Two-qubit entangling operators as chaos control in a discrete dynamic Cournot duopoly game

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The current trend in economics research is to incorporate quantum mechanical concepts to increase the security of business models. This interdisciplinary field of study represents real-world market dynamics more closely than do its classical counterparts. In this paper, we shed light on the significance of the two-qubit entangling operators in controlling chaos. We introduce a modified Eisert-Wilkens-Lewenstein scheme in a nonlinear Cournot duopoly game with complete and incomplete information. By doing so, the following interesting results are obtained: To begin, monopoly in a duopoly game can be avoided with the use of special perfect entanglers. Also, the stability analysis shows that there exists a class of entangling operators which can stabilize an unstable system and vice versa. Second, numerical analysis highlights the two-qubit entangling operators which can stabilize a chaotic system or at least delay chaos. Finally, we show that with an appropriate choice of initial state and speed of adjustments, entangling operators can decrease the sensitivity of the system. In short, while we know the importance of entangling operators in quantum game theory, in this paper we indicate the significance of operators in the context of a chaotic system.

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I. INTRODUCTION

Classically, game theory consists of two or more players who decide to maximize their outcome by implementing appropriate strategies [1]. Game theory finds its applications in various fields such as economics, biology, optimization problems, and many more [2,3]. One of the most important concepts in game theory is Nash equilibrium, wherein no player has an incentive to deviate from their strategy. The earliest version of this concept was proposed by Cournot in 1838 in his mathematical model of an oligopoly market [4]. The idea of introducing quantum mechanical principles in game theory was first proposed by Meyer [5]. In his seminal paper, he discussed how quantum strategies can outperform classical strategies in a single-penny flip game. This led to the formal introduction of quantum game theory. Since then, quantum game theory has been extensively studied to quantize complex classical games and to understand how quantum mechanical principles change the game dynamics [6–8].

One of the applications of quantum mechanics is in economics as it acts as a powerful tool to deal with the intrinsic randomness of the real market [9–12]. Any real-world market can be treated as a game where companies play the role of players and use the rate of production as strategies to outperform their opponents. The quantum analog of market games, namely oligopoly games, was introduced by Li, Du, and Massar in 2002 [13]. Since then, oligopoly and duopoly games have been widely studied in various market settings. Later, Iqbal and Toor [14] quantized duopoly games using Marinatto and Weber's scheme. This particular scheme, however, is not widely utilized to analyze market games. Later on, Frąckiewicz and co-workers introduced another quantization scheme wherein the strategies of the players are entangled [15,16]. All these schemes were initially structured to operate for linear systems with rational players and linear inverse demand and cost functions. A real market is usually nonlinear as there exist various nonlinear relationships between the expectations of the companies, consumer attitudes, and so on. Also, in actual market situations, players lack complete knowledge of their opponents. As a result, they adjust their profits in each period and slowly attain Nash equilibrium.

Nonlinearity in classical market games is well established. Rand, in 1978, was the first to introduce the concept of dynamics in duopoly games [17]. Later, Puu's study emphasized how the players' adjusting process can lead to chaos [18]. Refer to Refs. [19–24] for works on different types of classical nonlinear duopoly games. Recently, Ikeda and Aoki's work proved the importance of quantum information in economics, where money may be viewed as a quantum mechanical object [9]. There was not much research in quantum nonlinear duopoly games until Yang and Gong proposed the use of quantum entanglement in these games in 2018 [25]. They analyzed the quantum Cournot duopoly game with linear cost and demand function with bounded rational players (players with incomplete information). The expectation of the players is important in economics as it decides the number of equilibrium points. Recently, active research has been taking place in quantizing nonlinear oligopoly games to understand the effect of entanglement on the stability of the equilibrium points. Refer to Refs. [26–30] for recent works on quantum nonlinear market games. The quantum versions of these games fail to clearly show the role of entanglement, and all of them are quantized using Li, Du, and Massar's scheme. This quantization technique makes it difficult to understand the role of

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quantum operators, which can be realized as a quantum circuit with single- and two-qubit gates.

