its explicit expression is given in Eqs. (40) and (41), or also in Eq. (B10). The multinomial coefficient in Eq. (111) provides one power of N for each of the remaining $P_{ii'}$. To evaluate these probabilities at large N one first recalls that

$$\begin{aligned} NP_{aa'} &= NP(x > X_1, x' > X_1') \simeq g(z_1, z_1'; \tau) \\ &= e^{-z_1} + e^{-z_1'} - \Phi(z_1, z_1';\tau), \end{aligned} \tag{113}$$

where the function g is defined in Eq. (34) and its explicit expression is given in Eq. (B8). Next one expresses all the other probabilities in terms of this one, and of the single-time probabilities. For instance, one has

$$\begin{aligned} NP_{ab'} &= NP(x > X_1, X_2' < x' < X_1') \\ &= N\langle \theta_{x>X_1}(\theta_{x'>X_2'} - \theta_{x'>X_1'})\rangle \\ &\simeq g(z_1, z_2';\tau) - g(z_1, z_1';\tau), \\ NP_{ac'} &= NP(x > X_1, x' < X_2') = N\langle \theta_{x>X_1}(1 - \theta_{x'>X_2'})\rangle \\ &\simeq e^{-z_1} - g(z_1, z_2';\tau), \end{aligned} \tag{114}$$

and so on. This leads to the following multiple independent Poisson distribution

$$\begin{aligned} &\text{Prob}(n_{aa'}, n_{ab'}, n_{ac'}, n_{ba'}, n_{bb'}, n_{bc'}, n_{ca'}, n_{cb'}) \\ &= \frac{\lambda_{aa'}^{n_{aa'}}\lambda_{ab'}^{n_{ab'}}\lambda_{ac'}^{n_{ac'}}\lambda_{ba'}^{n_{ba'}}\lambda_{bb'}^{n_{bb'}}\lambda_{bc'}^{n_{bc'}}\lambda_{ca'}^{n_{ca'}}\lambda_{cb'}^{n_{cb'}}}{n_{aa'}!n_{ab'}!n_{ac'}!n_{ba'}!n_{bb'}!n_{bc'}!n_{ca'}!n_{cb'}!}e^{-\Phi(z_2, z_2', \tau)}, \end{aligned} \tag{115}$$

where the mean parameters, i.e., such that $\langle n_{ii'}\rangle = \lambda_{ii'}$, are given by

$$\begin{aligned} \lambda_{aa'} &= g(z_1, z_1';\tau), \\ \lambda_{ab'} &= g(z_1, z_2';\tau) - g(z_1, z_1';\tau), \\ \lambda_{ac'} &= e^{-z_1} - g(z_1, z_2';\tau), \\ \lambda_{ba'} &= g(z_2, z_1';\tau) - g(z_1, z_1';\tau), \\ \lambda_{bb'} &= g(z_2, z_2';\tau) - g(z_1, z_2';\tau) - g(z_2, z_1';\tau) + g(z_1, z_1';\tau), \\ \lambda_{bc'} &= e^{-z_2} - e^{-z_1} - g(z_2, z_2';\tau) + g(z_1, z_2';\tau), \\ \lambda_{ca'} &= e^{-z_1'} - g(z_2, z_1';\tau), \\ \lambda_{cb'} &= e^{-z_2'} - e^{-z_1'} - g(z_2, z_2';\tau) + g(z_2, z_1';\tau). \end{aligned} \tag{116}$$

By summing over the $\{n_{ii'}\}$ one can check that this distribution is correctly normalized to unity. That is, the sum of all the $\lambda_{ii'}$ equals exactly $\Phi(z_2, z_2';\tau)$ [using the second relation in Eq. (113)].

One can also check that the one-time result (19) is recovered. Indeed, one has $n_a = n_{aa'} + n_{ab'} + n_{ac'}$ is the sum of three independent Poisson variables, and the same for $n_b = n_{ba'} + n_{bb'} + n_{bc'}$, independent of n_a. The mean parameters simply add up and one can check that

$$\lambda_a = \lambda_{aa'} + \lambda_{ab'} + \lambda_{ac'} = e^{-z_1}, \tag{117}$$

$$\lambda_b = \lambda_{ba'} + \lambda_{bb'} + \lambda_{bc'} = e^{-z_2} - e^{-z_1}, \tag{118}$$

in agreement with Eq. (19). The same check can be performed for $\lambda_{a'}$ and $\lambda_{b'}$ with z_1, z_2 replaced by z_1', z_2'.

From the general result (115) the probability of various events can be computed. For instance, one obtains the joint "CDF" of the maximum and second maximum at two times. Indeed, one can check that the event where one has simultaneously

$$X^{(1)}(t) > X_1, X^{(2)}(t) < X_2, X^{(1)}(t') > X_1', X^{(2)}(t') < X_2' \tag{119}$$

is equivalent to the event

$$\begin{aligned} &n_{aa'} = 1 \text{ and } n_{ab'} = n_{ac'} = n_{ba'} = n_{bb'} \\ &\quad = n_{bc'} = n_{ca'} = n_{cb'} = 0, \\ \text{OR } &n_{ac'} = n_{ca'} = 1 \text{ and } n_{aa'} = n_{ab'} \\ &\quad = n_{ba'} = n_{bb'} = n_{bc'} = n_{cb'} = 0. \end{aligned} \tag{120}$$

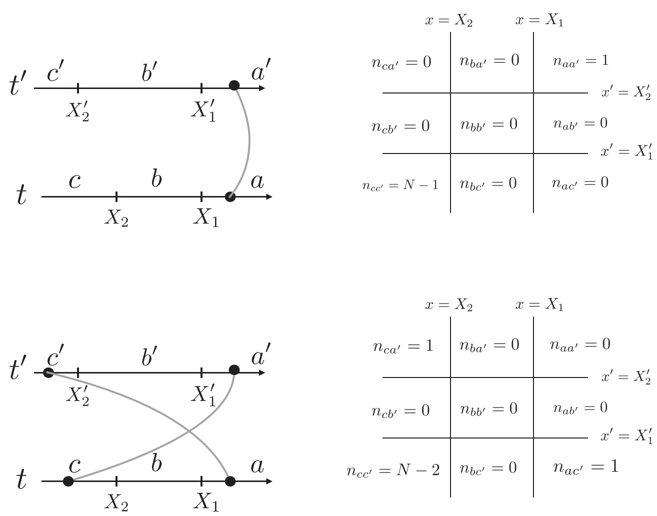

FIG. 3. Illustration of the two cases in Eq. (120) which occur in the calculation of the two-time maximum and second maximum CDF in Eq. (121). Top: first case. Left: the rightmost particle at t remains so at t'. Right: the values of the occupation numbers defined in the text, corresponding to this event. Since they are independent Poisson distributed, see Eq. (115), this case leads to the term $\lambda_{aa'}$ in Eq. (121). Bottom: second case. Left: the rightmost particle at t is different from the rightmost at t'. Since by definition of the CDF in Eq. (121) the middle interval is always empty (in both cases), these particles come from c, c' as illustrated. Right: the values of the occupation numbers corresponding to this event. This case leads to the term $\lambda_{ac'}\lambda_{ca'}$ in Eq. (121)

Thus, from Eq. (115) we obtain

$$\begin{aligned}
&\text{Prob}(X^{(1)}(t)\rangle X_1, X^{(2)}(t)\langle X_2, X^{(1)}(t')\rangle X_1', X^{(2)}(t')\langle X_2')\\
&\simeq (\lambda_{aa'} + \lambda_{ac'}\lambda_{ca'})e^{-\Phi(z_2,z_2';\tau)}\\
&= (g(z_1, z_1'; \tau) + (e^{-z_1} - g(z_1, z_2'; \tau))(e^{-z_1'} - g(z_2, z_1'; \tau)))\\
&\quad\times e^{-\Phi(z_2,z_2';\tau)}. \qquad (121)
\end{aligned}$$

Note that the first term λ_{aa} correspond to an event where the same particle is rightmost both at t and t', while the second one $\lambda_{ac'}\lambda_{ca'}$ corresponds to an event where the righmost particle has changed, and in both cases the middle interval (b and b') has remained empty. This is illustrated in Fig. 3.

Using the second relation in Eq. (113) the r.h.s. of Eq. (121) can be rewritten as

$$\begin{aligned}
&(e^{-z_1} + e^{-z_1'} - \Phi(z_1, z_1'; \tau) + (\Phi(z_1, z_2'; \tau) - e^{-z_2'})\\
&\quad(\Phi(z_2, z_1'; \tau) - e^{-z_2})))e^{-\Phi(z_2,z_2';\tau)}. \qquad (122)
\end{aligned}$$

Now taking four derivatives we obtain the joint PDF of the max and second max at two times, $q(z_1, z_2, z_1', z_2')$ in the variables z_i, z_i',

$$\begin{aligned}
q(z_1, z_2, z_1', z_2'; \tau) &= \partial_{z_1}\partial_{z_1'}\partial_{z_2}\partial_{z_2'} Eq.(122)\\
&= \partial_{z_1}\partial_{z_1'}\partial_{z_2}\partial_{z_2'}((\Phi(z_1, z_2'; \tau)\Phi(z_2, z_1'; \tau)\\
&\quad - \Phi(z_1, z_1'; \tau))e^{-\Phi(z_2,z_2';\tau)}), \qquad (123)
\end{aligned}$$

as we see that the contributions of the additional exponentials vanish. This is the result (107) announced above. This derivation is different, but equivalent to the one given in the Appendix D.

Another observable of interest, related to the two-time maximum, is the joint PDF of n_{X_1} and $n'_{X_1'}$, where n_X is the number of particles with $x_i(t) > X$ and $n'_{X'}$ is the number of particles with $x_i'(t) > X'$. One has

$$n_{X_1} = n_{aa'} + n_{ab'} + n_{ac'} = n_{aa'} + m, \qquad (124)$$

$$n'_{X_1} = n_{aa'} + n_{ba'} + n_{ca'} = n_{aa'} + m', \qquad (125)$$

where m, m' are two independent Poisson variables, independent of $n_{aa'}$, and of mean $\lambda = \lambda_{ab'} + \lambda_{ac'}$ and $\lambda' = \lambda_{ba'} + \lambda_{ca'}$, respectively. The couple $n_{X_1}, n'_{X_1'}$ thus obeys a bivariate Poisson distribution. Note that bivariate Poisson distributions also appear in the two-time counting statistics in the bulk, as discussed in Ref. [30]. Its distribution is

$$\begin{aligned}
&\text{Prob}(n_{X_1} = n_1, n'_{X_1'} = n_1')\\
&= \sum_{n_{aa}=0}^{\min(n_1,n_1')} \frac{\lambda^{n_1-n_{aa}}}{(n_1 - n_{aa})!}\frac{(\lambda')^{n_1'-n_{aa}}}{(n_1' - n_{aa})!}\frac{\lambda_{aa}^{n_{aa}}}{n_{aa}!}e^{-\lambda-\lambda'-\lambda_{aa}} \qquad (126)\\
&= \frac{(-1)^{n_1}}{n_1!}\frac{1}{n_1'!}\lambda_{aa}^{n_1}(\lambda')^{n_1'-n_1}U\left(-n_1, 1 - n_1 + n_1', -\frac{\lambda\lambda'}{\lambda_{aa}}\right)\\
&\quad\times e^{-\lambda-\lambda'-\lambda_{aa}}, \qquad (127)
\end{aligned}$$

where U is the confluent hypergeometric function and

$$\begin{aligned}
&\lambda = e^{-z_1} - g(z_1, z_1'; \tau), \quad \lambda' = e^{-z_1'} - g(z_1, z_1'; \tau),\\
&\lambda_{aa} = g(z_1, z_1'; \tau), \qquad (128)
\end{aligned}$$

with $\lambda + \lambda' + \lambda_{aa} = \Phi(z_1, z_1'; \tau) = e^{-z_1}\phi_\tau(z_1' - z_1)$. The characteristic function is

$$\langle e^{u_1 n_{X_1} + u_1' n_{X_1'}'}\rangle = e^{\lambda_{aa}(e^{u_1+u_1'}-1)+\lambda(e^{u_1}-1)+\lambda'(e^{u_1'}-1)}. \qquad (129)$$

One obtains in particular the two-time covariance of the number of particles

$$\begin{aligned}
\text{Cov}(n_{X_1}, n'_{X_1'}) &= \lambda_{aa} = g(z_1, z_1'; \tau)\\
&= e^{-z_1} + e^{-z_1'} - \Phi(z_1, z_1'; \tau), \qquad (130)
\end{aligned}$$

where we recall the asymptotics of the function g (see Appendix B),

$$g(z_1, z_1'; \tau) \simeq e^{-\max(z_1,z_1')}, \quad \tau \to 0, \qquad (131)$$

$$g(z_1, z_1'; \tau) = e^{-z_1}e^{-(z_1'-z_1+\tau)^2/(4\tau)}\chi_\tau(z_1' - z_1), \quad \tau \to +\infty, \qquad (132)$$

where the large τ behavior of $\chi_\tau(z)$ is given in Eq. (58).

One can further study the joint PDF of $n_{X_1}, n_{X_2}, n'_{X_1'}, n'_{X_2'}$. It is a multivariate Poisson distribution, which can be obtained from Eq. (115), with in addition to Eq. (124),

$$n_{X_2} = n_{X_1} + n_{ba'} + n_{bb'} + n_{bc'}, \qquad (133)$$

$$n'_{X_2} = n'_{X_1'} + n_{ab'} + n_{bb'} + n_{cb'}. \qquad (134)$$

Its characteristic function can be easily written as above, but we will not pursue this here. One can simply give the two-time covariance of the number of particles in $[X_2, X_1]$ at t and in $[X_2', X_1']$ at t', obtained as

$$\mathrm{Cov}\big((n_{X_2} - n_{X_1})(n'_{X_2'} - n'_{X_1'})\big) \simeq \mathrm{Var}(n_{bb'}) = \lambda_{bb'} \tag{135}$$

$$= g(z_2, z_2'; \tau) - g(z_1, z_2'; \tau) - g(z_2, z_1'; \tau) + g(z_1, z_1'; \tau), \tag{136}$$

using Eq. (116).

1. Subcase with only two disjoint intervals and two-time joint CDF of the second maximum

A subcase of the above result (115) is obtained by taking $X_1, X_1' \to +\infty$. One can then denote $X_2 = X$ and $X_2' = X'$. This amounts to divide the line into two disjoint intervals (b) $x > X$, (c) $x < X$ at time t, and into two disjoint intervals (b) $x > X$, (c) $x < X$ at time t'. In the large N limit, denoting z, z' the rescaled coordinates, one obtains the PDF

$$\mathrm{Prob}(n_{bb'}, n_{bc'}, n_{cb'}) = \frac{\lambda_{bb'}^{n_{bb'}} \lambda_{bc'}^{n_{bc'}} \lambda_{cb'}^{n_{cb'}}}{n_{bb'}! n_{bc'}! n_{cb'}!} e^{-\Phi(z,z',\tau)}, \tag{137}$$

with

$$\lambda_{bb'} = g(z, z'; \tau), \quad \lambda_{bc'} = e^{-z} - g(z, z'; \tau),$$
$$\lambda_{cb'} = e^{-z'} - g(z, z'; \tau), \tag{138}$$

e.g., from Eq. (116) taking $z_1, z_1' \to +\infty$, since the g function vanishes for any positive infinite argument (the other $\lambda_{ii'}$ vanish and the corresponding $n_{ii'}$ are frozen to 0). Note that the counting statistics discussed above for n_{aa}, m and m' in Eq. (124) is also recovered setting $(z, z') = (z_1, z_1')$, $bb' = aa'$, $\lambda_{bc'} = \lambda$ and $\lambda_{cb'} = \lambda'$.

