Martingale drift of Langevin dynamics and classical canonical spin statistics

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A martingale is a stochastic process that encodes a kind of fairness or unbiasedness, which is associated with a reference process. Here we show that, if the reference process x_t evolves according to the Langevin equation with drift $a(x)$ and if $a(x_t)$ is a martingale, then its amplitude is the Langevin function, which originally described the canonical response of a single classical Heisenberg spin under static field. Furthermore, the asymptotic limit of x_t/t obeys the ensemble statistics of such a Heisenberg spin.

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I. BACKGROUND

Langevin derived in 1905 what we nowadays call the Langevin function [1], which gives the canonical equilibrium response of paramagnetism under static magnetic field. It was based on the Boltzmann-Einstein statistics and explained Curie's law, which Pierre Curie had established experimentally 10 years before [2]. In modern language, a classical three-dimensional Heisenberg spin under a uniform magnetic field along an axis undergoes thermal fluctuation according to the Boltzmann weight, or the relative probability, $e^{x\cos\theta}$, where θ is the polar angle of the spin relative to the field axis and x is the strength of the magnetic field scaled by the temperature and magnetic moment of the spin. The mean polarization of the spin is given by the average of $\cos\theta$ weighted by the above probability and integrated over the solid angles. The result is the Langevin function, $\coth x - \frac{1}{x}$ [1]. That it behaves like $\frac{1}{3}x$ for small $|x|$ explained Curie's law.

Three years after Einstein [3] formulated Guoy's qualitative idea of Brownian motion [4,5], Langevin introduced, in 1908, the first stochastic differential equation that is nowadays called the Langevin equation [6]. In modern language a simple and generic form of the Langevin equation in d dimensions reads

$$\frac{d\vec{x}_t}{dt} = \vec{a}(\vec{x}_t) + \vec{\xi}_t, \tag{1}$$

which contains the drift $\vec{a}(\vec{x}_t)$ and the white Gaussian noise $\vec{\xi}$, with the zero mean and the unit diagonal covariance $\langle \vec{\xi}_t, \vec{\xi}_s\rangle = \mathbf{1}\delta(t-s)$, where $\mathbf{1}$ is the unit tensor. While we now know how this equation admits the first [7,8] and second [9,10] laws of thermodynamics, it was Langevin who revolutionized the notion of the evolution equation as a mapping between the path ensembles. In the 1900s, however, no link between the Langevin function and the Langevin equation was known, to the author's knowledge.

Around 1940 Viller and Doob modernized the historically old notion of a martingale as a powerful concept of modern probability theory; see, for example, Chap. 1 of [11] for a historical review. The martingale is the mathematical expression of the idea of the fairness for the future: The continuous-time stochastic process $\vec{Y}_t$ is said to be a martingale associated with the stochastic process $\{\vec{X}_s\}$ if (i) $\vec{Y}_t$ is causally determined by $\{\vec{X}_s\}_{0\leqslant s\leqslant t}$ and (ii) its conditional expectation for the future shows the fairness:

$$\langle \vec{Y}_t | \{\vec{X}_u\}_{0\leqslant u\leqslant s}\rangle = \vec{Y}_s \quad \forall\, t \geqslant s, \tag{2}$$

in addition to the other rigorous mathematical conditions. Here $\{\vec{X}_u\}_{0\leqslant u\leqslant s}$ denotes the history of $\vec{X}_u$ over the period $0 \leqslant u \leqslant s$, and $\langle R|\text{cond}\rangle$ indicates the expectation of the random variable R under the condition. For later convenience we introduce the differential version which follows from (2) applied to Y_{t+dt} as well as to Y_t:

$$\langle d\vec{Y}_t | \{\vec{X}_u\}_{0\leqslant u\leqslant s}\rangle = 0 \ \ \forall\, t \geqslant s. \tag{3}$$

As probability theory provides many useful theorems about martingales, efforts have often been made to convert a reference process of interest into a martingale process to discover new properties of X_t. The population dynamics or mathematical finances have developed that approach since longtime [11]. In the stochastic thermodynamics people recently recognized that exponentiated entropy production is the martingale process of the category called the path-probability ratio or Radon-Nikodym density process [12,13].

