Undular bore theory for the modified Korteweg–de Vries–Burgers equation

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We consider nonlinear wave structures described by the modified Korteweg–de Vries equation, taking into account a small Burgers viscosity for the case of steplike initial conditions. The Whitham modulation equations are derived, which include the small viscosity as a perturbation. It is shown that for a long enough time of evolution, this small perturbation leads to the stabilization of cnoidal bores, and their main characteristics are obtained. The applicability conditions of this approach are discussed. Analytical theory is compared with numerical solutions and good agreement is found.

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I. INTRODUCTION

The modified Korteweg-de Vries (mKdV) equation

$$u_t - 6\alpha u^2 u_x + u_{xxx} = 0 \tag{1}$$

appeared first in the study of the famous KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

related to Eq. (1) by the Miura transformation [1]. The existence of such a transformation allowed the pioneers of the inverse scattering transform method to discover this method [2–4] for the KdV equation, and it was extended later to many other equations, including the mKdV equation [5,6] (see also, e.g., the books in [7–9] and references therein). The mKdV equation is almost as widely used in physical applications as the KdV equation. Actually, the Gardner equation

$$u_t + 6\beta uu_x - 6\alpha u^2 u_x + u_{xxx} = 0, \tag{3}$$

combining the nonlinear terms of the KdV and mKdV equations, can be transformed into Eq. (1) by a simple change of variables. In addition, in physical applications, it often happens that the coefficient β is very small and can be neglected, so Eq. (3) reduces directly to the equation. The Gardner equation and its simplified mKdV version find applications to the theory of nonlinear waves in stratified fluids, for example, for the description of large-amplitude internal waves [10–12].

One of the most important and universal phenomena in nonlinear physics is the formation and evolution of dispersive shock waves (see, e.g., review articles in [13,14] and references therein). They are called undular bores in water wave physics and they were observed in both surface and internal waves. Their theory was initially developed by Gurevich and Pitaevskii [15], who represented such structures as modulated nonlinear periodic waves governed by the Whitham modulation equations [16,17]. They gave two typical examples of solutions that describe dispersive shock waves: the evolution of an initial discontinuity and the formation of a shock after generic wave breaking for the KdV equation case. The Whitham modulation equations for the mKdV case were derived in Ref. [18], but their application to the theory of dispersive shock waves turned out to be quite a difficult task even in the case of an initial discontinuity problem. The reason for this difficulty is that the mKdV equation is not genuinely nonlinear. (This notion was introduced by Lax in Ref. [19] for hyperbolic systems of first-order partial differential equations and it plays an important role in the classification of wave structures evolving from initial discontinuities in dispersive nonlinear systems; see, e.g., Ref. [20].) This means that in the dispersionless approximation, the nonlinear velocity $6\alpha u^2$ has an extremal (minimal for $\alpha > 0$) value at $u = 0$, whereas in the case of the genuinely nonlinear KdV equation, the nonlinear velocity $6u$ is everywhere a monotonic function of the wave amplitude u. As a result, in the KdV case an initial discontinuity can only evolve into two different structures (rarefaction waves or cnoidal undular bores), whereas in the mKdV case an initial discontinuity evolves into eight different wave structures depending on the parameters of the initial jump of u. Some particular results in this direction were obtained in Ref. [21] and the full solution was given in Ref. [22] in the context of the Gardner equation (3).

In Gurevich-Pitaevskii theory, dispersive shock waves are wave structures that expand with time, so in initial discontinuity-type problems, the change of modulation parameters per unit length decreases with time and can become, at large enough time, smaller than some other physical parameters that were neglected in the derivation of Eq. (1) or (2). For such large values of time, the neglected effects must be taken into account in the modulation theory. For example, small dissipation stops the infinite expansion of undular bores and their length is stabilized at some value inversely proportional to the viscosity coefficient in accordance with the early ideas of Refs. [23,24] about the structure of undular bores in water-wave physics and plasma. The corresponding modified Whitham equations for the KdV theory with weak Burgers dissipation were derived in Refs. [25,26] and were applied in these papers to the description of stationary dispersive shocks whose characteristic length is defined by the small viscosity coefficient γ in the KdV-Burgers equation

$$u_t + 6uu_x + u_{xxx} = \gamma u_{xx}. \quad (4)$$

The extension of this theory to the mKdV-Burgers (mKdVB) equation

$$u_t - 6\alpha u^2 u_x + u_{xxx} = \gamma u_{xx} \quad (5)$$

was discussed qualitatively in Ref. [27]; however, the modified Whitham equations were not obtained for this case and the quantitative theory was not developed. The main aim of this paper is to derive the Whitham modulation equations for the mKdVB case (5) and apply them to the theory of undular bores. To this end, we will use the direct Whitham method [16,18] developed further for the perturbed KdV equation in Ref. [28]. Its advantage is that it does not require the development of quite involved methods of the inverse scattering transform (see Ref. [29]). We obtain analytical formulas for the main characteristics of shock waves and confirm them by numerical solutions of Eq. (5).

We confine ourselves to the case of the defocusing mKdV equation with $\alpha > 0$, when dispersive shock wave are modulationally stable. The focusing mKdV equation with $\alpha < 0$ has modulationally unstable periodic solutions (see, e.g., Ref. [30] and references therein) and their slow modulations are also described by the Whitham modulation equations, but this theory is beyond the scope of the present paper.

II. ELEMENTARY WAVE STRUCTURES IN mKdVB EQUATION THEORY

Wave structures evolved from an initial discontinuity are typically combined from several types of elementary wave structures and we consider them briefly. For definiteness, we confine ourselves to the case of a positive coefficient $\alpha > 0$, although a similar theory can be developed for the case of negative α. Naturally, the viscosity coefficient γ is positive.

A. Rarefaction waves

First we consider situations when a wave connects two trivial solutions $u = u_-$ on the left and $u = u_+$ on the right from the initial discontinuity and assume that during the evolution the wave remains a smooth function of x. Then we can neglect dispersive and dissipative effects proportional to higher-order derivatives of x and describe such a wave in the simplest approximation taking into account only nonlinear effects proportional to the first-order space derivative

$$u_t - 6\alpha u^2 u_x = 0. \quad (6)$$

The boundary conditions suggest that there are two characteristic functions, one for the sound wave propagating along the plateau $u = u_-$, which has the characteristic $x_l = -6\alpha u_-^2 t$, and the other for the sound wave propagating along the plateau u_+, so this edge moves according to the equation $x_r = -6\alpha u_+^2 t$. Consequently, the solution consists of three parts: $u = u_-$ for $x < x_l$, $u = u_+$ for $x > x_r$, and between these two regions an evident self-similar solution of Eq. (6),

$$u(x,t) = \begin{cases} u_-, & x < x_l \\ \pm\sqrt{\frac{x}{-6\alpha t}}, & x_l < x < x_r \\ u_+, & x > x_r. \end{cases} \quad (7)$$

Obviously, such a solution exists only if both boundary values $u_\pm$ are lying in either of the monotonicity intervals $0 < u_+ < u_-$ or $0 > u_+ > u_-$. In both cases, these rarefaction waves propagate to the left with the left edge speed $s_l = -6\alpha u_-^2$ smaller than the right edge speed $s_r = -6\alpha u_+^2$. If the boundary values $u_\pm$ are lying in the different monotonicity intervals, then more complicated combined structures are generated, which we study in the following sections.

B. Periodic solutions

If the boundary values $u_\pm$ do not satisfy the above condition of belonging to the same monotonicity intervals, then the wave breaks and an undular bore forms. In the Gurevich-Pitaevskii approach [15], they are represented by modulated periodic solutions of Eq. (5), so first we have to describe the nonmodulated solutions for zero dissipation.

