Superstatistics from a dynamical perspective: Entropy and relaxation

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Distributions that deviate from equilibrium predictions are commonly observed across a broad spectrum of systems, ranging from laboratory experiments to astronomical phenomena. These distributions are generally regarded as a manifestation of a quasiequilibrium state and can very often be represented as a superposition of statistics, i.e., superstatistics. The underlying idea in this methodology is that the nonequilibrium system consists of a collection of smaller subsystems that remain infinitely close to equilibrium. This procedure has been effectively implemented in a kinetic setting, but thus far, only in the collisionless regime, limiting its scope of application. In this paper, we address the effect of collisions on the relaxation process and time evolution of superstatistical systems. After confronting the superstatistical distributions with experimental and simulation data, relevant to our analysis, we first study the effect of superstatistics on entropy. We explicitly show that, in the absence of long-range interactions, the extensivity of entropy is preserved, albeit influenced by the specific class of temperature fluctuations. Then, we examine how collisions drive the system towards a global equilibrium state, characterized by a maximum entropy, by employing the relaxation time approximation. This allows us to define a dynamical version of superstatistics, characterized by a singular time-varying parameter $q(t)$, which undergoes a continuous evolution towards equilibrium. We show how this approach enables the determination of the evolution of the underlying temperature distribution under the influence of collisions, which act as stochastic forces, gradually narrowing the temperature distribution over time.

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I. INTRODUCTION

Distributions deviating from those predicted by equilibrium statistical mechanics are commonly observed across a diverse spectrum of systems, encompassing nearly all scales, from laboratory-scale experiments to galaxy clusters. In experimental settings, one may mention experiments on cold atoms in optical lattices [1,2], trapped ions [3], driven dissipative dusty plasmas [4], spin glasses [5], particles coupled to an active bath [6], graphene membranes [7], cell monolayer systems [8], as well as high-energy collisional experiments [9,10]. Such nonequilibrium distributions are also commonplace in space environments, having been acknowledged since the late 1960s in space plasmas [11–15] and, subsequently, in gravity-dominated systems like stellar clusters and galaxy clusters [16–20]. Within the gravitational context (or more broadly in scenarios involving long-range interactions), the presence of such nonequilibrium distributions finds a simple explanation: the required (collisional) relaxation time is estimated to diverge approximately linearly with the number of particles [21,22]. Hence, sufficiently massive celestial objects (e.g., elliptical galaxies) would require a relaxation time greatly exceeding the age of the universe to reach equilibrium. Consequently, they remain trapped in quasiequilibrium states.

Determining the steady state of a nonequilibrium system is a very complex issue as it requires—at least in principle—the knowledge of the complete historical record of perturbations that the system has undergone. Several approaches have been proposed in the literature to address this problem. One extensively explored avenue is that of nonextensive statistical mechanics (NSM) [23], which originates from a generalized form of entropy (i.e., the Tsallis q-entropy), to derive generalized distributions. The distributions arising within this paradigm have been widely used in the literature due to their flexibility in modeling various observations, despite the ongoing debate over some conceptual issues within NSM [24–32] (see also Ref. [33] for a holistic perspective on the debate). Another approach, perhaps more ambitious, is that of superstatistics [34], which proposes modeling fluctuations in the local temperature of the nonequilibrium system, through a temperature distribution. This approach is somehow more deeply routed in the long tradition of statistical mechanics, and its essence can be traced back to the work of Kubo [35], Lavenda [36], and others [37–39] (see also Refs. [40,41] for related tools in the more general context of stochastic processes).

The latter approach has proved to be remarkably fruitful and has evolved into a standard paradigm for addressing nonequilibrium systems. The concept of superstatistics has found applications in physical scenarios as disparate as turbulence [42,43,51], plasmas [44–46], ultra-cold gases [47,48], self-gravitating systems [49,50], high-energy physics [52,53], and spin systems [54,55], among many others [56–58]. It has also been applied to other domains, extending its utility to areas beyond the scope of physics, such as traffic [59], power grid fluctuations [60], DNA architecture [61,62], rainfall statistics [63], air pollution [64], cognitive processes [65], etc.

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One interesting aspect of superstatistics lies in its ability to generate Tsallis distributions, known in NSM as a special instance of a broader family of distributions. In particular, three classes of (inverse) temperature distributions $f(\beta)$, namely the χ^2, the inverse-χ^2, and the log-normal classes, have shown compelling empirical support, across various physical contexts (see Sec. II for a detailed discussion). These three universality classes offer sufficient flexibility, extending beyond the domain of application of NSM. Note that the nonextensive nature of entropy is not explicitly invoked in this context, thus avoiding potential inconsistencies that remain subjects of ongoing debate [24–32].

When examining continuous media, like plasmas and gravitational systems, the concept of superstatistics can be easily implemented into kinetic equations (i.e., the Boltzmann equation in the classical case, or the Wigner-Moyal equation in the quantum case). This approach has demonstrated certain degrees of success [46,48,49]. However, to date, investigations have primarily focused on the collisionless regime, i.e., the Vlasov regime. When collisions come into play, the situation is different as collisional events (even rare) slowly drive the system irreversibly toward an equilibrium state. Hence, a natural question arises: Given a quasiequilibrium system, initially characterized by a superstatistical distribution, how does it evolve under the effect of collisions? This paper attempts to address this question.

More precisely, we study a kinetic model in which collisions are accounted for through the so-called relaxation time approximation (also known as the BGK approximation) [66,67]. This method has gained recognition as a simple and efficient way for addressing collisions across a wide spectrum of scenarios. Its applications span various problems, ranging from plasma physics [68,69] to self-gravitating systems [70] and high-energy physics [71–73], among many others [74–76]. Within this framework, we construct a dynamical class of superstatistical distributions, with a varying parameter $q(t)$, and show how this can be used to infer the evolution of the temperature distribution $f(\beta)$ over time, under the effect of collisions. This procedure provides insights into the underlying temperature dynamics and sheds light on the "history" of the observed nonequilibrium distributions.

The rest of this paper is organized as follows. In Sec. II, we provide a comprehensive overview of the superstatistics concept, with a particular emphasis on the three universality classes that have strong empirical support. We discuss the distributions that emerge within this framework and confront them with experimental and simulation data of granular gases and gas-solid flow [77–79]—a domain where, to our knowledge, their significance has not yet been recognized. In Sec. III, we examine their associated entropy and show that, while preserving the property of extensivity, temperature fluctuations enter the picture in a subtle way. We confirm that maximum entropy is attained in an equilibrium state, i.e., in the absence of temperature fluctuations. In Sec. IV, we study how collisions drive the superstatistical system to this equilibrium state, in the relaxation time approximation, and construct a dynamical version of superstatistics with a varying parameter $q(t)$. Finally, in Sec. V, we draw concluding remarks and outline future directions.

II. SUPERSTATISTICAL VELOCITY DISTRIBUTIONS

To set the stage, let us first outline the general situation we will be dealing with here. We are considering systems that are not in strict thermodynamic equilibrium, but exhibit only *local* equilibrium. In such a situation, the given system consists of small subsystems (or cells) in local equilibrium. These subsystems represent regions from which thermalization later spreads throughout the entire system. As the relaxation time τ_0 of a cell is much shorter than the relaxation time τ of the entire system (at least when long-range interactions are not involved), there exists a time scale t such that

$$\tau_0 \ll t \ll \tau, \tag{1}$$

on which the small cells are infinitely close to equilibrium, whereas the entire system has not yet reached global equilibrium. In the presence of collisions, the subsystems weakly exchange with each other, causing a slow drift of the entire system towards equilibrium. In these conditions the system is said to be in a quasiequilibrium state [49,81,82]. This scenario encompasses many relevant physical situations. For instance, one may think of space plasma environments (e.g., the solar wind) where binary collisions between particles are rare, or gravitational systems, which are characterized by a relaxation time τ exceeding the age of the universe.

