Exploring universality of the β-Gaussian ensemble in complex networks via intermediate eigenvalue statistics

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(Received 28 August 2023; accepted 8 November 2023; published 16 January 2024)

The eigenvalue statistics are an important tool to capture localization to delocalization transition in physical systems. Recently, a β-Gaussian ensemble is being proposed as a single parameter to describe the intermediate eigenvalue statistics of many physical systems. It is critical to explore the universality of a β-Gaussian ensemble in complex networks. In this work, we study the eigenvalue statistics of various network models, such as small-world, Erdős-Rényi random, and scale-free networks, as well as in comparing the intermediate level statistics of the model networks with that of a β-Gaussian ensemble. It is found that the nearest-neighbor eigenvalue statistics of all the model networks are in excellent agreement with the β-Gaussian ensemble. However, the β-Gaussian ensemble fails to describe the intermediate level statistics of higher order eigenvalue statistics, though there is qualitative agreement till $n < 4$. Additionally, we show that the nearest-neighbor eigenvalue statistics of the β-Gaussian ensemble is in excellent agreement with the intermediate higher order eigenvalue statistics of model networks.

DOI: 10.1103/PhysRevE.109.014218

I. INTRODUCTION

Random matrix theory serves as a powerful tool to study various phenomena in physical systems and finds its application in different fields of physics including statistical physics [1], quantum chaos [2], condensed matter physics [3], and high energy physics [4]. Random matrix theory (RMT) provides predictions for the spectral fluctuations of the systems [5,6]. There are exactly three matrix classes to which spectral statistics are compared: Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensemble. The three different classes are identified by the Dyson index, $\beta = 1, 2$, and 4 for GOE (real), GUE (complex), and GSE (quaternions), respectively. The Dyson index here is equivalent to the number of independent real variables needed to describe one entry of the random matrices. Further, Dumitriu and Edelman [7] introduce a β-Gaussian ensemble with tunable parameter $\beta \in (0, \infty)$ which not only covers GOE($\beta = 1$), GUE($\beta = 2$), and GSE($\beta = 4$) classes but also includes the Poisson level statistics ($\beta \rightarrow 0$).

In general, delocalized and localized phases are distinguished by the level statistics, fundamentally built on RMT. More specifically, the eigenvalues in the delocalized phase are correlated and follow GOE statistics. On the other hand, in the localized phase the eigenvalues are uncorrelated and follow Poissonian level statistics. Recently, it has been shown numerically that the β-Gaussian ensemble interpolates smoothly between the Poisson and the Wigner-Dyson level statistics in studies of many-body localization for the entire transition from delocalized to localized phase [8]. Here, the parameter β signifies the pairwise interactions of the eigenvalues taking all the pairs into account. Consequently, Sierant *et al.* [9] proposed the β-h model where the interactions between the eigenvalues are restricted to h nearest neighbors. The β-h model is also in good agreement with the intermediate level statistics of many-body localization and in fact it has been claimed to be more superior than the β-Gaussian ensemble.

On the other hand, network science has emerged as an effective framework to understand many real-world complex systems ranging from technological to social systems. In network theory, the constituent element of the systems are referred to as nodes and interactions between them are captured by links. There are three popular model networks proposed to capture the properties of real-world complex systems which are Erdös-Renyi random network [10], scale-free network [11], and small-world network [12]. In the last two decades, RMT has been able to show its potential to study and capture phase transitions in complex networks. For example, in Ref. [13] RMT has been used to study the localization to delocalization transition in Erdös-Renyi random network, Cayley tree, and Barabasi-Albert scale-free networks. In Ref. [14], RMT has been applied on random geometric graphs (RGG) to show the gradual transition from the Poisson to the GOE statistics with the change in tunable parameter. Moreover, Ref. [15] has shown the universality of eigenvalue spacing distribution with the network size.

In this article, we study the relevance of β-Gaussian ensemble in complex networks. We show that the β-Gaussian ensemble is in excellent agreement with the nearest-neighbor eigenvalue statistics of all the model networks considered here. However, it fails to describe the higher-order eigenvalue statistics though there is a qualitative agreement till $n < 4$.

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Furthermore, we find that the nearest-neighbor eigenvalue statistics of the β-Gaussian ensemble is in excellent agreement with the intermediate higher order eigenvalue statistics of model networks.