In this paper, a discrete dynamic Cournot duopoly game is quantized using a modified Eisert-Wilkens-Lewenstein (EWL) scheme [31] to investigate the role of quantum operators in nonlinear games. By quantizing and analyzing the game from the perspective of quantum operators, we add an additional dimension to Yang and Gong's work. The major goal of this paper is to demonstrate the importance of two-qubit nonlocal operators, namely special perfect entanglers [32], in a game with partial and imperfect information. Also, we aim to show the interplay of two-qubit operators and the speed of adjustment of the players in controlling chaos. Such an observation is of great interest in quantum nonlinear duopoly games. Furthermore, we wish to see the choice of entangling operators that can prevent monopoly and control chaos. The outline of the paper is as follows: The discrete dynamical nature of a Cournot duopoly game with different expectations is analyzed in Sec. III. Section IV confirms the chaotic nature of the system with bifurcation diagrams, strange attractors, and sensitive dependence on initial conditions. Finally, in Sec. V we highlight the importance and uniqueness of our results in detail.

II. COURNOT DUOPOLY GAME

"Cournot duopoly" refers to a game between two companies whose profits do not fluctuate over a period of time. However, it is important to note that the real economic market situation is dynamic and companies (firms) continuously adjust their strategies due to incomplete information about their opponents. Taking into account the actual market, we study the quantum version of the dynamic market game with a linear inverse demand and cost function. Classically, the profit function of the firms for a single-period game is given as [33]

$$\Pi_i = q_i[k_i - (q_1 + q_2)], \tag{1}$$

where $k_i = a - C_i$ for $i = 1, 2$ and $a > 0$ and where q_1 and q_2 are the quantities of production set by the firms. The cost function of the firms has the form $C_i = C_i q_i$. The classical version of this game is quantized using a modified EWL scheme.

III. DYNAMIC QUANTUM COURNOT DUOPOLY GAME

The modified EWL scheme provides a range of two-qubit nonlocal operators which were earlier not available using the Eisert-Wilkens-Lewenstein scheme [6] and the Marinatto-Weber scheme [34]. This is obtained by revoking the commutator conditions imposed by the EWL scheme. The general two-qubit entangling operator U is defined as [31]

$$U = \begin{bmatrix} e^{\frac{-ic_3}{2}}c^- & 0 & 0 & -ie^{\frac{-ic_3}{2}}s^- \\ 0 & e^{\frac{ic_3}{2}}c^+ & -ie^{\frac{-ic_3}{2}}s^+ & 0 \\ 0 & -ie^{\frac{-ic_3}{2}}s^+ & e^{\frac{ic_3}{2}}c^+ & 0 \\ -ie^{\frac{-ic_3}{2}}s^- & 0 & 0 & e^{\frac{-ic_3}{2}}c^- \end{bmatrix}, \tag{2}$$

where $c^{\pm} = \cos(\frac{c_1 \pm c_2}{2})$, $s^{\pm} = \sin(\frac{c_1 \pm c_2}{2})$, and c_1, c_2, and c_3 are the geometrical points of the Weyl chamber (geometrical representation of two-qubit operators) [32,35]. Considering $c_3 = 0$, we obtain the entangling operator U as defined in the modified EWL scheme. Now the quantum structure of the game is as follows: The game starts with an initially entangled state $|\psi_i\rangle = U|00\rangle$. This state is then given by the arbiter (in this case, the government) to the firms that produce homogeneous goods in the market. Let firm 1 and firm 2 initially pick quantities q_1 and q_2, respectively. These firms have an option to adhere to their initial decision or change it during the course of the game. Let us assume that firm 1 sticks to the decision (represented by an identity matrix) and firm 2 flips its decision using the Pauli σ_x operation. After the implementation of their strategies, a disentangling operator is applied to remove the initial entanglement. The final state of the game is as follows:

$$|\psi_f\rangle = U^{\dagger}(I \otimes \sigma_x)U|00\rangle. \tag{3}$$

This state is measured using the measurement operator $M_i(x_1, x_2)$ for $x_i \in [-1, 1]$. The measurement operator is given by Frąckiewicz [15] and is of the form

$$M_i(x_1, x_2) = \begin{cases} (x_1|0\rangle\langle 0| + x_2|1\rangle\langle 1|), & \text{if } i = 1 \\ (x_2|0\rangle\langle 0| + x_1|1\rangle\langle 1|), & \text{if } i = 2. \end{cases} \tag{4}$$