The two-interval counting statistics (137), Eq. (138) allows to obtain some two-time distributions. First, of course, the two-time joint CDF of the maximum is recovered as $\mathrm{Prob}(n_{bb'} = 0, n_{bc'} = 0, n_{cb'} = 0)$, setting $(z, z') = (z_1, z_1')$. The two-time joint CDF of the second maximum can also be obtained. Recall that in Sec. (II C) we noted that for a single time the event $X^{(2)}(t) < X$ is equivalent to $n_X = 0, 1$. Here one has $n_X = n_{bb'} + n_{bc'}$ and $n'_{X'} = n_{bb'} + n_{cb'}$. Hence, the event $X^{(2)}(t) < X$ and $X^{(2)}(t') < X'$ corresponds to $n_X = 0, 1$ and $n'_{X'} = 0, 1$, which corresponds to the union of events $(n_{bb'}, n_{bc'}, n_{cb'}) \in \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$. This leads to the two-time joint CDF of the second maximum

$$\begin{aligned}
&\mathrm{Prob}(X^{(2)}(t) < X, X^{(2)}(t') < X') \\
&\quad = 1 + \lambda_{bb'} + \lambda_{bc'} + \lambda_{cb'} + \lambda_{bc'}\lambda_{cb'} \\
&\quad = (1 + e^{-z} + e^{-z'} - g(z, z'; \tau) \\
&\qquad + (e^{-z} - g(z, z'; \tau))(e^{-z'} - g(z, z'; \tau)))e^{-\Phi(z,z';\tau)} \\
&\quad = (1 + \Phi(z, z'; \tau) + (\Phi(z, z'; \tau) - e^{-z'}) \\
&\qquad \times (\Phi(z, z'; \tau) - e^{-z}))e^{-\Phi(z,z';\tau)}.
\end{aligned} \tag{139}$$

Note that one can also find the two-time joint PDF of the second maximum from Eq. (121), as

$$\begin{aligned}
q(z_2, z_2'; \tau) &= \int_{z_2}^{+\infty} dz_1 \int_{z_2'}^{+\infty} dz_1' \partial_{z_1} \partial_{z_1'} \mathcal{Q}(z_1, z_2, z_1', z_2') \\
&= \mathcal{Q}(z_2, z_2, z_2', z_2'),
\end{aligned} \tag{140}$$

$$\mathcal{Q}(z_1, z_2, z_1', z_2') = \partial_{z_2}\partial_{z_2'} \mathrm{Eq.}\ (122), \tag{141}$$

since the boundary terms at infinity do not contribute as the function g vanish when any z argument goes to $+\infty$ (beware that $z_1 = z_2$ and $z_1' = z_2'$ should be taken only after taking the two derivatives $\partial_{z_2}\partial_{z_2'}$). We have checked that this calculation gives the same result as taking $\partial_{z_2}\partial_{z_2'}$ on the CDF (139).

The above calculation can be extended to obtain the two-time joint CDF of the kth maximum for any k. One must simply enumerate all the values of the triplet $(n_{bb'}, n_{bc'}, n_{cb'})$ such that $n_X \in \{0, 1, \dots, k-1\}$ and $n'_{X'} \in \{0, 1, \dots, k-1\}$. A similar method yields the joint PDF of second maximum at t and main maximum at t', or any other combination.

Finally, all the calculations in this section can be extended to any number of times, any rank order and any number of intervals, although it quickly becomes tedious. This is sketched in Appendix E where we give explicit formula, e.g., for the joint PDF of the two-time three first maxima and the three-time first two maxima.

VII. CONTINUOUS-TIME OBSERVABLES AND RESCALED PROCESS

A. Probability that the maximum remains below some curve for $t \in [t_1, t_2]$

The multitime joint PDF formula (24) is asking for a "path integral" generalization. Given again the maximum process $X(t) = X_N(t) = \max_{i=1,\dots,N} x_i(t)$, an interesting observable in that respect, which we study in this subsection, is

$$\mathrm{Prob}(X(t) < M(t),\ \forall t \in [t_1, t_2]) \tag{142}$$

upon proper rescaling of X, M and $t_2 - t_1$.

Let us make the following preliminary observations. Consider $n = 2$ and the factorized form (40). Considering z_1 and z_2 as random variables whose CDF is $Q_{<<}$ in Eq. (43), the factorization implies that for any real a, z,

$$\mathrm{Prob}(\max(z_1, z_2 - a) < z) = e^{-e^{-z}\phi_\tau(a)}. \tag{143}$$

Hence, the random variable $\max(z_1, z_2 - a)$ for fixed a is itself a Gumbel random variable, but shifted by $\ln \phi_\tau(a)$ (a deterministic quantity). Equation (143) gives another nice interpretation to the function $\phi_\tau(a)$. Recall from Sec. IV A 7 that the factorized form (40) *does not imply* that the PDF of z_1 conditioned to a given value of $z_{2,1}$ is Gumbel [its precise form is a bit different, see Eq. (77)].

For the original maximum process, the property (143) implies that the random variable $\max(X(t_1), X(t_2) - M)$, when properly scaled (and where M and $t_2 - t_1$ are properly scaled) is also a shifted Gumbel random variable.

Clearly the property (143) extends to any number of times and for any $n \geqslant 2$,

$$\max(z_1, z_2 - a_2, \dots, z_n - a_n) \quad \text{equal in law to} G + \ln \phi_{\tau_{2,1},\dots,\tau_{n,n-1}}(a_2, a_{3,2}, \dots, a_{n,n-1}), \tag{144}$$

where G is a Gumbel random variable.

To take the continuum limit one must consider the rescaled process. For this one must fix one-time t_1 and define the rescaled maximum process (or extremal process) $z(\tau)$ (a function of the rescaled time τ, which lives in the vicinity of time t_1) by the equivalence at large N,

$$X(t) \simeq \sqrt{2t \ln N}\left(1 + \frac{z(\tau) + c_N}{2 \ln N}\right), \quad t - t_1 = t_1 \frac{\tau}{\ln N}, \tag{145}$$

or, more properly as the process

$$z(\tau) = \lim_{N\to+\infty} \left[(2 \ln N)\left(\frac{X(t)}{\sqrt{2t \ln N}}|_{t=t_1(1+\frac{\tau}{\ln N})} - 1\right) - c_N\right]. \tag{146}$$

Such limits (and more general max stable processes) were considered rigorously in the statistics and probability literature, starting with the seminal work [15]. The one-time distribution of the process $z(\tau)$ is the Gumbel distribution. In the previous part of the paper we have studied the n time CDF's, $e^{-\Phi}$, of the process $z(\tau)$.

Let us return to the observable (142) and choose to scale

$$\begin{aligned} M(t) &= X_1 + \sqrt{\frac{t_1}{2 \ln N}}(m(\tau) + \tau), \\ X_1 &= \sqrt{2t_1 \ln N}(1 + \frac{z_1 + c_N}{2 \ln N}), \\ \tau &= \frac{t - t_1}{t_1} \ln N, \end{aligned} \tag{147}$$

where $m(\tau)$ is any function with $m(0) = 0$. Then one has at large N

$$\begin{aligned} &\text{Prob}(X(t) < M(t),\ \forall t \in [t_1, t_2]) \\ &\simeq \text{Prob}(z(\tau) < z_1 + m(\tau),\ \forall \tau \in [0, \tau_{2,1}]). \end{aligned} \tag{148}$$

Now we can guess the continuum limit from Eq. (25). Indeed, Φ is the expectation of $(1 - \prod_{i=1}^n \theta_{y_i<z_i})$ over a Brownian with diffusion coefficient $D = 2$ and drift -1, started at y_1 which is distributed with e^{-y_1}. The conjecture is thus

$$\text{Prob}(z(\tau) < z_1 + m(\tau),\ \forall \tau \in [0, \tau_{2,1}]) = e^{-\Psi(z_1;\tau_{2,1};m(\tau))}, \tag{149}$$

with

$$\begin{aligned} &\Psi(z_1; \tau_{2,1}; m(\tau)) \\ &= \int dy e^{-y_1}(1 - \theta(z_1 - y_1)\text{Prob}(y_1 + \sqrt{2}B(\tau) \\ &\quad - \tau < z_1 + m(\tau), \forall \tau \in [0, \tau_{2,1}])), \end{aligned} \tag{150}$$

where $B(\tau)$ is a standard Brownian. So it is expressed in terms of the probability that the above-mentioned Brownian does not hit the moving point $z_1 + m(\tau)$. One sees that, shifting $y_1 \to y_1 + z_1$, one has

$$\Psi(z_1, \tau_{2,1}; m(\tau)) = e^{-z_1}\Psi(\tau_{2,1}; m(\tau)), \tag{151}$$

$$\begin{aligned} \Psi(\tau_{2,1}; m(\tau)) = \int dy e^{-y_1}(1 - \theta(-y_1)\text{Prob}(y_1 + \sqrt{2}B(\tau) \\ - \tau < m(\tau),\ \forall \tau \in [0, \tau_{2,1}])). \end{aligned} \tag{152}$$

Hence, we can rewrite Eq. (149) as

$$\text{Prob}(z(\tau) - m(\tau) < z_1,\ \forall \tau \in [0, \tau_{2,1}]) = e^{-e^{-z_1}\Psi(\tau_{2,1};m(\tau))}, \tag{153}$$

which implies that the random variable

$$\max_{\tau\in[0,\tau_{2,1}]}(z(\tau) - m(\tau)) \text{ equal in law to} \quad G + \ln \Psi(\tau_{2,1}; m(\tau)), \tag{154}$$

where G is a Gumbel random variable. This is the continuum limit of Eq. (144). A scaled version can then be deduced for $X(t) - M(t)$.

One can derive this formula in the case where $m(\tau) = (w - 1)\tau$ is a linear function of τ. This is done in Appendix F 2. In that case we have, with $y_1 < 0$,

$$\begin{aligned} &\text{Prob}(y_1 + \sqrt{2}B(\tau) - \tau < (w - 1)\tau,\ \forall \tau \in [0, \tau_{2,1}]) \\ &= \text{Prob}\big(\mathsf{T}^{-w}_{-y_1} > \tau_{2,1}\big), \end{aligned} \tag{155}$$

where T^{-w}_z is the first passage time at level $z > 0$ for a Brownian starting at the origin with drift $-w$ and diffusion coefficient $D = 2$ (see Appendix F 1). This leads to

$$\text{Prob}(z(\tau) - (w - 1)\tau < z_1,\ \forall \tau \in [0, \tau_{2,1}]) = e^{-e^{-z_1}\Psi_w(\tau_{2,1})}, \tag{156}$$

with (changing variable to $y = -y_1$)

$$\Psi_w(\tau) = \int dy e^y\big(1 - \theta(y)\text{Prob}\big(\mathsf{T}^{-w}_y > \tau\big)\big). \tag{157}$$

This function can be computed explicitly for any w, the result is given in Eq. (F17) in Appendix F 2. Here we only display the result for $w = 1$, i.e., for $m(\tau) = 0$, which reads

$$\begin{aligned} \Psi_1(\tau) = \frac{1}{2}(\tau + 2)\left[\text{erf}\left(\frac{\sqrt{\tau}}{2}\right) + 1\right] + \frac{e^{-\tau/4}\sqrt{\tau}}{\sqrt{\pi}} \simeq_{\tau\to+\infty} \tau \\ + 2 + O(\tau^{-3/2}e^{-\tau/4}), \end{aligned} \tag{158}$$

which implies that $\max_{\tau\in[0,\tau_{2,1}]} z(\tau)$ is a Gumbel variable shifted by $\ln \Psi_1(\tau_{2,1})$. From Appendix F 2 we obtain that at large $\tau_{2,1}$ this shift saturates to a constant, $\ln(w/(w - 1))$ for $w > 1$, while it grows linearly with $\tau_{2,1}$, as $(1 - w)\tau_{2,1}$, for $w < 1$. Hence, there is a transition at $w = 1$ in the large time behavior of the shift.

Remark. An interesting question is what happens for a parabolic moving barrier $M(t) = a_N\sqrt{2t}$. For a single walker, $N = 1$, the probability to remain below the barrier defined in Eq. (142) decays as a power law of t_2 [32,33] (see also some different generalisations to many walkers [34,35]). One can ask the same question for $N \gg 1$ walkers. Note that because of the scaling considered here one should choose $a_N \simeq \sqrt{\ln N}(1 + \frac{b+c_N}{2\ln N})$, in which case $M(t) \simeq \sqrt{2t_1 \ln N}(1 + \frac{b+\tau+c_N}{2\ln N})$, i.e., the parabolic barrier becomes a linear barrier on the scales $t - t_1 = t_1\tau/\ln N$ with $\tau = O(1)$. It corresponds to the choice $z_1 + m(\tau) = b$ above, i.e., in terms of the process $z(\tau)$ it corresponds to the probability $\text{Prob}(z(\tau) < b, \forall\tau \in [0, \tau_{2,1}])$. The probability to remain below the barrier defined in Eq. (142) is thus given, in that scaling regime at large N by Eq. (156), setting $z_1 = b$ and $w = 1$ there, i.e., by Eq. (158). For large $\tau_{2,1}$ the probability in Eq. (142) thus behaves as $\exp(-e^{-b+\ln\tau_{2,1}})$ since it corresponds to a linear barrier at the transition point $w = 1$. Finally, note that another (unrelated) case of interest where the results of [32,33] may be relevant, from formula (152), is the choice $m(\tau) = -\tau + a\sqrt{\tau}$, left for future study.

B. Running maximum and arrival time of first particle: One-time distributions

Let us first consider a single standard Brownian $x(t)$, with $r(t) = \max_{0\leqslant t'\leqslant t} x(t')$ its running maximum, and the CDF for $R_1 > 0$,

$$\text{Prob}(r(t_1) < R_1) = \text{Prob}\big(T_{R_1} > t_1\big) = \mathcal{P}_1, \tag{159}$$

where T_R is the first passage time of the standard Brownian at level $R \geqslant 0$. It is given by (see Appendix F 1)

$$\mathcal{P}_1 = \text{Prob}(T_{R_1} > t_1) = \int_{-\infty}^{R_1} dx_1 \left(\frac{e^{-\frac{x_1^2}{2t_1}}}{\sqrt{2\pi t_1}} - \frac{e^{-\frac{(x_1-2R_1)^2}{2t_1}}}{\sqrt{2\pi t_1}} \right), \tag{160}$$

which is the probability that the Brownian with an absorbing wall at R_1 has survived up to t_1.

Let us now consider the *running maximum* $R(t)$ for N identical standard Brownian motions starting from the origin

$$R(t) = \max_{0\leqslant t'\leqslant t} X(t) = \max_i r_i(t), \quad r_i(t) = \max_{0\leqslant t'\leqslant t} x_i(t'). \tag{161}$$

Let us first study the one-time CDF of the running maximum (which is a standard calculation but which sets the stage for the multitime generalization given below). It is given by

$$\begin{aligned}\text{Prob}(R(t_1) < R_1) &= \text{Prob}(r(t_1) < R_1)^N = \text{Prob}\big(T_{R_1} > t_1\big)^N \\ &= \text{Prob}\big(T^{\min}_{R_1} > t_1\big),\end{aligned} \tag{162}$$

where the last term is equal to the probability that the minimum of the first passage times $T^{\min}_{R_1} = \min_i T^i_{R_1}$ at R_1 of N identical copies is larger than t_1. This is also the *arrival time at* R_1 *of the first particle*, an important quantity.