We look for traveling-wave solutions $u = u(\xi)$, $\xi = x - Vt$, of Eq. (5) with $\gamma = 0$ and after two integrations we get

$$u_\xi^2 = \alpha u^4 + Vu^2 + 2Bu - 2A, \quad (8)$$

where A and B are constants of integration. We assume that the polynomial on the right-hand side has four real roots ν_i, $i = 1, 2, 3, 4$, which are ordered according to inequalities $\nu_1 \leqslant \nu_2 \leqslant \nu_3 \leqslant \nu_4$, so Eq. (8) can be rewritten in the form

$$u_\xi^2 = \alpha(u-\nu_1)(u-\nu_2)(u-\nu_3)(u-\nu_4). \quad (9)$$

The constants in these two equations are related by the expressions

$$\begin{aligned} V &= \alpha(\nu_1\nu_2 + \nu_1\nu_3 + \nu_1\nu_4 + \nu_2\nu_3 + \nu_2\nu_4 + \nu_3\nu_4), \\ B &= -\frac{\alpha}{2}(\nu_1\nu_2\nu_3 + \nu_1\nu_2\nu_4 + \nu_1\nu_3\nu_4 + \nu_2\nu_3\nu_4), \\ A &= -\frac{\alpha}{2}\nu_1\nu_2\nu_3\nu_4 \end{aligned} \quad (10)$$

and the roots ν_i are not independent of each other but connected by the formula

$$\nu_1 + \nu_2 + \nu_3 + \nu_4 = 0. \quad (11)$$

Periodic real solutions can only exist when u oscillates between two consecutive roots where the potential curve is positive, that is, $\nu_2 \leqslant u \leqslant \nu_3$, as shown in Fig. 1. Integration of Eq. (9) with the initial condition $u = \nu_3$ at $\xi = \xi_0$ gives

$$\xi - \xi_0 = \int_u^{\nu_3} \frac{du}{\sqrt{\alpha(u-\nu_1)(u-\nu_2)(u-\nu_3)(u-\nu_4)}} \quad (12)$$

and standard calculation yields the expression

$$u = \frac{\nu_3(\nu_4-\nu_2) - \nu_4(\nu_3-\nu_2)\mathrm{sn}^2(\theta;m)}{(\nu_4-\nu_2) - (\nu_3-\nu_2)\mathrm{sn}^2(\theta;m)}, \quad (13)$$

where $\mathrm{sn}(\theta, m)$ is the Jacobi elliptic sinus function, with

$$\theta = \tfrac{1}{2}\sqrt{\alpha(\nu_3-\nu_1)(\nu_4-\nu_2)}\,\xi \quad (14)$$

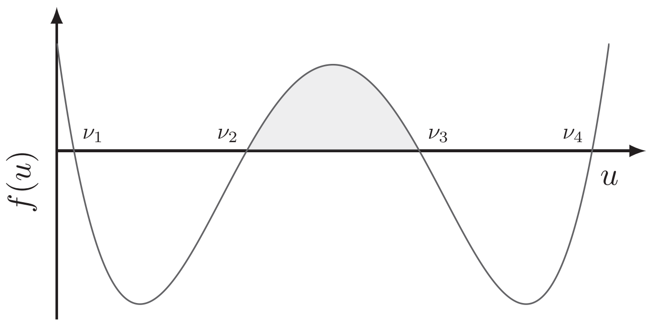

FIG. 1. Periodic solutions correspond to oscillations in the interval $\nu_2 \leqslant u \leqslant \nu_3$ where $f(u) \geqslant 0$.

and

$$m = \frac{(\nu_4 - \nu_1)(\nu_3 - \nu_2)}{(\nu_4 - \nu_2)(\nu_3 - \nu_1)}. \tag{15}$$

The expression (14) allows us to define the wave number and the frequency of the periodic wave in terms of parameters ν_i,

$$k = \sqrt{\alpha(\nu_3 - \nu_1)(\nu_4 - \nu_2)}, \quad \omega = kV, \tag{16}$$

where V is given by Eq. (10).

The cnoidal wave solution (13) reduces to important particular solutions in special limits. When $\nu_1 \to \nu_2$, and thus $m \to 1$ and $\mathrm{sn}(\theta; m) \to \tanh\theta$, we arrive at the bright soliton

$$u(\xi) = \nu_1 + \frac{\nu_3 - \nu_1}{\cosh^2\theta - \frac{\nu_3 - \nu_1}{\nu_4 - \nu_1}\sinh^2\theta}, \tag{17}$$

propagating along a constant background $u = \nu_1$. When $\nu_3 \to \nu_4$, we obtain the dark-soliton solution

$$u(\xi) = \nu_4 - \frac{\nu_4 - \nu_2}{\cosh^2\theta - \frac{\nu_4 - \nu_2}{\nu_4 - \nu_1}\sinh^2\theta}, \tag{18}$$

propagating along a constant background $u = \nu_4$. When $\nu_3 \to \nu_2$, we get $m \to 0$, so the elliptical sinus becomes the trigonometric one, $\mathrm{sn}(\theta; 0) = \sin\theta$, and we obtain a harmonic wave solution oscillating with very small amplitude around $u = \nu_2$,

$$u(\xi) = \nu_2 + \tfrac{1}{2}(\nu_3 - \nu_2)\cos 2\theta. \tag{19}$$

Finally, if we have simultaneously $\nu_1 \to \nu_2$ and $\nu_3 \to \nu_4$, it is convenient to change the initial condition in such a way that the integral (12) takes the form

$$\xi = \int_u^{\nu_3} \frac{du}{\sqrt{\alpha}(u - \nu_2)(u - \nu_4)} \tag{20}$$

and elementary integration yields

$$u = \frac{\nu_2 + \nu_4 \exp[\mp\sqrt{\alpha}(\nu_2 - \nu_4)(\xi - \xi_0)]}{1 + \exp[\mp\sqrt{\alpha}(\nu_2 - \nu_4)(\xi - \xi_0)]}. \tag{21}$$

It is important that due to Eq. (11) the parameters are related by the formula $\nu_2 + \nu_4 = 0$ and therefore the left and right limiting values of u have opposite signs and their absolute values are equal to each other. It is remarkable that an exact solution of this type exists for the full Eq. (5), taking into account dissipation [31]; we consider this modification of this so-called kink solution in the next section.

C. Kink

Here we find the kink solution of Eq. (5) with $\gamma \neq 0$. As usual, we look for a traveling-wave solution $u = u(\xi)$, $\xi = x - Vt$, and assume that $u \to u_-$ as $\xi \to -\infty$. Then trivial integration taking into account our boundary condition gives

$$u_{\xi\xi} = \gamma u_\xi + V(u - u_-) + 2\alpha(u^3 - u_-^3). \tag{22}$$

We also have $u \to u_+$ as $\xi \to +\infty$, as it should be for a kink solution. Then we immediately get the expression for the velocity

$$V = -2\alpha(u_-^2 + u_-u_+ + u_+^2); \tag{23}$$

substitution of this expression into Eq. (22) gives

$$u_{\xi\xi} = \gamma u_\xi + 2\alpha(u - u_-)(u - u_+)(u + u_- + u_+). \tag{24}$$

Now, following Ref. [31], we assume that this equation has an integral in the form

$$u_\xi = a(u - u_-)(u - u_+),$$

that is,

$$u_{\xi\xi} = \frac{du_\xi}{du}\frac{du}{d\xi} = a^2(2u - u_- - u_+)(u - u_-)(u - u_+).$$

Substitution of these expressions into Eq. (24) yields

$$a^2(2u - u_- - u_+) = \gamma a + 2\alpha(u + u_- + u_+).$$

A comparison of the coefficients before u gives $a^2 = \alpha$ or

$$a = \pm\sqrt{\alpha}. \tag{25}$$

Then the remaining terms give

$$u_- + u_+ = \mp\frac{\gamma}{3\sqrt{\alpha}}. \tag{26}$$

Finally, elementary integration of the equation

$$u_\xi = \pm\sqrt{\alpha}(u - u_-)(u - u_+) \tag{27}$$

yields

$$u(\xi) = \tfrac{1}{2}\{u_- + u_+ \pm (u_- - u_+)\tanh[\alpha(u_- - u_+)\xi)]\}. \tag{28}$$

As one can see, the upper sign corresponds to the growing kink with $u_+ > u_-$ and $u_+ + u_- = \gamma/3\sqrt{\alpha}$ and the lower sign corresponds to the decreasing kink with $u_+ < u_-$ and $u_+ + u_- = -\gamma/3\sqrt{\alpha}$. These are the separatrix solutions joining the stationary solutions of the second-order equation (24), so there are no other kink solutions of this equation.

