Letter | Editors' Suggestion

Universal to nonuniversal transition of the statistics of rare events during the spread of random walks

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Through numerous experiments that analyzed rare event statistics in heterogeneous media, it was discovered that in many cases the probability density function for particle position, $P(X,t)$, exhibits a slower decay rate than the Gaussian function. Typically, the decay behavior is exponential, referred to as Laplace tails. However, many systems exhibit an even slower decay rate, such as power-law, log-normal, or stretched exponential. In this study, we utilize the continuous-time random walk method to investigate the rare events in particle hopping dynamics and find that the properties of the hop size distribution induce a critical transition between the Laplace universality of rare events and a more specific, slower decay of $P(X,t)$. Specifically, when the hop size distribution decays slower than exponential, such as $e^{-|x|^\beta}$ ($\beta > 1$), the Laplace universality no longer applies, and the decay is specific, influenced by a few large events, rather than by the accumulation of many smaller events that give rise to Laplace tails.

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According to Wikipedia, "Rare or extreme events are events that occur with low frequency, and often refer to infrequent events that have widespread impact and might destabilize systems". Notable examples of rare events include the stock market crash [1], earthquakes [2], and cyclones [3]. The frequency and the scale of rare events in each field are very important. If the rare events are not "too rare" and large enough, then the usual statistical behavior is completely dictated by such rare events. A perfect example of situations when rare events fundamentally modify the nature of a physical process is sub-diffusion [4–10]. While for normal diffusion, the mean squared displacement (MSD) of a tracked particle grows linearly with time, for subdiffusion, the MSD grows in a nonlinear fashion, i.e., $\langle X^2\rangle \sim t^\alpha$, while $0<\alpha<1$. For systems where the transport is defined by the existence of trapping regions, e.g., glasses [11], filamentous networks [12] or living cells [13,14], the distribution of the trapping times defines the type of diffusion. When the mean trapping time is finite, diffusion is normal. But when the trapping time distribution decays as a power law, $\psi(\tau) \sim \tau^{-1-\alpha}$, subdiffusive behavior emerges when $\alpha < 1$. The rare events, in the form of long trapping times, are responsible for transforming the universal linear growth of the MSD to a specific, i.e., α-dependent, sublinear growth.

Subdiffusive behavior may stem from different sources, such as persistent long-range correlations between positional increments (as exhibited by models like fractional Brownian motion [15]) or large heterogeneity in the system [11] and a broad spectrum of interactions. However, in systems where subdiffusion arises from trapping events, the dynamics can give rise to unusual physical phenomena, such as aging [16–18], weak ergodicity breaking [16,19,20], and non-self-averaging [16,21–25]. Nevertheless, there are cases where such phenomena occur even when trapping times are not involved [26]. Theoretical models where trapping is present, like the continuous time random walk (CTRW) [27] and the quenched trap model [28] show that there is a critical transition to subdiffusion, aging, and nonergodic behavior, when the mean trapping time is diverging. This transition is also accompanied by a transformation of the positional probability density function (PDF). The universal Gaussian center of the positional PDF turns into an α-stable Lévy type [29,30], i.e., shape that depends on a specific parameter α of the trapping time distribution. This transition between a universal PDF that is determined by an accumulation of many events, and a nonuniversal PDF that follows the properties of one single rare event is a key feature associated with the appearance of new physical phenomena. Our study highlights the transition from the universal to the nonuniversal shape of the positional PDF, focusing on the decay properties at large $|X|$ values of the positional PDF. The identification of this transition within the tails of the PDF suggests the presence of an underlying critical transition, unveiling the potential for unforeseen physical phenomena. In the nonuniversal regime, we derive a general formula for the PDF that is applicable to many decay forms, including, power laws, log normal, and stretched exponential.

Recently, a huge number of experiments demonstrated exponential (and not Gaussian) decay of the PDF for large $|X|$. Such decay of the PDF is termed as Laplace tails [31,32]. Notable examples include colloidal beads [33], zooplanktons [34], glass-forming liquids [35], nanoparticles in polymer melts [36], particles at the liquid-solid interface [37], and close to glass and jamming transitions [38,39], etc. The CTRW model was used [40–42] to prove the suggested universal convergence [43] to Laplace tails for a broad class of processes, showing that the PDF tails are governed by

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the accumulation of multiple events and decay exponentially (with logarithmic corrections).