II. METHODS AND TECHNIQUES

The joint probability distribution for the eigenvalues $\{\lambda_i\}$ of N-dimensional matrices from the Gaussian ensembles is given by

$$\rho(\lambda_1, \ldots, \lambda_n) = C_{\beta,N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} e^{-\frac{\beta}{2}\lambda_i^2}, \quad (1)$$

where $C_{\beta,n}$ is a known normalization constant.

Dumitriu and Edelman [7] have studied the eigenvalue properties of a tridiagonal symmetric matrix ($T_{n\beta}$) whose diagonal elements are random numbers from standard normal distribution while the off-diagonal elements on both sides of the diagonal line are taken from χ distribution with a degree of freedom being equal to $(n-1)\beta$ for matrix elements $T_{n,n+1}$,

$$T = \frac{1}{\sqrt{\beta}} \begin{pmatrix} N(0,1) & \chi_{(n-1)\beta} & 0 & \ldots & 0 \\ \chi_{(n-1)\beta} & N(0,1) & \chi_{(n-2)\beta} & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \chi_\beta \\ 0 & \ldots & 0 & \chi_\beta & N(0,1) \end{pmatrix}. \quad (2)$$

The joint probability distribution of eigenvalues of such matrices can still be described using the same equation as that of random matrices[Eq. (1)], but the β value is not limited to $1, 2,$ and 4, and in fact, it can take on any continuous value within the range $\beta \in (0, \infty)$. There exist numerous works on the β ensemble in physical and mathematical contexts. The β-Gaussian ensemble has been extensively studied in terms of eigenvalues properties [16], eigenvector properties [17], fluctuations [18,19], and operators [20]. The distribution of the consecutive eigenvalue spacings is the widely used spectral measure to compare the spectral statistics in RMT. However, unfolding of the original eigenvalues is required to separate the global and smooth part [21]. To unfold the eigenvalues, there are no unique procedures and it may sometimes lead to misleading results [22]. To address this issue, Oganesyan and Huse [23] introduced a new parameter called the ratio of consecutive eigenvalue spacings (r). This parameter is not constrained by unfolding procedures and also incurs low computational cost. The ratio of consecutive eigenvalue spacing is defined as follows [23]:

$$r_i = \frac{\min(s_{i+1}, s_i)}{\max(s_{i+1}, s_i)}, \quad (3)$$

where $s_i = \lambda_{i+1} - \lambda_i$ is the spacing between eigenvalues λ_{i+1} and λ_i with i $\in$ (1,2,3, ..., N-1). In the subsequent years, Ref. [24] derived the exact distribution function of r for GOE, GUE, and GSE ensembles, and the distribution function [P(r)] is given by Eq. (4):

$$P(r) \sim \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+\frac{3}{2}\beta}}. \quad (4)$$

Further, the distribution function for Poisson statistics is given by Eq. (5):

$$P(r) \sim \frac{2}{1 + r^2}. \quad (5)$$

The theoretical average value of r for the GOE, GUE, and GSE classes is estimated to be $0.537, 0.60,$ and 0.67, respectively, while for Poisson statistics, the average value of r is equal to 0.38. Furthermore, we have also employed nonoverlapping higher order spacing ratio in our study which is defined as

$$r_i^n = \frac{\min(s_{i+n}^n, s_i^n)}{\max(s_{i+n}^n, s_i^n)} i, n = 1, 2, 3, \ldots, \quad (6)$$

where $s_i^n = \lambda_{i+n} - \lambda_i$. Additionally, the distribution function of higher order ratio $P(r^n)$ satisfies the following equations [25,26]:

$$P(r^n, \beta) = P(r^1, \beta'), \quad (7)$$

where

$$\beta' = \frac{n(n+1)\beta}{2} + (n-1). \quad (8)$$

III. MODEL

A network is made up of a set of nodes $V = \{v_1, v_2, v_3, \ldots, v_N\}$ and a set of links $E = \{e_1, e_2, e_3, \ldots, e_M\}$, where N and M represent the number of nodes and the number of links in the network, respectively. Conventionally, a network is expressed by its adjacency matrix A with $A_{ij} = 1$ if nodes i and j are connected, and zero otherwise. To study the localization to delocalization transition through eigenvalue statistics, we introduce disorder into the diagonal elements of the adjacency matrix. The diagonal elements, denoted as D_{ii}, are sampled from the uniform distribution $(-w, w)$ of width $2w$. We consider three popular network models in our analysis: (i) small-world networks, (ii) scale-free networks, and (iii) Erdős-Rényi random networks. We construct small-world networks using the Watts and Strogatz algorithm, as follows. We first build a regular network where each node has an equal number of neighbors or an equal degree. We then randomly rewire each link of the network with a probability p_r such that $0 < p_r \leqslant 1$. The rewiring procedure will alter two major structural features of the network: clustering coefficient and average shortest path length. Initially, for a small rewiring probability, the average shortest path length decreases drastically while the clustering coefficient remains at a very high value consistent with that of regular networks. As the rewiring probability increases further, the average shortest path length has attained the value of random networks, and the clustering coefficient also starts decreasing. Thus, the rewiring procedure transforms a regular network colored into a random network via a small-world network characterized by a very high clustering coefficient and a very small characteristic path length [12]. We construct scale-free networks by following a BA preferential model [11]. In this model, a network with m_0 nodes is first constructed and a new node at each time step having m connections is added to the network such that $m \leqslant m_0$. The new node's attachment to an existing node i

TABLE I. The value of $\langle r \rangle$ for various β values.

β	$\langle r \rangle$	β	$\langle r \rangle$
0.01	0.382	0.6	0.502
0.1	0.412	0.7	0.514
0.2	0.434	0.8	0.526
0.3	0.456	0.9	0.532
0.4	0.467	0.95	0.543
0.5	0.492	1	0.545

is determined by a probability proportional to k_i where k_i is the degree of the ith node and thus incorporates preferential attachment. It is easy to verify that after t time steps, the network will have $t + m_0$ nodes and mt connections, with degree distribution following power law. ER random networks are created using the ER model [27] as follows. For given network parameter N nodes and $\langle k \rangle$ average degree, each pair of the nodes can form links with a probability $p = \langle k \rangle / N$.

IV. RESULTS

In this section, we investigate the universality of β-Gaussian ensembles in networks through the intermediate level statistics of delocalization to localization transition. First, we generate the Gaussian beta ensemble following Eq. (2) for $N = 10^4$ with ten random realizations for various values of β. Next, we numerically diagonalize the β-Gaussian ensemble and compute its eigenvalues. Subsequently, we calculate the consecutive eigenvalue spacing ratio and nth order eigenvalue spacing ratio of β-Gaussian ensemble. Thus, for each value of β, we obtain the corresponding $\langle r \rangle$ value. It is important to note that, unlike the approach in [8], we do not unfold the eigenvalues and also consider all the eigenvalues of the spectrum to reduce computational costs. This leads to a slightly different correspondence between β and $\langle r \rangle$, compared to [8], as shown in Table I. If we limit our analysis to eigenvalues closer to zero, the relationship between $\langle r \rangle$ and β remains relatively stable as long as $0.20 < d\lambda$, where $d\lambda$ is the width on both sides of $\lambda \approx 0$. It is only when $d\lambda$ falls below 0.20 that noticeable variations in the relationship between $\langle r \rangle$ and β become evident. When dealing with small values of $d\lambda$, a substantial number of realizations are necessary to ensure the statistical robustness of the analysis, which is computationally exhaustive. We will use $\langle r \rangle$ to determine β and subsequently compare the intermediate level statistics of the model networks with that of the β-Gaussian ensemble.

Ratio of consecutive eigenvalue spacing: We first probe the small-world networks and study its eigenvalue statistics. We focus on eigenvalues within the central part of the spectrum, approximately extending 0.5 on both sides of $\lambda \approx 0$ which is a standard procedure in the study of eigenvalue statistics [28]. Further, we wish to emphasize here that a slight increase or decrease in the width does not affect the results and $\langle r \rangle$ remains constant for $0.25 \leqslant d\lambda \leqslant 3$. In [29], it was shown that increasing the diagonal disorder strength leads to gradual transition in the eigenvalue statistics of small-world networks from the GOE to Poisson statistics. Additionally, it was found that the critical disorder required to obtain the Poisson statistics increases with the increase in the rewiring probabilities.