III. WHITHAM MODULATION EQUATIONS FOR mKdVB THEORY

According to Whitham [16,17], the modulation theory can be based on averaging of the conservation laws for the equation under consideration over fast oscillations in the slightly modulated cnoidal wave. The perturbed theory of the Whitham modulation method for the mKdVB equation can be performed in the same way as it was done for the KdVB equation [28].

Due to condition (11), in this theory there are three independent parameters that can be chosen arbitrarily from the set ν_i, $i = 1, 2, 3, 4$. Therefore, we have to average three conservation laws. However, it is convenient to replace one of them by the universal law of conservation of the number of waves [16,17]. Indeed, a slightly modulated wave can be considered locally as a uniform one with the wave number and the frequency defined by the expressions

$$k = \theta_x, \quad \omega = -\theta_t. \tag{29}$$

Consequently, they satisfy the conservation law

$$k_t + \omega_x = 0, \tag{30}$$

where k plays the role of the density of waves and ω is their flux. They are still expressed in terms of the local values of the modulation parameters ν_i by Eqs. (16). Averaging can be performed over a wavelength due to the smallness of modulations,

$$\langle\phi\rangle = \frac{1}{L}\int_0^L \phi\, dx = \frac{1}{L}\oint \frac{\phi(x,t)}{\sqrt{f(u)}}du, \tag{31}$$

where $L = k^{-1}$ is the wavelength and $f(u) = u_x^2 = \alpha\prod(u-\nu_i)$. Thus, the averaged Eq. (30) can be written as

$$\langle k\rangle_x + \langle\omega\rangle_t = 0, \tag{32}$$

and it is easy to find two other conservation laws for the perturbed mKdV equation

$$u_t - 6\alpha u^2 u_x + u_{xxx} = R, \tag{33}$$

so in the averaged form they read

$$\begin{aligned} &\langle u\rangle_t + \langle -2\alpha u^3 + u_{xx}\rangle_x = \langle R\rangle,\\ &\langle u^2\rangle_t + \big\langle -3\alpha u^4 + 2uu_{xx} - u_x^2\big\rangle_x = 2\langle uR\rangle. \end{aligned} \tag{34}$$

In the case of Burgers friction we have $R = \gamma u_{xx}$ [see Eq. (5)], but to stress the generality of our derivation we keep it unspecified here.

Following Refs. [16–18,28], we express all averaged functions in terms of

$$\begin{aligned} \mathcal{W}(A,B,V) &= -\oint u_\xi du = -\oint \sqrt{f(u)}du\\ &= -\oint \sqrt{\alpha u^4 + Vu^2 + 2Bu - 2A}\, du \end{aligned} \tag{35}$$

so that

$$\begin{aligned} \mathcal{W}_A &= \oint \frac{du}{\sqrt{f(u)}} = \oint dx = L = k^{-1},\\ \mathcal{W}_B &= -\oint \frac{udu}{\sqrt{f(u)}},\\ \mathcal{W}_V &= -\frac{1}{2}\oint \frac{u^2du}{\sqrt{f(u)}}. \end{aligned} \tag{36}$$

Consequently, we get

$$\begin{aligned} \langle u\rangle &= k\oint \frac{udu}{\sqrt{f(u)}} = -k\mathcal{W}_B,\\ \big\langle \tfrac{1}{2}u^2\big\rangle &= \frac{k}{2}\oint \frac{u^2du}{\sqrt{f(u)}} = -k\mathcal{W}_V. \end{aligned} \tag{37}$$

In view of the relation $u_{xx} = \frac{1}{2}\frac{df}{du}$, we have $\langle u_{xx}\rangle = 0$. After simple transformations with the use of the mKdV equation, we can express all averaged quantities in terms of the above expressions and arrive at

$$\begin{aligned} (-k\mathcal{W}_B)_t + (-kV\mathcal{W}_B + B)_x &= \langle R\rangle,\\ (-k\mathcal{W}_V)_t + (-kV\mathcal{W}_V + A)_x &= \langle uR\rangle,\\ (\mathcal{W}_A)_t - V(\mathcal{W}_A)_x &= \mathcal{W}_A V_x. \end{aligned} \tag{38}$$

These equations can be rewritten in a more convenient form with the use of the differential operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x}$,

$$\begin{aligned} \frac{D\mathcal{W}_B}{Dt} &= \mathcal{W}_A\left(\frac{\partial B}{\partial x} - \langle R\rangle\right),\\ \frac{D\mathcal{W}_V}{Dt} &= \mathcal{W}_A\left(\frac{\partial A}{\partial x} - \langle uR\rangle\right),\\ \frac{D\mathcal{W}_A}{Dt} &= \mathcal{W}_A\frac{\partial V}{\partial x}. \end{aligned} \tag{39}$$

As we mentioned in the Introduction, the mKdV equation is not genuinely nonlinear. Therefore, as in the case of the Gardner equation [22], the relationship between physical parameters ν_i and the most convenient modulation parameters used in the Whitham equations is not single valued. In a similar way, the Whitham equations (39) can be transformed to the diagonal form for the Riemann invariants r_1, r_2, and r_3 which are related to ν_1, ν_2, ν_3, and ν_4 by two different sets of formulas. To obtain one such set of formulas, we choose ν_1, ν_2, and ν_3 as dependent variables, so ν_4 is given by Eq. (11) and $d\nu_4 = -(d\nu_1 + d\nu_2 + d\nu_3)$. Then differentials dV, dA, and dB of the modulation parameters used in Eqs. (39) are equal to

$$\begin{aligned} dV &= \alpha[(\nu_4-\nu_1)d\nu_1 + (\nu_4-\nu_2)d\nu_2 + (\nu_4-\nu_3)d\nu_3],\\ dB &= -\frac{\alpha}{2}[(\nu_4-\nu_1)(\nu_2+\nu_3)d\nu_1 + (\nu_4-\nu_2)(\nu_1+\nu_3)d\nu_2\\ &\quad + (\nu_4-\nu_3)(\nu_1+\nu_2)d\nu_3],\\ dA &= -\frac{\alpha}{2}[\nu_2\nu_3(\nu_4-\nu_1)d\nu_1 + \nu_1\nu_3(\nu_4-\nu_2)d\nu_2\\ &\quad + \nu_1\nu_2(\nu_4-\nu_3)d\nu_3]. \end{aligned} \tag{40}$$

Introducing the variables $w_i = \nu_4 - \nu_i$, we write Eq. (39) in the form

$$\begin{aligned} \sum_{i=1}^{3}\mathcal{W}_{A,\nu_i}\frac{D\nu_i}{Dt} &= \alpha\mathcal{W}_A(w_1\nu_{1,x} + w_2\nu_{2,x} + w_3\nu_{3,x}),\\ \sum_{i=1}^{3}\mathcal{W}_{B,\nu_i}\frac{D\nu_i}{Dt} &= -\frac{\alpha}{2}\mathcal{W}_A[w_1(\nu_2+\nu_3)\nu_{1,x} + w_2(\nu_1+\nu_3)\nu_{2,x}\\ &\quad + w_3(\nu_1+\nu_2)\nu_{3,x}] - \mathcal{W}_A\langle R\rangle,\\ \sum_{i=1}^{3}\mathcal{W}_{V,\nu_i}\frac{D\nu_i}{Dt} &= -\frac{\alpha}{2}\mathcal{W}_A(\nu_2\nu_3w_1\nu_{1,x} + \nu_1\nu_3w_2\nu_{2,x}\\ &\quad + \nu_1\nu_2w_3\nu_{3,x}) - \mathcal{W}_A\langle uR\rangle. \end{aligned} \tag{41}$$