After measurement, the quantities of the companies are determined as $q_i = \mathrm{Tr}(M_i\rho_i)$, where ρ_i is a reduced density matrix of the final state. As a result, the quantities can be written as follows:

$$\begin{aligned} q_1 &= lx_1 + mx_2, \\ q_2 &= lx_2 + mx_1. \end{aligned} \tag{5}$$

Here, $l = \cos^2 c_1$, $m = \sin^2 c_2$, and x_1 and x_2 are the continuous strategic space available to firms with $x_i \in [-1, 1]$. Substituting Eq. (5) into Eq. (1), one can easily obtain the new profits of the firms after quantization. In the entire analysis, we consider $c_2 = c_1$ for the choice of entangling operators to satisfy the condition $q_1 + q_2 = x_1 + x_2$.

Nonlinearity in a duopoly game arises when either the cost function or the inverse demand function is nonlinear or when the players possess incomplete information. We consider the following cases: (i) Both firms possess incomplete information, and (ii) firms have partial information. Sections III A and III B provide brief descriptions of these two cases.

A. Dynamic game with homogeneous players

Let us start with a dynamic Cournot duopoly game wherein the firms produce homogeneous goods and offer them at discrete time intervals. We consider the firms to be bounded rational players who have limited information about their opponent's moves. They adjust their strategies by either increasing or decreasing their level of production at period $t + 1$ in response to the marginal profit observed at period t. Since both firms are homogeneous, the discrete dynamic logistic map can be written as [21,22]

$$\begin{aligned} x_1(t+1) &= x_1(t) + \alpha\frac{\partial \Pi_1}{\partial x_1}, \\ x_2(t+1) &= x_2(t) + \beta\frac{\partial \Pi_2}{\partial x_2}. \end{aligned} \tag{6}$$

Here, α and β represent the speed of adjustments of firm 1 and firm 2, respectively. Setting $x_i(t+1) = x_i(t) = x_i$ for $i = 1, 2$ in the above logistic map, the following equilibrium points are obtained: $E_0 = (0, 0)$, $E_1 = (\frac{k_1}{2}, 0)$, $E_2 = (0, \frac{k_2}{2})$, and $E_3 = (\frac{l(2k_1l-k_2)}{4l^2-1}, \frac{l(2k_2l-k_1)}{4l^2-1})$. Here, E_1 and E_2 are boundary equilibrium points (also known as monopoly equilibrium points), whereas E_3 corresponds to the quantum Nash equilibrium point. Note that in the absence of entanglement, a single firm dominates the market at the equilibrium points E_1 and E_2. This form of monopoly in the market can be avoided in the quantum version with an appropriate choice of entangling operator $U(\pi/4, \pi/4, 0)$.

The stability of these equilibrium points is analyzed by calculating the eigenvalues of the Jacobian matrix constructed using Eq. (6). Note that the equilibrium point is stable if and only if the eigenvalues of the Jacobian matrix $|\lambda_i| < 1$. The Jacobian matrix of Eq. (6) is written as

$$J(x_1, x_2) = \begin{bmatrix} 1+\alpha(lk_1 - 4lx_1 - x_2) & -\alpha x_1 \\ -\beta x_2 & 1+\beta(lk_2 - 4lx_2 - x_1) \end{bmatrix}. \tag{7}$$

Substituting E_0 into Eq. (7), the following eigenvalues are obtained: $\lambda_1 = 1 + \alpha lk_1$ and $\lambda_2 = 1 + \beta lk_2$, wherein both the eigenvalues λ_1 and λ_2 are greater than zero. This implies that the trivial equilibrium point stays unstable independent of the choice of entangling operators.