At large N, we will scale R_1 as usual as $R_1 = \sqrt{2t_1 \ln N}(1 + \frac{z_1+c_N}{2\ln N})$ so that $1 - \mathcal{P}_1 = O(1/N)$ and

$$\text{Prob}(R(t_1) < R_1) = \text{Prob}\big(T_{R_1} > t_1\big)^N \simeq e^{-N(1-\mathcal{P}_1)}. \tag{163}$$

Let us now estimate, from Eq. (160), using similar manipulations as in Appendix A

$$\begin{aligned}N(1-\mathcal{P}_1) &= N\int dx_1 \frac{e^{-\frac{x_1^2}{2t_1}}}{\sqrt{2\pi t_1}}\bigg(1-\theta(R_1-x_1) \\ &\quad\times(1-e^{-\frac{2R_1(R_1-x_1)}{t_1}})\bigg) \\ &\simeq \int dy_1 e^{-y_1}(1-\theta(z_1-y_1)(1-e^{-2(z_1-y_1)})) \\ &= e^{-z_1}\int dy e^{y}(1-\theta(y)(1-e^{-2y})) \\ &= e^{-z_1}\left(\int_{y>0} e^{y} + \int_{y<0} e^{-y}\right) = 2e^{-z_1},\end{aligned} \tag{164, 165}$$

where we have changed variables denoting $x_1 = \sqrt{2t_1\ln N}(1 + \frac{y_1+c_N}{2\ln N})$, followed by $y_1 = z_1 - y$. Note that the term $e^{-y_1} - e^{2z_1-y_1}$ upon expanding the middle line can be interpreted as the stationary measure of the diffusion with negative drift in presence of a hard wall at z_1. At the end, not surprisingly, at large N the running maximum has a Gumbel distribution

$$\text{Prob}(R(t_1) < R_1) \simeq e^{-2e^{-z_1}} \simeq \text{Prob}(X(t_1) < R_1)^2, \tag{166}$$

with, however, a shift as compared to the instantaneous maximum, i.e., $z_1 = G + \ln 2$.

Since one can read Eq. (162) both ways (i.e., for the running maximum or for the arrival time of the first particle), the above result also implies that

$$\text{Prob}\big(T^{\min}_{R_1} > t_1\big) \simeq e^{-2e^{-z_1}}, \tag{167}$$

where the arrival time of the first particle [see Appendix F 3 for details]

$$T^{\min}_{R_1} = t_1 = \frac{R_1^2}{2\ln N}\left(1 - \frac{z_1+c_N}{\ln N})\right), \tag{168}$$

where from Eq. (167) $z_1 = G + \ln 2$ and G is Gumbel distributed.

C. Running maximum: Two-time distribution

We can now ask about the two-time joint CDF of the running maximum, at two given times, $t_2 > t_1$

$$\begin{aligned}&\text{Prob}(R(t_1) < R_1, R(t_2) < R_2) \\ &\quad= \text{Prob}(r(t_1) < R_1, r(t_2) < R_2)^N \\ &\quad= \text{Prob}\big(T_{R_1} > t_1, T_{R_2} > t_2\big)^N,\end{aligned} \tag{169}$$

with $R_2 > R_1$, which now involves the two-time joint "CDF" of the first passage times of a single Brownian at R_1 and R_2. The latter is given by

$$\begin{aligned}\mathcal{P} &= \text{Prob}\big(T_{R_1} > t_1, T_{R_2} > t_2\big) \\ &= \int_{-\infty}^{R_1} dx_1 \int_{-\infty}^{R_2} dx_2 \left(\frac{e^{-\frac{x_1^2}{2t_1}}}{\sqrt{2\pi t_1}} - \frac{e^{-\frac{(x_1-2R_1)^2}{2t_1}}}{\sqrt{2\pi t_1}} \right) \\ &\quad\times\left(\frac{e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}} - \frac{e^{-\frac{(x_2+x_1-2R_2)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}} \right),\end{aligned} \tag{170}$$

which is the probability that the Brownian with an absorbing wall at R_1 for $t \in [0, t_1]$ and an absorbing wall at R_2 for $t \in [t_1, t_2]$ has survived up to t_2.

Note that for $N = 1$, i.e., for a single Brownian $x(t)$, the two-time PDF of $R(t_1) = R_1$ and $R(t_2) = R_2$ was obtained in Ref. [36] [their Eq. (6)]. This PDF vanishes for $R_2 < R_1$ since the running maximum can only increase with time, but there is however a $\delta(R_2 - R_1)$ component in the PDF. Its weight corresponds to the probability that $x(t)$ reaches $R(t_1)$ at some time before t_1, but never crosses again the level $R(t_1)$ for $t \in [t_1, t_2]$, so that $R(t_2) = R(t_1)$. As we will see below there is a similar feature for $N > 1$.

In the large N limit we will scale as usual

$$R_i = \sqrt{2t_i \ln N}\left(1 + \frac{z_i + c_N}{2 \ln N}\right), \quad t_2 - t_1 = t_1 \frac{\tau}{\ln N}, \tag{171}$$

and insert in Eq. (170). The calculation is sketched in the Appendix F 4. One finds that at large N the two-time CDF of the running maximum takes the form, for $R_1 \leqslant R_2$, which corresponds to $z_{2,1} = z_2 - z_1 \geqslant -\tau$,

$$\text{Prob}(R(t_1) < R_1, R(t_2) < R_2) \simeq e^{-\Gamma(z_1, z_2; \tau)} = e^{-e^{-z_1} \gamma_\tau(z_{21})}, \tag{172}$$

where for $z \geqslant -\tau$

$$\gamma_\tau(z) = \int dy_1 \int dy_2\, e^{y_1} \frac{e^{-\frac{(z - y_{2,1} + \tau)^2}{4\tau}}}{\sqrt{4\pi\tau}} \bigg(1 - \theta(y_1)\theta(y_2) \times (1 - e^{-2y_1})\bigg(1 - e^{-\frac{(z+\tau+y_1)y_2}{\tau}}\bigg)\bigg), \tag{173}$$

a generalization of Eq. (165). It turns out that this integral can be computed explicitly (see Appendix F 4) and one finds

$$\gamma_\tau(z) = 2\text{erf}\left(\frac{\tau + z}{2\sqrt{\tau}}\right) + e^{-z}\text{erfc}\left(\frac{z - \tau}{2\sqrt{\tau}}\right) + e^{2\tau + z}\text{erfc}\left(\frac{3\tau + z}{2\sqrt{\tau}}\right), \quad z \geqslant -\tau. \tag{174}$$

Since the running maximum always increases, for $R_2 \leqslant R_1$ one has

$$\text{Prob}(R(t_1) < R_1, R(t_2) < R_2) = \text{Prob}(R(t_2) < R_2) \simeq e^{-2e^{-z_2}}. \tag{175}$$

The boundary case $R_2 = R_1$ corresponds to $z_1 = z_2 + \tau$, i.e., $z_{2,1} = -\tau$. Then we see that Eq. (175) is consistent with the boundary value $\gamma_\tau(z = -\tau) = 2e^{\tau}$ which one obtains from Eq. (174). Thus, one can extend Eq. (172) to any $z_{2,1}$ if one defines

$$\gamma_\tau(z) = 2e^{-z} \quad \text{for} \quad z \leqslant -\tau. \tag{176}$$

One can check that $\gamma_\tau(z)$ is a decreasing function of z, with $\gamma_\tau(z) > \phi_\tau(z)$, which is consistent with the fact that the running maximum is always larger than the instantaneous maximum.

Let us study the asymptotic behaviors of $\gamma_\tau(z)$. The large z behavior is

$$\gamma_\tau(z) = 2 + e^{-\frac{(z+\tau)^2}{4\tau}} \frac{16\tau^{5/2}}{\sqrt{\pi} z^3}\left(1 - \frac{3\tau}{z} + \frac{2\tau(5\tau - 6)}{z^2} + O(z^{-3})\right), \quad z \to +\infty. \tag{177}$$

The limit $\gamma_\tau(+\infty) = 2$ is consistent with the one-time result (166). The large τ limit is

$$\gamma_\tau(z) = 2(1 + e^{-z}) - e^{-\frac{\tau}{4} - \frac{z}{2}} \frac{16}{3\sqrt{\pi\tau}}\left(1 - \frac{80 + 3z(4 + 3z)}{36\tau} + O(\tau^{-2})\right), \quad \tau \to +\infty. \tag{178}$$

The asymptotic value $2(1 + e^{-z})$ yields $e^{-e^{-z_1}\gamma_\tau(z_{21})} \to e^{-2e^{-z_1}} e^{-2e^{-z_2}}$, i.e., $R(t_1)$ and $R(t_2)$ become statistically independent and one recovers the product of the one-time distributions.

Consider now z_1 and $z_{2,1} = z_2 - z_1$, i.e., the scaled positions of the running maximum from Eq. (171), as random variables. Their exponential moments can be computed as in Eq. (C5), replacing $\phi_\tau(z)$ by $\gamma_\tau(z)$. Similarly on has $1/\gamma_\tau(z)^{a+b} \to 2^{-(a+b)}$ as as $z \to +\infty$. Hence, Eq. (49), as well as Eq. (51), hold with the replacement $\phi_\tau(z)$ by $\gamma_\tau(z)$. One notes from Eq. (176) that the combination which appears in the formula

$$1 + \frac{\gamma_\tau'(z)}{\gamma_\tau(z)} = 0 \quad \text{for} \quad z < -\tau \tag{179}$$

vanishes for $z < -\tau$, and converges to 1 exponentially fast as $z \to +\infty$. However, this combination vanishes discontinuously at $z = -\tau$. Indeed, one finds

$$1 + \frac{\gamma_\tau'(z)}{\gamma_\tau(z)}\Big|_{z=-\tau^+} = \text{Erfc}(\sqrt{\tau}). \tag{180}$$

Hence, the marginal PDF of $z = z_{2,1} = z_2 - z_1$, which we denote by $\tilde{P}_\tau^{(2,1)}(z)$, now acquires a δ function part and is given by, for $z \geqslant -\tau$,

$$\tilde{P}_\tau^{(2,1)}(z) = q_\tau \delta(z + \tau) + \partial_z^2 \ln \gamma_\tau(z), \quad q_\tau = \text{Erfc}(\sqrt{\tau}), \quad z \geqslant -\tau, \tag{181}$$

and vanishes for $z < -\tau$. The second term is smooth: it has a finite value $1 - \text{Erfc}(\sqrt{\tau})^2$ at $z = -\tau$, with positive first derivative, so $\tilde{P}_\tau^{(2,1)}(z)$ has a maximum for some τ-dependent value of z. As explained above for $N = 1$, the weight of the δ part in Eq. (181) corresponds to the probability q_τ that $R(t_2) = R(t_1)$. Specifically, it corresponds to events such that the running maximum $R(t_1)$ was achieved by one particle at some time before t_1 and that all particles have remained below that level for $t \in [t_1, t_2]$. Note that when $\tau \to 0$ the δ part in Eq. (181) dominate the smooth part. Finally, the CDF of $z_{2,1}$ is given by

$$\text{Prob}(z_{2,1} < z) = 1 + \partial_z \ln \phi_\tau(z) \tag{182}$$

and exhibits a jump q_τ at $z = -\tau$.

The PDF in Eq. (181) is plotted in Fig. 4 and has the following asymptotic behaviors. For fixed τ and $z \to +\infty$ it decays as

$$\tilde{P}_\tau^{(2,1)}(z) = e^{-\frac{(z+\tau)^2}{4\tau}} \frac{2\sqrt{\tau}}{z\sqrt{\pi}}\left(1 - \frac{\tau}{z} + \frac{\tau(5\tau - 2)}{z^2} + O(z^{-3})\right), \quad z \to +\infty, \tag{183}$$

while for fixed z and $\tau \to +\infty$ it behaves as

$$\tilde{P}_\tau^{2,1}(z) = \frac{1}{4\cosh^2(\frac{z}{2})} + \frac{e^{-\tau}}{\sqrt{\pi\tau}}\delta(z + \tau) + \frac{e^{-\frac{\tau}{4}}}{\sqrt{\pi\tau}}\left(\frac{3 - \cosh(z)}{6\cosh^3(\frac{z}{2})} + O\left(\frac{1}{\tau}\right)\right), \quad \tau \to +\infty, \tag{184}$$

which should be compared with the result (59) for the instantaneous maximum. Hence, at large τ, $z_{2,1}$ is distributed again

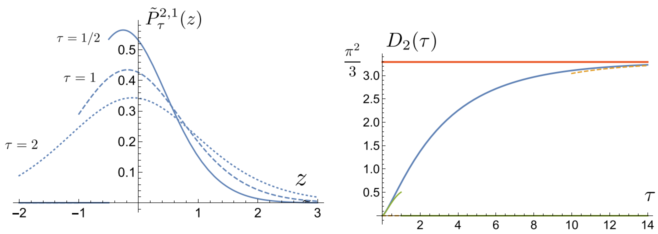

FIG. 4. Left: marginal PDF of $z = z_{2,1} = z_2 - z_1$, the scaled distance traveled by the running maximum, with $R(t_2) - R(t_1) \simeq \sqrt{\frac{t_1}{2\ln N}}(\tau + z)$. It is plotted for $\tau = 1/2$ (plain), $\tau = 1$ (dashed), $\tau = 2$ (dotted), and vanishes discontinuously for $z < -\tau$. In addition there is a δ function at $z = -\tau$, not shown, of amplitude $q_\tau = \mathrm{Erfc}(\sqrt{\tau}) \approx 0.32, 0.16, 0.05$, corresponding to the event $R(t_2) = R(t_1)$, see Eq. (181). Right: its second moment $\langle z_{2,1}^2\rangle = D_2(\tau)$, plotted versus τ (blue, plain). The first two terms in the small τ asymptotics (192) (plain, red, lowest curve), and the first three terms in the large τ asymptotics (187) (dashed, red) are also plotted, together with the limiting value $\pi^2/3$ (horizontal line). Convergence to the asymptotics is slower than for $A_2(\tau)$. Recall that $\pi^2/3 - D_2(\tau)$, i.e., the curve reflected versus $\pi^2/3$ describes the two-time covariance of the running maximum, see Eq. (194).

as the difference of two Gumbel random variables, the even moments have exactly the same limit as in Eq. (61), and the odd ones tend to zero.

The integer moments of the random variable $z_{2,1}$ are obtained as, for $k \geqslant 1$,

$$\begin{aligned}\langle z_{2,1}^k\rangle = D_k(\tau) &= \int_{-\tau}^{+\infty} dz z^k P_\tau^{2,1}(z)\\ &= q_\tau(-\tau)^k + \int_{-\tau}^{+\infty} dz z^k \partial_z^2 \ln \gamma_\tau(z). \end{aligned} \tag{185}$$

The function $D_2(\tau)$ is plotted in Fig. 4 using that formula. An alternative expression can be obtained upon integration by parts

$$\begin{aligned}\langle z_{2,1}^k\rangle = D_k(\tau) &= (1-k)(-\tau)^k + k(k-1)\\ &\times \int_{-\tau}^{+\infty} dz z^{k-2} \ln\left(\frac{\gamma_\tau(z)}{2}\right),\end{aligned} \tag{186}$$

where we have used Eq. (180) and $\gamma_\tau(-\tau) = 2e^\tau$, $\gamma_\tau(+\infty) = 2$ and that $\gamma_\tau'(z)/\gamma_\tau(z)$ decays exponentially fast at $z \to +\infty$. For $k = 1$ this implies that the first moment vanishes, $\langle z_{2,1}\rangle = 0$, as it should since $\langle z_1\rangle = \langle z_2\rangle = \ln 2 + \gamma_E$ from the one-time result (166).