To diagonalize the last system, we multiply the first, second, and third lines by the constant parameters p, q, and r, respectively, sum the resulting equations, and choose p, q, and r in such way that the coefficient of $\nu_{1,x}$ on the right-hand side vanishes and the coefficients of $\nu_{2,x}$ and $\nu_{3,x}$ are equal to each other. These conditions determine p, q, and r up to a numerical factor, and we take the following values:

$$\begin{aligned} p &= -(\nu_2+\nu_3)(\nu_1\nu_4+\nu_2\nu_3),\\ q &= -2(\nu_1\nu_4-\nu_2\nu_3),\\ r &= -4(\nu_2+\nu_3). \end{aligned} \tag{42}$$

After elementary transformations the resulting right-hand side of the sum takes the form

$$\mathcal{W}_A\bigg(\alpha(\nu_2-\nu_1)(\nu_3-\nu_1)(\nu_4-\nu_2)(\nu_4-\nu_3)\frac{\partial(\nu_2+\nu_3)}{\partial x} + 2(\nu_1\nu_4-\nu_2\nu_3)\langle R\rangle + 4(\nu_2+\nu_3)\langle uR\rangle\bigg). \tag{43}$$

Calculation of the coefficient before $D\nu_1/Dt$ gives

$$\begin{aligned} K_1 &= p\mathcal{W}_{A,\nu_1} + q\mathcal{W}_{B,\nu_1} + r\mathcal{W}_{V,\nu_1}\\ &= -\frac{\nu_4-\nu_1}{2}\oint\frac{(p-qu-ru^2/2)du}{\sqrt{\alpha(u-\nu_1)^3(u-\nu_2)(u-\nu_3)(u-\nu_4)^3}}\\ &= -(\nu_4-\nu_1)\oint\frac{d}{du}\sqrt{\frac{(u-\nu_2)(u-\nu_3)}{\alpha(u-\nu_1)(u-\nu_4)}} = 0. \end{aligned} \tag{44}$$

Similar calculation of the coefficient before $D\nu_2/Dt$ gives

$$\begin{aligned} K_2 &= p\mathcal{W}_{A,\nu_2} + q\mathcal{W}_{B,\nu_2} + r\mathcal{W}_{V,\nu_2}\\ &= (\nu_4-\nu_2)(\nu_4-\nu_3)I_1, \end{aligned} \tag{45}$$

where

$$I_1 = \oint\sqrt{\frac{u-\nu_1}{\alpha(u-\nu_2)(u-\nu_3)(u-\nu_4)^3}}. \tag{46}$$

As one can see, this expression is symmetrical with respect to the interchange of ν_2 and ν_3, so $K_3 = p\mathcal{W}_{A,\nu_3} + q\mathcal{W}_{B,\nu_3} + r\mathcal{W}_{V,\nu_3} = K_2$. Consequently, we obtain one of the modulation equations in the form

$$\begin{aligned} &(\nu_4-\nu_2)(\nu_4-\nu_3)I_1\bigg(\frac{\partial(\nu_2+\nu_3)}{\partial t} + V\frac{\partial(\nu_2+\nu_3)}{\partial x}\bigg)\\ &= \mathcal{W}_A\bigg(\alpha(\nu_2-\nu_1)(\nu_3-\nu_1)(\nu_4-\nu_2)(\nu_4-\nu_3)\frac{\partial(\nu_2+\nu_3)}{\partial x} + 2(\nu_1\nu_4-\nu_2\nu_3)\langle R\rangle + 4(\nu_2+\nu_3)\langle uR\rangle\bigg), \end{aligned} \tag{47}$$

and the other two equations can be obtained by cyclic permutations of ν_1, ν_2, and ν_3.

The terms that do not depend on R have a diagonal form with respect to derivatives, so the three values of any function of $\nu_1+\nu_2$, $\nu_1+\nu_3$, and $\nu_2+\nu_3$ can serve as the Riemann invariants of the resulting Whitham modulation equations. It is convenient to define them as

$$r_1 = \tfrac{1}{4}(\nu_2+\nu_3)^2,\quad r_2 = \tfrac{1}{4}(\nu_1+\nu_3)^2,\quad r_3 = \tfrac{1}{4}(\nu_1+\nu_2)^2 \tag{48}$$

and

$$\begin{aligned} &\nu_1 = \sqrt{r_1}-\sqrt{r_2}-\sqrt{r_3},\quad \nu_2 = -\sqrt{r_1}+\sqrt{r_2}-\sqrt{r_3},\\ &\nu_3 = -\sqrt{r_1}-\sqrt{r_2}+\sqrt{r_3},\quad \nu_4 = \sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}. \end{aligned} \tag{49}$$

The Riemann invariants r_i are positive and we assume that they are ordered according to the inequalities $0 < r_1 \leqslant r_2 \leqslant r_3$. Then the parameters ν_i are ordered as

$$\nu_1 \leqslant \nu_2 \leqslant \nu_3 < 0 < \nu_4. \tag{50}$$

The phase velocity V and elliptic modulus m reduce to

$$V = -2\alpha(r_1+r_2+r_3),\quad m = \frac{r_3-r_2}{r_3-r_1} \tag{51}$$

and the wavelength is given by the formula

$$L = \frac{2}{\sqrt{\alpha(r_3-r_1)}}K(m), \tag{52}$$

with $K(m)$ the complete elliptic integral of the first kind. The integral (46) can also be expressed in terms of the Riemann invariants,

$$I_1 = 2(\sqrt{r_2}-\sqrt{r_1})(\sqrt{r_3}-\sqrt{r_1})\frac{\partial L}{\partial r_1}, \tag{53}$$

and similar expressions can be obtained for its counterparts for equations derived from Eq. (47) by cyclic permutations of ν_1, ν_2, and ν_3. As a result, we arrive at the form of the Whitham equations for the perturbed mKdV theory,

$$\frac{\partial r_i}{\partial t} + v_i\frac{\partial r_i}{\partial x} = \frac{L}{\partial L/\partial r_i}\frac{\sqrt{r_1r_2r_3}\langle R\rangle - r_i\langle uR\rangle}{\prod_{j\neq i}(r_i-r_j)}, \tag{54}$$

where

$$v_i = \bigg(1-\frac{L}{\partial L/\partial r_i}\frac{\partial}{\partial_{r_i}}\bigg)V = V + \frac{2\alpha L}{\partial L/\partial r_i} \tag{55}$$

are the standard Whitham velocities for the unperturbed mKdV equation [18,32].