The concept of superstatistics presents itself as a powerful technique of systematically handling such situations. The concept is centered around this simple idea: At a small scale, the system may relax toward thermodynamic equilibrium, and its local statistical properties are given by equilibrium Boltzmann statistics, with a well-defined local inverse temperature $\beta \equiv 1/T$ (throughout the paper we set the Boltzmann constant k_B to unity). The local probability of finding the system at some energy ϵ reads as $p(\epsilon) \propto \exp(-\beta\epsilon)$. At the level of the entire system, however, the (inverse) temperature is not constant but fluctuates from cell to cell. If the fluctuation operates over a large spatiotemporal scale, then one may assign a distribution $f(\beta)$ to the inverse temperature, and the statistical properties of the entire system read as[1]

$$B(\epsilon) = \int_0^\infty d\beta f(\beta) \frac{\exp(-\beta\epsilon)}{Z(\beta)}, \tag{2}$$

where $Z(\beta)$ is the (local) partition function. Note that $B(\epsilon)$ is simply the Laplace transform of the (rescaled) temperature distribution

$$\tilde{f}(\beta) \equiv \frac{f(\beta)}{Z(\beta)}. \tag{3}$$

[1]One may also think about this in terms of the *adiabatic ansatz* [80]: During its evolution, the system travels within its state space X which is divided up into small cells, each characterized by a constant value of some parameter β. Within each of these cells, the system is described by the conditional probability $p(A|\beta)$ to be found in a specific state $A \in X$. As β varies adiabatically across these cells, the joint distribution of finding the system in the state A with a given value of β reads as $p(A, \beta) = p(A|\beta)p(\beta)$ (i.e., the *De Finetti-Kolmogorov relation*). The probability $p(A)$ for finding the system in the state A is obtained by summing over all possible values of β, resulting in Eq. (2).

That is, $B(\epsilon) = \mathcal{L}\{\tilde{f}\}(\epsilon)$, and conversely, $\tilde{f}(\beta) = \mathcal{L}^{-1}\{B\}(\beta)$. This means that the distribution $B(\epsilon)$ is fully determined by the temperature distribution and *vice versa*.

An essential ingredient in Eq. (2) is the temperature distribution $f(\beta)$. In principle, the latter may be any normalized distribution of a positive variable. However, aside from very simple models (e.g., a two-level distribution), three main classes of $f(\beta)$ emerge as universal limit statistics in the majority of known systems. These classes are strongly supported by substantial empirical evidence, and their emergence can be explained through probabilistic arguments, relying on the central limit theorem (CLT)[2] [85] or the maximum entropy principle [86]. While our focus will naturally be on these three classes, our findings can readily extend to other distributions $f(\beta)$ that could be employed to model some specific situations.

As we will be mainly interested in velocity distributions, we shall assume, locally, a Maxwell-Boltzmann (MB) distribution

$$f(v) = \left(\frac{\beta m}{2\pi}\right)^{d/2} \exp\left[-\frac{\beta m v^2}{2}\right], \tag{4}$$

for a d-dimensional system, and discuss the corresponding superstatistical probability distribution functions (PDFs). The three universality classes of superstatistics are the following:

(1) χ^2 *superstatistics.* In this case, β is assumed to follow a χ^2 distribution of degree n:

$$f_1(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{n}{2\beta_0}\right)^{n/2} \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}, \tag{5}$$

where $\beta_0 \equiv \langle\beta\rangle$ is the average of β. The corresponding superstatistical velocity distribution follows from Eq. (2) as[3]

$$B(v) = \left(\frac{\beta_0 m}{\pi n}\right)^{d/2} \frac{\Gamma\left(\frac{n+d}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{\beta_0}{n} m v^2\right)^{-\frac{n+d}{2}}. \tag{6}$$

Interestingly, Eq. (6) is formally identical to the so-called q-Gaussian distribution, known in NSM [23]. This connection can be made more transparent upon defining an entropic index $\widetilde{q}$ and an effective inverse temperature $\widetilde{\beta}$ as follows:

$$\widetilde{q} \equiv 1 + \frac{2}{n+d} \quad \text{and} \quad \widetilde{\beta} \equiv \frac{(n+d)\beta_0}{n}. \tag{7}$$

Equation (6) can then be rewritten in the more standard form, used in NSM, namely

$$B(v) \propto \left(1 + (\widetilde{q} - 1)\frac{\widetilde{\beta} m v^2}{2}\right)^{\frac{1}{1-\widetilde{q}}}. \tag{8}$$

Note that for large values of $|v|$, Eq. (6) behaves as a power law, i.e., $B(v) \sim |v|^{-(n+d)}$. This property makes it extremely useful for modeling various space and astrophysical observations, which frequently exhibit suprathermal tails that decrease following a power law. In fact, such quasi-power-law distributions are commonly observed in various physical contexts, such as space [11,12,15] and laboratory [4] plasmas, and stellar systems [16–19]. They are also prevalent in diverse experimental realizations, such as cold atoms in optical traps [1,2], and high energy collisions [9,10].

(2) *Inverse-*χ^2 *superstatistics.* In this scenario, it is not the inverse temperature β that follows a χ^2 distribution but rather the temperature β^{-1}. In turn, β follows an inverse-χ^2 distribution:

$$f_2(\beta) = \frac{\beta_0}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{n\beta_0}{2}\right)^{n/2} \beta^{-n/2-2} e^{-\frac{n\beta_0}{2\beta}}. \tag{9}$$

The corresponding velocity distribution follows in this case from Eq. (2) as

$$B(v) = \frac{2\beta_0}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{2\pi}\right)^{d/2}\left(\frac{\beta_0 n}{2}\right)^{n/2}\left(\frac{m v^2}{\beta_0 n}\right)^{\frac{2-d+n}{4}} \times \mathcal{K}_{\frac{2-d+n}{2}}\left(\sqrt{nm\beta_0}|v|\right), \tag{10}$$

$\mathcal{K}_\alpha(x)$ being the modified Bessel function of the second kind. Asymptotically, Eq. (10) exhibits exponential tails in the velocity. This type of exponential behavior has been observed in several nonequilibrium problems, such as vortex glasses/liquids [87], fusion plasmas [39], and in the case of harmonic oscillators coupled to solvent baths [88]. Similar trends have also been documented in other problems, outside of the scope of physics, such as cancer disease-specific mortality distributions [89].

(3) *Log-normal superstatistics.* In this case, β follows a lognormal distribution,

$$f_3(\beta) = \frac{1}{\sqrt{2\pi} s\beta} \exp\left\{\frac{-(\ln\beta - \mu)^2}{2s^2}\right\}, \tag{11}$$

with an average of β given by $\beta_0 = \mu e^{s^2/2}$. In this last situation, a closed-form expression for the corresponding velocity distribution is currently unavailable, but it can be readily computed numerically. This class of superstatistics is also strongly supported by experimental findings. Empirical evidence of log-normal superstatistics has been found for instance in Lagrangian and Eulerian turbulence [43,51,85], in space plasmas [46], in stellar systems [50], and in other contexts [60].

Figure 1 displays examples of the superstatistical velocity PDFs, produced by the three universality classes of superstatistics. For future convenience, and to enable easy comparison among them, the PDFs have been (re)parameterized using a single parameter defined as $q := \langle\beta^2\rangle/\langle\beta\rangle^2$, which can be

[2]We note in passing the compelling formal results [83,84] that explore generalizations of the CLT, using the language of NSM. The rationale here is that, while the standard CLT applies to independent random variables, one may relax this condition by allowing correlations between them. From one perspective, such generalized forms of the CLT may offer insights into the emergence of superstatistics in the presence of correlations among the microscopic random variables contributing to β. From another perspective, exploring potential extensions of the generalized CLT to other classes of superstatistics might yield valuable formal results.

[3]In the statistics literature, distributions in the form of Eq. (6) are referred to as Students t-distributions. They represent a special case of the Burr-type III distribution.