A distinct type of martingale other than the path-probability ratio has also been found in what we call progressive quenching [14–17]. Starting from an equilibrium many spin system, we fix the spins, one after another, in the instantaneous state they take. Through such an unbiased protocol, the martingale process was found not in the progressively fixed spins, but in the mean spin *to be fixed.* Those already fixed spins influence the latter through the molecular field. We have coined the name "hidden martingale" for it because, in its continuous-time counterpart, it is not $\vec{x}_t$ in (1), but the drift $\vec{a}(\vec{x}_t)$, that should be the martingale—this is the

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starting point of the present study. In Sec. II we show that, if the drift $\vec{a}(\vec{x}_t)$ in (1) is a martingale associated with the process (1) itself, then the drift is the Langevin function [18]. The emergence of a spin-related function is surprising because the Langevin equation (1) works in a noncompact space. However, a further surprise in Sec. III is that the canonical statistics of such a spin emerges in the ensemble of the long-time asymptotes of x_t/t.

II. LANGEVIN FUNCTION AS SELF-HARMONIC DRIFT

A. Family of harmonic functions associated with a drift

When the process $\vec{Y}_t = \vec{h}(\vec{X}_t)$ is a martingale associated with the process $\{\vec{X}_s\}$, the function $\vec{h}$ is said to be harmonic [11]. In particular, if the process $\{\vec{X}_s\} = \{\vec{x}_t\}$ is generated by (1) and $\vec{h}(\vec{x}_t)$ takes a value in the same space as $\vec{x}_t$ does, the condition for the harmonicity reads

$$(\vec{a}\cdot\nabla)\vec{h} + \tfrac{1}{2}\Delta\vec{h} = 0. \tag{4}$$

To see this, it suffices to develop $d\vec{h}(\vec{x}_t)$ up to the order $O(dt)$, that is,

$$d\vec{h}(\vec{x}_t) = (d\vec{x}_t \bullet \nabla)\vec{h} + \tfrac{1}{2}(d\vec{x}_t \bullet \nabla)^2\vec{h}, \tag{5}$$

where $\bullet$ is the Itô-type product [11]. We substitute into (5) the stochastic differential version of (1),

$$d\vec{x}_t = \vec{a}(\vec{x}_t)dt + d\vec{W}_t, \tag{6}$$

where $\vec{W}_t$ is the d-dimensional standardized Brownian motion, i.e., the Wiener process, such that $d\vec{W}_t d\vec{W}_t = \mathbf{1}dt$, with $\mathbf{1}$ being the unit tensor. The imposition of the martingale condition (3) on $\vec{Y}_t = \vec{h}(\vec{x}_t)$ then leads to (4). The form (4) explains the denomination "harmonic" because, in the absence of drift, $\vec{a} \equiv 0$, Eq. (4) is the vector Laplace equation.

Given a drift $\vec{a}$, we can conceive the family of harmonic functions associated with the process obeying (1), We shall denote this family by $\mathcal{H}_{\vec{a}}$:

$$\mathcal{H}_{\vec{a}} = \left\{\vec{h}\ ;\ (\vec{a}\cdot\nabla)\vec{h} + \tfrac{1}{2}\Delta\vec{h} = 0\right\}. \tag{7}$$

B. Langevin function as fixed point

Among the families associated with different drifts, the self-referential condition

$$\vec{a}^* \in \mathcal{H}_{\vec{a}^*} \tag{8}$$

defines the special drift $\vec{a}^*$ as a kind of fixed point, which we shall call the *self-harmonic* drift when it exists. More concretely,

$$(\vec{a}^*\cdot\nabla)\vec{a}^* + \tfrac{1}{2}\Delta\vec{a}^* = \vec{0}. \tag{9}$$

This is the key equation of the present paper.

Asymmetric solutions of (9) are possible. Apparently, the cylindrical lift-up of the solution for lower dimensions is an anisotropic solution. Nevertheless, we here seek the solutions which are isotropic with respect to the origin, $\vec{x} = 0$. (Note that the process $\vec{x}_t$ can, nevertheless, start from any $\vec{x}$ off the origin.)

We then assume the form [19]

$$\vec{a}^*(\vec{x}) = L_d(\|\vec{x}\|)\hat{x}, \tag{10}$$

where $\hat{x} \equiv \vec{x}/\|\vec{x}\|$ is the unit vector along $\vec{x}$ and the suffix d stands for the space dimension. Then (9) implies

$$L_d''(x) + 2L_d(x)L_d'(x) + \frac{d-1}{x}\left(L_d'(x) - \frac{L_d(x)}{x}\right) = 0. \tag{11}$$