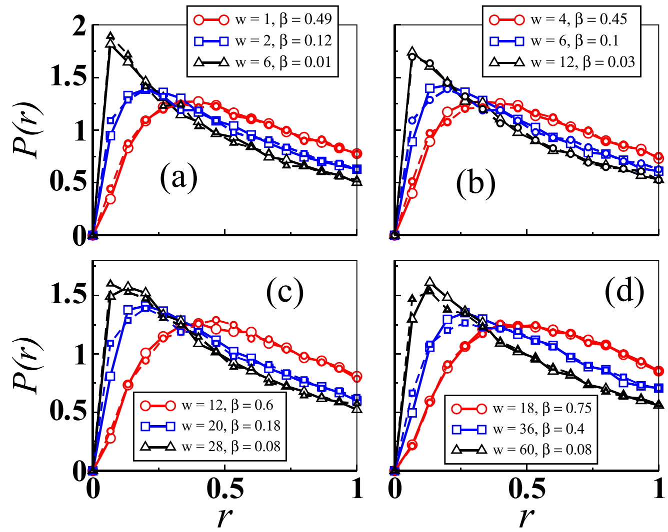

FIG. 1. Distributions of consecutive eigenvalue spacing ratios of the small-world networks at various values of disorder strength denoted with solid lines. The corresponding distributions for the β-Gaussian ensemble are represented by dashed lines with an identical scheme of symbols and colors. (a) $p_r = 0.001$, (b) $p_r = 0.01$, (c) $p_r = 0.1$, and (d) $p_r = 1$. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations.

Note that the critical disorder required to obtain the localization transition is usually obtained by performing finite size analysis with various system sizes. In the absence of finite size scaling analysis, for a fixed $N = 2000$, we consider the critical disorder (w_c) to be the value at which eigenvalue statistics exhibit Poisson statistics. In Fig. 1, we plot the probability distribution $P(r)$ of the consecutive eigenvalue spacing ratios of the small-world networks for various rewiring probabilities. In the same figure (Fig. 1), we also plot the corresponding probability distribution $P(r)$ of the β-Gaussian ensemble. It is clearly visible from the figure that the level statistics of the β-Gaussian ensemble is in excellent agreement with the level statistics of the small-world networks. Next, in Fig. 2, we plot the probability distribution $P(r)$ of the Erdős-Rényi random and scale-free networks. These two network types have been

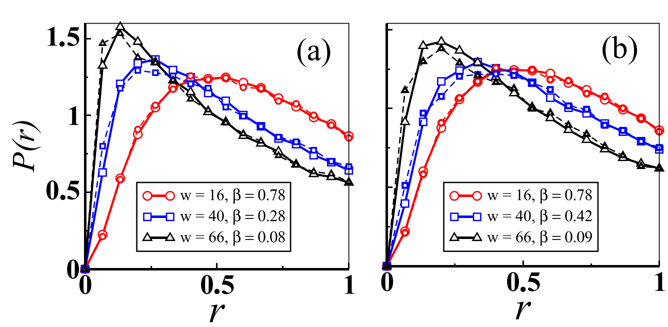

FIG. 2. Distributions of consecutive eigenvalue spacing ratios of the ER and SF networks at various values of disorder strength denoted with solid lines. The corresponding distributions for the β-Gaussian ensemble are represented by dashed lines with an identical scheme of symbols and colors. (a) ER networks and (b) SF networks. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations.

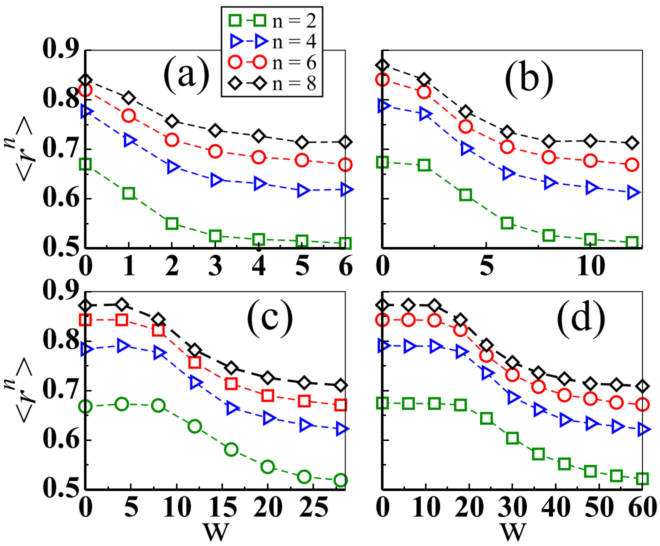

FIG. 3. $\langle r^n \rangle$ against diagonal disorder for various rewiring probabilities of small-world networks. (a) $p_r = 0.001$, (b) $p_r = 0.01$, (c) $p_r = 0.1$, and (d) $p_r = 1$. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations.

extensively studied and exhibit localization transition captured by the level statistics [13,30,31]. It is evident from Fig. 2 that the level statistics of the β-Gaussian ensemble provide an accurate description of the intermediate level statistics of the model networks.