The definitions (48) and (49) of the Riemann invariants imply that in this case a modulated wave oscillates in the region $\nu_2 \leqslant u \leqslant \nu_3 < 0$ of its amplitude [see Eq. (50)]. To get modulation equations for bores with positive values of the amplitude, it is convenient to take ν_2, ν_3, and ν_4 as the modulation parameters, so that $\nu_1 = -(\nu_2+\nu_3+\nu_4)$, and to define the Riemann invariants by the formulas

$$r_1 = \tfrac{1}{4}(\nu_2+\nu_3)^2,\quad r_2 = \tfrac{1}{4}(\nu_2+\nu_4)^2,\quad r_3 = \tfrac{1}{4}(\nu_3+\nu_4)^2 \tag{56}$$

and

$$\begin{aligned} &\nu_1 = -\sqrt{r_1}-\sqrt{r_2}-\sqrt{r_3},\quad \nu_2 = \sqrt{r_1}+\sqrt{r_2}-\sqrt{r_3},\\ &\nu_3 = \sqrt{r_1}-\sqrt{r_2}+\sqrt{r_3},\quad \nu_4 = -\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}. \end{aligned} \tag{57}$$

For $0 < r_1 \leqslant r_2 \leqslant r_3$ the parameters ν_i are ordered according to

$$\nu_1 < 0 < \nu_2 \leqslant \nu_3 \leqslant \nu_4 \tag{58}$$

and the variable u takes positive values in the interval

$$0 < \nu_2 \leqslant u \leqslant \nu_3. \tag{59}$$

The Whitham equations (54) for this definition of the Riemann invariants remain the same. Consequently, one solution of the Whitham modulation equations describes two different modulated wave structures, which is a characteristic feature of not-genuinely nonlinear wave equations (other examples of such a behavior can be found in Refs. [22,33,34]).

IV. STATIONARY BORES IN mKdVB THEORY

As mentioned in the Introduction, after a long enough time of evolution, small dissipation stops expansion of undular bores and they acquire stationary profiles. The corresponding theory for the KdV-Burgers equation was developed in Refs. [25,26,28]. Here we obtain similar solutions for the case of the mKdVB theory, following mainly the method of Ref. [28].

A stationary bore propagates with constant velocity V without a change of the profile determined by the modulation variables $r_i = r_i(\xi)$, $\xi = x - Vt$. Such a stationary profile is supported by the difference of the values of the wave variable u at two infinities,

$$u(x,0) \to \begin{cases} u_- & \text{as } x \to -\infty \\ u_+ & \text{as } x \to +\infty. \end{cases} \tag{60}$$

If there were no dispersion effects, we would get a jumplike viscous shock with velocity determined by the Rankine-Hugoniot conditions (see, e.g., Ref. [17]). Dispersion effects transform a jumplike transition between two levels of the u variable into an oscillatory bore, but the Rankine-Hugoniot conditions are still applicable [27]. Following Whitham's theory of weak shocks [17], we introduce the flux function $Q = -2\alpha u^3$ so that the dispersionless limit of the mKdV equation takes the form the conservation law

$$u_t + Q_x = 0 \tag{61}$$

and then a shock wave propagates with velocity

$$V = \frac{Q(u_-) - Q(u_+)}{u_- - u_+} = -2\alpha(u_-^2 + u_- u_+ + u_+^2). \tag{62}$$

[It is worth noting that it coincides with the velocity of kinks (23) calculated taking into account viscosity, which confirms the generality of the above argumentation.] This velocity must coincide with the constant velocity V of the bore given by Eq. (51),

$$V = -2\alpha(r_1 + r_2 + r_3). \tag{63}$$

Thus, in stationary solutions, the sum of three Riemann invariants is constant and Eq. (54) reduces to

$$\frac{dr_i}{d\xi} = \frac{\sqrt{r_1 r_2 r_3}\langle R\rangle - r_i\langle uR\rangle}{2\alpha\prod_{i\neq j}(r_j - r_i)}, \quad i = 1, 2, 3. \tag{64}$$

It is convenient to introduce symmetric functions of the Riemann invariants,

$$\sigma_1 = r_1 + r_2 + r_3, \quad \sigma_2 = r_1 r_2 + r_1 r_3 + r_2 r_3, \quad \sigma_3 = r_1 r_2 r_3. \tag{65}$$

It is not hard to derive equations for them,

$$\frac{d\sigma_1}{d\xi} = 0, \quad \frac{d\sigma_2}{d\xi} = \frac{1}{2\alpha}\langle uR\rangle, \quad \frac{d\sigma_3}{d\xi} = \frac{\sqrt{\sigma_3}}{2\alpha}\langle R\rangle. \tag{66}$$

Consequently, σ_1 is an integral of motion, as it should be. The theory greatly simplifies if $\langle R\rangle = 0$. In particular, it takes place for the Burgers viscosity $\langle u_{xx}\rangle = (1/L)(u_x)|_0^L = 0$ due to periodicity of u in the main approximation. Then $\sigma_3 = \text{const}$ is also an integral of motion and we get an ordinary differential equation for a sole dependent variable σ_2 or any other variable changing along the bore. It is convenient to choose as such a variable the modulus m. The Riemann invariants can be expressed as functions of m in the following way. The first and third of Eqs. (65) give r_1 and r_2 as functions of r_3:

$$\begin{aligned} r_1 &= \tfrac{1}{2}\left[\sigma_1 - r_3 - \sqrt{(\sigma_1 - r_3)^2 - 4\sigma_3/r_3}\right], \\ r_2 &= \tfrac{1}{2}\left[\sigma_1 - r_3 + \sqrt{(\sigma_1 - r_3)^2 - 4\sigma_3/r_3}\right]. \end{aligned} \tag{67}$$

Then, with the use of Eq. (51) for m, we find the formula

$$m = \frac{3r_3 - \sigma_1 - \sqrt{(\sigma_1 - r_3)^2 - 4\sigma_3/r_3}}{3r_3 - \sigma_1 + \sqrt{(\sigma_1 - r_3)^2 - 4\sigma_3/r_3}}, \tag{68}$$

which defines in an implicit form the function $r_3 = r_3(m)$ so that substitution of this function into Eqs. (67) gives the functions $r_1 = r_1(m)$ and $r_2 = r_2(m)$. Differentiation of m by ξ and substitution of Eq. (64) with $\langle R\rangle = 0$ yield the equation for m:

$$\frac{dm}{d\xi} = -\Phi(m). \tag{69}$$

Consequently, we obtain the solution in the implicit form

$$\xi - \xi_0 = \int_m^1 \frac{dm}{\Phi(m)}, \tag{70}$$

with

$$\Phi(m) = \frac{r_1(r_2 - r_3)^2 + r_2(r_1 - r_3)^2 + r_3(r_1 - r_2)^2}{2\alpha(r_1 - r_2)(r_1 - r_3)^3(r_2 - r_3)}\langle uR\rangle, \tag{71}$$

where $\langle uR\rangle$ can also be expressed in terms of the Riemann invariants, that is, as a function of m (ξ_0 is the position of the soliton edge of the bore with $m = 1$ at the initial moment of time). This completes, in principle, the solution of the Whitham equations for a stationary bore. When the function $m = m(\xi)$ is found, it means that the dependence of the Riemann invariants r_1, r_2, and r_3 on ξ is also known. Substitution of these functions into the two sets (49) and (57) gives us two different dependences of the parameters ν_i, $i = 1, 2, 3, 4$, on ξ. This means that their substitution into the solution (13) yields two different modulated bores. We distinguish the correct solution by the boundary conditions. Thus, now we are in a position to classify all possible wave structures supported by boundary conditions at infinities in the mKdV theory taking into account the small Burgers viscosity.