In the two-dimensional family of solutions of (11) the invariance under the similarity transformation $L_d(x) \to \alpha L_d(\alpha x)$ provides one parameter. If we write the first integral of (11) as

$$L_d'(x) + [L_d(x)]^2 + \frac{d-1}{x}L_d(x) = \alpha^2, \tag{12}$$

then we can reduce the problem to finding $L_d(x)$ for (12) with $\alpha = 1$. Then in the remaining one-parameter family of solutions for (12), some analysis using *Mathematica* indicates that, at least for d =2, 3, and 4, there is a unique solution which does not diverge at $x = 0$. For example, in $d = 3$, the solution $L_3^{(\beta)}(x) \equiv \coth(x+\beta) - \frac{1}{x}$ is regular only with $\beta = 0$. For $d = 1$ we can solve (9) directly so that $\alpha = 1$ in (12) and we can appropriately choose the origin. Altogether, the solutions thus identified are

$$\begin{aligned}
L_1(x) &:= \tanh(x),\\
L_2(x) &:= \frac{I_1(x)}{I_0(x)},\\
L_3(x) &:= \coth x - \frac{1}{x},\\
L_4(x) &:= \frac{x[I_0(x) + I_2(x)] - 2I_1(x)}{2xI_1(x)}
\end{aligned} \tag{13}$$

for $x \neq 0$ and $L_d(0) = 0$ for any dimension d, where $I_n(x)$ are the nth modified Bessel functions of the first kind. A standard but inspiring approach to obtain these solutions from (12) with $\alpha = 1$ is to use the (Riccati) transformation

$$L_d \equiv \frac{Z_d'}{Z_d} = \frac{d}{dx}[\ln Z_d(x)], \tag{14}$$

which renders (12) to the linear equation,

$$\frac{d^2 Z_d}{dx^2} + \frac{d-1}{x}\frac{dZ_d}{dx} = Z_d. \tag{15}$$

By noticing the radial part of the Laplacian operator on the left-hand side of (15), we find

$$Z_d(\|\vec{x}\|) \propto \oint_{\|\hat{S}\|=1} e^{\hat{S}\cdot\vec{x}} d\Omega_S, \tag{16}$$

which is the partition function for a single classical Heisenberg spin under the nondimensionalized external field $\vec{x}$. Through (14) the drift is therefore the canonical average of the spin $\hat{S}$ under this field. All $L_d(x)$ are odd in x, and their graphs look similar to the simplest one, $L_1(x)$, and the original Langevin function, $L_3(x)$. They also have the unique limit $\lim_{x\to\infty} L_d(x) = 1$. We may call $L_d(x)$ the *d-dimensional Langevin function.* In any case the Langevin equation meets here with the Langevin functions in the context of the martingale. For completeness we give $Z_d(x)$ with the numerical coefficient so that $Z_d(0)$ is the surface area of a d-dimensional unit hypersphere:

$$(Z_1, Z_2, Z_3, Z_4) = \left(2\cosh(x), 2\pi I_0(x), 4\pi\frac{\sinh(x)}{x}, 4\pi^2\frac{I_1(x)}{x}\right). \quad (17)$$

Remark on the solution family. The change in the units of length and time causes the rewriting of (6). If we introduce $\vec{y}$ and τ through $\vec{x} = \alpha\vec{y}$ and $t = \alpha^2 C\tau$, then (1) with (10) for $\vec{a}(\vec{x}_t)$ leads to

$$d\vec{y}_\tau = C\alpha\, L_d(\alpha\|\vec{y}_\tau\|)\hat{y}_\tau d\tau + \sqrt{C}d\vec{W}_\tau, \quad (18)$$

where $\hat{y}_\tau \equiv \vec{y}_\tau/\|\vec{y}_\tau\|$ and we have used the statistical equivalence $d\vec{W}_{\kappa t} \simeq \sqrt{\kappa}\, d\vec{W}_t$ for the standard d-dimensional Wiener process. In (18) we see that, apart from the flexibility of the self-harmonic function $\alpha L_d(\alpha\|\vec{y}_\tau\|)$, there is a specific scale relationship between the drift and diffusion for the drift to be a martingale. See Appendix A for more discussion.

III. STATISTICAL PROPERTY OF THE ASYMPTOTIC LIMITS : MICROSCOPE

In general the Langevin functions $L_d(x)$ can appear to be unrelated to the canonical statistics of a Heisenberg spin. For example, $L_2(x)$ has appeared to be the (1+1)-dimensional nucleation-controlled polymer crystal growth rate [20,21], with no relevance to the XY spin. In this section we show that the statistics of a Heisenberg spin indeed appears in the asymptotic behavior of the Langevin dynamics with self-harmonic drift.