Higher order spacing ratios: In this section, we explore the universality of β-Gaussian ensemble in describing the higher-order level statistics. We obtain the higher order spacing ratio (r^n) using Eq. (6) and compute $\langle r^n \rangle$. The higher order eigenvalue statistics have also been found to be useful in understanding the behavior of the level statistics [8,32–34]. In Fig. 3, we first present the behavior of $\langle r^n \rangle$ for various values of n with the change in diagonal disorder. Notably, for given network parameters, $N = 2000$ and $\langle k \rangle = 10$, it requires a very large value of disorder strength to reach Poisson statistics. Furthermore, as the disorder strength increases, the number of eigenvalues closer to zero also decreases. Thus, to keep the analysis statistically sound, we restrict to $n \leqslant 8$. We find that in the absence of diagonal disorder ($w = 0$) and

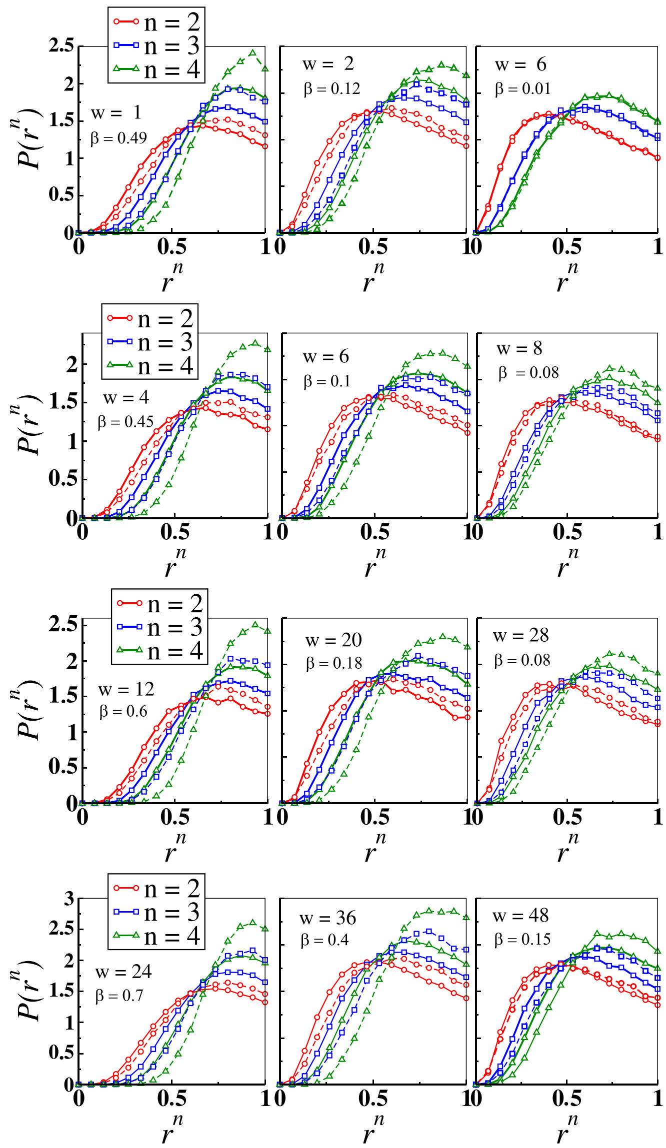

FIG. 5. Distributions of higher-order spacing ratios of small-world networks at various values of disorder strength are denoted with solid lines. The corresponding distributions for the β-Gaussian ensemble are represented by dashed lines with an identical scheme of symbols and colors. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations. The first, second, third, and fourth rows correspond to the cases when p_r equals 0.001, 0.01, 0.1, and 1, respectively.

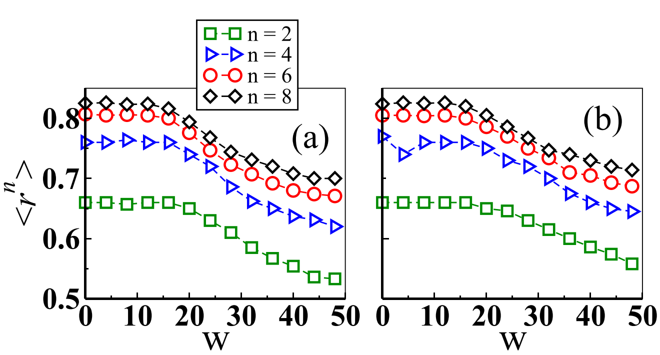

FIG. 4. $\langle r^n \rangle$ against diagonal disorder for ER and SF networks. (a) ER networks and (b) SF networks. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations.