To obtain the large time asymptotics of the moments we use Eq. (184) (to a higher order, not shown). Inserting into Eq. (185) the last term of Eq. (184) leads to a convergent integral on $z \in]-\infty, +\infty[$. The term $q_\tau(-\tau)^k \simeq (-\tau)^k e^{-\tau}/\sqrt{\pi\tau}$ is subdominant as compared to the leading decay $\sim e^{-\tau/4}$. One finds $\langle z_{2,1}\rangle = D_1(\tau) = 0$ and

$$\langle z_{2,1}^2\rangle = D_2(\tau) = \frac{\pi^2}{3} - \frac{16\sqrt{\pi}e^{-\tau/4}}{3\sqrt{\tau}}\left(1 - \frac{\frac{20}{9}+\frac{\pi^2}{4}}{\tau} + \frac{\frac{364}{27}+\frac{5\pi^2}{3}+\frac{5\pi^4}{32}}{\tau^2} + O\left(\frac{1}{\tau^3}\right)\right), \tag{187}$$

$$\langle z_{2,1}^3\rangle = D_3(\tau) = \frac{16\pi^{5/2}e^{-\tau/4}}{3\tau^{3/2}}\left(1 - \frac{5}{12\tau}(16+3\pi^2) + \frac{\frac{1820}{27}+\frac{125\pi^2}{9}+\frac{61\pi^4}{32}}{\tau^2} + O\left(\frac{1}{\tau^3}\right)\right), \tag{188}$$

$$\langle z_{2,1}^4\rangle = D_4(\tau) = \frac{7\pi^4}{15} - \frac{32\pi^{5/2}e^{-\tau/4}}{\sqrt{\tau}}\left(1 - \frac{5}{36\tau}(16+9\pi^2) + \frac{\frac{364}{27}+\frac{25\pi^2}{3}+\frac{61\pi^4}{32}}{\tau^2} + O\left(\frac{1}{\tau^3}\right)\right). \tag{189}$$

Note that the third moment of $z_2 - z_1$, which is also the third cumulant vanishes at large τ, as all the odd moments, and its leading order is one order lower than the corrections to the even moments.

Let us study now the close time asymptotics. At short-time difference $\tau \ll 1$, we can scale

$$z_{2,1} = -\tau + w\sqrt{\tau}, \quad w \geqslant 0, \tag{190}$$

where w is a $O(1)$ positive random variable. This is a bit different from the case of the instantaneous maximum. Upon this scaling we find that the PDF $p_\tau(w)$ of the random variable w admits the following small τ expansion

$$\begin{aligned}p_\tau(w) &= \left(1 - \frac{2\sqrt{\tau}}{\sqrt{\pi}}\left(1 - \frac{\tau}{3} + O(\tau^2)\right)\right)\delta(w) + \sqrt{\tau}\,\mathrm{erfc}\left(\frac{w}{2}\right)\\ &+\tau^{3/2}\left(\frac{e^{-\frac{w^2}{4}}w\left(5\mathrm{erfc}\left(\frac{w}{2}\right)-1\right)}{\sqrt{\pi}} + \frac{1}{2}\mathrm{erfc}\left(\frac{w}{2}\right)\left(-(3w^2+2)\mathrm{erfc}\left(\frac{w}{2}\right) + w^2 + 2\right) - \frac{4e^{-\frac{w^2}{2}}}{\pi}\right) + O(\tau^{5/2}),\end{aligned} \tag{191}$$

which is normalized to unity order by order in τ. Since $\langle z_{2,1}\rangle = 0$, one must have $\langle w\rangle = \int_0^{+\infty} dw\, w\, p_\tau(w) = \sqrt{\tau}$, which is indeed satisfied by Eq. (191) to the order displayed. Note that in the small τ limit $z_{2,1}$ has *an intermittent behavior*, it is equal to $-\tau$ with probability $1 - O(\sqrt{\tau})$ [which corresponds to the event $R(t_2) = R(t_1)$] and is of order $O(\sqrt{\tau})$ with probability $O(\sqrt{\tau})$ [which corresponds to the event $R(t_2) > R(t_1)$].

From Eq. (191) one finds the small τ expansion of the moments of $z_{2,1}$ of lowest order, as well as the skewness Sk

$$\langle z_{2,1}^2\rangle = \langle z_{21}^2\rangle^c = D_2(\tau) = \tau\langle w^2\rangle - \tau^2 = \tau\langle w^2\rangle^c = \frac{8}{3\sqrt{\pi}}\tau^{3/2} - \tau^2 + \frac{8(4\sqrt{2}-5)}{15\sqrt{\pi}}\tau^{5/2} + O(\tau^{7/2}),$$

$$\langle z_{2,1}^3\rangle = \langle z_{21}^3\rangle^c = D_3(\tau) = \tau^{3/2}\langle w^3\rangle - 3\tau^2\langle w^2\rangle + 2\tau^3 = \tau^{3/2}\langle w^3\rangle^c$$

$$= 3\tau^2 - \frac{8\tau^{5/2}}{\sqrt{\pi}} + \frac{8\tau^3}{\pi} - \frac{8(4\sqrt{2}-5)\tau^{7/2}}{5\sqrt{\pi}} + O(\tau^4),$$

$$\mathrm{Sk} = \frac{D_3(\tau)}{D_2(\tau)^{3/2}} = \frac{9\sqrt{3}\pi^{3/4}}{16\sqrt{2}\tau^{1/4}}\left(1 + \frac{(27\pi-64)\sqrt{\tau}}{24\sqrt{\pi}} + \left(-\frac{3}{2} - \frac{6\sqrt{2}}{5} + \frac{8}{3\pi} + \frac{135\pi}{128}\right)\tau + O(\tau^{3/2})\right). \quad (192)$$

The large and small τ asymptotics are shown in Fig. 4. As compared to $A_2(\tau)$, one needs larger values of τ (respectively, smaller) to approximate the function by the first few terms of the series.

Finally, let us recall that in the original variables for the running maximum $R(t_i) = R_i$, from Eq. (171) one has

$$R_2 - R_1 \simeq \sqrt{\frac{t_1}{2\ln N}}(\tau + z_{2,1}), \quad \tau = \frac{t_2 - t_1}{t_1}\ln N = O(1). \quad (193)$$

The one-time distributions of z_1 and z_2 are shifted Gumbel, hence their variance are the same as for the instantaneous maximum, i.e., one has $\mathrm{Var}R_1 = \mathrm{Var}R_2 \simeq \frac{t_1}{2\ln N}\frac{\pi^2}{6}$. This leads to the two-time covariance of the running maximum

$$\mathrm{Cov}(R(t_1), R(t_2)) \simeq \frac{t_1}{4\ln N}\left(\frac{\pi^2}{3} - D_2(\tau)\right) \simeq_{\tau\to+\infty} \times \frac{t_1}{4\ln N}\frac{16\sqrt{\pi}e^{-\tau/4}}{3\sqrt{\tau}}, \quad (194)$$

which can be compared to Eq. (73)

D. Arrival time of the first particle: Two-time distribution

Consider now, for fixed $R_2 > R_1$, the joint distribution of $T_{R_1}^{\min}$ and $T_{R_1}^{\min}$, the arrival times of the first particle, respectively, at $x = R_1$ and $x = R_2$. We can introduce again the rescaled variables z_1 and z_2 as

$$T_{R_1}^{\min} = t_1 = \frac{R_1^2}{2\ln N}\left(1 - \frac{z_1 + c_N}{\ln N}\right), \quad (195)$$

$$T_{R_2}^{\min} = t_2 = \frac{R_2^2}{2\ln N}\left(1 - \frac{z_2 + c_N}{\ln N}\right). \quad (196)$$

Clearly, if R_2 and R_1 are sufficiently separated, then z_1 and z_2 (seen as random variables) will be two independent Gumbel variables, each with the one-time CDF (167). To see how close R_2 and R_1 must be so that nontrivial correlations exist we look at the ratio

$$\frac{t_2 - t_1}{t_1} = \frac{R_2^2 - R_1^2}{R_1^2} - \frac{z_2 - z_1}{\ln N}\frac{R_2^2}{R_1^2} + O\left(\frac{1}{(\ln N)^2}\right). \quad (197)$$

Since we want this ratio to be of order $1/\ln N$ we need to choose

$$\frac{R_2 - R_1}{R_1} = \frac{\rho}{2\ln N}, \quad (198)$$

where $\rho = O(1)$ is a fixed number. Hence, we can approximate $\frac{R_2^2 - R_1^2}{R_1^2} \simeq \frac{2(R_2 - R_1)}{R_1}$ in the first term in the r.h.s. of Eq. (197), and $R_2/R_1 \simeq 1$ in the second term there, and our usual variable τ becomes

$$\tau = \frac{t_2 - t_1}{t_1}\ln N \simeq \rho - z_{2,1}, \quad (199)$$

but we have to remember that τ it is now fluctuating. Hence, the variable τ is related to the variable $z_{2,1}$ of the previous section. Note, in particular, that since $\tau > 0$, one must have $z_{2,1} < \rho$.

We can now use the results of the previous section and obtain from Eq. (172), the joint "CDF" of the first particles arrival times $T_{R_1}^{\min}, T_{R_2}^{\min}$, for fixed dimensionless distance $\rho > 0$ defined in Eq. (198),

$$\mathrm{Prob}\left(T_{R_1}^{\min} > t_1, T_{R_2}^{\min} > t_2\right) = \mathrm{Prob}(R(t_1) < R_1, R(t_2) < R_2) \quad (200)$$

$$\simeq Q_{<<}(z_1, z_2) = e^{-e^{-z_1}\sigma_\rho(z_{2,1})},$$
$$\sigma_\rho(z) = \gamma_{\tau=\rho-z}(z), \quad (201)$$

where $Q_{<<}(z_1, z_2)$ is the CDF in the rescaled variables. For large ρ, using (178), one finds $Q_{<<}(z_1, z_2) \to e^{-2e^{-z_1} - 2e^{-z_2}}$ which corresponds to two independent shifted Gumbel random variables, as expected.

In fact, it is more convenient to eliminate z_2 and use z_1 and τ as the basic random variables. It means that we write

$$T_{R_1}^{\min} = t_1 = \frac{R_1^2}{2\ln N}\left(1 - \frac{z_1 + c_N}{\ln N}\right),$$

$$\frac{T_{R_2}^{\min} - T_{R_1}^{\min}}{T_{R_1}^{\min}} \simeq \frac{T_{R_2}^{\min} - T_{R_1}^{\min}}{\left(R_1^2/2\ln N\right)} = \frac{\tau}{\ln N}, \quad (202)$$

hence τ has the interpretation of the (rescaled) delay time between detecting a first particle at R_1 and detecting a first particle at R_2 (which, of course, may not be the same particle). Note that in the second equation in Eq. (202), $T_{R_1}^{\min}$ in the denominator can be approximated by its leading order value $R_1^2/2\ln N$, to the same order at large N.

The random variables z_1 and $\tau > 0$ defined in Eq. (202) are distributed with the joint PDF (using the change of variable (C1) with $\partial_{z_{21}} = -\partial_\tau$),

$$q(z_1, \tau) = -\partial_\tau(\partial_{z_1} + \partial_\tau)e^{-e^{-z_1}\Sigma_\rho(\tau)},$$
$$\Sigma_\rho(\tau) = \gamma_\tau(\rho - \tau), \quad \rho > 0, \quad \tau > 0, \quad (203)$$

where the function $\Sigma_\rho(\tau)$ has the explicit form, from Eq. (174),

$$\Sigma_\rho(\tau) = 2\mathrm{erf}\left(\frac{\rho}{2\sqrt{\tau}}\right) + e^{\tau-\rho}\mathrm{erfc}\left(\frac{\rho - 2\tau}{2\sqrt{\tau}}\right) + e^{\rho+\tau}\mathrm{erfc}\left(\frac{\rho + 2\tau}{2\sqrt{\tau}}\right). \quad (204)$$

It obeys the identity $\Sigma_\rho(\tau) - \Sigma'_\rho(\tau) = 2\,\mathrm{erf}(\frac{\rho}{2\sqrt{\tau}})$. We recall that the parameter ρ is defined in Eq. (198) and is proportional to the spatial separation of the two points R_1, R_2 where the arrival times are measured.

Below we will need the small and large τ asymptotics of $\Sigma_\rho(\tau)$, which we now study. For $\tau \to 0$ at fixed ρ one finds

$$\Sigma_\rho(\tau) = 2 + \frac{16\tau^{5/2}}{\sqrt{\pi}\rho^3}\left(1 - \frac{12\tau}{\rho^2} + \frac{4(\rho^2+45)\tau^2}{\rho^4} + O(\tau^3)\right) \times \exp\left(-\frac{\rho^2}{4\tau}\right), \tag{205}$$

so that for $\tau \to 0$ at fixed ρ one has $\frac{\Sigma'_\rho(\tau)}{\Sigma_\rho(\tau)} \to 0$ exponentially fast. To study large τ at fixed ρ it is more convenient to write $\Sigma_\rho(\tau)$ in the equivalent form

$$\Sigma_\rho(\tau) = 2e^{\tau-\rho} + 2\mathrm{erf}\left(\frac{\rho}{2\sqrt{\tau}}\right) + e^{\tau+\rho}\mathrm{erfc}\left(\frac{2\tau+\rho}{2\sqrt{\tau}}\right) - e^{\tau-\rho}\mathrm{erfc}\left(\frac{2\tau-\rho}{2\sqrt{\tau}}\right). \tag{206}$$

From it one obtains the large τ asymptotics

$$\Sigma_\rho(\tau) = 2e^{\tau-\rho} + \frac{2\rho}{\sqrt{\pi\tau}} \times \left(1 - \frac{\rho^2+6}{12\tau} + \frac{\rho^4+20\rho^2+120}{160\tau^2} + O(\tau^{-3})\right). \tag{207}$$

In that limit the following combination, useful below, vanishes as

$$1 - \frac{\Sigma'_\rho(\tau)}{\Sigma_\rho(\tau)} = e^{\rho-\tau}\,\mathrm{erf}\left(\frac{\rho}{2\sqrt{\tau}}\right) + O(e^{-2\tau}) = e^{\rho-\tau}\,\frac{\rho}{\sqrt{\pi\tau}}\left(1 - \frac{\rho^2}{12\tau} + \frac{\rho^4}{160\tau^2} + O(\tau^{-3})\right). \tag{208}$$

Let us now turn to the exponential moments and to the marginal PDF of the delay time τ. One finds, by a similar calculation as in Eq. (C5),

$$\langle e^{-sz_1 - b\tau}\rangle = -\Gamma(1+s)\int_0^{+\infty} d\tau e^{-b\tau}\partial_\tau \times \left(\left(1 - \frac{\Sigma'_\rho(\tau)}{\Sigma_\rho(\tau)}\right)\frac{1}{\Sigma_\rho(\tau)^s}\right). \tag{209}$$

For $b = 0$, the integrand is a total derivative with boundary values 0 at $\tau = +\infty$ and 2^{-s} at $\tau = 0$ (from the above asymptotics), and one recovers the one-time result, $\langle e^{-sz_1}\rangle = \Gamma(1+s)2^{-s}$. Inserting $s = 0$ one finds the PDF $q(\tau)$ and the "CDF" of the (scaled) delay time, i.e., the random variable τ,

$$q(\tau) = -\partial_\tau\left(1 - \frac{\Sigma'_\rho(\tau)}{\Sigma_\rho(\tau)}\right) = \partial_\tau^2 \ln \Sigma_\rho(\tau),$$
$$\mathrm{Prob}(\tau > t) = 1 - \partial_\tau \ln \Sigma_\rho(\tau). \tag{210}$$

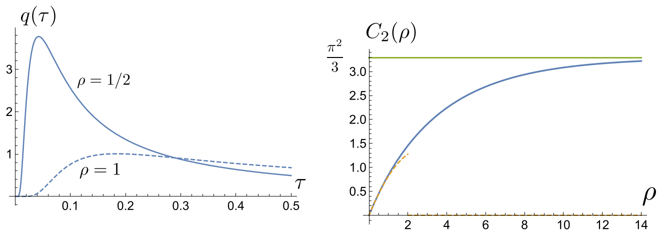

FIG. 5. Left: PDF $q(\tau)$ of the scaled time delay τ between the arrival of the first particles at R_1 and at R_2, with $\frac{T^{\min}_{R_2} - T^{\min}_{R_1}}{T^{\min}_{R_1}} \simeq \frac{\tau}{\ln N}$, for some values of the parameter ρ, $\rho = 1/2$ (plain), $\rho = 1$ (dotted), with $\frac{R_2 - R_1}{R_1} = \frac{\rho}{2\ln N}$. Right: its second cumulant $\langle\tau^2\rangle_c = C_2(\rho)$, plotted versus ρ (plain). The small ρ asymptotics (dashed) and the the limiting value $\pi^2/3$ at large ρ (horizontal line) are also shown.