A. Convergence of the self-harmonic drift and asymptotic behavior of the Langevin process

Since the velocity $d\vec{x}_t/dt$ is, on average, oriented along $\vec{x}_t$ [see (10)] and $L_d(\|\vec{x}_t\|)$ is non-negative for any nonzero $\vec{x}_t$, we may expect that $\vec{x}_t$ most likely grows unbounded for a long time. However, further details are not clear at first glance. We therefore focus first on the evolution of the drift $\vec{a}^*(\vec{x}_t)$. Because of the martingale condition (9) the development of $d\vec{a}^*(\vec{x}_t)$ contains no (drift) term for $O(dt)$ [see (5)], and the result reads

$$d\vec{a}^*(\vec{x}_t) = d\vec{W}_t \bullet (\nabla\vec{a}^*), \quad (19)$$

$$\nabla\vec{a}^* = \frac{dL_d(\chi)}{d\chi}\hat{a}^*\hat{a}^* + \frac{L_d(\chi)}{\chi}(\mathbf{1} - \hat{a}^*\hat{a}^*), \quad (20)$$

where $\hat{a}^* \equiv \vec{a}^*/\|\vec{a}^*\|$ and χ is inversely determined so that $L_d(\chi) = \|\vec{a}^*\|$. Thus, $(\nabla\vec{a}^*)$ is a function of $\vec{a}^*$. The process generated by $d\vec{a}_t = d\vec{W}_t \bullet \mathsf{M}(\vec{a}_t)$, with $\mathsf{M}(\vec{a})$ being any rank-2 tensor as a function of $\vec{a}$, is a martingale associated with the Wiener process, $\vec{W}_s$ $(0 \leqslant s < \infty)$. The self-harmonic drift $\vec{a}^*$ defined above, however, has several additional particular properties, which we will discuss below.

We notice that $\vec{a}^*(\vec{x}_t)$ of each realization converges for $t \to \infty$ and the limit $\vec{a}^*_\infty \equiv \lim_{t\to\infty}\vec{a}^*(\vec{x}_t)$ is on the hypersphere, $\|\vec{a}^*_\infty\| = 1$. The case of $d = 1$ is particularly simple: Eqs. (19) and (20) read

$$da^*_t = \left(1 - a^{*2}_t\right) \bullet dW_t \quad (21)$$

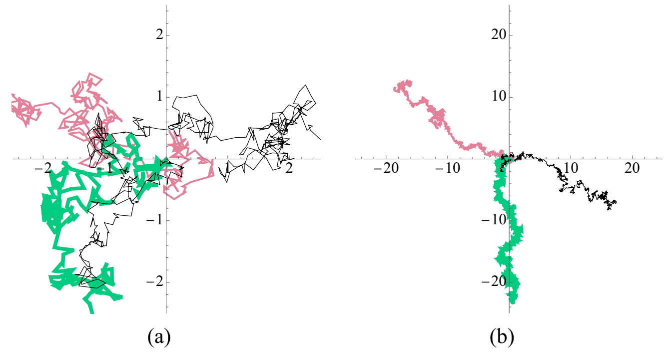


FIG. 1. Two-dimensional trajectories from $\vec{x}_0 = (0, 0)$. (a) Close-up view near the origin. (b) Whole trajectories up to $t = 16$. The corresponding curves for different magnifications are identifiable by their color and/or thickness.

because $(\tanh x)' = 1 - \tanh^2 x$. The multiplicative noise term on the right-hand side causes, as the second order effect of dW_t, a drift of the empirical probability density $P(\alpha, t) := \delta(a^*_t - \alpha)$, which reads $dP(\alpha, t) = -\partial_\alpha[P(\alpha, t)(1 - \alpha^2)] \bullet dW_t + (1/2)\partial^2_\alpha[P(\alpha, t)(1 - \alpha^2)^2]dt$, where Itô's formula has been used. Therefore, the Fokker-Planck (FP) equation for $\langle P(\alpha, t)\rangle$ equivalent to (21) has the potential $\ln[(1 - \alpha^2)^2]$, which causes drift, and the diffusion coefficient $\frac{1}{2}(1 - \alpha^2)^2$. The variable a^*_t should be pushed towards ± 1 and held there. In fact, the singular densities $\delta(a^*_\infty \mp 1)$ and their linear combinations are all stationary solutions of that FP equation.