$n > 2$, $\langle r^n \rangle$ increases with the increase in the rewiring probability up to $p_r \leqslant 0.01$. However, for $p_r > 0.01$, it reaches a constant value and no longer changes with the further adjustments to rewiring probabilities. This behavior can be explained as follows. With an increase in the rewiring probability, the randomness in the network architecture increases, leading to correlations between eigenvalues even at the longer ranges [35]. Consequently, $\langle r^n \rangle$ exhibits an increasing trend. However, when the network achieves a sufficient level of randomness ($p_r > 0.01$), $\langle r^n \rangle$ becomes constant and remains unaffected by further changes in rewiring probability. When diagonal disorder is introduced, it is found that $\langle r^n \rangle$ decreases with increasing disorder strength and becomes saturated after

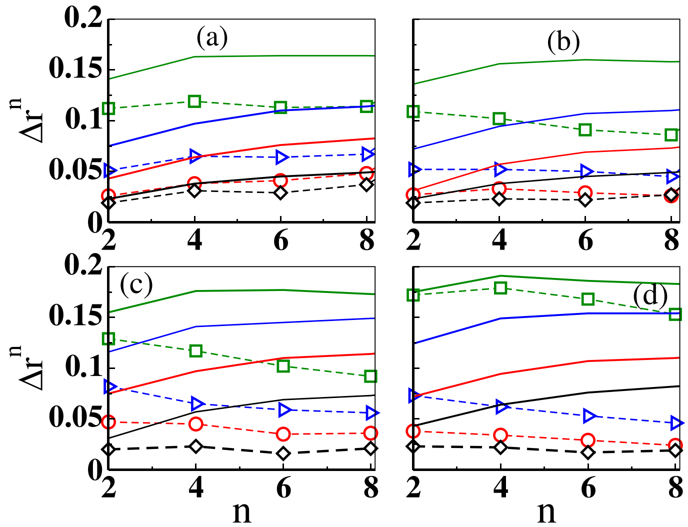

FIG. 6. Δr^n against n for various values of diagonal disorder of small-world networks. (a) $p_r = 0.001$, (b) $p_r = 0.01$, (c) $p_r = 0.1$, and (d) $p_r = 1$. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations. Diagonal disorder from top to bottom (a) $w = 1, 2, 3, 4$, (b) $w = 4, 6, 8, 10$, (c) $w = 12, 16, 20, 28$, and (d) $w = 18, 36, 48, 60$. The corresponding β-Gaussian prediction is a solid line with an identical color scheme.

the critical disorder (w_c). It is worth noting that for each rewiring probability, $\langle r^n \rangle(w_c)$ takes on the same constant value, similar to $\langle r^1 \rangle(w_c)$, which follow the Poissonian value. We find that $\langle r^n \rangle(w_c) \approx 0.510 \pm 0.01, 0.619 \pm 0.01, 0.669 \pm 0.01, 0.715 \pm 0.01$ for $n = 2, 4, 6, 8$ for all rewiring probabilities. Further, in Fig. 4, we also plot the behavior of $\langle r^n \rangle$

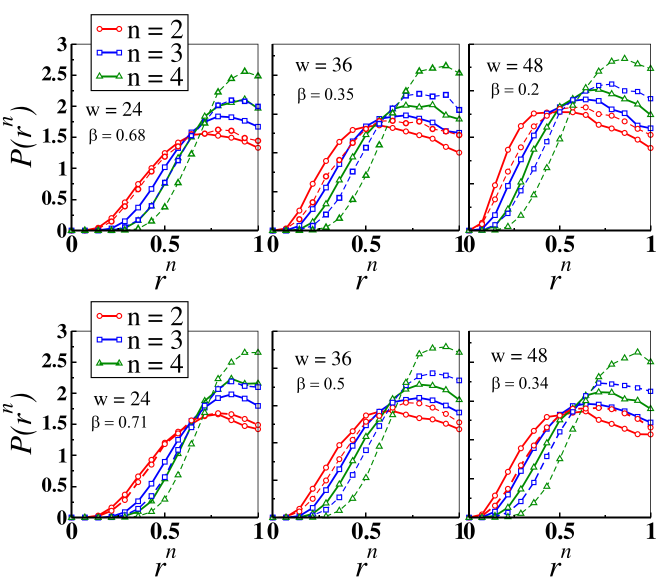

FIG. 7. Distributions of higher-order spacing ratios of ER and SF networks at various values of disorder strength are denoted with solid lines. The corresponding distributions for the β-Gaussian ensemble are represented by dashed lines with an identical scheme of symbols and colors. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations. The first and second rows correspond to ER and SF networks, respectively.