This PDF is plotted in Fig. 5. The PDF $q(\tau)$ vanishes exponentially fast for $\tau \to 0$,

$$q(\tau) = \frac{\rho e^{-\frac{\rho^2}{4\tau}}}{2\sqrt{\pi}\tau^{3/2}}\left(1 + \frac{4\tau^2}{\rho^2} - \frac{8\tau^3}{\rho^4} + \frac{16(\rho^2+3)\tau^4}{\rho^6} + O(\tau^5)\right). \tag{211}$$

It is easy to see that the leading behavior is exactly the PDF of the first passage time of a single (symmetric standard) Brownian [using the definitions (198) and (199) of ρ and τ]. This is because for small τ the rightmost particle at t_1 is also the rightmost particle at t_2. The PDF $q(\tau)$ exhibits a maximum for some value of τ, and then decreases exponentially at large τ as

$$q(\tau) = \frac{\rho}{\sqrt{\pi\tau}}e^{\rho-\tau}\left(1 + \frac{6-\rho^2}{12\tau} + \frac{\rho^2(\rho^2-20)}{160\tau^2} + O(\tau^{-3})\right) + O(e^{-2\tau}). \tag{212}$$

Using integration by parts and the above asymptotics one finds that the average scaled delay time is simply

$$M_1(\rho) = \langle\tau\rangle = \int_0^{+\infty} d\tau\, \tau\, q(\tau) = -\int_0^{+\infty} d\tau\, \tau\, \partial_\tau\left(1 - \frac{\Sigma'_\rho(\tau)}{\Sigma_\rho(\tau)}\right) = \rho, \tag{213}$$

which in the original variables means

$$\left\langle\frac{T^{\min}_{R_2} - T^{\min}_{R_1}}{T^{\min}_{R_1}}\right\rangle \simeq 2\frac{R_2 - R_1}{R_1}. \tag{214}$$

Similarly, one finds by integration by parts, the second moment and the second cumulant,

$$M_2(\rho) = \langle\tau^2\rangle = \int_0^{+\infty} d\tau\, \tau^2$$
$$q(\tau) = 2\int_0^{+\infty} d\tau\, \ln\left(\frac{\Sigma_\rho(\tau)}{2e^{\tau-\rho}}\right),$$
$$C_2(\rho) = \langle\tau^2\rangle_c = M_2(\rho) - \rho^2. \tag{215}$$

As one can see in Fig. 5 the second cumulant reaches a finite value at large ρ, again equal to $C_2(+\infty) = \pi^2/3$. At small ρ it is well approximated by $C_2(\rho) \simeq \rho - 0.181\rho^2$.

To understand the large ρ limit, one can decompose τ into its average $\langle\tau\rangle = \rho$ and its fluctuating part δ, $\tau = \rho + \delta$, and one finds that for large ρ at fixed δ,

$$\Sigma_\rho(\tau = \rho + \delta) = 2 + 2e^\delta - e^{-\frac{\rho}{4}+\frac{\delta}{4}}\frac{16}{3\sqrt{\pi\rho}}(1 + O(\rho^{-1})), \tag{216}$$

which leads to

$$q(\tau = \rho + \delta) = \frac{1}{4\cosh^2(\delta/2)} + \frac{(e^\delta(22 - 9e^\delta) - 1)e^{\frac{\delta-\rho}{4}}}{6\sqrt{\pi\rho}(e^\delta + 1)^3} \times (1 + O(\rho^{-1})). \tag{217}$$

The leading term is again the PDF of the difference between two independent Gumbel random variables that we encountered several times previously. The correction term correctly integrates to zero for $\delta \in]-\infty, +\infty[$, with also zero first moment. The second moment gives

$$C_2(\rho) = \frac{\pi^2}{3} - \frac{\gamma}{6\sqrt{\pi\rho}}e^{-\frac{\rho}{4}+\frac{\delta}{4}}(1 + O(\rho^{-1})), \quad \rho \to +\infty, \tag{218}$$

with $\gamma \approx 142.172$. We see that the second cumulant $\langle\tau^2\rangle_c$, which is infinite for the first passage time of a symmetric Brownian, here is finite and dominated at large ρ by the $O(1)$ vicinity of $\tau = \rho$.

Finally, using similar methods as in this and the previous sections, one can obtain a formula for the multitime CDF of the running maximum and of the arrival time of the first particle, $\text{Prob}(R(t_1) < R_1, \dots, R(t_n) < R_n) = \text{Prob}(T^{\min}_{R_1} > t_1, \dots, T^{\min}_{R_n} > t_n)$, for $n \geqslant 3$ times. It is given in Appendix F 5.

VIII. CONCLUSION

In this paper, using simple methods of statistical physics, we have studied the dynamics of a cloud of a large number of independent identical Brownian particles near its edge in one dimension. We have focused on the few rightmost particles, i.e., with the largest positions (the outliers). To probe their dynamics we have computed the joint distribution of the maximum position at a set of different times, and extended it to the maximum and second maximum, and eventually to any finite rank, although the formula quickly become complicated. For the maximum itself we have obtained distributions which appeared before in some form in probability theory and statistics. We have found a physically appealing derivation using the diffusion equation which naturally leads to a recursive construction of these distributions. For the outliers, a useful tool was the counting statistics, which, for independent particles, leads to multivariate Poisson distributions. In a second part we have studied other properties of the rescaled maximum process, such as the probability that it remains below some space-time curve. We have studied the multitime statistics of the running maximum of the cloud, that is the maximum of all positions up to time t. Since the running maximum is intimately related to the first passage time, we have also obtained the statistics of the "arrival times of the first particle," at several locations. In particular we obtained an explicit formula for the distribution of delay time between the first detection of a particle at two different neighboring locations.

We believe that the above result will be of interest for numerics or experiments probing the behavior of a cloud of diffusing particles. We have restricted here to the case of identical Brownian particles all starting from the origin, but the study can be extended to more general initial conditions, nonidentical particles, or even to more general Gaussian processes, e.g., as considered in Ref. [30]. It would be of great interest to extend these results to diffusion in presence of a random environment, e.g., as discussed in the introduction. The method based on the diffusion equation may provide a route in that direction.

There are possible applications to a number of other problems. One is single-file diffusion, i.e., Brownian motions which do not interact except that they reflect on each others at each collision [30,37,38]. It amounts to considering the ordered set of positions in our problem, $x_i(t) \to x^{(i)}(t)$, and the present results immediately apply to the dynamics at the edge. This would describe the dynamics of a gas at very high temperature with hard core repulsion.

Another example is related to the random energy model (REM). Consider a portfolio with N stocks, each performing independent Black-Scholes geometric Brownian motions, of total value $Z = \sum_{i=1}^N e^{x_i(t)}$, where $x_i(t)$ are the positions of the particles in our Brownian cloud model. One can scale $t = \tilde{t}\ln N$ and use the results of the present paper since it is just a uniform rescaling of the t_i's. The rescaled time plays the role of an inverse temperature $\beta = \sqrt{\tilde{t}}$. It is well known [39–41] that (minus) the intensive free energy $f = \frac{1}{\ln N\sqrt{\tilde{t}}}\ln Z(\tilde{t})$ exhibits a REM freezing transition from a high temperature phase for $\tilde{t} < 2$ with $f = \frac{1}{\sqrt{\tilde{t}}} + \frac{\sqrt{\tilde{t}}}{2}$, towards a glass phase for $\tilde{t} > 2$, with $f = \sqrt{2}$. In the zero temperature limit $\tilde{t} \to +\infty$ one has $f = \max_i x_i(t)/(\sqrt{t\ln N})$ and the free energy of the REM identifies with the maximum of N Gaussian random variables (properly scaled), while the extensive free energy $F = f\ln N$ has $O(1)$ Gumbel distributed fluctuations. The multitime distribution of the maximum discussed in this paper thus describes the time evolution of the portfolio at large $\tilde{t}$, with correlations existing in small time windows. It would be interesting to compute the analog of the multitime distributions studied here, but for finite $\tilde{t}$, i.e., in the finite temperature regime for the REM. Some results were obtained in [42,43], within a different scaling.

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APPENDIX A: DERIVATION OF THE MULTITIME JOINT CDF

Here we provide a simple derivation of the results (24) and (25) in the text. Since the walkers are independent one has

$$\mathrm{Prob}(X(t_1) < X_1, \dots, X(t_n) < X_n) = \mathrm{Prob}(x(t_1) < X_1, \dots, x(t_n) < X_n)^N = e^{N \ln \mathrm{Prob}(x(t_1)<X_1,\dots,x(t_n)<X_n)}. \tag{A1}$$

Now one has, from normalization

$$\mathrm{Prob}(x(t_1) < X_1, \dots, x(t_n) < X_n) = \int_{x_1<X_1} \dots \int_{x_n<X_n} p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\dots p_{t_n-t_{n-1}}(x_n-x_{n-1}) = 1 - I(X_1,\dots,X_n;t_1,\dots t_n),$$

$$I(X_1,\dots,X_n;t_1,\dots t_n) = \int dx_1\dots dx_n\left(1-\prod_{i=1}^n \theta(x_i<X_i)\right)p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\dots p_{t_n-t_{n-1}}(x_n-x_{n-1}). \tag{A2}$$

Hence, one has

$$\mathrm{Prob}(X(t_1) < X_1, \dots, X(t_n) < X_n) = e^{N\ln(1-I(X_1,\dots,X_n;t_1,\dots t_n))}. \tag{A3}$$

Until now this is exact for any N. Let us now consider $N \gg 1$.

For large N the probability remains of order unity when $I(X_1,\dots,X_n;t_1,\dots t_n)) = O(1/N)$. The change of variable from X_j to z_j and from t_j to τ_j will produce exactly the correct factor. Let us recall that

$$t_j = t_1\left(1+\frac{\tau_{j,1}}{\ln N}\right), \tag{A4}$$

with the notation $\tau_{i,j} = \tau_i - \tau_j$ and $\tau_1 = 0$. In the expression for $I(X_1,\dots,X_n;t_1,\dots t_n)$ one also performs the change of variables

$$X_j = \sqrt{2t_j}\sqrt{\ln N}\left(1+\frac{z_j+c_N}{2\ln N}\right) \simeq \sqrt{2t_1}\sqrt{\ln N}\left(1+\frac{z_j+\tau_{j,1}+c_N}{2\ln N}\right), \tag{A5}$$

$$x_j = \sqrt{2t_j}\sqrt{\ln N}\left(1+\frac{y_j+c_N}{2\ln N}\right) \simeq \sqrt{2t_1}\sqrt{\ln N}\left(1+\frac{y_j+\tau_{j,1}+c_N}{2\ln N}\right). \tag{A6}$$

Hence, we have

$$dx_j = \frac{\sqrt{2t_j}}{2\sqrt{\ln N}}dy_j \simeq \frac{\sqrt{2t_1}}{2\sqrt{\ln N}}dy_j. \tag{A7}$$

Consider now, for $j = 1,\dots,n-1$,

$$\begin{aligned} p_{t_{j+1}-t_j}(x_{j+1}-x_j) &= \frac{1}{\sqrt{2\pi(t_{j+1}-t_j)}}e^{-\frac{(x_{j+1}-x_j)^2}{2(t_{j+1}-t_j)}} \simeq \frac{1}{\sqrt{2\pi t_1\frac{\tau_{j+1,j}}{\ln N}}}e^{-\frac{\left(\frac{\sqrt{2t_1}}{2\sqrt{\ln N}}(y_{j+1}+\tau_{j+1,1}-(y_j+\tau_{j,1}))\right)^2}{\frac{\tau_{j+1,j}}{\ln N}}} \\ &= \frac{\sqrt{\ln N}}{\sqrt{2\pi t_1\tau_{j+1,j}}}e^{-\frac{(y_{j+1}-y_j+\tau_{j+1,j})^2}{4\tau_{j+1,j}}} = \frac{\sqrt{2\ln N}}{\sqrt{t_1}}G(y_{j+1,j},\tau_{j+1,j}). \end{aligned} \tag{A8}$$

Hence,

$$dx_{j+1}p_{t_{j+1}-t_j}(x_{j+1}-x_j) \simeq dy_{j+1}G(y_{j+1,j},\tau_{j+1,j}). \tag{A9}$$

Finally, from Eq. (6),

$$Np_{t_1}(x_1)dx_1 = e^{-y_1}dy_1, \tag{A10}$$

putting all the factors together we obtain

$$I(X_1,\dots,X_n;t_1,\dots t_n) \simeq \frac{1}{N}\int_{y_1,\dots,y_n}\left(1-\prod_{i=1}^n\theta_{y_i<z_i}\right)e^{-y_1}G(y_{2,1},\tau_{2,1})\dots G(y_{n,n-1},\tau_{n,n-1}), \tag{A11}$$

leading to the result (25) in the text.

APPENDIX B: CALCULATIONS OF SOME INTEGRALS

In this Appendix we compute the functions $g(z_1,z_2;\tau)$, $\Phi(z_1,z_2;\tau)$ and $\phi_\tau(z)$ defined in the text, show a symmetry property, and discuss their limit as $\tau \to 0$.

From its definition in Eq. (34) one has

$$g(z_1,z_2;\tau_{2,1}) = \int_{z_1<y_1,z_2<y_2} e^{-y_1}G(y_{2,1},\tau) = e^{-z_1}I(z_{2,1},\tau), \tag{B1}$$

in terms of the integral

$$I(z,\tau)=\int_{y_1>0,y_2>0} e^{-y_1}G(y_{2,1}+z,\tau), \tag{B2}$$

where we recall that $z_{2,1}=z_2-z_1$ and $y_{2,1}=y_2-y_1$ and the last equality in Eq. (B1) is obtained by the shift of integration variables $y_i\to y_i+z_i$. One first recalls that for any $b>0$,

$$\int\frac{dk}{2\pi}\frac{e^{-k^2\tau-ikx}}{ik+b}=\int\frac{dk}{2\pi}\int_{y>0}e^{-k^2\tau-ik(y+x)-by}=\int_{y>0}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x+y)^2}{4\tau}-by}=\frac{1}{2}e^{b^2\tau+bx}\text{erfc}\left(\frac{x+2b\tau}{2\sqrt{\tau}}\right). \tag{B3}$$

Hence, one has

$$I(z,\tau)=\int\frac{dk}{2\pi}e^{-k^2\tau-ik(z+\tau)}\int_{y_1>0,y_2>0}e^{-y_1-iky_{2,1}} \tag{B4}$$

$$=\int\frac{dk}{2\pi}e^{-k^2\tau-ik(z+\tau)}\frac{1}{1-ik}\frac{1}{ik+0^+}=g_\tau(z+\tau), \tag{B5}$$

where we have defined

$$g_\tau(a):=\int\frac{dk}{2\pi}e^{-k^2\tau-ika}\frac{1}{1-ik}\frac{1}{ik+0^+}=\int\frac{dk}{2\pi}e^{-k^2\tau-ika}\left(\frac{1}{1-ik}+\frac{1}{ik+0^+}\right) \tag{B6}$$

$$=\frac{1}{2}e^{\tau-a}\text{erfc}\left(\frac{2\tau-a}{2\sqrt{\tau}}\right)+\frac{1}{2}\text{erfc}\left(\frac{a}{2\sqrt{\tau}}\right). \tag{B7}$$

We finally obtain

$$g(z_1,z_2;\tau)=e^{-z_1}g_\tau(z_{2,1}+\tau)=\frac{1}{2}\left(e^{-z_1}\text{erfc}\left(\frac{z_2-z_1+\tau}{2\sqrt{\tau}}\right)+e^{-z_2}\text{erfc}\left(\frac{z_1-z_2+\tau}{2\sqrt{\tau}}\right)\right). \tag{B8}$$

Let us also recall the definitions of the functions Φ and ϕ_τ, from Eqs. (32) and (40),

$$\Phi(z_1,z_2;\tau)=e^{-z_1}+e^{-z_2}-g(z_1,z_2;\tau)=e^{-z_1}\phi_\tau(z_{2,1}). \tag{B9}$$

This leads to

$$\Phi(z_1,z_2;\tau)=\frac{1}{2}\left(e^{-z_1}\text{erfc}\left(\frac{z_1-z_2-\tau}{2\sqrt{\tau}}\right)+e^{-z_2}\text{erfc}\left(\frac{z_2-z_1-\tau}{2\sqrt{\tau}}\right)\right), \tag{B10}$$

where we used that $\text{erfc}(x)+\text{erfc}(-x)=2$, since one has $\text{erfc}(x)=1-\text{erf}(x)$ and $\text{erf}(-x)=-\text{erf}(x)$. From this one obtains the explicit form (41) for $\phi_\tau(z)$ given in the text. It can also be obtained by noting that

$$\phi_\tau(z)=1+e^{-z}-g_\tau(z+\tau) \tag{B11}$$

and using Eq. (B7).