While the behavior of $\vec{a}^*(\vec{x}_t)$ for $d > 1$ is less clear than that for $d = 1$, the fact that ∇a^* vanishes on the hypersphere $\|\vec{a}^*\| = 1$ (i.e., at $\chi = +\infty$) indicates that the hypersphere is the natural absorbing boundary for the evolution of $\vec{a}^*(\vec{x}_t)$. A more formal and general argument for the existence of the limit is provided by the *convergence theorem of (sub)martingales* (see, e.g., Sec. 4.1.5 of [11]). According to this theorem, because $\vec{a}^*(\vec{x})$ is bounded, i.e., $\|\vec{a}^*\| \leqslant 1$, the martingale process $\vec{a}^*(\vec{x}_t)$ should have a limit:

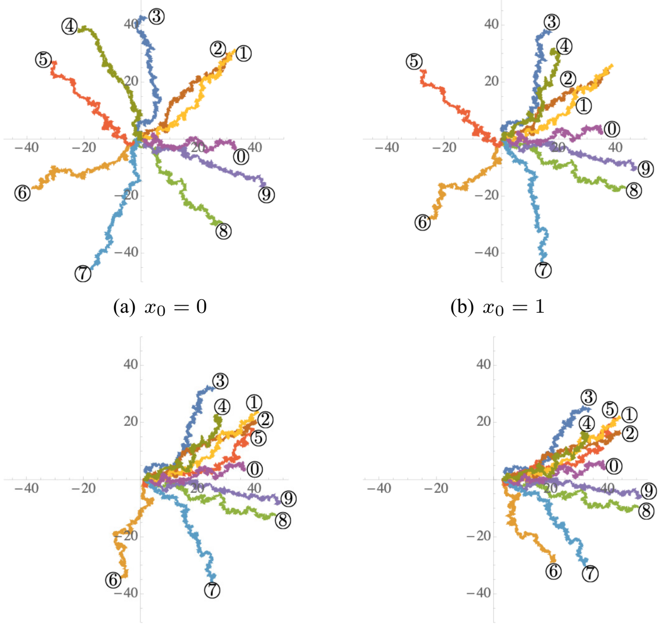
$$\vec{a}^*_\infty \equiv \lim_{t\to\infty}\vec{a}^*(\vec{x}_t). \quad (22)$$

On the other hand, if the limit $\vec{a}^*_\infty$ were not on the natural boundary, the noise for $\vec{x}_t$ could still relocate it. Therefore, $\|\vec{a}^*_\infty\| = 1$ is concluded.

Next, we consider the trajectory of $\vec{x}_t$ that leads to $\lim_{t\to\infty}\vec{a}^*(\vec{x}_t) = \vec{a}^*_\infty$. Figure 1 shows three samples with $d = 2$, starting from the origin, $\vec{x}_0 = 0$. Figure 1(a) is a close-up view of the initial part of the trajectories, and Fig. 1(b) shows the entire trajectory up to $t = 16$, including Fig. 1(a) [22]. In the early stage with $\|\vec{x}_t\| \lesssim 1$, the noise term $\vec{\xi}_t$ dominates over the drift; then there is a slow crossover to the long-time ballistic behavior. Using this observation together with (1) or (6), we understand that $\vec{x}_t$ becomes asymptotically ballistic in the sense that

$$\lim_{t\to\infty}\frac{\vec{x}_t}{t} = \vec{a}^*_\infty, \quad (23)$$

which also implies $\lim_{t\to\infty}\frac{\|\vec{x}_t\|}{t} = 1$. In Appendix B we show (23) in more detail. Note that each realization of the trajectory ends up with a particular orientation of $\vec{a}^*_\infty$.



(a) $x_0 = 0$ (b) $x_0 = 1$ (c) $x_0 = 2$ (d) $x_0 = 4$

FIG. 2. Two-dimensional trajectories from $\vec{x}_0 = (x_0, 0_+)$ with (a) $x_0 = 0$, (b) $x_0 = 1$, (c) $x_0 = 2$, and (d) $x_0 = 4$. In all cases the duration is up to $t = 40$, and we used the identical set of noise histories for $\vec{\xi}_t$ in (1), identifiable by the circled numbers (0–9).

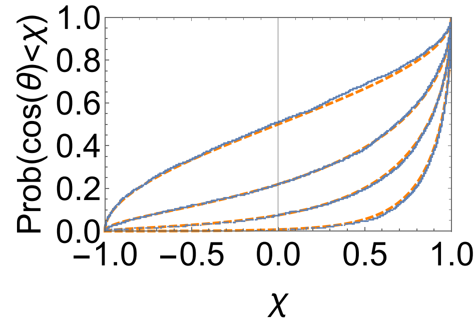


FIG. 3. The two-dimensional orientational distributions of $\lim_{t\to\infty}(\vec{x}_t/t) = \vec{a}^*_\infty$ for the different starting points, $x_0 = 0$, 1, 2, and 4, from top to bottom. We represent the orientation of $\vec{a}^*_\infty$ using $\cos\theta$, where θ is the angle between $\vec{a}^*_\infty$ and $\vec{x}_0 = (x_0, 0)$, and we show the distributions by the empirical cumulative probability (trains of blue points). Numerically, the ensemble of $\vec{a}^*_\infty$ is approximated by 3000 realizations of $\vec{x}_t/t|_{t=40}$. The orange dashed curves represent the canonical equilibrium distribution of a single unitary spin under the nondimensionalized field $\vec{x}_0$ [see (25) in the main text].