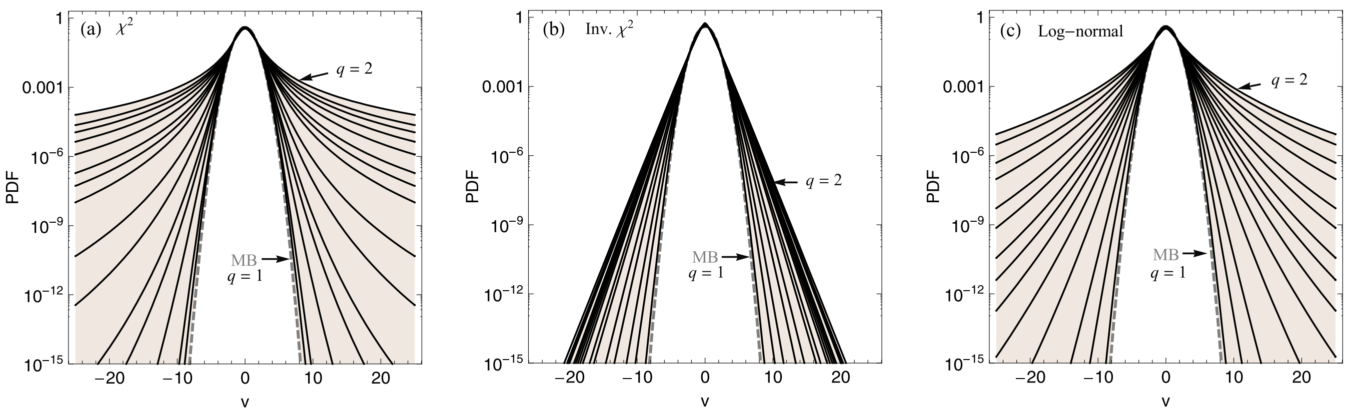

FIG. 1. Velocity probability distributions $B(v)$ corresponding to χ^2 (a), inverse χ^2 (b), and log-normal (c) superstatistics, for different values of $q := \langle\beta^2\rangle/\beta_0^2$. The velocity has been normalized to the thermal velocity, i.e., $\mathrm{v} \equiv \beta_0 v/2m$.

expressed, for the three classes of $f(\beta)$, as

$$q := \frac{\langle\beta^2\rangle_{f_1}}{\beta_0^2} = 1 + \frac{2}{n} \quad (n > 2),$$
$$q := \frac{\langle\beta^2\rangle_{f_2}}{\beta_0^2} = \frac{n}{n-2}, \tag{12}$$
$$q := \frac{\langle\beta^2\rangle_{f_3}}{\beta_0^2} = e^{s^2}.$$

The latter can be regarded as an extension of the Tsallis index to the other classes of superstatistics. Note, however, that beyond the χ^2 class, there is no direct correspondence between $q := \langle\beta^2\rangle/\beta_0^2$ [Eq. (12)] and the entropic index used in NSM [23]. In our notation, this parameter quantifies the departure of a system from (global) equilibrium. In fact, regardless of the class of superstatistics, one has $q \geqslant 1$, with $q = 1$ corresponding to equilibrium (i.e., a vanishing variance of $f(\beta)$, implying a constant temperature across the entire system).

One may see from Fig. 1 that the produced PDFs exhibit the typical profile found in most physically relevant scenarios. To demonstrate more clearly the empirical validity of superstatistics, it is worthwhile to compare them with recent experimental and simulation results. We present two examples where the relevance of superstatistics has not been recognized before, specifically in the context of gas-solid flows and granular gases velocity distributions. We have employed a nonlinear regression method to fit the three universality classes of superstatistics with nonequilibrium velocity distributions computed from the direct numerical simulations of a gas-solid flow conducted by Liu *et al.* [77,78], and the experimental velocity distributions of (electrostatically driven) granular gases due to Kohlstedt *et al.* [79]. The results are reported in Fig. 2. While a closer analysis and more data might be necessary for a thorough differentiation between the three classes in this specific situation, one may clearly see that the observed deviations from the MB distribution are effectively captured by the three universality classes of superstatistics.

III. SUPERSTATISTICS AND ENTROPY

Before examining the impact of collisions on the evolution of a superstatistical system towards equilibrium, it is essential to discuss the concept of entropy within the superstatistics framework. This discussion is crucial because the process involves a change in entropy, with its maximum value occurring in the final equilibrium state. This is particularly relevant as the extensivity of entropy has been widely questioned in the context of the theoretical foundation for nonequilibrium distributions [90–94], even in the absence of interactions. For simplicity, we consider a stationary, isolated medium, with vanishing spatial gradients, i.e., we assume spatial homogeneity. In such conditions the density n does not depend on the location. Let us denote $f(\mathbf{r}, \mathbf{v}; t)$ the phase space distribution function of N particles, and compute the entropy for a distribution corresponding to the superstatistical distributions derived in Sec. II. While $B(v)$ is normalized to unity, $f(v)$ is normalized to the number of particles N. That is,

$$\iint f(v) d^3r d^3v = n \iint B(v) d^3r d^3v = N. \tag{13}$$

We consider a general definition of entropy S, valid for equilibrium and nonequilibrium systems, namely

$$S = -\iint f[\ln(f) - 1] d^3r d^3v - N \ln\left(\frac{h^3}{m^3}\right), \tag{14}$$

with h being the Planck constant and m the mass of the particles composing the system. This formulation traces its origins to the works of Boltzmann [95] and Gibbs [96], and has been revisited more recently, in particular in the context of plasma physics [97–99] and for non-Gaussian distributions [100]. We note that Eq. (14) incorporates the quantum mechanical restriction on the minimum phase-space volume occupied by an individual particle and the Gibbs factor, avoiding therefore the so-called Gibbs paradox associated with particle indistinguishability. It is important to note that the formalism of superstatistics does not make explicit assumptions about the form of entropy, and alternative entropic forms beyond Eq. (14) warrant attention. In particular, generalized entropies are important elements for establishing a structural foundation for superstatistics, based on the maximum entropy principle

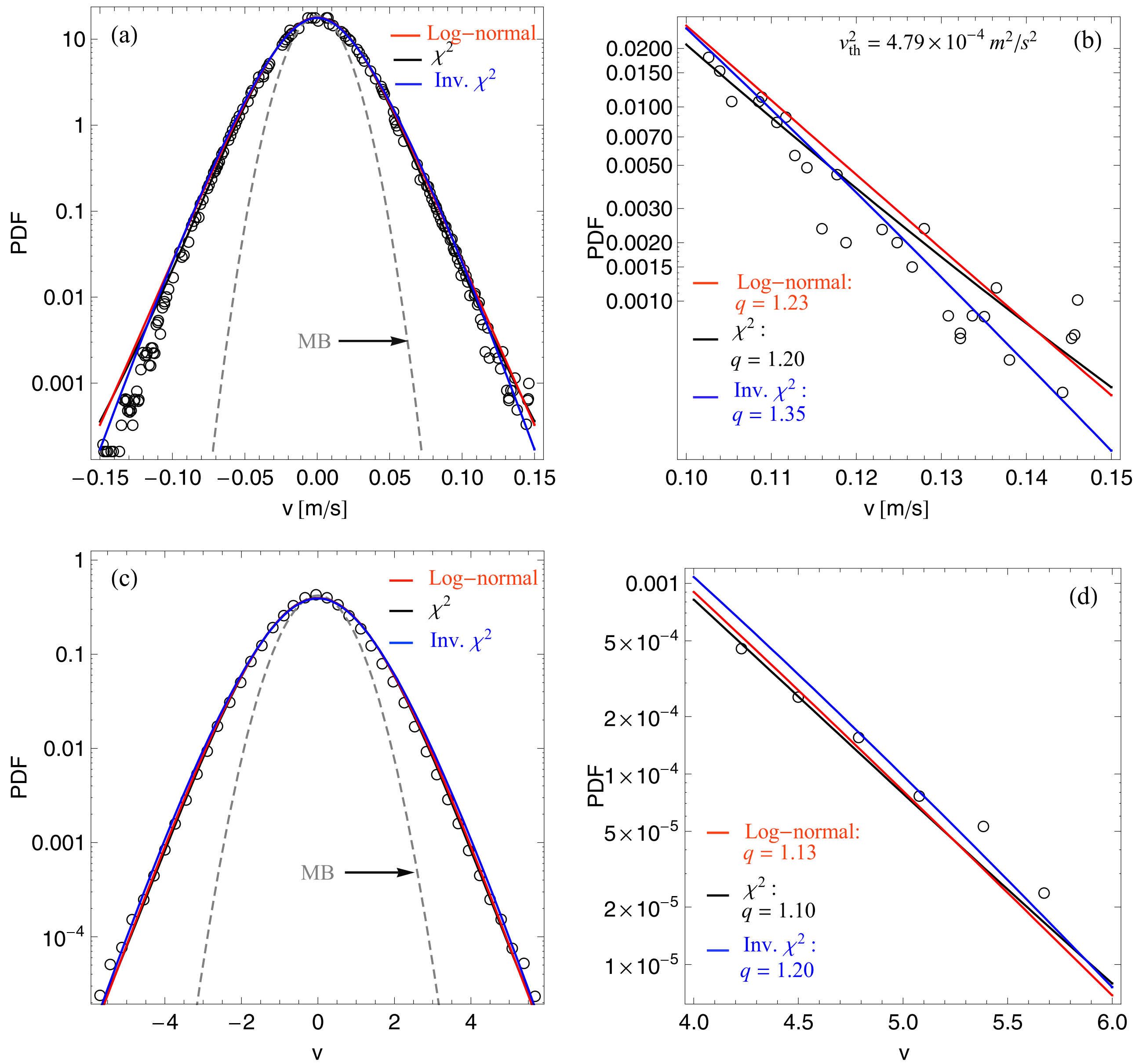

FIG. 2. Direct numerical simulation data of gas-solid flow nonequilibrium velocity distributions [77] (upper panel) and experimental velocity distributions of driven granular gases [79] (lower panel), fitted with the three universality classes of superstatistics. The left panel (a), (c) shows the whole domain while the right panel (b), (d) shows only the overpopulated high energy tail.