For the equilibrium point E_1 the eigenvalues are $\lambda_1 = 1 - \frac{\alpha}{2}k_2 + \alpha k_1 l$ and $\lambda_2 = 1 - \beta k_2 l$. This point can be both stable and unstable depending on the choice of the entangling operators. For the choice of entangling operators $U(0 \leqslant c_1 \leqslant \frac{\pi}{4},\ c_2 = c_1,\ c_3 = 0)$, the eigenvalues are $\lambda_1 > 1$ and $\lambda_2 < 1$ (unstable equilibrium point). However, for the choice of entangling operators $U(\frac{\pi}{4} < c_1 \leqslant \frac{\pi}{2},\ c_2 = c_1,\ c_3 = 0)$, both the eigenvalues are negative, and hence the equilibrium point is stable. Such an observation holds true for the equilibrium point E_2 with the eigenvalues $\lambda_1 = 1 - \alpha k_1 l$ and $\lambda_2 = 1 - \frac{\beta}{2}k_1 + \beta k_2 l$. Once again the equilibrium point E_2 is unstable for the choice of entangling operators $U(0 \leqslant c_1 \leqslant \frac{\pi}{4},\ c_2 = c_1,\ c_3 = 0)$.

The local stability of the Nash equilibrium point E_3 is analyzed by checking the following Jury conditions [36,37]:

$$\begin{aligned} 1 + \mathrm{Tr}(J) + \mathrm{Det}(J) &= \frac{l^2(2\alpha k_1 l - \alpha k_2 - 4)(2\beta k_2 l - \beta k_1 - 4) - 4}{4l^2 - 1} > 0, \\ 1 - \mathrm{Tr}(J) + \mathrm{Det}(J) &= \frac{l^2\alpha\beta(2k_1 l - k_2)(2k_2 l - k_1)}{4l^2 - 1} > 0, \\ 1 - \mathrm{Det}(J) &= \frac{l^2[2\alpha(k_1 l - k_2) + \beta(2k_2 l - k_1)(2 - \alpha(2k_1 l - k_2))]}{4l^2 - 1} > 0. \end{aligned} \tag{8}$$

Here, J is the Jacobian matrix given in Eq. (7). The Nash equilibrium point is stable only for the choice of entangling operators $U(0 \leqslant c_1 \leqslant \frac{\pi}{4},\ c_2 = c_1,\ c_3 = 0)$ irrespective of the choice of control parameters α and β.

B. Dynamic game with heterogeneous players

In this section, we focus on the chaotic dynamics of heterogeneous players. We continue to consider firm 1 as the bounded rational player, whereas we now consider firm 2 as a naive player. A naive player is one who believes that the production of the opponent firm in period $t + 1$ is the same as that in the last period, t, in a dynamic game setting. The two-dimensional discrete dimensional map for these players is defined as [24]

$$\begin{aligned} x_1(t+1) &= x_1(t) + \alpha\frac{\partial \Pi_1}{\partial x_1}, \\ x_2(t+1) &= \frac{(a - C_2)l - x_1(t)}{2l}. \end{aligned} \tag{9}$$

Again, α is the speed of adjustment of firm 1. Setting $x_i(t+1) = x_i(t) = x_i$, the following equilibrium points are obtained: $E_0 = (0, \frac{k_2}{2})$ and $E_1 = (\frac{l(2k_1l-k_2)}{4l^2-1}, \frac{l(2k_2l-k_1)}{4l^2-1})$. Here, E_0 is a monopoly equilibrium point, and E_1 is the Nash equilibrium point. The Jacobian matrix of the two-dimensional logistic map given in Eq. (9) is

$$J(x_1, x_2) = \begin{bmatrix} 1+\alpha[l(k_1 - 4x_1) - x_2] & -\alpha x_1 \\ -\frac{1}{2l} & 0 \end{bmatrix}. \tag{10}$$

Substituting E_0 into the above matrix, the following eigenvalues are obtained: $\lambda_1 = 0$ and $\lambda_2 = 1 - \alpha\frac{k_2}{2} + l\alpha k_1$. For these eigenvalues, the equilibrium point is stable but not asymptotically stable for the choice of entangling operator $U(\frac{\pi}{4} < c_1 \leqslant \frac{\pi}{2},\ c_2 = c_1,\ c_3 = 0)$. However, for any other choice of entangling operators, E_0 is unstable.