B. Distribution of trajectories at $t \to \infty$ as a spin microscope

Each trajectory of $\vec{x}_t$ or of $\vec{a}^*(\vec{x}_t)$ has the limiting value $\vec{a}^*_\infty$ as a random variable, and by the definition of self-harmonicity, we have the martingality:

$$\langle \vec{a}^*_\infty | \vec{x}_0 \rangle = \vec{a}^*(\vec{x}_0). \tag{24}$$

This is a constraint on the statistics of $\vec{a}^*_\infty$. However, it is only for $d = 1$ that the probability of realizing $a^*_\infty = \pm 1$ and the initial data $a^*(x_0)$ are trivially related by (24),

$$\mathrm{Prob}(a^*_\infty = s) = \frac{1 + s\, a^*(x_0)}{2}, \qquad s = \pm 1.$$

By contrast, the statistics of $\vec{a}^*_\infty$ on the $(d > 1)$-dimensional hypersphere surface is by no means trivial. Figure 2 gives a qualitative idea of how the trajectories of $\vec{x}_t$ up to $t = 40$ depend on the initial data, $\vec{x}_0$. Roughly, the larger is $\|x_0\|$ is, the more polarized the orientation of $\vec{x}_t$ is.

We shall parametrize the orientation of $\vec{a}^*_\infty$ by its orthogonal projection, $\cos\theta \equiv \vec{a}^*_\infty \cdot (\vec{x}_0/\|\vec{x}_0\|)$, onto the axis along $\vec{x}_0$, where $0 \leqslant \theta \leqslant \pi$. When $\vec{x}_0 = 0$, the distribution of $\vec{a}^*_\infty$ through (23) must be isotropic, and the cumulative probability distribution of $\cos\theta$ should rigorously obey $\mathrm{Prob}^{(\mathrm{can})}(\cos\theta < \chi) = \frac{1}{\pi}\int_{-1}^{\chi} \frac{d\xi}{\sqrt{1-\xi^2}} = 1 - \frac{1}{\pi}\arccos\chi$. In Fig. 3 the top orange dashed curve represents this formula. The train of blue dots along this curve shows the results obtained from the numerical data over 3000 trajectories, where $\vec{a}^*_\infty = \lim_{t\to\infty} \frac{\vec{x}_t}{t}$ is approximated by $\frac{\vec{x}_t}{t}|_{t=40}$. The deviations from the theoretical curve show the errors due to the finiteness of sampling. The other three trains of blue dots in Fig. 3 represent the numerically obtained cumulative probabilities $\cos\theta$ for $x_0 = 1, 2$, and 4, respectively, in descending order. (Note that we took the cumulative probability because it can be empirically reconstructed just by plotting the normalized rank $r/3000$ vs the corresponding value of $\cos\theta_r$ without any binning or smoothing.)

Quite surprisingly, the results almost surely obey the canonical statistics of a Heisenberg spin under the nondimensionalized field, $\vec{x}_0 = (x_0, 0)$, which leads to the cumulative probability,

$$\mathrm{Prob}^{(\mathrm{can})}(\cos\theta < \chi\,|\vec{x}_0) = \frac{1}{\pi I_0(x_0)} \int_{-1}^{\chi} \frac{e^{x_0\xi}\, d\xi}{\sqrt{1-\xi^2}}. \tag{25}$$

This formula is shown by the dashed orange curves in Fig. 3. Thus, the thermal distribution of the (fictitious) spin orientations that would yield the expectation $\vec{a}^*(\vec{x}_0)$ has been mapped onto the static distribution of the "spin" $\vec{a}^*_\infty$ characterizing the trajectory of $\vec{x}_t$ in the limit of $t \to \infty$. Being consistent with (24), our proposition for the probability density $\rho(\vec{a}^*_\infty)$ is therefore

$$\rho(\vec{a}^*_\infty) = \frac{e^{\vec{x}_0 \cdot \vec{a}^*_\infty}}{Z_d(\|\vec{x}_0\|)}, \qquad \vec{a}^*_\infty \in S^{d-1}, \tag{26}$$

where $\vec{a}^*_\infty$ means $\lim_{t\to\infty} \frac{\vec{x}_t}{t}$ and S^{d-1} is the $(d-1)$ surface of the d-dimensional unit sphere. This mapping from $\vec{x}_0$ to the distribution of $\lim_{t\to\infty} \frac{\vec{x}_t}{t}$ serves as a *microscope,* which allows us to assess $\vec{x}_0$ through the repeated measurements of $\vec{x}_t$ at sufficiently large t. Notice that the mapping $\vec{x}_0 \mapsto \mathrm{Prob}^{(\mathrm{can})}(\cos\theta < \chi\,|\vec{x}_0)$ is asymptotically independent of micro-macro ratios such as $\|\vec{x}_0\|/\|\vec{x}_t\|$. This is in contrast to the case of assessing the initial position $\vec{x}_0$ of a Brownian particle from its position at a later time $\vec{x}_t$. In the latter case the signal-to-background ratio lessens with time. See Appendix C for further explanation.