[101–104]. From an axiomatic point of view, these entropies satisfy the first three ShannonKhinchin axioms, but violate the fourth axiom associated with the composability of statistical systems. Such constructs are particularly useful in the context of non-Markovian or nonergodic complex systems. Nonetheless, given the scenario considered here and the pragmatic nature of our approach, the entropy (14) is appropriate as it remains valid for nonequilibrium situations.

From Eqs. (14) and (13), one may write

$$S = -\ln(n)\iint f(v)d^3r d^3v - \iint f(v)\ln[B(v)]d^3r d^3v + \iint f(v)d^3r d^3v - N\ln\left(\frac{h^3}{m^3}\right), \tag{15}$$

which can be recast as

$$S = N\left[1+\ln\left(\frac{m^3}{nh^3}\right)+\Sigma_q\right], \tag{16}$$

where we have defined the "scaled entropy" as follows:

$$\Sigma_q := -\int B(v)\ln[B(v)]d^3v. \tag{17}$$

The latter encodes all the information on the temperature distribution [note that the form of $B(v)$ is entirely determined by $f(\beta)$], and the degree of departure from equilibrium, through the parameter q. As q approaches its equilibrium value $q = 1$, $B(v)$ reduces to the MB distribution, and the scaled entropy

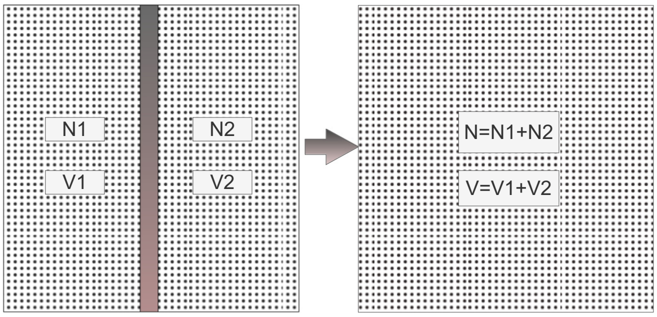

FIG. 3. A group of free particles confined within a box (left) eventually split into two separate subsystems (right).

reduces to

$$\lim_{q\to 1} \Sigma_q = \frac{1}{2}\left[3 + \ln\left(\frac{8\pi^3 T_0^3}{m^3}\right)\right]. \quad (18)$$

In this limit, the entropy (16) reduces to the standard Sackur-Tetrode equation, for an ideal (monoatomic, classical, and nonrelativistic) gas at equilibrium, namely

$$S = -N \ln\left(\frac{nh^3}{(2\pi m T_0)^{3/2}}\right) + \frac{5}{2}N = N\left[\ln\left(\frac{1}{n\lambda^3}\right) + \frac{5}{2}\right], \quad (19)$$

where $\lambda = h/\sqrt{2\pi m T_0}$ is the thermal de Broglie wavelength defined at the mean temperature $T_0 \equiv 1/\beta_0$.

An interesting feature of Eq. (16) is that nonequilibrium effects are contained in the quantity Σ_q. Otherwise, the entropy preserves the same properties as the equilibrium Sackur-Tetrode entropy. In particular, extensivity is preserved, regardless of the class of superstatistics, as long as $\Sigma_q \neq \infty$ [note that all quantities in the square bracket in Eq. (16) are independent of N].

This can be made more explicit by considering the case of two volumes V_1 and V_2, filled, respectively, with N_1 and N_2 particles, of the same species (see Fig. 3). The two subsystems are assumed to have the same density $n = N_1/V_1 = N_2/V_2$ and the same mean temperature T_0. These two subsystems are then mixing and, eventually, fill the total volume $V = V_1 + V_2$ with $N_1 + N_2$ particles. Note that when two systems described by the same class of superstatistics have equal mean temperature and equal density, it implies that the two distributions for each subsystem share the same value of q. The entropy of each subsystem follows from Eq. (16) as

$$S_1 = N_1\left[1 + \ln\left(\frac{m^3 V_1}{h^3 N_1}\right) + \Sigma_q\right],$$
$$S_2 = N_2\left[1 + \ln\left(\frac{m^3 V_2}{h^3 N_2}\right) + \Sigma_q\right]. \quad (20)$$

As the two subsystems are assumed to have the same density, one has

$$\frac{N_1}{V_1} = \frac{N_2}{V_2} = \frac{N_1 + N_2}{V_1 + V_2}, \quad (21)$$

then the total entropy, after the two subsystems have merged, reads as

$$S_{1\oplus 2} = (N_1 + N_2)\left\{1 + \ln\left[\frac{m^3(V_1 + V_2)}{h^3(N_1 + N_2)}\right] + \Sigma_q\right\} = S_1 + S_2, \quad (22)$$

which corresponds, in fact, to the sum of the two entropies S_1 and S_2.

A word of caution is appropriate at this point: Note that the preceding discussion has been limited to systems without interactions. Its primary objective is to demonstrate that nonequilibrium distributions, by themselves, do not imply entropy nonextensivity. However, when (long-range) interactions are involved, this may change the picture. This is where the recently introduced concept of "entropy defect" [105,106] may come into play. The latter measures the change in entropy (leading to nonextensivity) due to the order induced in the system through the correlations among its constituents—an effect bearing some analogies with the mass defect that takes place when nuclear particle systems are assembled.

Along similar lines, it is important to clarify the (sometimes overlooked) distinction between additivity and extensivity. Additivity asserts that the total entropy of a given system is identical to the sum of the entropies of its (probabilistically independent) components, whereas extensivity corresponds to the requirement that $S(N) \propto N$ for $N \to \infty$. Of the two properties—additivity and extensivity—extensivity is the essential one. The insistence on the (thermodynamic) entropy of a given system being extensive is crucial due to its connection to the Legendre transformations structure in thermodynamics (see, e.g., discussion in [107]). Although the extensivity of entropy can be anticipated in the present scenario, more complex classes of systems could require a nonadditive entropic form to ensure extensivity. Compelling arguments supporting this perspective can be found in Refs. [108–110].

Having confirmed the extensive nature of entropy in the superstatistics scenario, one may ask how do temperature fluctuations affect entropy. To elucidate this, we have numerically computed the scaled entropy (17) for the three universality classes of superstatistics (see Fig. 4). Here, two complementary interpretations can help understand how temperature fluctuations enter the picture:

In Fig. 4(a), the scaled entropy has been computed by imposing a fixed mean temperature T_0. One may see that the presence of fluctuations around T_0 results in an increase in entropy. As one compares between the three universality classes, one may observe that, for the same value of $q := \langle\beta^2\rangle/\beta_0^2$ (i.e., the same degree of fluctuations), the class of χ^2 superstatistics is the one that has the most significant effect on the entropy, followed by the log-normal class, and then the inverse-χ^2 class. This general tendency can nicely be explained from a *Bayesian* perspective, where entropy is regarded as a measure of ones uncertainty about the measurable properties of a system. In fact, if one has access to the mean temperature T_0, the presence of fluctuations around this value introduces an additional source of uncertainty.