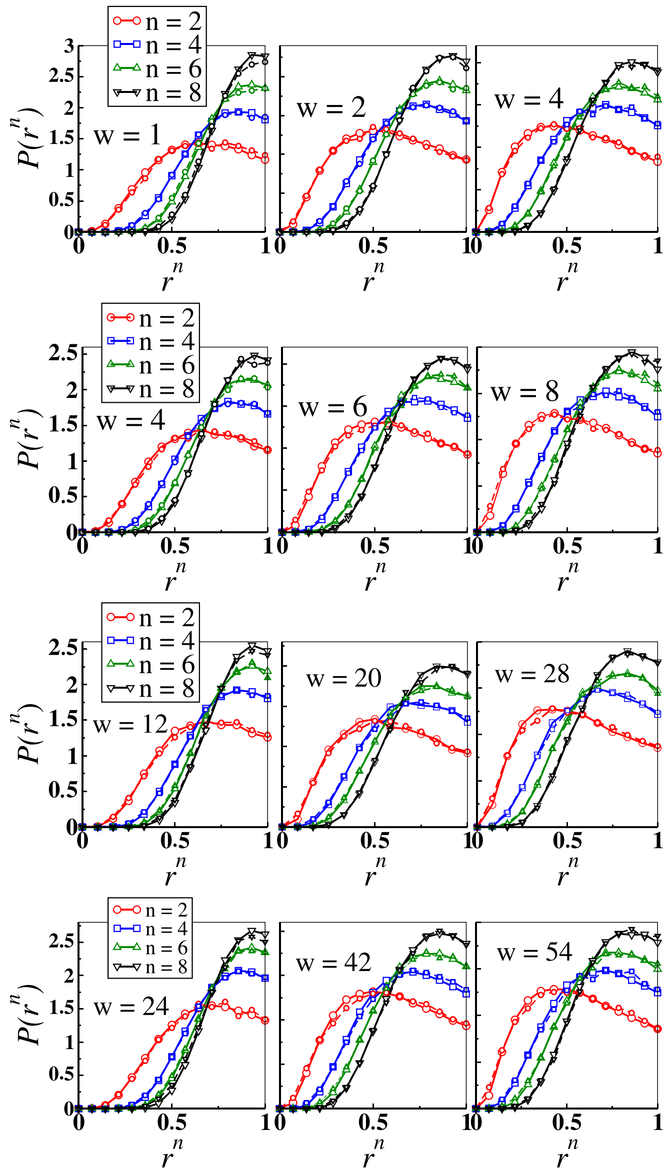

FIG. 8. Distributions of higher-order spacing ratios of small-world networks at various values of disorder strength are denoted with solid lines. The corresponding nearest-neighbor distributions for the β-Gaussian ensemble are represented by dashed lines with an identical scheme of symbols and colors. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations. The first, second, third, and fourth rows correspond to the cases when p_r equals 0.001, 0.01, 0.1, and 1, respectively.

against diagonal disorder strength for Erdős-Rényi random and scale-free networks. It is evident from the figure that, similar to the small-world networks, $\langle r^n \rangle$ decreases with increasing disorder strength and gets saturated for higher diagonal disorder strength.

To describe the level statistics of the physical system, typically, $P(r^n)$ of the random matrix model (in our case, its β-Gaussian ensemble) is compared with that of $P(r^n)$ of the physical systems under consideration by finding out the β value from corresponding $\langle r^1 \rangle$ value [8,9,36]. We plot the distribution of higher-order ratios, $P(r^n)$, for various disorder