IV. DISCUSSION

The present work leads to two things. First, we have found a connection between the Langevin equation and the Langevin functions through the martingale process. Usually, the relationship between the Langevin equation and the canonical distribution is through Einstein's relation by which the stationary state of the Langevin equation becomes canonical (if the drift has a potential). In the present case, however, neither the evolution of $\vec{x}_t$ nor that of $\vec{a}(\vec{x}_t)$ has *regular* stationary density; $\vec{x}_t$ grows unboundedly, while $\vec{a}(\vec{x}_t)$ is trapped asymptotically at a point on the surface of the unit hypersphere. Nevertheless, the canonical spin statistics, which could give the Langevin function as a response, emerges in the distribution of the asymptotic limit of $\vec{x}_t/t$. Second, our report is another example in which the self-referential or fixed-point condition leads to a nontrivial outcome. We note that the celebrated diagonal arguments by Cantor and by Gödel were later reformulated from a unified viewpoint using the self-reference and fixed points [23,24]. Usually, the martingale process Y_t is constructed from a reference process X_t so that we can apply useful theorems of the martingale theory to the former and, eventually, we gain insights about the reference process. The present setup asks, instead, with what drift $\vec{a}^*$ defining the reference process X_t does the former play the role of Y_t. The underlying physics by which the martingale brings the Langevin function is unknown. That the linearizing transformation of the Riccati equation, $L_d(x) = Z_d'(x)/Z_d(x)$, takes the form of the canonical response might be a clue to this mystery. Also unknown is how the martingale constraint transmits the initial molecular field $\vec{x}_0$ to the asymptotic distribution of spin $\vec{a}^*_\infty$. In short, the present findings bring us questions, rather than answers, about the relevance of martingales in physics beyond being a mere mathematical tool. We therefore would like to conclude our paper with several open questions: Can the space-time harmonic function (see, e.g., Sec. 3.2.5.3 of [11]) also be a fixed point? Can the microscope idea developed below (26) be generalizable to nonmartingale drift? Are there links to the stochastic thermodynamics [25–27]? Can there be a parallel framework for the state space with topologies other than the Euclidean one? Will there be a quantum counterpart? Does the self-harmonic drift optimize some physical entity or information?

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APPENDIX A: FAMILY OF SELF-HARMONIC DRIFTS OBTAINED BY RESCALING

For different interpretations of $\vec{x}_t$ the self-harmonicity of the drift implies different constraints between the drift and noise terms. For example, when $\vec{y}_t$ represents a position of a particle in a heat bath of temperature T and friction coefficient γ, the choice $(\alpha, C) = (1, 2\mathcal{D})$, with $\mathcal{D} = \frac{k_B T}{\gamma}$ in (18), leads to

$$d\vec{y}_\tau = \frac{2k_B T}{\gamma} L_d(\|\vec{y}_\tau\|)\hat{y}_\tau d\tau + \sqrt{\frac{2k_B T}{\gamma}}\, d\vec{W}_\tau. \tag{A1}$$

Since $L_d(\|\vec{y}_\tau\|)\hat{y}_\tau = \nabla \ln Z_d(\|\vec{y}\|)$, this equation describes the motion in the potential $[-2k_B T \ln Z_d(\|\vec{y}\|)]$. To ensure the self-harmonicity the energy scale of the potential is thus constrained.