V. CLASSIFICATION OF WAVE STRUCTURES FOR JUMPLIKE BOUNDARY CONDITIONS

In the region of applicability of the Gurevich-Pitaevskii theory based on the Whitham method of slow modulations of periodic solutions of the mKdV equation, the general diagram of possible wave structures coincides qualitatively with the diagram obtained in Ref. [22] for the related Gardner equation without viscosity (see also Ref. [27]). Taking viscosity into account leads to two modifications: (i) Undular bores become stationary and (ii) the kinks' parameters are slightly changed, as shown in Sec. II C. The resulting diagram is shown in Fig. 2 and here we derive analytical formulas for the main characteristics of the wave structures and compare them with numerical solutions of the mKdVB equation. All numerical simulations in this section are performed with the use

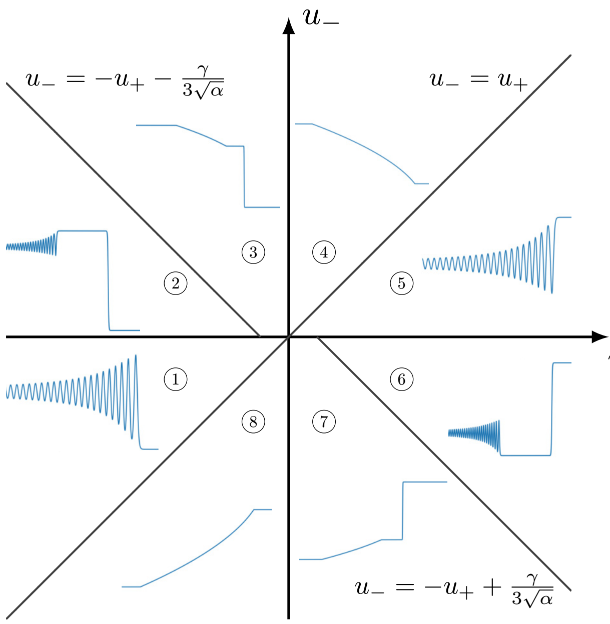

FIG. 2. Wave structures supported by the boundary conditions u_- as $x \to -\infty$ and u_+ as $x \to +\infty$.

of the split-step Fourier technique with fourth-order accuracy similar to the approach in Refs. [35,36]. To avoid interference caused by the periodicity of the boundary conditions in this method, we use a very large x domain.

In regions 1 and 5 in Fig. 2, we get just undular bores of different polarities. Let us consider first region 1 where $u_+ < u_- < 0$ so that u oscillates in the negative interval $\nu_2 \leqslant u \leqslant \nu_3 < 0$. Correspondingly, we have to use the formulas (48) and (49) relating ν_i and r_j. In the small-amplitude limit $x \to -\infty$ we have $\nu_2 = \nu_3 = u_-$ and $m \to 0$, that is, $r_2 \to r_3$. Consequently, we get at the left edge of the bore $r_1^- = u_-^2$ and $r_2^- = r_3^-$, that is,

$$\sigma_1 = u_-^2 + 2r_2^-, \quad \sigma_3 = u_-^2 (r_2^-)^2. \tag{72}$$

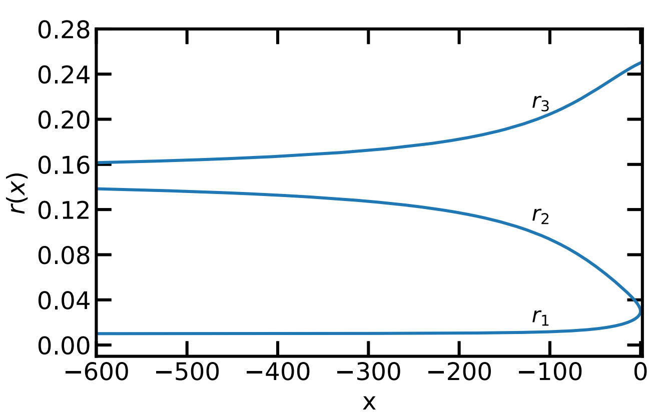

FIG. 3. Riemann invariants for the bores in regions 1 and 5 and the boundary conditions $u_- = -0.1$ and $u_+ = -0.5$ in region 1 and $u_- = 0.1$ and $u_+ = 0.5$ in region 5. The parameters of the equations are equal to $\alpha = 0.2$ and $\gamma = 0.01$.

At the soliton edge we have $m = 1$ and $r_2 = r_1$, that is, $\nu_1 = \nu_2 = -\sqrt{r_3} = u_+$ (or $r_3^+ = u_+^2$), so

$$\sigma_1 = 2r_2^+ + u_+^2, \quad \sigma_3 = (r_2^+)^2 u_+^2. \tag{73}$$

The values of these two constants of motion must be the same at both edges of the bore, so simple calculations give the limiting expressions for the Riemann invariants at the small-amplitude edge,

$$r_1^- = u_-^2, \quad r_2^- = r_3^- = \tfrac{1}{2} u_+ (u_+ + u_-), \tag{74}$$

and at the soliton edge,

$$r_1^+ = r_2^+ = \tfrac{1}{2} u_- (u_- + u_+), \quad r_3^+ = u_+^2. \tag{75}$$

Naturally, their substitution into Eq. (63) reproduces the expression (62) for the velocity of the bore. In addition, we obtain the necessary expressions for the constants of motion

$$\sigma_1 = u_-^2 + u_- u_+ + u_+^2, \quad \sigma_3 = \tfrac{1}{4} u_-^2 u_+^2 (u_- + u_+)^2. \tag{76}$$

To average the Burgers friction term with $uR = \gamma u u_{xx}$, it is convenient to make the replacement $u \to 2v - s_1$, where $s_1 = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}$. The variable v oscillates in the interval $\sqrt{r_2} \leqslant v \leqslant \sqrt{r_3}$, so we obtain the expression

$$\langle u u_{xx} \rangle = -\frac{16}{L} \int_{\sqrt{r_2}}^{\sqrt{r_3}} \sqrt{Q(v)} dv, \tag{77}$$

where $Q(v) = \alpha(v - \sqrt{r_1})(v - \sqrt{r_2})(v - \sqrt{r_3})(v - s_1)$. The integral here can be expressed in terms of the Jacobi elliptic integrals, but it is convenient enough for practical calculations to keep it in this nonintegrated form.

To find the criterion of applicability of our theory, we note that it is correct as long as the length l of the whole bore is much greater than a typical local wavelength L inside it. To estimate these two parameters, we turn to the small-amplitude limit $\xi \to -\infty$ where the Riemann invariants are given by the formulas (74). Then Eq. (69) reduces to

$$\frac{dm}{d\xi} = 4\gamma m, \quad m \propto \exp(4\gamma\xi), \tag{78}$$

so the bore's length can be estimated as

$$l \sim (4\gamma)^{-1}. \tag{79}$$

Substitution of Eqs. (74) into Eq. (52) gives, according to the standard definition $L = 2\pi/k$, the wavelength

$$L = \frac{\pi}{\sqrt{r_3 - r_1}} = \frac{\sqrt{2}\pi}{\sqrt{(u_- - u_+)|u_+ + 2u_-|}}. \tag{80}$$

Then the condition $L \ll l$ can be written in the form

$$u_- - u_+ \ll \frac{32\pi^2\gamma^2}{|u_+ + 2u_-|}. \tag{81}$$

On the axis $u_- = 0$ we get $-u_+ \ll u_c = 4\sqrt{2}\pi\gamma$, and for $|u_-| \gg u_c$ we obtain

$$u_- - u_+ \ll \frac{u_c^2}{3|u_+|} \sim \frac{\gamma^2}{|u_+|}. \tag{82}$$

Thus, the applicability region is separated from the line $u_+ = u_-$ by a narrow strip formed by the hyperbola boundary (81).