1. Symmetry property

The function $g_\tau(z)$ obeys an interesting identity. Consider the form (B5). From it, it is immediate to see that $\partial_\tau g_\tau(z+\tau)=\frac{-1}{\sqrt{4\pi\tau}}e^{-\frac{(z+\tau)^2}{4\tau}}$. Taking into account the boundary conditions, we obtain

$$g_\tau(z+\tau)=\int_\tau^\infty\frac{dt}{\sqrt{4\pi t}}e^{-\frac{(t+z)^2}{4t}}. \tag{B12}$$

On this expression, using that $\frac{(t+z)^2}{4t}-\frac{(t-z)^2}{4t}=z$ we obtain

$$g_\tau(-z+\tau)=\int_\tau^\infty\frac{dt}{\sqrt{4\pi t}}e^{-\frac{(t-z)^2}{4t}}=e^z\int_\tau^\infty\frac{dt}{\sqrt{4\pi t}}e^{-\frac{(t+z)^2}{4t}}=e^zg_\tau(z+\tau). \tag{B13}$$

From this we have

$$\phi_\tau(-z)=1+e^z-g_\tau(-z+\tau)=1+e^z-e^zg_\tau(z+\tau)=e^z(1+e^{-z}-g_\tau(z+\tau))=e^z\phi_\tau(z), \tag{B14}$$

which is the symmetry property (42) discussed in the text. Note that this symmetry is equivalent to the fact that $\Phi(z_1,z_2;\tau)$ is symmetric in z_1,z_2 as can be seen on its explicit form (B10).

2. Limit $\tau \to 0$

For small time difference $\tau \to 0$ one has

$$\lim_{\tau\to 0} g_\tau(z+\tau) = \int_0^\infty \frac{dt}{\sqrt{4\pi t}} e^{-\frac{(t+z)^2}{4t}} = e^{-z}\theta(z) + \theta(-z). \tag{B15}$$

Hence,

$$\lim_{\tau\to 0} \phi_\tau(z) = \theta(z) + e^{-z}\theta(-z), \quad \lim_{\tau\to 0} \Phi(z_1, z_2, \tau) = e^{-\min(z_1,z_2)}, \tag{B16}$$

as it should since for $\tau \to 0$, $X(t_2) \to X(t_1)$.

APPENDIX C: SOME DETAILS OF CALCULATIONS FOR $n = 2$

Let us compute the exponential moments associated to the joint CDF (43) in the text. For the calculations we consider z_1 and $z_{21} = z_2 - z_1$ as independent real variables. The derivatives should be replaced as follows:

$$\partial_{z_1} \to \partial_{z_1} - \partial_{z_{21}}, \quad \partial_{z_2} \to \partial_{z_{21}}. \tag{C1}$$

Let us consider the expectation value of exponentials (note that we abusively denote by the same letter the random variable and a real integration variable)

$$\langle e^{-az_1-bz_2}\rangle = \int dz_1 dz_2 e^{-az_1-bz_2} \partial_{z_1}\partial_{z_2} Q_{<<}(z_1,z_2) \tag{C2}$$

$$= \int dz_{21} e^{-bz_{21}} \partial_{z_{21}} \int dz_1 e^{-(a+b)z_1} (\partial_{z_1} - \partial_{z_{21}})\, e^{-e^{-z_1}\phi_\tau(z_{21})} \tag{C3}$$

$$= \int dz_{21} e^{-bz_{21}} \partial_{z_{21}} \left((\phi_\tau(z_{21}) + \phi'_\tau(z_{21})) \int dz_1 e^{-(a+b+1)z_1}\, e^{-e^{-z_1}\phi_\tau(z_{21})} \right) \tag{C4}$$

$$= \Gamma(1+a+b) \int dz e^{-bz} \partial_z \left(\left(1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)}\right) \frac{1}{\phi_\tau(z)^{a+b}} \right), \tag{C5}$$

where we have set $z_{21} = z$ and performed the integration over z_1 using that $\int dz_1 e^{-Az_1} e^{-pe^{-z_1}} = p^{-A}\Gamma(A)$. This is equivalent to Eq. (49) in the text, with $a + b = s$.

Let us examine the asymptotic behavior of the terms which appear in the integral (C5). One has, using Eq. (44),

$$1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)} = 1 + \frac{-e^{-z} + \psi'_\tau(z)}{e^{-z} + \psi_\tau(z)} = \frac{\psi_\tau(z) + \psi'_\tau(z)}{e^{-z} + \psi_\tau(z)} \simeq e^z \psi'_\tau(z) = -e^{-z} e^{-\frac{(z+\tau)^2}{4\tau}} \left(\frac{\tau^{1/2}}{z\sqrt{\pi}} + O(z^{-2}) \right), \quad z \to -\infty,$$

$$1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)} \simeq 1 + \psi'_\tau(z) = 1 - e^{-\frac{(z+\tau)^2}{4\tau}} \left(\frac{\tau^{1/2}}{z\sqrt{\pi}} + O(z^{-2}) \right), \quad z \to +\infty, \tag{C6}$$

and

$$\frac{1}{\phi_\tau(z)^{a+b}} \simeq e^{(a+b)z}, \quad z \to -\infty, \quad \frac{1}{\phi_\tau(z)^{a+b}} \simeq 1, \quad z \to +\infty. \tag{C7}$$

Using these asymptotics, one checks that setting $b = 0$ in Eq. (C5) the integrand is a total derivative and one obtains

$$\langle e^{-az_1}\rangle = \Gamma(1+a) \int dz \partial_z \left(\left(1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)}\right) \frac{1}{\phi_\tau(z)^a} \right) = \Gamma(1+a) \tag{C8}$$

as required since the PDF of z_1 is the Gumbel distribution. Similarly, setting $a = 0$ one obtains

$$\langle e^{-bz_2}\rangle = \Gamma(1+b) I_b, \quad I_b = \int dz e^{-bz} \partial_z \left(\left(1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)}\right) \frac{1}{\phi_\tau(z)^b} \right) \tag{C9}$$

In fact, one can show that $I_b = 1$, which leads to $\langle e^{-bz_2}\rangle = \Gamma(1+b)$ as required since the PDF of z_2 is also the Gumbel distribution. This is indeed a consequence of the symmetry (42) which also implies

$$\frac{e^{-bz}}{\phi_\tau(z)^b} \left(1 + \frac{\phi'_\tau}{\phi_\tau}(z)\right) = -\frac{\phi'_\tau}{\phi_\tau}(-z) \frac{1}{\phi_\tau(-z)^b} \tag{C10}$$

Hence, for $b > 0$ integrating by part, using the symmetry and changing $z \to -z$

$$I_b = b \int dz e^{-bz} \left(1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)}\right) \frac{1}{\phi_\tau(z)^b} = -b \int dz \frac{\phi'_\tau}{\phi_\tau}(z) \frac{1}{\phi_\tau(z)^b} = \left[\frac{1}{\phi_\tau(z)^b} \right]_{-\infty}^{+\infty} = 1 \tag{C11}$$

One can further simplify the expression of the exponential moments. Indeed, for $b > 0$, using the above asymptotics one see that one can integrate by part and get

$$\langle e^{-az_1 - bz_2} \rangle = b\,\Gamma(1+a+b) \int dz e^{-bz} \left(1 + \frac{\phi'_\tau(z)}{\phi_\tau(z)}\right) \frac{1}{\phi_\tau(z)^{a+b}} \tag{C12}$$

For $a, b > 0$ one can perform another integration by part and obtain

$$\langle e^{-az_1 - bz_2} \rangle = \frac{ab}{a+b} \Gamma(1+a+b) \int dz \frac{e^{-bz}}{\phi_\tau(z)^{a+b}} \tag{C13}$$

Although this closed expression is easy to integrate numerically, it is tricky to obtain the moments from it, and we refer to the discussion in the text.

Finally, for three times one performs the change $\partial_{z_1} \to \partial_{z_1} - \partial_{z_{21}}$, $\partial_{z_2} \to \partial_{z_{21}} - \partial_{z_{32}}$, $\partial_{z_3} \to \partial_{z_{32}}$ which leads to Eqs. (81) and (82).

APPENDIX D: MULTITIME ORDER STATISTICS: COMBINATORICS

To display more conveniently the combinatorics associated to the multitime order statistics, it is useful to consider the case where each particle $i = 1, \dots, N$ is described by a distinct one-time PDF, which in this Appendix we denote as $p_i(x)$, and a CDF $P_i(x < X)$, and a two-time CDF denoted $P_i(x < X, x' < X')$. For two-time we denote with prime the quantities at time $t' > t$.

As a warmup let us recall the PDF and CDF of the maximum at one time

$$\begin{aligned} q(X_1) &= \sum_i p_i(X_1) \prod_{j \neq i} P_j(x < X_1) = \partial_{X_1} Q(X_1), \\ Q(X_1) &= \prod_\ell P_\ell(x < X_1) \end{aligned} \tag{D1}$$

and the joint PDF of the maximum and the second maximum at one time

$$q(X_1, X_2) = \theta_{X_2 < X_1} \sum_{i \neq j} p_i(X_1) p_j(X_2) \prod_{k \neq i,j} P_k(x < X_2). \tag{D2}$$

It can also be retrieved from a joint "CDF", since for $X_2 < X_1$ one has

$$\begin{aligned} &\text{Prob}(X^{(1)}(t) > X_1, X^{(2)}(t) < X_2) \\ &= \sum_i P_i(x > X_1) \prod_{k \neq i} P_k(x < X_2) \end{aligned} \tag{D3}$$

and taking $-\partial_{X_1}\partial_{X_2}$ it recovers the above expression for $q(X_1, X_2)$. It turns out that a generalization of this "CDF" is convenient to obtain the two-time distribution for the maximum and second maximum.

As a first exercise, let us now write the PDF of the maximum at two times, and in a second stage check consistency with the result given in the text for the CDF. There are two possibilities, either particle i is the rightmost for both times, of it is the rightmost only at the first time, but at the second time particle j has become the rightmost. This leads to the expression of the joint PDF

$$\begin{aligned} q(X_1, X'_1) &= \sum_i p_i(X_1, X'_1) \prod_{k \neq i} P_k(x < X_1, x' < X'_1) \\ &\quad + \sum_{i \neq j} P_i(X_1, x' < X'_1) P_j(x < X_1, X'_1) \end{aligned} \tag{D4}$$

$$\begin{aligned} &\prod_{k \neq i,j} P_k(x < X_1, x' < X'_1) = \partial_{X_1}\partial_{X'_1} Q(X_1, X'_1), \\ &Q(X_1, X'_1) = \prod_\ell P_\ell(x < X_1, x' < X'_1) \end{aligned} \tag{D5}$$

Let us now consider the case of identical particles ($p_i = p$ independent of i and so on) and estimate (D4) at large N, introducing the associated variables z_1, z'_1 and $\tau_{2,1} = \tau$ as before. The two terms in Eq. (D4) have factors N and $N(N-1) \simeq N^2$, respectively. One uses the limits

$$\begin{aligned} NP(x > X_1, x' > X'_1) &\simeq g(z_1, z'_1; \tau) \\ &= e^{-z_1} + e^{-z'_1} - \Phi(z_1, z'_1; \tau) \end{aligned} \tag{D6}$$

$$P(x < X_1, x' < X'_1)^N \simeq e^{-\Phi(z_1, z'_1; \tau)} \tag{D7}$$

where g and Φ are defined in Eq. (34) and given explicitly in Eqs. (B8) and (B10). By two differentiation of the first line, this leads to

$$N p(X_1, X'_1) dX_1 dX'_1 \simeq -\partial_{z_1}\partial_{z'_1} \Phi(z_1, z'_1; \tau) dz_1 dz'_1 \tag{D8}$$

To evaluate the mixed PDF/CDF $p(X_1, x' < X'_1)$ we first differentiate the first line of Eq. (D6) w.r.t. X_1 which gives

$$N p(X_1, x' > X'_1) dX_1 \simeq -\partial_{z_1} g(z_1, z'_1; \tau) dz_1 \tag{D9}$$

which can be rewritten as

$$N(p(X_1) - p(X_1, x' < X'_1)) dX_1 = -\partial_{z_1} g(z_1, z'_1; \tau) dz_1 \tag{D10}$$

Hence, using that $N p(X_1) dX_1 \simeq e^{-z_1} dz_1$ one obtains

$$\begin{aligned} N p(X_1, x' < X'_1) dX_1 &= e^{z_1} + \partial_{z_1} g(z_1, z'_1; \tau) \\ &= -\partial_{z_1} \Phi(z_1, z'_1; \tau) dz_1 \end{aligned} \tag{D11}$$

Putting all together we obtain from Eq. (D6) the joint PDF of the maximum at two times

$$\begin{aligned} &q(X_1, X'_1) dX_1 dX'_1 \\ &\simeq (-\partial_{z_1}\partial_{z'_1} \Phi(z_1, z'_1; \tau) + \partial_{z_1} \Phi(z_1, z'_1, ; \tau) \partial_{z'_1} \Phi(z_1, z'_1; \tau)) \\ &\quad \times e^{-\Phi(z_1, z'_1; \tau)} dz_1 dz'_1 \end{aligned} \tag{D12}$$

$$= \partial_{z_1}\partial_{z'_1} e^{-\Phi(z_1, z'_1; \tau)} dz_1 dz'_1 \tag{D13}$$

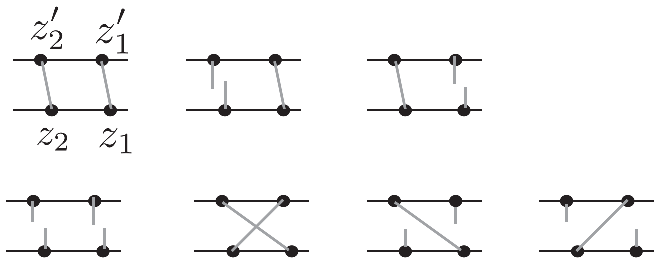

FIG. 6. Interpretation of the different terms in Eq. (D18) in the order in which they appear, left to right and top to bottom. Only the maximum and second maximum are shown at each time. In the second diagram the particle which realizes the maximum at t also realizes the maximum at t', but the particle which realizes the second maximum at t is neither maximum nor second maximum at t', and so on.

in agreement with the result in the text for the joint CDF. This is an equivalent derivation to the one given in Appendix in the case of $n = 2$.