Another case to apply (18) is through an analogy with the progressive quenching of d-dimensional Heisenberg spins on a complete network [14–17], which we mentioned in the last paragraph of Sec. I. In this context the appropriate choice is $(\alpha, C) = (\frac{J}{k_B T}, \frac{k_B T}{J})$, which brings (18) into the following:

$$d\vec{y}_\tau = L_d\left(\frac{J\|\vec{y}_\tau\|}{k_B T}\right)\hat{y}_\tau d\tau + \sqrt{\frac{k_B T}{J}}\, d\vec{W}_\tau. \tag{A2}$$

In this picture we regard $L_d(J\|\vec{y}_\tau\|/k_B T)\hat{y}_\tau$ on the right-hand side of (A2) as the equilibrium mean of a Heisenberg spin under the molecular field $J\vec{y}_\tau$, with J being a spin-spin coupling constant and $k_B T$ being the temperature. Then a new fragment of spin $d\vec{y}_\tau$ appears at every interval $d\tau$ and joins the quenched part $\vec{y}_\tau$. While the conditional mean of $d\vec{y}_\tau$ follows the canonical statistics, there is a Wiener noise $\propto d\vec{W}_\tau$ on top of it. To ensure the self-harmonicity the amplitude of the noise must be $\sqrt{k_B T/J}$.

APPENDIX B: LONG-TIME CONVERGENCE OF $\vec{x}_t/t$

In the main text we saw that, for every particular trajectory of $\vec{a}^*(\vec{x}_t)$ starting from $x_t = x_0$, there is a limit $\vec{a}^*_\infty$ with $\|\vec{a}^*_\infty\| = 1$. Below we will see that the ratio $\vec{x}_t/t$ also converges to $\vec{a}^*_\infty$.

Once we admit the existence of the limit $\vec{a}^*_\infty$, a u exists such that $\|\vec{a}^*(x_t) - \vec{a}^*_\infty\| < \epsilon\ \forall t \geqslant u$ for a given constant ϵ (>0). When we integrate (6) over the interval $[u, t]$ as

$$\vec{x}_t = \vec{x}_u + \int_u^t \vec{a}^*(\vec{x}_s)ds + \int_{s=u}^{s=t} dW_s, \tag{B1}$$

we see that $\vec{x}_u/t \sim t^{-1}$ and $\int_{s=u}^{s=t} dW_s/t \sim t^{-1/2}$ for $t \to \infty$. We therefore focus on $\int_u^t \vec{a}^*(\vec{x}_s)ds$, which reads

$$\int_u^t \vec{a}^*(\vec{x}_s)ds = (t-u)\vec{a}^*_\infty + \int_u^t [\vec{a}^*(\vec{x}_s) - \vec{a}^*_\infty]ds.$$

While the magnitude of the integrand on the right-hand side is already bounded by ϵ, we expect the integrand to decay almost like s^{-1} because for large $\vec{x}_s$ the Langevin equation (1) means roughly $d\vec{x}_s/ds \simeq [1 - O(1/\|\vec{x}_s\|)]\hat{x}_s + d\vec{W}_t$, where we notice $L_d(x) = 1 - O(x^{-1})$ for $x \to \infty$. Therefore, the time integral on the right-hand side is dominated by $(t-u)\epsilon$. Dividing each term in (B1) by t, we reach the claimed result:

$$\lim_{t\to\infty} \frac{\vec{x}_t}{t} = \vec{a}^*_\infty, \quad \lim_{t\to\infty} \vec{a}^*(\vec{x}_t) = \vec{a}^*_\infty.$$

APPENDIX C: INFERRING THE INITIAL POSITION OF A BROWNIAN PARTICLE *A POSTERIORI*

When a Brownian particle starts from $\vec{x}_0$, its position at time t, that is, $\vec{x}_t = \vec{x}_0 + \vec{W}_t$, is a martingale associated with the Wiener process $\vec{W}_t$. While the mean $\langle \vec{x}_t | \vec{x}_0 \rangle$ remains $\vec{x}_0$, the variance $\langle (\vec{x}_t - \vec{x}_0)^2 | \vec{x}_0 \rangle$ grows linearly in time. Therefore, unlike the case of self-harmonic drift, the inference of $\vec{x}_0$ through the observations of $\vec{x}_t$ at a "macroscopic" distance R ($\gg \|\vec{x}_0\| \geqslant 0$) is hard to realize under the signal-to-background ratio decaying with R. In fact, through the analysis of the exit problem [28], the detected position on the circle, $\|\vec{x}\| = R$, obeys the cumulative probability

$$\begin{aligned} \text{Prob}(\cos\theta < \chi) &= \frac{1-\epsilon^2}{2}\left(\frac{1}{\sqrt{1-2\epsilon\chi+\epsilon^2}} - \frac{1}{1+\epsilon}\right) \\ &= \frac{1+\chi}{2}\left[1 - \frac{3(1-\chi)}{2}\epsilon + O(\epsilon^2)\right], \end{aligned} \tag{C1}$$

where $\epsilon \equiv \|\vec{x}_0\|/R$. We see that the signal, the term $-\frac{3(1-\chi)}{2}\epsilon$, vanishes in the macroscopic limit $\epsilon \to 0$.

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