This can be made more explicit by computing the so-called *thermal uncertainty*, which has a close connection with entropy [111]. To have an insight, let us consider a minimalistic version of the system under discussion, i.e., the case of a free

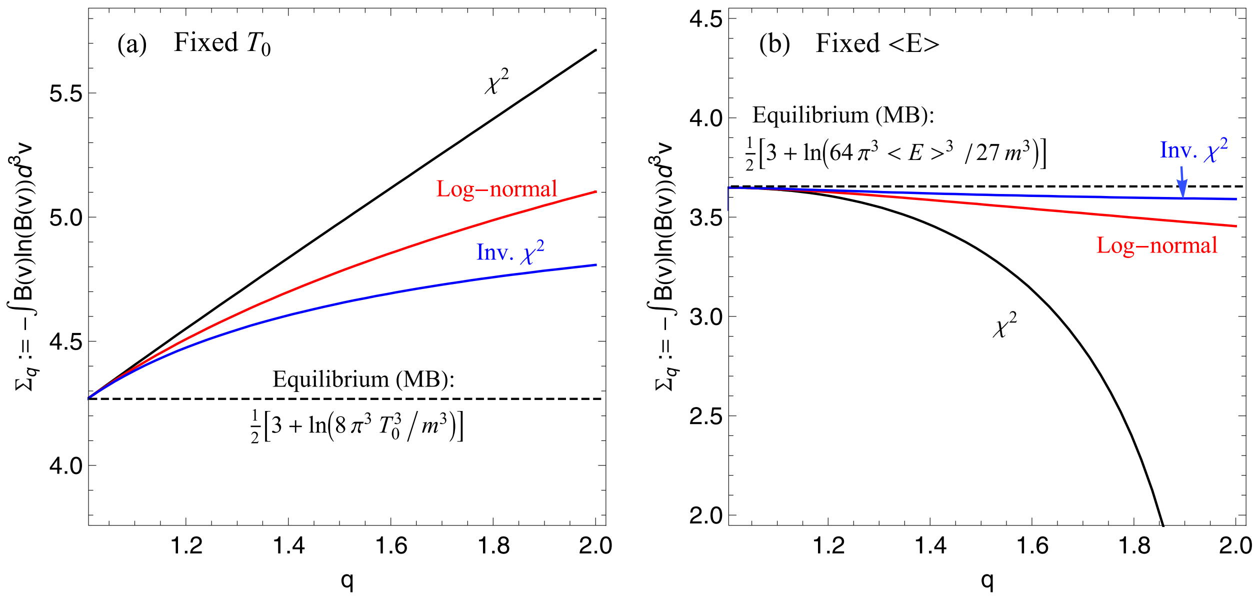

FIG. 4. The scaled entropy Σ_q [Eq. (17)], computed numerically for the three universality classes of superstatistics, under two conditions: (a) with a fixed mean temperature T_0 and (b) with a fixed mean energy $\langle E\rangle$.

particle in a one-dimensional box of length L. In this scenario, the thermal uncertainty relation, obtained by averaging the Heisenberg uncertainty over thermal (equilibrium) statistics, reads as [111]

$$\langle(\Delta x)(\Delta p)\rangle \equiv \frac{1}{Z(\beta)}\sum_{n=1}^{\infty}(\Delta x)(\Delta p)e^{-\beta\epsilon_n} \approx \frac{\sqrt{3}L}{6\pi}\sqrt{\frac{2\pi m}{\beta}}, \tag{23}$$

where the approximation consists of taking the continuous limit. Generalizing this result to the context of superstatistics is a simple exercise. For superstatistical distributions in the generic form of (2), which are essentially Boltzmann distributions averaged over $f(\beta)$, we can combine Eq. (23), with the moments of $f(\beta)$, given by

$$\begin{aligned}\langle\beta^l\rangle_{f_1} &= \frac{\Gamma\left(\frac{n}{2}+l\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{2}{n}\right)^l\beta_0^l,\\ \langle\beta^l\rangle_{f_2} &= \frac{\Gamma\left(\frac{n}{2}+1-l\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{n}{2}\right)^{l-1}\beta_0^l,\\ \langle\beta^l\rangle_{f_3} &= e^{l(l-1)s^2/2}\beta_0^l,\end{aligned} \tag{24}$$

for the three universality classes, and use Eq. (12) to find

$$\langle\langle(\Delta x)(\Delta p)\rangle\rangle_{f_i(\beta)} = \alpha_i(q)\langle(\Delta x)(\Delta p)\rangle, \tag{25}$$

where we have defined the following auxiliary functions:

$$\begin{aligned}\alpha_1(q) &= \frac{\Gamma\left[\frac{1}{q-1}-\frac{1}{2}\right]}{\Gamma\left[\frac{1}{q-1}\right]\sqrt{q-1}},\\ \alpha_2(q) &= \frac{\Gamma\left[\frac{q}{q-1}+\frac{3}{2}\right]}{\Gamma\left[\frac{q}{q-1}\right]}\left(\frac{q-1}{q}\right)^{3/2},\\ \alpha_3(q) &= q^{3/8},\end{aligned} \tag{26}$$

with $i = 1, 2, 3$ corresponding, respectively, to χ^2, inverse-χ^2, and log-normal superstatistics. Figure 5 shows $\alpha_i(q)$ as a function of q, for the three classes of superstatistics. One may see that the presence of temperature fluctuations induces an increase in the thermal uncertainty, following the same pattern observed in the scaled entropy Σ_q.

From a different angle, the fact that nonequilibrium effects increase Σ_q, and consequently the total entropy S, leads to an apparent paradox. In fact, the entropy is expected to reach its maximum at equilibrium, which corresponds to $q = 1$. This

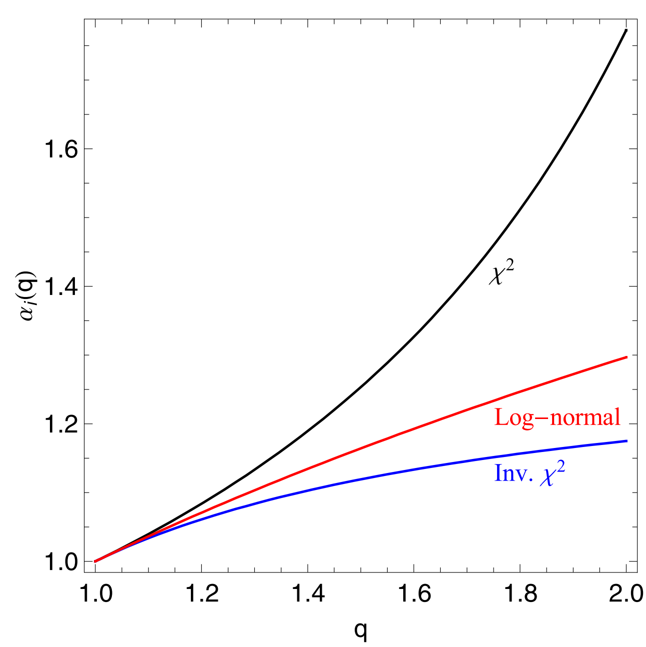

FIG. 5. The normalized thermal uncertainty $\alpha_i(q)$ [viz. Eq. (26)] as a function of q, for the three universality classes of superstatistics.

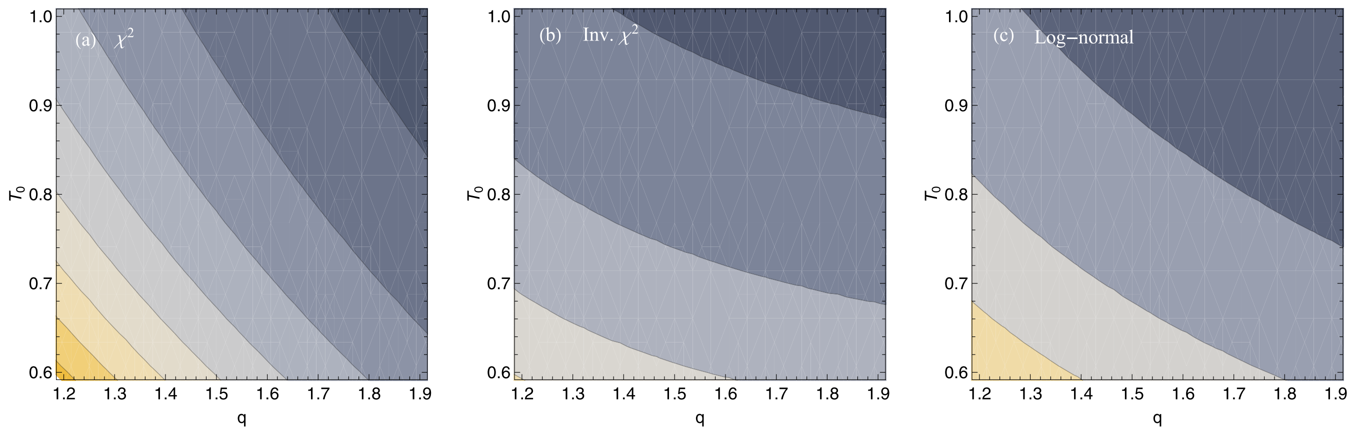

FIG. 6. The V-T_0-q diagram, namely the volume V as a function of the average temperature T_0 and $q := \langle\beta^2\rangle/\beta_0^2$, corresponding to a constant entropy, for χ^2 (a), inverse χ^2 (b), and log-normal (c) superstatistics. For $q = 1$, one recovers the adiabatic equation, i.e., $V \sim T_0^{-3/2}$.