We can now turn to the joint PDF $q(X_1, X_2, X_1', X_2')$ of (i) the maximum (variable X_1) and second maximum (X_2) at time t (ii) the maximum (variable X_1') and second maximum (X_2') at time t'. There are several possible cases, depending on which particle realizes the maximum or second maximum at each of the two times. We can indicate them schematically as

$$(X_1, X_1', X_2, X_2') = (i, i, j, j), \quad j \neq i \tag{D14}$$

$$(X_1, X_1', X_2, X_2') = (i, i, j, j'), \quad j \neq i, j \neq j', j' \neq i \tag{D15}$$

$$(X_1, X_1', X_2, X_2') = (i, i', j, j), \quad i \neq j, i' \neq i, j \neq i' \tag{D16}$$

$$(X_1, X_1', X_2, X_2') = (i, i', j, j'), \quad i \neq i', j \neq j', i \neq j, i' \neq j' \tag{D17}$$

Note that in the last case $i = j'$ is possible and so is $i' = j$ and so is both at the same time. So in total there are seven terms which read (the terms are illustrated in Fig. 6)

$$\begin{aligned}
q(X_1, X_2, X_1', X_2') = &\sum_{i\neq j} p_i(X_1, X_1')p_j(X_2, X_2') \prod_{r\neq i,j} P_r(x < X_2, x' < X_2') \\
&+ \sum_{i\neq j, i\neq j', j\neq j'} p_i(X_1, X_1')p_j(X_2, x' < X_2')p_{j'}(x < X_2, X_2') \prod_{r\neq i,j,j'} P_r(x < X_2, x' < X_2') \\
&+ \sum_{i\neq i', i\neq j, i'\neq j} p_i(X_1, x' < X_2')p_{i'}(x < X_2, X_1')p_j(X_2, X_2') \prod_{r\neq i,i',j} P_r(x < X_2, x' < X_2') \\
&+ \sum_{i,j,i',j' \text{all distinct}} p_i(X_1, x' < X_2')p_{i'}(x < X_2, X_1')p_j(X_2, x' < X_2')p_{j'}(x < X_2, X_2') \prod_{r\neq i,i',j,j'} P_r(x < X_2, x' < X_2') \\
&+ \sum_{i\neq j} p_i(X_1, X_2')p_j(X_2, X_1') \prod_{r\neq i,j} P_r(x < X_2, x' < X_2') \\
&+ \sum_{i\neq i', i\neq j, i'\neq j} p_i(X_1, X_2')p_{i'}(x < X_2, X_1')p_{j'}(X_2, x' < X_2') \prod_{r\neq i,j,j'} P_r(x < X_2, x' < X_2') \\
&+ \sum_{i\neq i', i\neq j, i'\neq j} p_i(X_1, x' < X_2')p_{i'}(X_2, X_1')p_{j'}(x < X_2, X_2') \prod_{r\neq i,j,j'} P_r(x < X_2, x' < X_2')
\end{aligned} \tag{D18}$$

We can now estimate each term in the large N limit (for identical particles) introducing the variables z_1, z_2, z_1', z_2', and using the same rules for the asymptotics of the various PDF, CDF and mixed PDF/CDF as explained above. One obtains, for each term of Eq. (D18) in the same order (we abusively denote by the same letter q the two joint PDF)

$$q(X_1, X_2, X_1', X_2')dX_1 dX_2 dX_1' dX_2' \simeq q(z_1, z_2, z_1', z_2')dz_1 dz_2 dz_1' dz_2' \tag{D19}$$

$$\begin{aligned}
q(z_1, z_2, z_1', z_2') = &(\partial_{z_1}\partial_{z_1'}\Phi(z_1, z_1')\partial_{z_2}\partial_{z_2'}\Phi(z_2, z_2') - \partial_{z_1}\partial_{z_1'}\Phi(z_1, z_1')\partial_{z_2}\Phi(z_2, z_2')\partial_{z_2'}\Phi(z_2, z_2') \\
&- \partial_{z_1}\Phi(z_1, z_2')\partial_{z_1'}\Phi(z_2, z_1')\partial_{z_2}\partial_{z_2'}\Phi(z_2, z_2') + \partial_{z_1}\Phi(z_1, z_2')\partial_{z_1'}\Phi(z_2, z_1')\partial_{z_2}\Phi(z_2, z_2')\partial_{z_2'}\Phi(z_2, z_2') \\
&+ \partial_{z_1}\partial_{z_2'}\Phi(z_1, z_2')\partial_{z_2}\partial_{z_1'}\Phi(z_2, z_1') - \partial_{z_1}\partial_{z_2'}\Phi(z_1, z_2')\partial_{z_1'}\Phi(z_2, z_1')\partial_{z_2}\Phi(z_2, z_2') \\
&- \partial_{z_1}\Phi(z_1, z_2')\partial_{z_2}\partial_{z_1'}\Phi(z_2, z_1')\partial_{z_2'}\Phi(z_2, z_2'))e^{\Phi(z_2, z_2')}
\end{aligned} \tag{D20}$$

where for clarity we have made the time argument implicit, i.e., $\Phi(z, z') \equiv \Phi(z, z', \tau)$. Recall that each term gives the respective probabilities of how the particle realizing the maximum and second maximum change from time t to time t'. For instance, the first term corresponds to events where both the rightmost particle and the second rightmost at t have remained so at t', and so on.

It turns out that the rather bulky expression (D19) and (D20) is a total derivative, i.e., one can check that, for $z_1 > z_2$, $z_1' > z_2'$,

$$q(z_1, z_2, z_1', z_2') = \partial_{z_1}\partial_{z_1'}\partial_{z_2}\partial_{z_2'}(-\Phi(z_1, z_1') + \Phi(z_1, z_2')\Phi(z_2, z_1'))e^{\Phi(z_2,z_2')} \tag{D21}$$

which is the result (107) displayed in the text, where it is also derived by a different method using counting statistics.

APPENDIX E: MULTITIME, MULTISPACE COUNTING STATISTICS

One can generalize the arguments in Sec. VI B. Let us illustrate for $n = 3$ times and an arbitrary number of points k, k' and k''. Here we denote $t' - t = t\frac{\tau}{\ln N}$ and $t'' - t' = t\frac{\tau'}{\ln N}$, with $\tau, \tau' = O(1)$. Consider the three sequences of points $\{X_j\}_{j=1,\dots,k}$, $\{X'_{j'}\}_{j'=1,\dots,k'}$, $\{X''_{j''}\}_{j''=1,\dots,k''}$, each being in decreasing order. For each time the real axis is the union of $k+1$ (and then $k'+1$ and $k''+1$) contiguous intervals, e.g., at time t these are $[X_j, X_{j-1}]$, $j = 1, k+1$ with by convention $X_0 = +\infty$ and $X_{k+1} = -\infty$, and similarly for t' and t''. Let $n_{j,j',j''}$, for $j = 1, \dots, k+1$, $j' = 1, \dots, k'+1$, $j'' = 1, \dots, k''+1$, be the numbers of particles which are in $[X_j, X_{j-1}]$ at t and in $[X'_{j'}, X'_{j'-1}]$ at t' and in $[X''_{j''}, X''_{j''-1}]$ at t''. The set of these numbers obey a multinomial distribution. In the large N limit, in the (multitime) edge regime, defined such that the corresponding variables $z_j, z'_{j'}, z''_{j''}$ are all of order $O(1)$, all of these numbers are of order $O(1)$ with the exception of $n_{k+1,k'+1,k''+1} \simeq N$. Then this reduced set of numbers are independent Poisson variables, each with mean parameter $\lambda_{j,j',j''}$. These parameters can be related to the functions defined in this paper as follows

$$\lambda_{j,j',j''} = \langle(\theta_{x>X_j} - \theta_{x>X_{j-1}})(\theta_{x'>X'_{j'}} - \theta_{x'>X'_{j'-1}}) \times (\theta_{x''>X'''_{j''}} - \theta_{x''>X'''_{j''-1}})\rangle \tag{E1}$$

$$\simeq \sum_{\ell=0,1}\sum_{\ell'=0,1}\sum_{\ell''=0,1}(-1)^{\ell+\ell'+\ell''}g_3(z_{j-\ell}, z'_{j'-\ell'}, z''_{j''-\ell''};\tau,\tau') \tag{E2}$$

where g_3 was defined in Eq. (35). We recall the convention $z_0 = z_0' = z_0'' = +\infty$ and $z_{k+1} = z'_{k'+1} = z''_{k''+1} = -\infty$. As mentioned in Sec. III, g_3 vanishes when any of the z argument is taken to $+\infty$, and reduces to $g_2 = g$ when any of the z argument is taken to $-\infty$, more precisely one has $g_3(-\infty, z', z'', \tau, \tau') = g(z', z'', \tau')$, $g_3(z, -\infty, z'', \tau, \tau') = g(z, z'', \tau + \tau')$, $g_3(z, z', -\infty, \tau, \tau') = g(z, z', \tau)$, and similarly for $g_2 = g$ which reduces to $g_1(z) = e^{-z}$. One can check that the sum of all the $\lambda_{j,j',j''}$ (over all indices, not including $(j, j', j'') = (k+1, k'+1, k''+1)$) equals $\Phi(z_k, z'_{k'}, z''_{k''};\tau,\tau')$, yielding the normalization factor $e^{-\Phi(z_k, z'_{k'}, z''_{k''};\tau,\tau')}$ for the multiple independent Poisson distribution of the $n_{j,j',j''}$.

Two times, three first maxima. Let us first return to the case of 2 times. The probability that the maximum is in $[X_1, +\infty]$ and the second maximum is in $]-\infty, X_2]$ at t, and

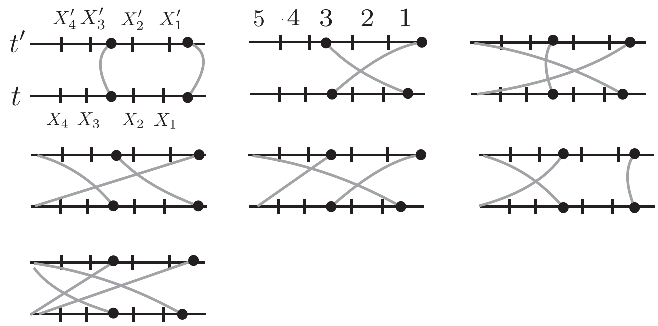

FIG. 7. Interpretation of the different terms in Eq. (E5) in the order in which they appear, left to right and top to bottom. Only the maximum and second maximum and their trajectories are shown at each time.

the maximum is in $[X_1', +\infty]$ and the second maximum is in $]-\infty, X_2']$ at t' was given in Eq. (121), and reads, translated in the present notations

$$(\lambda_{11} + \lambda_{13}\lambda_{31})e^{-\Phi(z_2, z_2';\tau)} \tag{E3}$$

which is a sum over the two permutations of 2 elements. Upon taking the derivatives $\partial_{z_1}\partial_{z_2}\partial_{z_1'}\partial_{z_2'}$ yields the two-time joint PDF of the maximum and second maximum.

This can be generalized. For instance, consider $n = 2$ and $k = k' = 4$ and the following joint probability

$$\begin{aligned}\mathcal{P} = \text{Prob}(&X^{(1)}(t) > X_1, X^{(2)}(t) \\ &\in [X_3, X_2], X^{(3)}(t) < X_4, X^{(1)}(t') > X_1', X^{(2)}(t') \\ &\in [X_3', X_2'], X^{(3)}(t') < X_4')\end{aligned} \tag{E4}$$

From $\mathcal{P}$ one can obtain by differentiation the two-time joint PDF of the maximum, second maximum and third maximum. The various cases are shown in Fig. 7 and one obtains

$$\mathcal{P} \simeq (\lambda_{11}\lambda_{33} + \lambda_{13}\lambda_{31} + \lambda_{15}\lambda_{51}\lambda_{33} + \lambda_{13}\lambda_{35}\lambda_{51} + \lambda_{15}\lambda_{53}\lambda_{31} + \lambda_{11}\lambda_{35}\lambda_{53} + \lambda_{15}\lambda_{51}\lambda_{35}\lambda_{53})e^{-\Phi(z_3, z_3',\tau)} \tag{E5}$$

which, apart from the last term, is a sum over the six permutations of three elements. Note that the intervals $[X_2, X_1]$ and $[X_4, X_3]$ remain empty. The two-time three-order statistics PDF is obtained as (in the z variables)

$$q(z_1, z_2, z_3, z_1', z_2', z_3') = \partial_{z_1}\partial_{z_2}\partial_{z_3}\partial_{z_1'}\partial_{z_2'}\partial_{z_3'}\mathcal{P} \tag{E6}$$

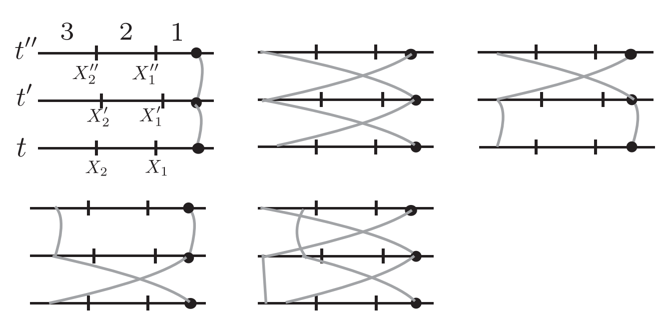

FIG. 8. Interpretation of the different terms in Eq. (E7) in the order in which they appear, left to right and top to bottom. Only the maximum and its trajectory is shown at each time.

Three times, two first maxima. One can also obtain the three-time joint PDF of the maximum and second maximum. From Fig. 8 we see that, for $z_1 > z_2, z_1' > z_2', z_1'' > z_2''$,

$$q(z_1, z_2, z_1', z_2', z_1'', z_2'') = \partial_{z_1}\partial_{z_2}\partial_{z_1'}\partial_{z_2'}\partial_{z_1''}\partial_{z_2''}((\lambda_{111} + \lambda_{131}\lambda_{313} + \lambda_{113}\lambda_{331} + \lambda_{133}\lambda_{311} + \lambda_{133}\lambda_{331}\lambda_{313})e^{-\Phi(z_2,z_2',z_2'';\tau,\tau')}) \tag{E7}$$

which has $2!^2$ terms (two successive permutations of 2 elements). To be specific we give

$$\lambda_{111} = g_3(z_1, z_1', z_1''; \tau, \tau') \tag{E8}$$

$$\lambda_{113} = g(z_1, z_1', \tau) - g_3(z_1, z_1', z_2''; \tau, \tau') \tag{E9}$$

$$\lambda_{331} = e^{-z_1''} - g_2(z_2, z_1''; \tau + \tau') - g(z_2', z_1'', \tau') + g_3(z_2, z_2', z_1''; \tau, \tau') \tag{E10}$$

$$\lambda_{133} = e^{-z_1} - g(z_1, z_2'; \tau) - g(z_1, z_2''; \tau + \tau') + g_3(z_1, z_2', z_2''; \tau, \tau') \tag{E11}$$

$$\lambda_{311} = g(z_1', z_1'', \tau') - g_3(z_2, z_1', z_1''; \tau, \tau') \tag{E12}$$

$$\lambda_{131} = g(z_1, z_1'', \tau + \tau') - g_3(z_1, z_2', z_1''; \tau, \tau') \tag{E13}$$

$$\lambda_{313} = e^{-z_1'} - g(z_2, z_1'; \tau) - g(z_1', z_2''; \tau') + g_3(z_2, z_1', z_2''; \tau, \tau') \tag{E14}$$

APPENDIX F: CONTINUUM TIME OBSERVABLES: CALCULATIONS

1. First passage time

Let us recall the following. Let $B(t)$ the standard Brownian motion (i.e., with $B(0) = 0$) and $W(t) = \sqrt{D}B(t) + \mu$ a Brownian with drift μ and diffusion coefficient D. Let us denote $T_z^{\mu,D}$ the first passage time of $W(t)$ at level z. For $z > 0$ the PDF of $T = T_z^{\mu,D}$ is

$$p_z(T) = \frac{z}{\sqrt{2\pi D}T^{3/2}} e^{-\frac{(z-\mu T)^2}{2DT}} + \delta_{+\infty}(T)(1 - e^{2\mu z/D})\theta(-\mu) \tag{F1}$$

One has upon integration

$$\text{Prob}\big(T_z^{\mu,D} > t\big) = \frac{1}{2}\left(\text{erfc}\left(\frac{\mu t - z}{\sqrt{2Dt}}\right) - e^{2\mu z/D}\text{erfc}\left(\frac{\mu t + z}{\sqrt{2Dt}}\right)\right) \tag{F2}$$

For $\mu = 0$ it simplifies into

$$\text{Prob}\big(T_z^{0,D} > t\big) = \text{erf}\left(\frac{z}{\sqrt{2Dt}}\right) \tag{F3}$$

Eq. (F2) is also the probability that a Brownian survives up to time t in presence of an absorbing wall at $x = z$ and can thus also be obtained from the image method. Indeed, one has, denoting x the position of the Brownian at time t,

$$\text{Prob}\big(T_z^{\mu,D} > t\big) = \int_{-\infty}^{z} dx \left(\frac{e^{-(x-\mu t)^2/(2Dt)}}{\sqrt{2\pi Dt}} - e^{2\mu z/D}\frac{e^{-(x-2z-\mu t)^2/(2Dt)}}{\sqrt{2\pi Dt}}\right) \tag{F4}$$

which upon integration recovers (F2).