Furthermore, the Nash equilibrium point E_1 is stable if and only if the following Jury conditions are satisfied:

$$\begin{aligned} 1 + \mathrm{Tr}(J) + \mathrm{Det}(J) &= \frac{16l^2 - 4 + \alpha(k_2 - 2lk_1)(1 + 4l^2)}{8l^2 - 2} > 0, \\ 1 - \mathrm{Tr}(J) + \mathrm{Det}(J) &= \alpha\left(k_1 l - \frac{k_2}{2}\right) > 0, \\ 1 - \mathrm{Det}(J) &= \frac{8l^2 + 2\alpha l k_1 - \alpha k_2 - 2}{8l^2 - 2} > 0. \end{aligned} \tag{11}$$

Here again, J is the Jacobian matrix given in Eq. (10) at the Nash equilibrium point. Unlike homogeneous players, the stability of the Nash equilibrium is determined by the choice of entangling operator and the speed with which firm 1 adjusts. Furthermore, the Nash equilibrium is stable only for the choice of entangling operator $U(\frac{\pi}{4} < c_1 \leqslant \frac{\pi}{2},\ c_2 = c_1,$

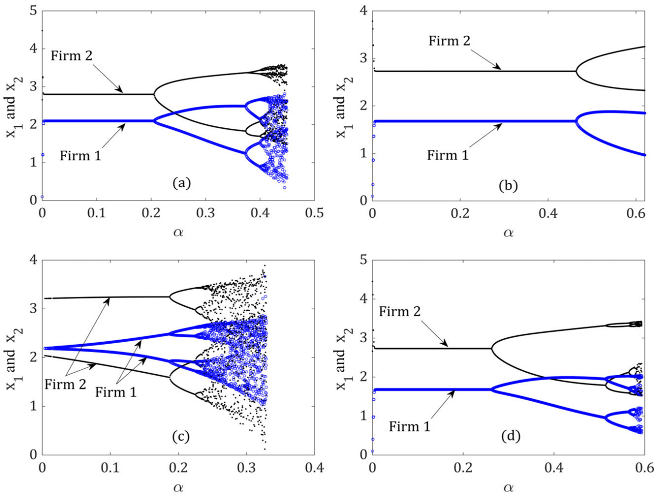

FIG. 1. Bifurcation diagrams for $\beta = 0.3$ (a) without quantum entanglement and (b) with quantum entanglement $U(\pi/6, \pi/6, 0)$ and for $\beta = 0.4$ (c) without quantum entanglement and (d) with quantum entanglement for homogeneous players.

$c_3 = 0$) and for the choice of the control parameter $0 < \alpha < 0.4$. It is to be noted that the identified entangling operator U belongs to the class of perfect entanglers [35].

IV. NUMERICAL ANALYSIS

In the context of nonlinear dynamical systems, it is not feasible to obtain exact analytical solutions. Hence we have shown the chaotic nature of the system by using bifurcation diagrams, Lyapunov dimensions, and the sensitive dependence on the initial condition. Throughout the analysis, we set the cost function and marginal cost as follows: $a = 10$, $C_1 = 3$, and $C_2 = 2.3$. In addition, we assume initial conditions of $x_1 = 0.1$ and $x_2 = 0.1$.

A. Bifurcation and chaos

1. *For homogeneous players*

The bifurcation diagrams for different adjustment speeds of firm 2 are shown in Figs. 1(a)–1(d). Figure 1(a) indicates that the Nash equilibrium point is stable for $\alpha < 0.25$ in the absence of entanglement when the speed of adjustment of firm 2 is set at $\beta = 0.3$. As the adjustment speed of firm 1 increases ($\alpha > 0.25$), bifurcation occurs with a period-doubling route to chaos. However, for the choice of entangling operator $U(c_1 = \pi/6,\ c_2 = c_1,\ c_3 = 0)$ as in Fig. 1(b), the system is stable up to $\alpha < 0.5$. Furthermore, for $\alpha > 0.5$ the system just bifurcates with two periods.