equilibrium state is associated with the MB distribution in the case of an ideal gas considered here. This puzzle can be resolved by noting that, for an isolated system, it is not the mean temperature T_0 but rather the energy $\langle E\rangle$ that is conserved. For the three universality classes, one may express the energy $\langle E\rangle = m\langle v^2\rangle/2$ in terms of T_0, and *vice versa*, by using the moments (24) and Eq. (12), as follows:

$$\begin{aligned}\langle E\rangle_1 &= \frac{d}{4-2q}T_0 \Leftrightarrow T_0 = (4-2q)\frac{\langle E\rangle_1}{d},\\ \langle E\rangle_2 &= d\frac{2q-1}{2q}T_0 \Leftrightarrow T_0 = \left(\frac{2q}{2q-1}\right)\frac{\langle E\rangle_2}{d},\\ \langle E\rangle_3 &= q\frac{d}{2}T_0 \Leftrightarrow T_0 = \frac{2}{q}\frac{\langle E\rangle_3}{d},\end{aligned} \tag{27}$$

d being the number of degrees of freedom [see Eq. (4)] and $i = 1, 2, 3$ corresponding, respectively, to χ^2, inverse-χ^2, and log-normal superstatistics. Figure 4(b) displays the scaled entropy (17) for a fixed energy $\langle E\rangle$. One may observe that Σ_q, and consequently the total entropy S, remains lower than that of the equilibrium state. The latter is correctly reproduced in the limit $q \to 1$.

Observing Figs. 4 and 5, an interesting trend emerges: the three classes of superstatistics tend to align as the value of q approaches unity. This pattern reflects a broader phenomenon discussed in Ref. [34], that all superstatistics exhibit a universal behavior when converging towards equilibrium distributions (for q close to 1). That is, regardless of the class of fluctuations (even beyond the three universality classes considered here), distributions (2) display a quadratic correction $(1 + \frac{1}{2}\sigma^2\epsilon^2)$ for small fluctuations, where σ represents the standard deviation of $\tilde{f}(\beta)$. It is worth mentioning that such distributions featuring a quadratic correction beyond the equilibrium distribution frequently occur in plasma physics. These are commonly referred to as nonthermal or Cairns distributions in the plasma physics literature [112,113].

Another interesting result that can be extracted from the entropy (16) concerns isentropic processes, i.e., processes in which the entropy of the system remains unchanged. In equilibrium, an isentropic compression or expansion process of a parcel of gas implies that the argument of the logarithm in the SackurTetrode equation (19) remains constant. It follows that $V \sim T^{-3/2}$. This is the standard $V - T$ adiabatic equation (for three degrees of freedom, i.e., those of the monoatomic ideal gas). It is interesting to explore how this may change in the superstatistical scenario. By rewriting the entropy (16) as

$$S = N\left[1 + \ln\left(\frac{Vm^3\,e^{\Sigma_q}}{Nh^3}\right)\right], \tag{28}$$

one observes that, for an isentropic process, $V\,e^{\Sigma_q}$ must be constant. It is only in the case $q \to 1$ that one recovers the V-T adiabatic equation, namely that $VT^{3/2}$ = constant. In the general context, the lines of constant entropy correspond to a nontrivial relation between V, T_0, and q. Figure 6 shows the volume as a function of T_0 and q, corresponding to an isentropic process, for the three universality classes of superstatistics, obtained by numerically solving Eq. (28). This represents a generalization of the standard V-T adiabatic equation for superstatistical systems.

IV. EVOLUTION TOWARD EQUILIBRIUM: THE RELAXATION TIME APPROXIMATION

We now turn our attention to the relaxation process of a system, initially in a quasiequilibrium state described by a superstatistical distribution, under the effect of collisions. To explore this situation, we adopt a kinetic approach, considering the phase space spanned by the space and velocity coordinates. The state of the system is defined by the distribution function $f(\mathbf{r}, \mathbf{v}; t)$, whose evolution over time is governed by the Boltzmann equation. That is,

$$\frac{df(\mathbf{r}, \mathbf{v}; t)}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}f + \frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}}f = C[f], \tag{29}$$

where $\nabla_{\mathbf{r}}$ and $\nabla_{\mathbf{v}}$ stand for the partial derivatives with respect to position $\mathbf{r}$ and velocity $\mathbf{v}$, respectively. Above, $\mathbf{F}$ represents the external force and $C[f]$ is the collision operator. For simplicity, we assume homegeneity (i.e., $\nabla_{\mathbf{r}}f = 0$) and the absence of external forces (i.e., $\mathbf{F} = 0$). In these conditions, the Boltzmann equation (29) reduces to

$$\frac{df(\mathbf{r}, \mathbf{v}; t)}{dt} = \frac{\partial f}{\partial t} = C[f(t)]. \tag{30}$$

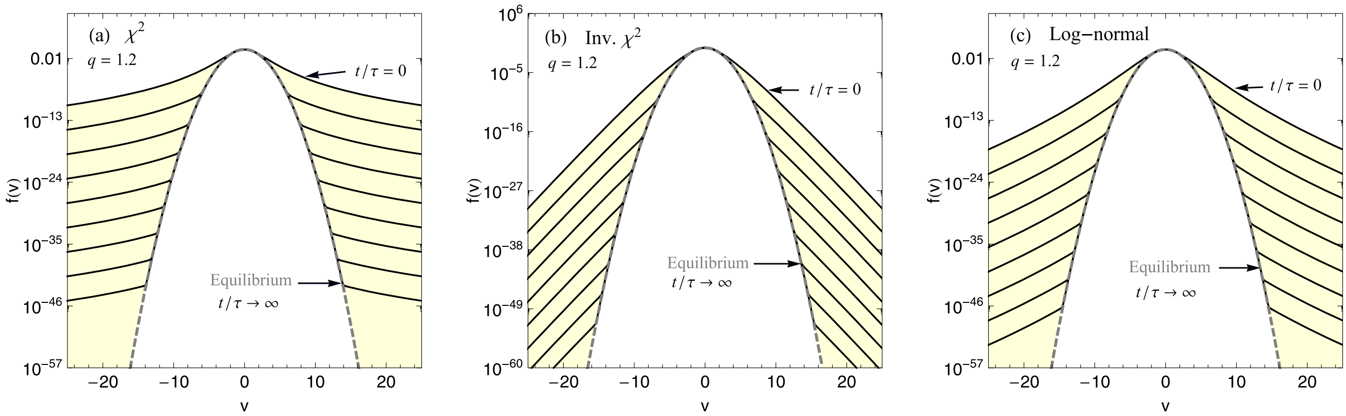

FIG. 7. Velocity probability distributions resulting from the relaxation time approximation scenario calculated using Eq. (34), for an initial distribution corresponding to χ^2 (a), inverse χ^2 (b), and log-normal (c) superstatistics, with $q := \langle\beta^2\rangle/\beta_0^2 = 1.2$. The velocity has been normalized to the thermal velocity, i.e., $\mathrm{v} \equiv v/2mT_0$.

This corresponds to the situation discussed in Sec. III. We note that, even with these simplifications, the equation remains quite general as we have not specified the form of the collision operator yet. For the exact form of $C[f(t)]$, Eq. (30) corresponds to an integro-differential equation, making it highly challenging to tackle analytically. It is a common practice to use model collision operators, namely simplified operators preserving the main qualitative properties of the original exact collision operator, while omitting fine details. One noteworthy approximation of the collision operator that allows gaining analytical insight is the relaxation time approximation (also known as the BGK-model). This model was independently introduced by Bhatnagar, Gross, and Krook [66], and by Welander [67], as a simplified Boltzmann-like model. In this approximation, $C[f(t)]$ reads as

$$C[f] = \frac{f_{eq} - f}{\tau}, \tag{31}$$

where f_{eq} is the equilibrium distribution (i.e., the MB distribution in our scenario) and τ is the relaxation time, i.e., the time taken by a nonequilibrium system to reach (global) equilibrium. While the relaxation time approximation appears to primarily account for small deviations from equilibrium, it has been acknowledged [114] that this approximation remains effective well beyond its theoretical limits, as long as the relaxation time can capture the relevant physics. This approximation has proven effective in facilitating the analytical treatment of various problems, raging from plasma physics [68,69] to self-gravitating systems [70] and high energy collisions [71–73], among many others [74–76]. With this approximation, Eq. (30) becomes

$$\frac{\partial f}{\partial t} = \frac{f_{\mathrm{eq}} - f}{\tau}, \tag{32}$$

which can be solved, by assuming some initial distribution $f(t=0) \equiv f_{\mathrm{in}}$, to give

$$f(v) = f_{\mathrm{eq}} + (f_{in} - f_{\mathrm{eq}}) \exp\left(-\frac{t}{\tau}\right). \tag{33}$$

This can be identified with the nonequilibrium distribution observed at time t, with $t \ll \tau$. Eq. (33) can be expressed in a more intuitive form as follows

$$f(v) = f_{\mathrm{in}}(v) \exp\left(-\frac{t}{\tau}\right) + f_{\mathrm{eq}}(v)\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \tag{34}$$

emphasizing that, at any given moment, the distribution corresponds to a superposition of the two distributions, f_{in} and f_{eq}, with a "weight" controlled by the dimensionless parameter t/τ.