While the exponential decay of PDF tails has established itself as a universality class [40,41], the decay behavior of $P(X,t)$ for a wide class of systems is even slower. For example, a power-law decay of the PDF $P(X,t)$ has been observed in nonequilibrium cytoquake dynamics in cytoskeletal stabilization [44], heavy-tailed fluctuations of active arrays in cytoskeleton [45], swimming algal cells on two-dimensional films [46], active steps of Janus particles under chemotaxis [47], displacement patterns of private vehicles in Italy [48], extreme price fluctuations in bitcoin markets [49] and the famous inverse-cube laws of econophysics [50,51]. Another class of heavy tailed distributions like the log-normal have been observed for example, in the individual modes of human transportation [52], chromatin dynamics [53], intra-cellular reaction dynamics [54], single molecule fluorescence bursts [55], multimedia file size [56], post lengths involving internet discussions [57], wealth distribution of low-middle income groups [58], and the returns distribution in Nairobi securities exchange [59], while stretched-exponential decay has been observed in local intermittent movement in a sporting arena [60], internet media access patterns [61], empirical distribution of stock returns [62], and avalanche sizes in superconducting vortices [63], to mention a few. In heterogeneous media power-law fluctuations can emerge as a result of force dipoles [64] while stretched-exponentials can arise due to multiplicative processes [65,66]. Following the above discussion we ask: do the universal Laplace tails act as an attractor even when we take into account heavy tailed jumps? Or there is a breakdown of universality similar to the critical transition from diffusion to subdiffusion? In other words, are the PDF tails of a tracer particle determined by the occurrence of one single rare event [67–71] or they represent an accumulation of many realizations of not-so-large but more frequent events [72–76]? Similar to the case of subdiffusion to diffusion transition, we use the celebrated CTRW model, originally exploited for explaining transport in amorphous materials [26,77–79]. In CTRW, a particle performs random jumps in space and waits for a random amount of time between jumps. All the jumps and waiting times are independent and identically distributed (IID) random variables. The distribution of a jump x is given by $f(x)$, while each waiting time τ has a distribution $\psi(\tau)$. The position of the process X at time t is determined by the random number of jumps N_t performed by time t, i.e., $X = x_1 + \cdots + x_{N_t}$ where x_i, is the size of a single jump. The positional PDF $P(X,t)$ is readily obtained in terms of the subordination equation by conditioning on the number of jumps N [4,28,80,81],

$$P(X,t) = \sum_{N=0}^{\infty} P_N(X) Q_t(N), \tag{1}$$

where $P_N(X)$ is the distribution of X for a given $N(=N_t$ for a fixed measurement time $t)$ and $Q_t(N)$ is the distribution of the number of jumps up to time t. The mentioned phenomena, like anomalous transport and aging, appear for CTRWs with $\psi(\tau) \sim \tau^{-1-\alpha}$ $(\tau \to \infty)$ when $\alpha < 1$. For such $\psi(\tau)$, the sum $\tau_1 + \cdots + \tau_{N_t}$ is dominated (in the $t \to \infty$ limit) by the maximal summand [82], as can be observed from the behavior of

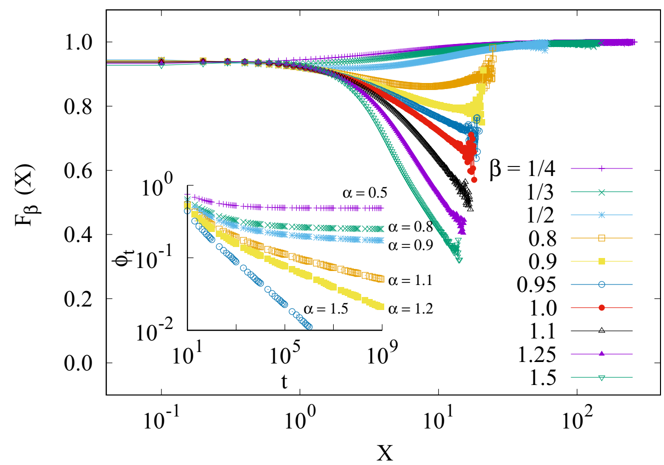

FIG. 1. $F_\beta(X)$ for a CTRW with jumps $x \sim f(x)$ with mean zero and variance one as a function of β. The waiting time distribution $\psi(\tau) = e^{-\tau}$ and the measurement is done at $t = 0.6$ for trajectories reaching a position $\pm X$. The inset shows the behavior of $\phi_\alpha(t)$ for $\psi(\tau) \sim \tau^{-1-\alpha}$ for large τ for different values of α from top to bottom.