Figures 1(c) and 1(d) show the bifurcation diagrams when the speed of adjustment of firm 2 is set at $\beta = 0.4$. Figure 1(c) indicates that the Nash equilibrium is not stable even for low values of α. The dynamics of the game starts with two periods, and the system becomes chaotic for $\alpha > 0.2$ through period doubling. In the presence of quantum entanglement as in Fig. 1(d), Nash equilibrium is stable up to $\alpha < 0.3$ for the same β, beyond which bifurcation occurs with a period-doubling route to chaos. Overall, the system stability for bounded rational players is controlled by two intrinsic parameters, β and α, and a two-qubit nonlocal entangling operator U. An appropriate choice of these three parameters can delay the onset of chaos.

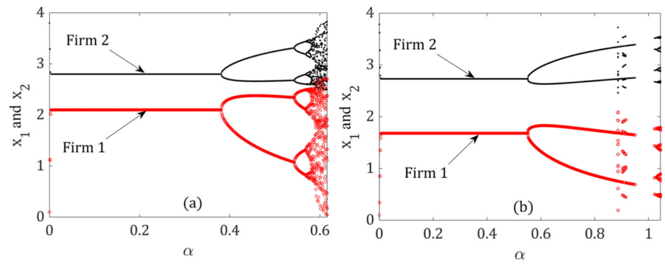

FIG. 2. (a) Without quantum entanglement and (b) with quantum entanglement $U(\pi/6, \pi/6, 0)$ for heterogeneous players.

2. *For heterogeneous players*

The bifurcation diagrams without and with quantum entanglement for bounded rational and naive players are shown in Figs. 2(a) and 2(b). In the absence of entanglement, the Nash equilibrium of the system is stable for $\alpha < 0.4$. Beyond that, the system experiences period doubling, and for $\alpha > 0.5$ the system approaches chaos. Furthermore, Fig. 2(b) shows that the Nash equilibrium is stable up to $\alpha < 0.6$ in the presence of entangling operator $U(\pi/6,\ c_2 = c_1,\ c_3 = 0)$, beyond which period doubling leads to chaos for $\alpha > 0.8$. Therefore we can state that an appropriate choice of α and entanglement improves the stability of an otherwise chaotic system in a heterogeneous game.

B. Strange attractors

Chaotic strange attractors have a local topological structure and are characterized by fractal dimension. In 1979, Kaplan and Yorke proposed a relationship between the fractal dimension and the Lyapunov spectrum called the Lyapunov dimension [38]. The Lyapunov dimension of a two-dimensional logistic map is defined as

$$D_{KL} = 1 + \frac{\lambda_1}{|\lambda_2|}; \quad \lambda_1 > 0 \quad \text{and} \quad \lambda_2 < 0. \tag{12}$$

In this context, we calculate the Lyapunov dimension for a two-dimensional logistic map of homogeneous and heterogeneous players. For the parametric values ($\alpha = 0.32,\ \beta = 0.4,\ a = 10,\ C_1 = 3,\ C_2 = 2.3$), two Lyapunov exponents, $\lambda_1 = 0.2160$ and $\lambda_2 = -0.2009$, exist when there is no entanglement. The Lyapunov dimension for the homogeneous players with no entanglement assumes $D_{KL} = 2.0752$. Furthermore, in the presence of quantum entanglement $U(\pi/6, \pi/6, 0)$, two Lyapunov exponents, $\lambda_1 = 0.1485$ and $\lambda_2 = -0.6371$, exist for the parametric values ($\alpha = 0.35,\ \beta = 0.4,\ a = 10,\ C_1 = 3,\ C_2 = 2.3$). As a result, the Lyapunov dimension is $D_{KL} = 1.2331$ which is less than that of its classical analog. This difference in the Lyapunov dimension indicates that quantum entanglement reduces the chaotic nature of the system. The strange attractors for these parametric values for homogeneous players are shown in Fig. 3.

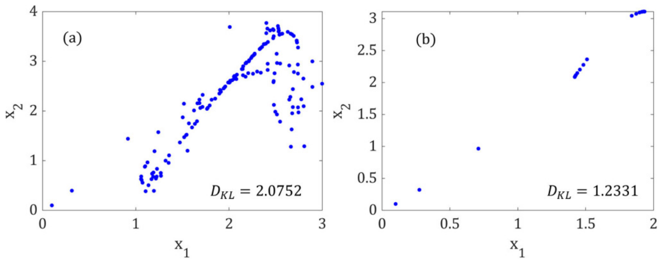

FIG. 3. Strange attractors for bounded rational players (a) without quantum entanglement and (b) with quantum entanglement $U(\pi/6, \pi/6, 0)$.