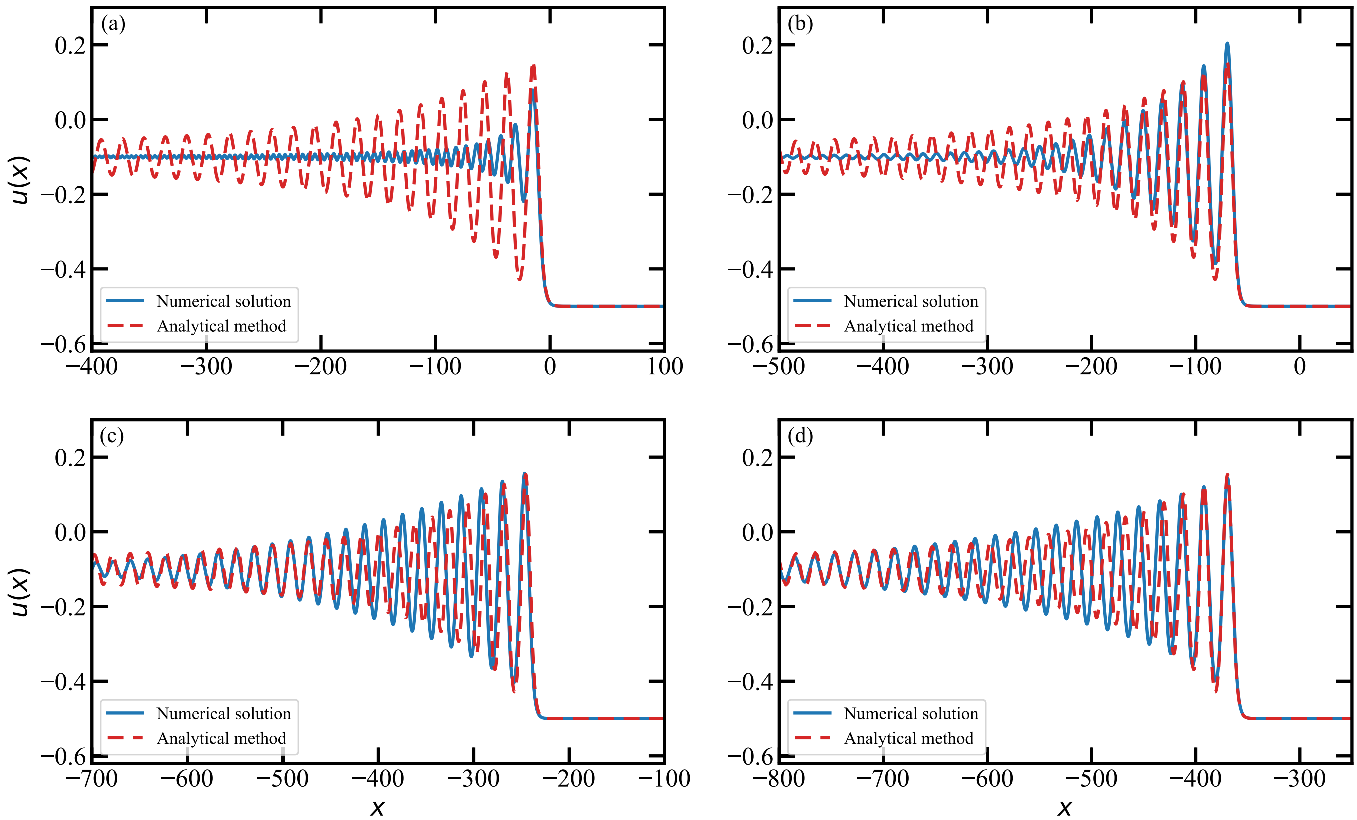

FIG. 4. Numerical evolution of undular bores (blue solid lines) and the stationary asymptotic solution (70) (red dashed lines) at (a) $t = 50$, (b) $t = 100$, (c) $t = 2000$, and (d) $t = 3000$. The parameters of the equations are equal to $\alpha = 0.2$ and $\gamma = 0.01$ and the boundary conditions are $u_- = -0.1$ and $u_+ = -0.5$.

In a similar way, in region 5, where u oscillates in the positive interval $0 < \nu_2 \leqslant u \leqslant \nu_3$, we have to use the formulas (56) and (57) relating the Riemann invariants to the physical parameters of the wave. We obtain the same formulas (74) and (75) for the limiting values of the Riemann invariants, but to average the viscosity term, we make the replacement $u = -2v + s_1$ and obtain again the same formula (77).

If we take symmetrical boundary conditions in regions 1 and 5 that differ only by signs, then in both cases we get the same function $m = m(\xi)$ [see Eq. (70)] and the same plots of the Riemann invariants $r_1(\xi)$, $r_2(\xi)$, and $r_3(\xi)$ shown in Fig. 3. Their substitution into Eqs. (49) or (57) gives the dependences $\nu_i = \nu_i(\xi)$, $i = 1, 2, 3, 4$, for the modulation parameters of the bores in regions 1 and 5, correspondingly. These functions $\nu_i = \nu_i(\xi)$ substituted into Eq. (15) yield the profiles of bores in these two regions shown in Fig. 4 by red dashed lines. They are compared with numerical solutions of the mKdVB equation for different times of evolution and quite good agreement is found for large values of time, especially for the positions and amplitudes of the leading solitons. In agreement with the qualitative estimates of Ref. [26], the leading soliton reaches its stationary state at the characteristic time of order of magnitude of approximately γ^{-1}. At the same time, quite a slow convergence to the stationary profile is observed at the small-amplitude tail of the dispersive shock. The numerically found velocity of the shock agrees very well with the analytical formula (62) for the asymptotic state. The resulting undular bore wave structures for regions 1 and 5 are shown in Figs. 5(a) and 5(b), respectively.

As was shown in Ref. [22], we cannot join the boundaries $u_- > 0$ and $u_+ < 0$ by a single undular bore solution because the mKdV equation is not genuinely nonlinear. In this case, the wave structure must contain a kink solution as shown in Fig. 2 for region 2 and for the symmetrical region 6. In region 2, we have a decreasing kink joining the right boundary $u_+ < 0$ with the intermediate plateau

$$u_* = -u_+ - \frac{\gamma}{3\sqrt{\alpha}} > u_-. \tag{83}$$

This plateau is connected with the left boundary $u_- < u_*$ by the negative undular bore, whose profile can be found in the same way as above by means of the replacement $u_+ \mapsto u_*$. In particular, velocities of the kink and the bore are equal to

$$\begin{aligned} V_{\text{kink}} &= -2\alpha(u_*^2 + u_* u_+ + u_+^2), \\ V_{\text{bore}} &= -2\alpha(u_-^2 + u_- u_* + u_*^2). \end{aligned} \tag{84}$$

For separation of these two constituents of the whole wave structure in space, the difference

$$V_{\text{kink}} - V_{\text{bore}} = 2\alpha(u_- - u_+)\left(u_- - \frac{\gamma}{3\sqrt{\alpha}}\right)$$