$$\phi_\alpha(t) = \left\langle \frac{\max\{\tau_1, \ldots, \tau_{N_t}\}}{\sum_{i=1}^{N_t} \tau_i} \right\rangle \tag{2}$$

for large t. While the maxima of a set of random variables has been extensively studied [83–86], the definition of $\phi_\alpha(t)$ has the advantage that it belongs to the unit interval [0, 1], making it an appropriate order parameter. For more details see the Supplementary Material (SM) [87]. We see in Fig. 1 (inset) that $\phi_\alpha(t)$ saturates to a finite value for $\alpha = 0.5,\ 0.8,\ 0.9$ while for $\alpha = 1.1,\ 1.2,\ 1.5$ exhibits a decaying behavior at large times. This difference in the properties of $\phi_\alpha(t)$ follows from the fact that for $\alpha < 1$, $\langle \tau \rangle = \int_0^\infty d\tau\, \tau\, \psi(\tau) \to \infty$, while it is finite for $\alpha > 1$. This implies that as we move from $\alpha > 1$ to $\alpha < 1$, the rare fluctuations in the sequence of waiting times $\{\tau_1, \ldots, \tau_{N_t}\}$ exhibit qualitatively different behaviors. Consequently, the thermodynamic limit in which the system behaves extensively (all waiting times τ_i are of the same order of magnitude) ceases to exist for $\alpha < 1$. Can we see similar behavior if we focus on spatial fluctuations? This question comes up naturally once we realize that the system has access to the entire phase space in the thermodynamic limit. The phase space can be swept by focusing on large spatial fluctuations without going to the $t \to \infty$ limit. Namely, we take the large $|X|$ limit at fixed t and expect the system to visit many states during its evolution.

Motivated by this, let us look at a CTRW described by Eq. (1) with jumps following $f(x) \sim e^{-|x|^\beta}$ with $\beta > 0$ [88] for reasons that will be clear shortly. In analogy with $\phi_\alpha(t)$ we define,

$$F_\beta(X) = \left\langle \frac{\max\{|x_1|, \ldots, |x_{N_t}|\}}{\sum_{i=1}^{N_t} |x_i|} \right\rangle. \tag{3}$$

We see in Fig. 1 that even for a well behaved $\psi(\tau)$ like the exponential distribution, $\lim_{|X|\to\infty} F_\beta(X)$ decreases monotonically for $\beta \geqslant 1$ and seemingly takes a finite value for trajectories with $\beta < 1$. Now, for large X, $F_\beta(X) \approx \sum_{N=1}^{\infty} Q_t(N) \frac{1}{XP(X,t)} \int_{x_m} dx_m x_m p(x_m, X|N)$ with $X > 0$ [87]. Here $x_m = \max\{x_1, \dots, x_N\}$ and $p(x_m, X|N)$ is the joint distribution of x_m and X such that $\sum_{i=1}^{N} x_i = X$. As x_i are IID, $p(x_m, X|N) = Nf(x_m)p_{N-1}(X - x_m)$ where $p_{N-1}(X - x_m)$ is the distribution of the sum $\sum_{i=1}^{N-1} x_i$ such that $x_i \leqslant x_m \, \forall \, i = 1, ..., N-1$. This is where the regions $\beta < 1$ and $\beta \geqslant 1$ become different. For $\beta < 1$ the function $x_m p(x_m, X|N)$ is maximal around $x_m \sim X$ and thus, the integral above can be extended to the whole real line leading to $\int_{x_m} dx_m x_m p(x_m, X|N) \approx XP_N(X) \Rightarrow F_\beta(X) \approx 1$. On the other hand, for $\beta \geqslant 1$ the function is peaked at $x_m \approx X/N$. However, as $x_m \geqslant X/N$ only the tail part of $x_m p(x_m, X|N)$ contributes to the integral, leading to smaller values as more jumps are taken into account. In the limiting cases, $F_\beta(X) \to 0$ (see the SM for details [87]). In other words,

$$\lim_{|X| \to \infty} F_\beta(X) \approx \begin{cases} 1, & \beta < 1 \\ 0, & \beta \geqslant 1, \end{cases} \tag{4}$$

reminiscent of a jump transition with the transition point at $\beta = 1$. While the analysis above has been performed in the limit of $|X| \to \infty$ and fixed t, a pertinent question at this point is how does the transition modify at large times, i.e., fixing $|X|$ and taking $t \to \infty$. In this limit, the process will reach X executing a large number of jumps such that $X/N \ll 1$. As a result, $p_{N-1}(X - x_m)$ can be easily approximated by a Gaussian (following the central limit theorem), and due to its rapidly decaying nature, the integral $\int_{x_m} dx_m \, x_m p(x_m, X|N)$ exhibits a saturating behavior only for sufficiently small β like $\beta = 1/2$ or 0.6, while $F_\beta(X)$ for higher values of β exhibit a decaying behavior for large X (see the SM for details [87]). This suggests that for large t the transition boundary will no longer be sharp as compared to the previously discussed case for small t. Generally speaking, in the limit of large t the critical transition will be smeared out.