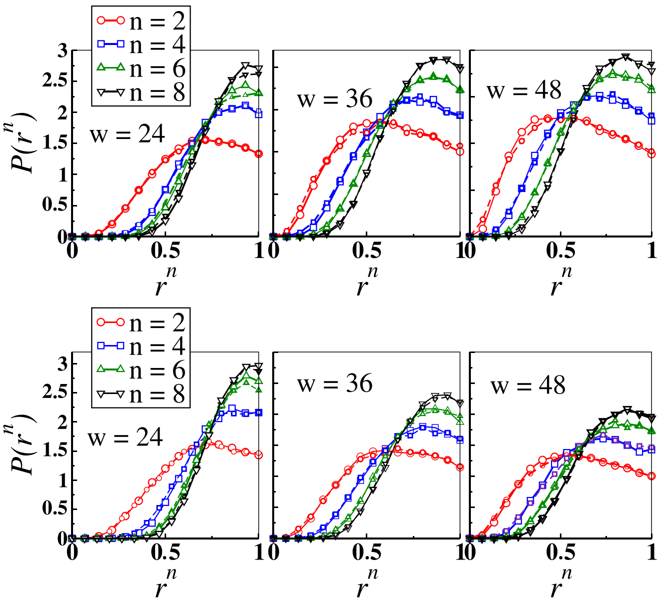

FIG. 9. Distributions of higher-order spacing ratios of ER and SF networks at various values of disorder strength are denoted with solid lines. The corresponding nearest-neighbor distributions for the β-Gaussian ensemble are represented by dashed lines with an identical scheme of symbols and colors. We consider $N = 2000$ and $\langle k \rangle = 10$ with 100 realizations. The first and second rows correspond to ER and SF networks, respectively.

strengths in small-world networks in Fig. 5. It is important to note that the corresponding level statistics of $P(r^n)$ for the β-Gaussian ensemble is also plotted in Fig. 5. It is visible from the figure that $P(r^n)$ of the β-Gaussian ensemble is in qualitative agreement with the intermediate level statistics of the small-world networks for $n < 4$ for all values of disorder strength, consistent with the findings in [8]. To get a more quantitative insight, we calculate $\Delta r^n = \langle r^n \rangle - \langle r^n_{PS} \rangle$, where $\langle r^n_{PS} \rangle$ represents the nth order spacing of Poissonian statistics. Note that Δr^n measures the degree of correlation between the eigenvalues at the nth scale [9]. From Fig. 6, it is evident that for $n \geqslant 2$, Δr^n is overestimated when compared to predictions from the β-Gaussian ensemble. Furthermore, in Fig. 7, we plot the $P(r^n)$ for various disorder strengths in Erdős-Rényi random and scale-free networks, respectively. The result is consistent with that of small-world networks, and we find that $P(r^n)$ of the β-Gaussian ensemble is only in qualitative agreement for $n < 4$.

As according to Eq. (6), we have $P(r^n, \beta) = P(r^1, \beta')$ with the relation followed by Eq. (8). Hence, we are also interested in comparing the higher-order level statistics of the model networks with the nearest-neighbor eigenvalue statistics of the β-Gaussian ensemble, i.e., comparing $P(r^n)$ of the model networks with the corresponding $P(r^1)$ of the β-Gaussian ensembles. We first calculate $\langle r^n \rangle$ and then determine the corresponding β value to obtain the level statistics of corresponding β ensembles. Intriguingly, we find that the nearest-neighbor eigenvalue statistics of β-Gaussian ensembles closely align with the higher-order level statistics of the small-world networks, as evident in Fig. 8. Next, in Fig. 9, we check the results for Erdős-Rényi and scale-free networks and it is found that the results are in good agreement similar to the small-world networks.

V. CONCLUSION

To conclude, we have explored the universality of β-Gaussian ensemble in describing the intermediate level statistics of the localization transition in complex networks. We have made several noteworthy observations which are as follows. For the nearest-neighbor distribution, the level statistics of the β-Gaussian ensemble perfectly describes the intermediate level statistics of the model networks. However, when it comes to higher-order spacing ratios, our findings indicate that the level statistics of a β-Gaussian ensemble can only be compared at a qualitative level until $n < 4$. Furthermore, we find that the nearest-neighbors eigenvalue statistics of the β-Gaussian ensemble is in excellent agreement with the intermediate higher order eigenvalue statistics of model networks. It is important to note that various models apart from the β-Gaussian ensemble have also been proposed such as a β-h model [9], mixed (Brownian) ensemble [37], Pechukas-Yukawa distribution [38], and short-range plasma model [36], to capture the intermediate level statistics of the many physical systems.

ACKNOWLEDGMENTS

This work was supported by the Singapore Ministry of Education Academic Research Fund (AcRF) Tier 2 (Grant No. MOE-T2EP50120-0021).

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