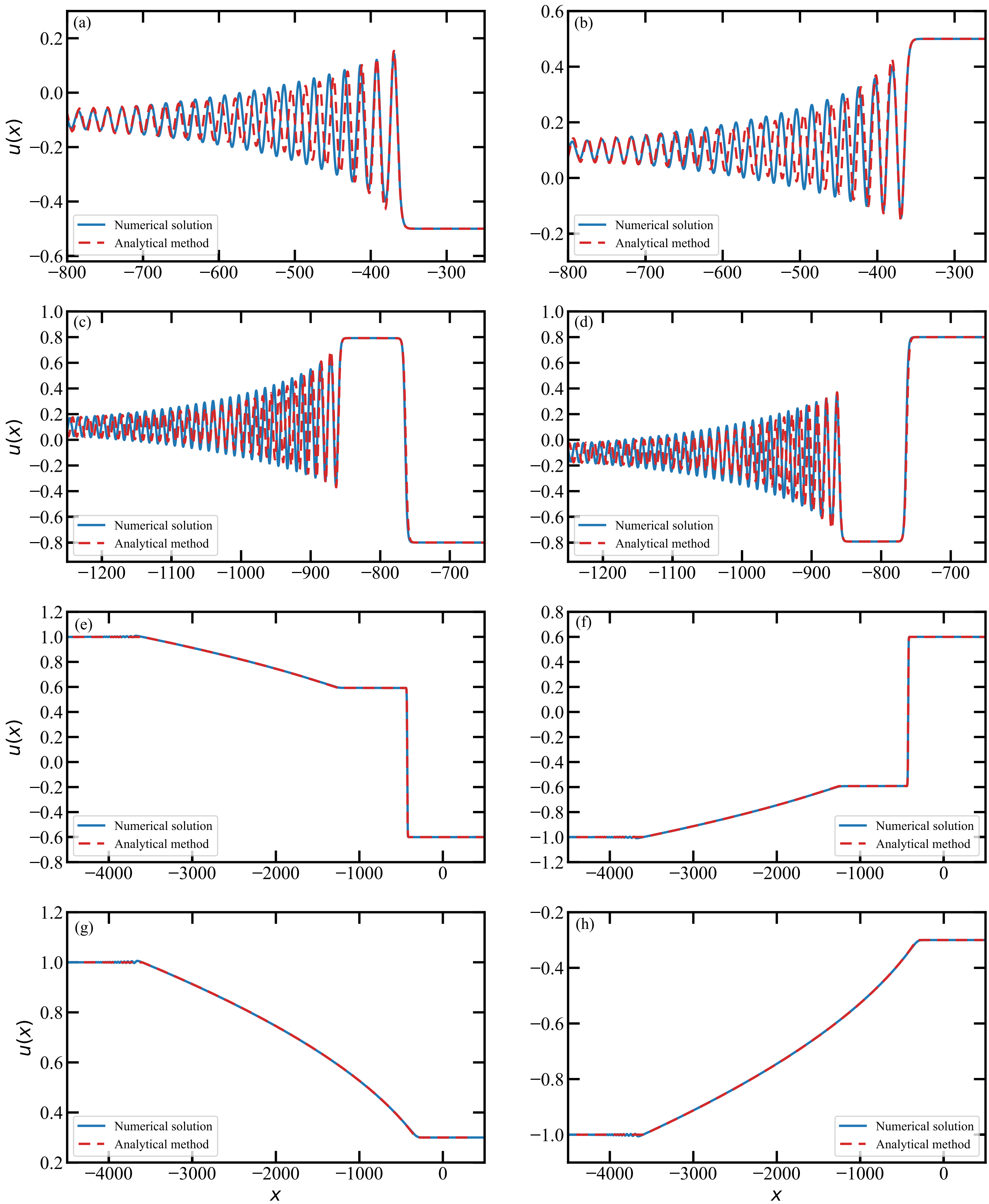

FIG. 5. Wave structures calculated numerically (blue solid lines) and analytically (red dashed lines) for $\alpha = 0.2$, $\gamma = 0.01$, and the evolution time $t = 3000$. The boundary conditions correspond to different regions in the diagram of Fig. 2: (a) region 1, $u_- = -0.1$ and $u_+ = -0.5$; (b) region 5, $u_- = 0.1$ and $u_+ = 0.5$; (c) region 2, $u_- = 0.1$ and $u_+ = -0.8$; (d) region 6, $u_- = -0.1$ and $u_+ = 0.8$; (e) region 3, $u_- = 1.0$ and $u_+ = -0.6$; (f) region 7, $u_- = -1.0$ and $u_+ = 0.6$; (g) region 4, $u_- = 1.0$ and $u_+ = 0.3$; and (h) region 8, $u_- = -1.0$ and $u_+ = -0.3$.

must be positive. Hence, to realize such a structure the left boundary must satisfy the additional condition

$$u_- > \frac{\gamma}{3\sqrt{\alpha}}. \tag{85}$$

If this condition is not fulfilled, then a combined rarefaction wave matched with a kink is formed (see the discussion of such situations in Ref. [27]).

In region 6 with $u_- < 0$ and $u_+ > 0$ we get a structure with a growing kink, so the intermediate plateau has the amplitude

$$u_* = -u_+ + \frac{\gamma}{3\sqrt{\alpha}} < u_-, \tag{86}$$

and such a structure is realized for

$$u_- < -\frac{\gamma}{3\sqrt{\alpha}}. \tag{87}$$

We compare analytical and numerical solutions for regions 2 and 6 in Figs. 5(c) and 5(d), respectively. Again, quite satisfactory agreement is observed.

It is clear that when u_- reaches the level $u_- = u_*$, the cnoidal bore disappears and the wave structure reduces to a sole kink. After a further increase of u_- we get into region 3 where the left boundary u_- is joined with the plateau u_* by a rarefaction wave (7). Its left edge propagates with velocity $V_{rw}^- = -6\alpha u_-^2$ and its right edge propagates with velocity $V_{rw}^+ = -6\alpha u_*^2$, which must be smaller than the kink's velocity. This gives the condition

$$u_+ < -\frac{2\gamma}{3\sqrt{\alpha}} \quad \text{or} \quad 0 > u_+ > -\frac{\gamma}{6\sqrt{\alpha}} \tag{88}$$

for the realization of such a structure in region 3. A similar structure in the symmetrical region 7 is realized for

$$u_+ > \frac{2\gamma}{3\sqrt{\alpha}} \quad \text{or} \quad 0 < u_+ < \frac{\gamma}{6\sqrt{\alpha}}. \tag{89}$$

As one can see in Figs. 5(e) and 5(f), the analytical theory agrees very well with the numerical solutions for these two regions.

Finally, in regions 4 and 8 the boundary values $u_\pm$ have the same signs, so they are connected by standard rarefaction waves with negligible influence of the Burgers friction [see Figs. 5(g) and 5(h)]. This completes the classification of possible wave structures supported by different boundary conditions in the theory of the mKdVB equation.

VI. WHITHAM EQUATIONS FOR CYLINDRICAL AND SPHERICAL mKdV EQUATIONS

We obtained the Whitham modulation equations in a quite general form (54) where the expression for the perturbation term R in Eq. (33) was not specified. Therefore, this universal form of the Whitham equations can be applied to other problems of the dynamics of mKdV dispersive shock waves. In particular, when we consider cylindrical or spherical dispersive shock waves whose width is much smaller than the radius of the whole wave structure, the curvature of the shock can be treated as a small parameter of the theory and Eq. (54) becomes applicable. Cylindrical or spherical mKdV equations were derived, for example, in Ref. [37] and they can be written in the form

$$u_t - 6\alpha u^2 u + u_{xxx} = -\frac{d}{2(t+t_0)}u, \tag{90}$$

where $d = 1$ or 2 for cylindrical or spherical geometry, respectively. For a large enough time of evolution $t_0 \gg 1$ the perturbative right-hand side term is small, so the dispersive shock wave solutions to this equation can be approximated by periodic solutions of the standard mKdV equation with slowly changing parameters, whose evolution is governed by Eq. (54) with

$$\langle R\rangle = -\frac{d}{2(t+t_0)}k\oint\frac{u\,du}{\sqrt{f(u)}},$$
$$\langle uR\rangle = -\frac{d}{2(t+t_0)}k\oint\frac{u^2du}{\sqrt{f(u)}}. \tag{91}$$

The integrals here can be expressed in terms of standard Jacobi elliptic integrals of first, second, and third kinds, so we arrive quite easily at the Whitham equations derived earlier by different methods in Ref. [38] for cylindrical cases and in Ref. [39] for spherical cases, respectively. Thus, the Whitham equations (54) can find various applications besides consideration of the effects of the small viscosity.

VII. CONCLUSION

The above theory confirms the general statement that weak dissipative effects stabilize the expanding evolution of dispersive shock waves, so after a long enough time, they converge to stationary structures characterized by some finite length, which is inversely proportional to the viscosity coefficient. The appearance of the new parameter leads to some limitations for the applicability of the Whitham method used in the Gurevich-Pitaevskii approach to description of bores. In particular, the condition that the size of the whole shock is much greater than the typical wavelength inside the shock demands that the jump between the boundary conditions is large enough. Since the mKdV equation is not genuinely nonlinear, we get combined wave structures consisting of a kink and a cnoidal bore or a rarefaction wave. Small viscosity leads to modification of the kink solution found in Ref. [31] and the condition that the two structural elements of a combined structure propagate separately from each other also leads to some limitations for boundary conditions. Although in the case of small viscosity these restrictions are not essential, one should keep in mind their existence in the practical application of the theory.

ACKNOWLEDGMENTS

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