To show how this applies to a quasiequilibrium system described by superstatistics, we associate our initial distribution f_{in} to one of the three universality classes of superstatistical velocity distributions, discussed in Sec. II, and f_{eq} to the equilibrium MB distribution (for simplicity, considering one degree of freedom). In Fig. 7, we depict the evolution of superstatistical velocity distributions, obtained from Eq. (34). One may see that, as time goes by, the distinctive heavy tails characteristic of quasiequilibrium states fade away, and the distribution smoothly transforms into the equilibrium MB distribution (see Fig. 1 for comparison).

In this picture, the system is characterized, at any given moment, by a superposition of the distributions f_{in} and f_{eq}, with the nonequilibrium distribution gradually evolving toward f_{eq}. Now, one may observe that, in the superstatistics scenario, the shape of $f(v)$ is entirely determined by the temperature distribution $f(\beta)$ across the system (indeed, the temperature distribution $f(\beta)$ can be univocally determined from $f(v)$). At this level of description, the evolution of $f(v)$ translates into an evolution for the temperature distribution itself: The temperature distribution starts with a given shape (in principle, determined by some initial conditions) and progressively evolves towards $f(\beta) \equiv \delta(\beta - \beta_0)$, corresponding to an equilibrium state, with a constant temperature $T_0 \equiv 1/\beta_0$. Put otherwise, the constituent cells of our nonequilibrium system progressively thermalize, forming larger regions at uniform temperature. At the end of this process, the whole system has thermalized and is described by the equilibrium distribution (as illustrated in Fig. 8).

As the distribution smoothly evolves from a specific initial nonequilibrium state, corresponding to a particular universality class of superstatistics, toward an equilibrium distribution, one may define a dynamical factor $q(t)$, with f_{in}

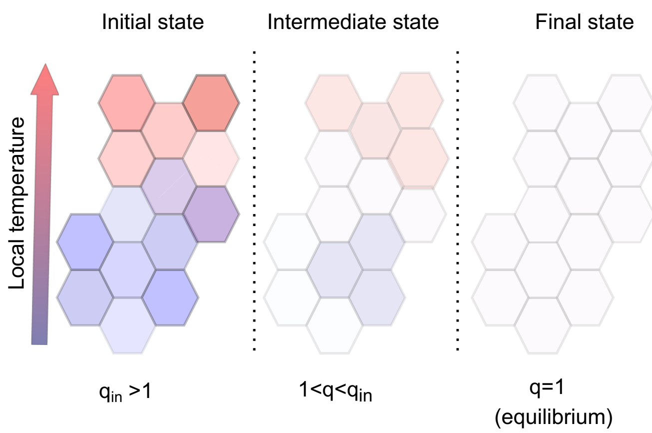

FIG. 8. Schematic illustration of the relaxation process: The initial nonequilibrium system is divided up into small cells, each exhibiting nearly constant temperature. This initial temperature distribution is characterized by an index $q_{\rm in}$. As time goes by, neighboring cells gradually reach thermal equilibrium, resulting in the emergence of larger areas with uniform temperature. At the end of the process (i.e., for $t/\tau \to \infty$), the entire system attains a uniform temperature, corresponding to $q := \langle \beta^2 \rangle / \beta_0^2 = 1$.

corresponding to some initial value $q_{\rm in} > 1$ and $f_{\rm eq}$ corresponding to $q(t/\tau \to \infty) = 1$ (in which case the temperature distribution shrinks to a Dirac δ distribution). Note that a similar scenario has been nicely elaborated for NSM, in the context of multiparticle production processes, involving a dynamical nonextensivity index [72,73]. The present discussion can be regarded as an extension to cover the entire range of superstatistics universality classes. We emphasize that for inverse χ^2 and log-normal superstatistics, there is no direct correspondence between $q := \langle \beta^2 \rangle / \beta_0^2$ and the entropic index used in NSM. Such correspondence can only be established in the case of the χ^2 class (see Ref. [23]).

Now, we can determine the evolution of $q(t)$ by using Eq. (33). Although the equation can be numerically solved in the general case, a more insightful approach can be employed by eliminating as much of irrelevant information as possible from the description, enabling us to retain an analytical treatment. This is particularly relevant in the context of log-normal superstatistics, where a closed-form expression for the distribution is not available, but its moments can be computed analytically. Taking advantage of this, Eq. (33) can be multiplied by v^l ($l \in \mathbb{N}$) and integrated over velocity space, providing us with the evolution of the velocity moments $\langle v^l \rangle$.

For superstatistical distributions in the generic form of (2), which are merely MB distributions averaged over $f(\beta)$, the velocity moments can be written as

$$\langle v^l \rangle \equiv \int v^l B(v) dv = \langle \langle v^l \rangle_{\rm MB} \rangle_{f(\beta)}. \tag{35}$$

Using the moments of the three universality classes of $f(\beta)$ [Eq. (24)], one may express the velocity moments as

$$\langle v^l \rangle_{f_1} = \frac{\Gamma\left[\frac{1}{q-1} - \frac{l}{2}\right]}{\Gamma\left[\frac{1}{q-1}\right](q-1)^{l/2}} \langle v^l \rangle_{\rm MB},$$

$$\langle v^l \rangle_{f_2} = \frac{\Gamma\left[\left(\frac{1}{q-1}\right) + 2 + \frac{l}{2}\right]}{\Gamma\left[\left(\frac{1}{q-1}\right) + 2\right]} \left(\frac{q-1}{q}\right)^{l/2} \langle v^l \rangle_{\rm MB},$$

$$\langle v^l \rangle_{f_3} = q^{\frac{l}{4}\left(\frac{l}{2}+1\right)} \langle v^l \rangle_{\rm MB}, \tag{36}$$

where we have used Eq. (12), to express them in terms of q. In particular, for the lowest moment $l = 2$ (note that all odd moments vanish for symmetry reason), one has

$$\langle v^2 \rangle_{f_1} = \frac{1}{2-q} \langle v^2 \rangle_{\rm MB} \quad (1 \leqslant q < 2),$$

$$\langle v^2 \rangle_{f_2} = \frac{2q-1}{q} \langle v^2 \rangle_{\rm MB}, \tag{37}$$

$$\langle v^2 \rangle_{f_3} = q \langle v^2 \rangle_{\rm MB}.$$

Using this and Eq. (33), we find the dynamical expression of $q(t)$ for the three classes as follows:

$$q_1(t) = 1 + \frac{q_{\rm in} - 1}{q_{\rm in} - 1 - (q_{\rm in} - 2)e^{t/\tau}},$$

$$q_2(t) = \frac{q_{\rm in} e^{t/\tau}}{1 + q_{\rm in}(e^{t/\tau} - 1)}, \tag{38}$$

$$q_3(t) = 1 + (q_{\rm in} - 1)e^{-t/\tau}.$$

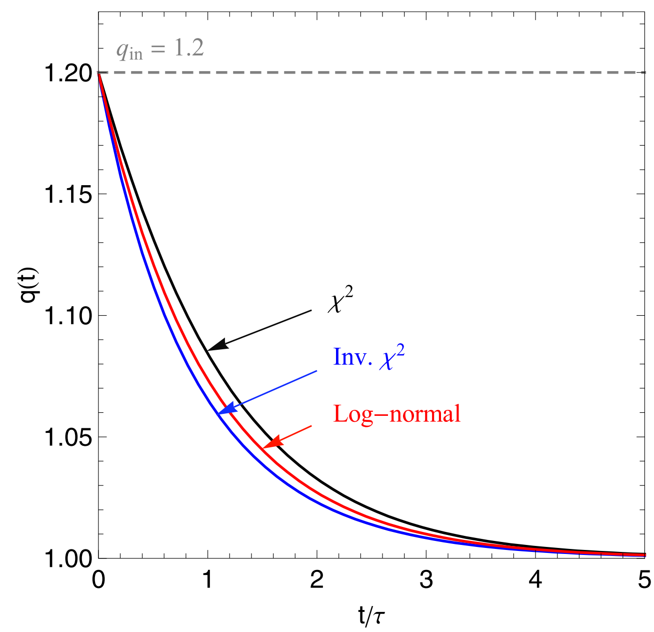

FIG. 9. The evolution of $q(t)$, starting from an initial value $q_{\rm in} = 1.2$, for the three universality classes of superstatistics.