In the case of heterogeneous players, with the parametric values ($\alpha = 0.6,\ a = 10,\ C_1 = 3,\ C_2 = 2.3$), two Lyapunov exponents, $\lambda_1 = 0.1862$ and $\lambda_2 = -0.5980$, exist when there is no entanglement. The Lyapunov dimension for these fixed points is $D_{KL} = 1.3113$. Furthermore, for the choice of entangling operator $U(\pi/6, \pi/6, 0)$, for the parametric values ($\alpha = 0.6,\ a = 10,\ C_1 = 3,\ C_2 = 2.3$), two Lyapunov exponents, $\lambda_1 = 0.1945$ and $\lambda_2 = -0.5889$, exist. The Lyapunov dimension with quantum entanglement is $D_{KL} = 1.0554$. Once again, a smaller value of Lyapunov dimension with quantum entanglement indicates that the system is less chaotic. The strange attractors for heterogeneous players are shown in Fig. 4. Both Figs. 3 and 4 indicate the separation of the two strange attractors triggered by the presence of quantum entanglement.

C. Sensitive dependence on initial conditions

In this section, sensitivity to initial conditions is identified for homogeneous and heterogeneous players. The difference between the two trajectories $(x_1(0), x_2(0))$ and $(x_1(0) = 0.001,\ x_2(0))$ is observed for the initial conditions $x_1(0) = 0.1$ and $x_2(0) = 0.1$. From Fig. 5(a) for the conditions ($\alpha = 0.6,\ a = 10,\ C_1 = 3,\ C_2 = 2.3$), it is observed that in the absence of entanglement, the two trajectories are indistinguishable in the beginning. After $t > 20$ iterations, the difference between the two trajectories builds up rapidly. Such an observation is referred to as the *butterfly effect*, wherein a small change in initial conditions can have far-reaching consequences over time [39]. Note that in the presence of an entangling operator $U(c_1 = \pi/6,\ c_2 = c_1,\ c_3 = 0)$, the difference between the two trajectories becomes negligible and the system becomes periodic for the iterations $t > 70$.

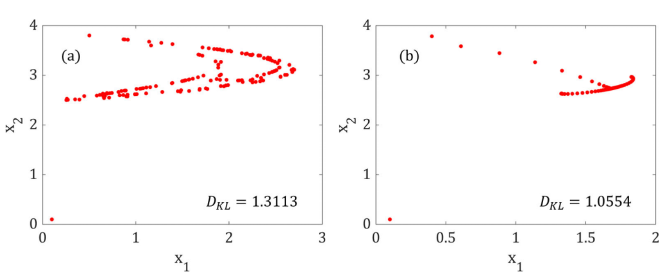

FIG. 4. Strange attractors for heterogeneous players (a) without quantum entanglement and (b) with quantum entanglement $U(\pi/6, \pi/6, 0)$.

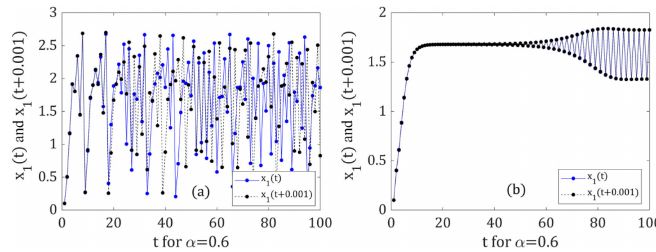

FIG. 5. Sensitive dependence on initial conditions for heterogeneous players (a) without quantum entanglement and (b) with quantum entanglement $U(\pi/6, \pi/6, 0)$.