2. Probability that the maximum remains below a straight line

Let us consider the following observable for the maximum process $X(t) = X^{(1)}(t)$ of N standard Brownian motions started at the origin

$$\text{Prob}(X(t) < X_1 + v(t - t_1),\ \forall t \in [t_1, t_2]) = \text{Prob}(x(t) < X_1 + v(t - t_1),\ \forall t \in [t_1, t_2])^N \tag{F5}$$

Now one has

$$\text{Prob}(x(t) < X_1 + v(t - t_1),\ \forall t \in [t_1, t_2]) = \int_{-\infty}^{X_1} dx_1 \frac{e^{-x_1^2/(2t_1)}}{\sqrt{2\pi t_1}} \text{Prob}\big(T_{X_1 - x_1}^{-v} > t_2 - t_1\big) \tag{F6}$$

where here we must set $D = 1$. Indeed, asking that a standard Brownian motion starting at x_1 at time t_1, remains below $X_1 + v(t - t_1)$ until t_2 is equivalent to asking that a standard drifted Brownian of drift $-v$ remains below X_1 until t_2, which is also equivalent to asking that the first passage time at level X_1 of a drifted Brownian of drift $-v$ started at x_1 at time t_1 is larger than $t_2 - t_1$. Using Eq. (F2) one obtains

$$\mathcal{P} := \text{Prob}(x(t) < X_1 + v(t - t_1),\ \forall t \in [t_1, t_2]) = \int_{-\infty}^{X_1} dx_1 \frac{e^{-x_1^2/(2t_1)}}{\sqrt{2\pi t_1}} \frac{1}{2}\left(\text{erfc}\left(\frac{-v(t_2 - t_1) - (X_1 - x_1)}{\sqrt{2(t_2 - t_1)}}\right) - e^{-2(X_1 - x_1)v}\text{erfc}\left(\frac{-v(t_2 - t_1) + (X_1 - x_1)}{\sqrt{2(t_2 - t_1)}}\right)\right). \tag{F7}$$

We will now scale X near the edge, and scale the time window as well, i.e., choose

$$X_1 = \sqrt{2t_1 \ln N}\left(1 + \frac{z_1 + c_N}{2\ln N}\right), \quad \frac{t_2 - t_1}{t_1} = \frac{\tau_{2,1}}{\ln N}, \tag{F8}$$

and we will need to scale the slope v as

$$v = w\sqrt{\frac{\ln N}{2t_1}}, \quad w = O(1). \tag{F9}$$

In that region $\mathcal{P}$ is close to unity, with $1 - \mathcal{P} = O(1/N)$ and

$$\text{Prob}(X(t) < X_1 + v(t - t_1),\ \forall t \in [t_1, t_2]) = \mathcal{P}^N \simeq e^{-N(1-\mathcal{P})}. \tag{F10}$$

Using that the edge coordinates satisfy

$$N p_{t_1}(x_1) dx_1 = e^{-y_1} dy_1, \quad X_1 - x_1 = \frac{\sqrt{2t_1}}{2\sqrt{\ln N}}(z_1 - y_1). \tag{F11}$$

We can write

$$N(1-\mathcal{P}) = N\int_{-\infty}^{+\infty} dx_1 \frac{e^{-x_1^2/(2t_1)}}{\sqrt{2\pi t_1}} \tag{F12}$$

$$\times\left(1 - \theta(X_1 - x_1)\frac{1}{2}\left(\text{erfc}\left(\frac{-v(t_2-t_1) - (X_1 - x_1)}{\sqrt{2(t_2-t_1)}}\right) - e^{-2(X_1-x_1)v}\text{erfc}\left(\frac{-v(t_2-t_1) + (X_1-x_1)}{\sqrt{2(t_2-t_1)}}\right)\right)\right)$$

$$\simeq \int dy_1 e^{-y_1}(1 - \theta(z_1 - y_1)\frac{1}{2}\left(\text{erfc}\left(\frac{-w\tau_{2,1} - (z_1 - y_1)}{2\sqrt{\tau_{2,1}}}\right) - e^{-(z_1-y_1)w}\text{erfc}\left(\frac{-w\tau_{2,1} + (z_1-y_1)}{2\sqrt{\tau_{2,1}}}\right)\right)\right)$$

$$= \int dy e^{-y_1}\big(1 - \theta(z_1 - y_1)\text{Prob}(\mathsf{T}^{-w}_{z_1 - y_1} > \tau_{2,1})\big) = e^{-z_1}\int dy e^{y}\big(1 - \theta(y)\text{Prob}\big(\mathsf{T}^{-w}_{y} > \tau_{2,1}\big)\big), \tag{F13}$$

where in the last line we have set $y_1 = z_1 - y$ and where $\mathsf{T}^{-w}_z = T^{-w,D=2}_z$ is the first passage time for a Brownian with drift $-w$ and diffusion coefficient $D = 2$, with

$$\text{Prob}\big(\mathsf{T}^{-w}_y > \tau\big) = \frac{1}{2}\left(\text{erfc}\left(\frac{-w\tau - y}{2\sqrt{\tau}}\right) - e^{-yw}\text{erfc}\left(\frac{-w\tau + y}{2\sqrt{\tau}}\right)\right)\Bigg). \tag{F14}$$

In summary, we find that

$$\text{Prob}(X(t) < X_1 + v(t - t_1),\ \forall t \in [t_1, t_2]) \simeq e^{-e^{-z_1}\Psi_w(\tau_{2,1})}, \tag{F15}$$

where

$$\Psi_w(\tau) = \int dy e^{y}\big(1 - \theta(y)\text{Prob}\big(\mathsf{T}^{-w}_y > \tau\big)\big) = 1 + \int_{y>0} dy\, e^{y}\, \text{Prob}\big(\mathsf{T}^{-w}_y < \tau\big). \tag{F16}$$

An explicit calculation and one finds, for $w \neq 1$

$$\Psi_w(\tau) = \frac{e^{-\tau w}\left(w e^{\tau w}\left(\text{erf}\left(\frac{\sqrt{\tau}w}{2}\right) + 1\right) + e^{\tau}(w-2)\text{erfc}\left(\frac{1}{2}\sqrt{\tau}(w-2)\right)\right)}{2(w-1)}, \tag{F17}$$

with $\Psi_w(0) = 1$. For $w = 1$ it gives Eq. (158) in the text, and for $w = 0$ one has

$$\Psi_0(\tau) = e^{\tau}(\text{erf}(\sqrt{\tau}) + 1) \simeq_{\tau\to+\infty} 2e^{\tau} + O\left(\frac{1}{\tau^{1/2}}\right). \tag{F18}$$

For general w, $w \neq 0, 1$, the large τ asymptotics is

$$\Psi_w(\tau) = \frac{w}{w-1}\theta(w) + \frac{2-w}{1-w}e^{(1-w)\tau}\theta(w < 2) + O(e^{-w^2\tau/4}\tau^{-3/2}). \tag{F19}$$

Hence, it saturates to a constant for $w > 1$, while it diverges exponentially for $w < 1$.

Now under the rescaling described here and the definition of $z(\tau)$ in the text one finds that

$$\text{Prob}(X(t) < X_1 + v(t - t_1), \forall t \in [t_1, t_2]) \simeq \text{Prob}(z(\tau) + \tau < z_1 + w\tau), \tag{F20}$$

which shows the result (157) conjectured in the text.

3. Running maximum and arrival time of the first particle: One time

Here we derive the one-time distributions by a more detailed calculation. For the standard Brownian motion, we see from Eq. (F3) that Eq. (159) becomes

$$\mathrm{Prob}(r(t_1) < R_1) = \mathrm{Prob}\big(T_{R_1} > t_1\big) = \mathcal{P}_1 = \mathrm{erf}\left(\frac{R_1}{\sqrt{2t_1}}\right). \tag{F21}$$

This can be either interpreted as the CDF for the running maximum at fixed t_1 or for the "CDF" of the first passage time at fixed R_1. Similarly, for N Brownian one has

$$\mathrm{Prob}(R(t_1) < R_1) = \mathrm{Prob}\big(T_{R_1}^{\min} > t_1\big), \tag{F22}$$

which can be interpreted either as the CDF of $R(t_1) = \max_i r_i(t_1)$ at fixed t_1 or the "CDF" of the arrival time of the first particle, $T_{R_1}^{\min} = \min_i T_{R_1}^{(i)}$ at fixed R_1. At large N (F22) is estimated as

$$\simeq e^{-N(1-\mathcal{P}_1)} = e^{-N\left(1-\mathrm{erf}\left(\frac{R_1}{\sqrt{2t_1}}\right)\right)} \simeq e^{-N\frac{\sqrt{2t_1}}{\sqrt{\pi}R_1}e^{-\frac{R_1^2}{2t_1}}} = e^{-e^{-z}}, \tag{F23}$$

where

$$\frac{R_1^2}{2t_1} = \ln N + z - \ln\frac{\sqrt{\pi}R_1}{\sqrt{2t_1}} \simeq \ln N + z + c_N', \quad c_N' = -\frac{1}{2}\ln(\pi\ln N) = c_N + \ln 2. \tag{F24}$$

If we are interested in the running maximum at fixed t_1, then we obtain from this estimate

$$R(t_1) = R_1 = \sqrt{2t_1\ln N}\left(1 + \frac{z_1 + c_N}{2\ln N})\right), \quad z_1 = z + \ln 2, \tag{F25}$$

in agreement with the text, where z is Gumbel distributed [by definition from Eq. (F23)], and the shift of $\ln 2$ agrees with the one obtained in Eq. (166). However, if one is interested in the arrival time of the first particle, then one obtains from the same estimate

$$T_{R_1}^{\min} = t_1 = \frac{R_1^2}{2\ln N}\left(1 - \frac{z + c_N'}{\ln N})\right), \tag{F26}$$

where z is Gumbel distributed. One may ask why is the arrival time of the first particle also distributed with (minus) Gumbel, since the distribution of the first passage time is very different from a Gaussian, see Eq. (F1). To see that immediately one can consider that $\min_i T_R^{(i)} = 1/(\max_i U_i)$ where $U = 1/T_R$ has a distribution with an exponential tail which clearly belongs to Gumbel class.

4. Running maximum and arrival time of the first particle: Two time

We give here more details of the calculation of the two-time joint CDF of the running maximum depicted in the text. Let us start from the exact expression for $\mathcal{P}$ in Eq. (170). In the large N limit, with the scaling (171), using similar estimates as in Appendix A, we obtain

$$\begin{aligned} N(1-\mathcal{P}) &= N\int dx_1\int dx_2\left(\frac{e^{-\frac{x_1^2}{2t_1}}}{\sqrt{2\pi t_1}}\frac{e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}}(1-\theta(R_1-x_1)\theta(R_2-x_2)(1-e^{-\frac{2R_1(R_1-x_1)}{t_1}})(1-e^{-\frac{2(R_2-x_1)(R_2-x_2)}{t_2-t_1}})\right)\\ &\simeq \int dy_1\int dy_2\frac{e^{-\frac{(y_2-y_1+\tau)^2}{4\tau}}}{\sqrt{4\pi\tau}}(e^{-y_1} - \theta(z_1-y_1)\theta(z_2-y_2)(e^{-y_1}-e^{-(2z_1-y_1)})(1-e^{-\frac{(z_2+\tau-y_1)(z_2-y_2)}{\tau}}), \end{aligned} \tag{F27}$$

where we have changed variables denoting $x_i = \sqrt{2t_i\ln N}(1+\frac{y_i+c_N}{2\ln N})$. As for the functions Φ we can give some interpretation to this formula in terms of the Brownian motion with unit negative drift and diffusion coefficient $D = 2$, with however an important difference. The factor $(e^{-y_1} - e^{-(2z_1-y_1)})\theta(z_1-y_1)$ is the stationary measure in the presence of an absorbing wall at z_1. The other factor can be written as

$$\theta(z_2-y_2)\left(\frac{e^{-\frac{(y_2-y_1+\tau)^2}{4\tau}}}{\sqrt{4\pi\tau}} - \frac{e^{-\frac{(y_2-(2(z_2+\tau)-y_1)+\tau)^2}{4\tau}}}{\sqrt{4\pi\tau}}\right), \tag{F28}$$

which vanishes at $y_2 = z_2$ but is not exactly the propagator in presence of a fixed absorbing wall, since the wall is effectively moving. Performing the change of variable $y_1 \to z_1 - y_1$ and $y_2 \to z_2 - y_2$ in the last line of Eq. (F27) one obtains

$$\mathrm{Prob}(R(t_1) < R_1, R(t_2) < R_2) \simeq e^{-N(1-\mathcal{P})} \simeq e^{-e^{-z_1}\gamma_\tau(z_{21})}, \tag{F29}$$

with the function $\gamma_\tau(z)$ displayed in Eq. (173) in the text. To obtain the explicit expression (174) one splits the integral into $y_1 > 0$ and $y_1 < 0$ and use that $\int_{y_1<0} \int dy_2 e^{y_1} G(z - y_{2,1}, \tau) = 1$. The part $y_1 > 0, y_2 < 0$ is found to equal $\phi_\tau(z) - 1$. The remaining part $y_1 > 0, y_2 > 0$, which is positive, can be integrated explicitly.

5. Multitime CDF for the running maximum

The two-time calculation of the previous section can easily be extended to any number of times n. Here we just give the result. One finds, under the scaling (171) and $t_j - t_{j-1} = t_1 \tau_{j,j-1}/\ln N$,

$$\mathrm{Prob}(R(t_1) < R_1, \dots, R(t_n) < R_n) = \mathrm{Prob}\big(T^{\min}_{R_1} > t_1, \dots, T^{\min}_{R_n} > t_n\big) \simeq e^{-\Gamma(z_1,\dots,z_n;\tau_{2,1},\dots,\tau_{n,n-1})}, \tag{F30}$$

where

$$\begin{aligned}\Gamma(z_1, \dots, z_n; \tau_{2,1}, \dots, \tau_{n,n-1}) = & \int dy_1 \int dy_2 \dots dy_n\, e^{-y_1} G(y_{2,1}, \tau_{2,1}) \dots G(y_{n,n-1}, \tau_{n,n-1}) \\ & \times \left(1 - \prod_{i=1}^{n} \theta(z_i - y_i)(1 - e^{-2(z_1-y_1)})(1 - e^{-\frac{(z_2+\tau-y_1)(z_2-y_2)}{\tau}}) \dots (1 - e^{-\frac{(z_n+\tau_{n,n-1}-y_n)(z_n-y_n)}{\tau_{n,n-1}}})\right).\end{aligned} \tag{F31}$$

Similarly, the result (F30) and (F31) can be read as a result for the multitime joint "CDF" of the arrival times $T^{\min}_{R_i}$ of the first particle at R_i, which allows to obtain the joint distribution of z_1, and of the scaled delay times $\tau_{2,1}, \dots, \tau_{n,n-1}$, defined as in Eq. (202) in the text.

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