with $i = 1, 2, 3$ corresponding, respectively, to χ^2, inverse χ^2, and log-normal superstatistics. The evolution of $q(t)$, as determined by Eq. (38), is shown in Fig. 9, with an initial value $q_{\rm in} = 1.2$, chosen for illustrative purposes. Note that, for the three universality classes, the equilibrium value $q \to 1$ is reached in the limit $t/\tau \to \infty$. It is instructive to observe that, for log-normal superstatistics, Eq. (38) implies that $q(t)$ obeys the same evolution equation as $f(v)$ in the relaxation

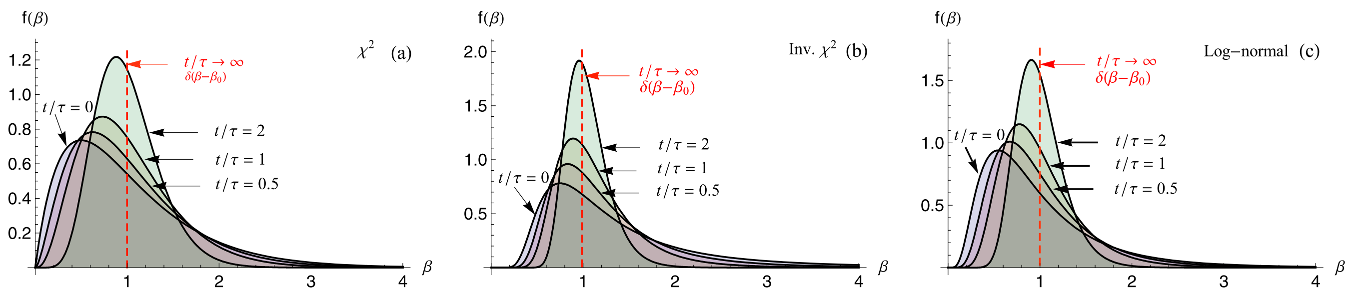

FIG. 10. Temperature distributions $f(\beta)$ corresponding to χ^2 (a), inverse-χ^2 (b), and log-normal (c) superstatistics, for different t/τ, determined using Eq. (38). The initial state has been set to $q_{\rm in} \equiv q(t/\tau = 0) = 1.4$. As t/τ increases, the distribution becomes progressively narrower, eventually approaching a Dirac δ distribution in the limit $t/\tau \to \infty$.

time approximation (32). That is,

$$\frac{dq}{dt} = \frac{q_{\rm eq} - q}{\tau}, \tag{39}$$

with $q_{\rm eq} = 1$.

Now, note that the dynamics of $q(t)$ fully determines the dynamics of the temperature distribution $f(\beta)$. This allows determining the changes occurring within the underlying temperature distribution, as driven by collisions, and to capture its specific form at any given moment. This process is portrayed in Fig. 10, for the three universality classes of superstatistics. Starting from a given temperature distribution, characterized by an initial values $q_{\rm in} = 1.4$ (for the sake of illustration), the distribution becomes progressively narrower and eventually shrinks to a Dirac δ distribution for $t/\tau \to \infty$.

It is instructive to note that, despite the simplifications we have applied here (homogeneity and the absence of an external force), the general tendency outlined in Eq. (38) finds support in empirical observations. An appealing example is that of stellar astrophysics, where the relaxation time approximation may be used to effectively model close stellar encounters (see, e.g., Ref. [115]). In this context, it has long been known that Tsallis distributions [16–18] and, more broadly, superstatistical distributions [50], accurately model the observed distributions of radial and rotational velocities of stars. Extensive analyses reveal that the deviation from the MB (equilibrium) distribution correlates *inversely* with the age of the stellar cluster, with equilibrium distributions observed for ages greater than approximately 170 million years (Myr). In Ref. [116], this tendency is attributed to a memory loss" effect while in our interpretation, it can be explained by the influence of collisions (close stellar encounters in this context) gradually driving the stellar system toward equilibrium distributions over a long period of time.

Before concluding this section, it is important to note that, in the situation considered here with a fixed number of particles N, one would naturally expect an evolution towards equilibrium. Nonetheless, certain complex systems, especially those involving long-range interactions, may exhibit nonuniform convergence as $(N, t) \to (\infty, \infty)$. In this context, we highlight the findings of Ref. [117], where it has been demonstrated, without making any a priori hypotheses about entropy or other thermodynamic quantities, that in the ordering $\lim_{N\to\infty} \lim_{t\to\infty}$, Maxwellian distributions dominate. Conversely, in the $\lim_{t\to\infty} \lim_{N\to\infty}$ ordering, the crossover occurs in the opposite direction, with q-Gaussian distributions (corresponding to the χ^2 universality class of superstatistics) becoming dominant.

V. CONCLUSIONS

In this paper, our primary focus has been on quasiequilibrium systems, characterized by local equilibrium, yet not reaching global equilibrium. These systems can be effectively modeled using a superposition of statistics, i.e., superstatistics. We first investigated the effect of superstatistics on entropy. We showed that, although temperature fluctuations enter the picture, the extensivity of entropy is preserved (at least when long-range interactions are not involved). Subsequently, we studied a kinetic model, accounting for collisions via the relaxation time approximation, to explore the evolution of superstatistical systems towards equilibrium. This allowed us to define a dynamical version of superstatistics, smoothly evolving towards equilibrium distributions. This dynamics is governed by a single time-varying parameter $q(t)$, whose rate of change is controlled by the relaxation time τ. Within this framework, collisions assume the role of a stochastic force, causing temperature fluctuations, and gradually narrowing the temperature distribution over time.

The present approach presents interesting prospects for future research. In fact, the concept of superstatistics has found interesting applications in the study of continuous media, where a kinetic treatment is appropriate. However, prior investigations have been somewhat limited in scope, primarily focusing on the collisional regime. The present approach offers a pathway to extend this methodology and explore the influence of collisions on a broader spectrum of phenomena, such as waves and instabilities in various media.

The approach discussed here, nonetheless, retains its limitations, especially due to the absence of long-range interactions and the reliance on the relaxation time approximation. Potential future avenues of research may involve examining the influence of interactions and considering more general forms of the collision operator. While the relaxation time approximation is versatile and accommodating for a wide range of scenarios, there is room for exploration of more realistic collision operator forms (see, e.g., Ref. [118] and references therein). This will shed light on the extent to which the present conclusions remain applicable in a broader context. Of course, at the technical level, one anticipates important challenges when dealing with more sophisticated collisional operator forms but the underlying rationale remains the same, at the conceptual level. In the same context, we stress that the relaxation time approximation employed here inherently produces an exponential dependence on time, which is well-suited for modeling rapid relaxation processes. Nevertheless, in certain experimentally relevant situations, the initially rapid relaxation undergoes a gradual slowdown, leading to nonexponential behavior. In such instances, the current model does not fully capture the dynamics, and one expects a time dependence following more general functions. Examples of such functions include power laws [119] or stretched exponentials [120].

Finally, note that the outlined reasoning can be reversed to explore how collisions might give rise to superstatistical distributions. In fact, while collisions typically tend to drive the system toward equilibrium, they can, under certain conditions, lead to the generation of heavy-tailed distributions that are characteristic of quasiequilibrium states. This phenomenon has recently been observed in the velocity distributions of radio-frequency ion traps [121]. In this scenario, collisions cause an increase in the high-energy tails of the velocity distribution, for sufficiently high densities. In Ref. [121], this effect is attributed to the reduction of the mean-free-path to a scale smaller than the trap diameter, comparable to the distances where radio-frequency heating occurs. Consequently, a "runaway" heating of ions near the trap edges takes place, resulting in an enhanced high-energy tail.

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