Figure 6 shows the sensitive dependence for homogeneous players for the conditions ($\alpha = 0.32,\ \beta = 0.4,\ a = 10,\ C_1 = 3,\ C_2 = 2.3$). Again in the absence of quantum entanglement, there exists a difference between the two trajectories after $t > 20$ iterations. Furthermore, for the choice of entangling operator $U(c_1 = \pi/6,\ c_2 = c_1,\ c_3 = 0)$, there is no difference in the trajectories of the system, and it is periodic for $t > 5$ iterations. Such an observation holds good for firm 2 as well. The sensitive dependence plots for the two types of players illustrate that an appropriate choice of entangling operator can make the two trajectories identical. Thus, in the presence of quantum entanglement, the system loses its sensitivity to the initial conditions and after some iterations becomes periodic.

V. DISCUSSION AND CONCLUSION

Nonlinearity in duopoly games has received a lot of attention in the context of classical economic market situations. The quantum version of this game is less explored with little or no significance of quantum entanglement. In this paper, we address the existing research gap pertaining to quantum nonlinear discrete dynamical systems. We highlight the significance of the two-qubit nonlocal entangling operators available due to a modified EWL scheme in controlling chaos. The following noteworthy insights are found by quantizing the linear Cournot duopoly game with dynamic players.

First of all, local stability analysis of the equilibrium points shows that there exists monopoly in a duopoly game. This can be prevented by the arbiter (or government) by employing a double controlled-NOT (DCNOT) operation. Double CNOT $U(\pi/4, \pi/4, 0)$ is a special perfect entangler which can maximally entangle a full product basis [35]. Also there

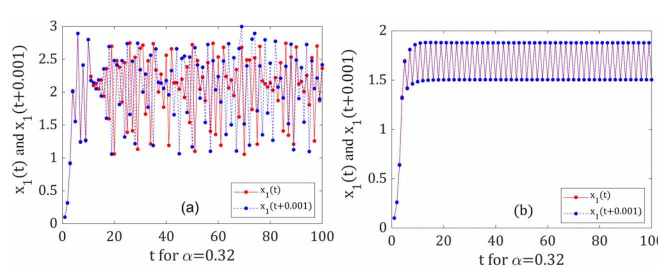

FIG. 6. Sensitive dependence on initial conditions for homogeneous players (a) without quantum entanglement and (b) with quantum entanglement $U(\pi/6, \pi/6, 0)$.

exists a choice of entangling operator U and intrinsic control parameters α and β that adds stability to an unstable equilibrium point.

Secondly, the bifurcation diagrams for particular initial conditions ($x_1 = 0.1,\ x_2 = 0.1$) highlight how quantum entanglement can bring order in a discrete dynamical system. This is proved by the observation that no bifurcation occurs when the entangling operator is a DCNOT. This is true for all initial conditions with stable Nash equilibrium throughout. For all choices of entangling operators $U(0 < c_1 < \pi/4,\ c_2 = c_1,\ c_3 = 0)$, the stability of the Nash equilibrium can be tuned by delaying bifurcation and thereby preventing chaos in the system. It is also to be noted that the same choice of entangling operator that can prevent chaos can also prevent monopoly by equalizing the profits of the firms. Furthermore, the strange attractors emphasize how quantum entanglement changes the fractal dimensions. The small positive and negative Lyapunov exponent values indicate a weakly chaotic system. Furthermore, the sensitive dependence on initial conditions indicates that the two trajectories with small perturbations of the order of 0.001 are distinguishable in the absence of entanglement. Whereas, for the choice of entangling operator $U(\pi/6, \pi/6, 0)$, the trajectories are indistinguishable, and the system eventually becomes periodic. In other words, entangling operators can make the system less sensitive and control chaos.

To conclude, quantum entanglement in a discrete dynamical system with other control parameters α and β (speed of adjustments of the firms) can delay bifurcation and hence benefit the players in a real market to attain stability. We add significance to the two-qubit entangling operators in a discrete dynamical quantum Cournot duopoly game. Notably, quantum entanglement as chaos control requires further investigation with nonlinear inverse demand and cost functions in different game settings. It is known from earlier works [31,40] that entangling operators can increase or decrease the payoff of the players under certain game settings. In this paper, we highlight that entangling operators can prevent or delay the chaotic nature of the dynamical systems. It remains interesting to see the crossroads of quantum games and the chaotic nature of the system at the junction of entangling operators. To be precise, it is worth investigating the class of entangling operators, which can increase the payoff and at the same time control the chaotic features of the dynamical system.

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