To summarize, in the limit of large $|X|$ and fixed t for $\beta < 1$, a single maximal jump contributes to the fraction significantly, while for $\beta \geqslant 1$ its contribution is comparable to all the other jumps of the CTRW. While the transition for $\phi_\alpha(t)$ occurs when the the temporal moments ($\langle \tau^n \rangle$) diverge, in the case of $F_\beta(X)$ all the spatial moments ($\langle X^n \rangle$) are finite. Nevertheless, the effective behavior of $\lim_{t\to\infty} \phi_\alpha(t)$ and $\lim_{|X|\to\infty} F_\beta(X)$ is similar. As a function of the parameters α/β, there is a transition from a state defined by the accumulation of many events to a state dominated by a single large event.

At this point, it is worth noticing that the role played by $\phi_\alpha(t)$ for describing temporal fluctuations is taken over by $F_\beta(X)$ for spatial fluctuations. While the emergence of a non-Gaussian center accompanying the transition of $\phi_\alpha(t)$ is well understood [29,30] the corresponding behavior of $P(X, t)$ accompanying the transition of $F_\beta(X)$ is not known. It was shown in Ref. [40] that $P(X, t)$ exhibits universal tails exhibiting exponential decay for $\beta > 1$ and $\psi(\tau)$ analytic near $\tau = 0$. Furthermore, just like $\phi_\alpha(t) \to 0$ marks the emergence of a universal Gaussian center following the central limit theorem, $F_\beta(X) \to 0$ at $|X| \to \infty$ characterizes the universal exponential tails of $P(X, t)$ [40]. On the other hand, a finite value of $\phi_\alpha(t)$ for $\alpha < 1$ is reminiscent of the α dependent Lévy PDF characterized solely by the rare fluctuations in the temporal domain. Given that we observed the convergence of $F_\beta(X)$ to a nonzero value as $X \to \infty$ for $\beta < 1$, we aim to explore the properties of the tails of $P(X, t)$ within the $\beta \leqslant 1$ regime.

For $\beta < 1$ the distribution of jumps belongs to the class of stretched exponential distributions [67] which possesses heavy tails as $\int_0^\infty dx \, e^{\lambda x} f(x) = \infty \, \forall \, \lambda > 0$ and does not admit a large deviation form [76]. But, the family of stretched exponential distributions satisfies the big jump principle $P(x_1 + \cdots + x_N \geqslant X) \overset{|X|\to\infty}{\sim} P(\max\{x_1, ..., x_N\} \geqslant X)$[67,89–92], with the distribution of maxima evaluating to $1 - [1 - \int_X^\infty dx f(x)]^N$ for IID x_i. Hence, from Eq. (1) we have

$$\int_X^\infty dX' \, P(X', t) \overset{|X|\to\infty}{\sim} 1 - G_t\left(1 - \int_X^\infty dx \, f(x)\right), \tag{5}$$

where $G_t(z) = \sum_{N=0}^{\infty} z^N Q_t(N)$. Furthermore, for large $|X|$ we have $\int_X^\infty dx \, f(x) \sim 0$, as a result we can analyze $P(X, t)$ in terms of the behavior of $G_t(z)$ for z in the neighborhood of unity. Now $G_t(1 - \eta) \approx G_t(1) - \frac{\partial G}{\partial z}|_{z=1} \eta$ for small η and $\frac{\partial G}{\partial z} = \sum_{N=1}^{\infty} NQ_t(N) z^{N-1}$. This implies $G_t(1 - \int_X^\infty dx \, f(x)) \approx 1 - \langle N_t \rangle \int_X^\infty dx \, f(x)$ and from here it follows that ([87])

$$P(X, t) \overset{|X|\to\infty}{\sim} \langle N_t \rangle f(X). \tag{6}$$

Equation (6) implies that the probability of being at a location X at time t equals the mean number of jumps $\langle N_t \rangle$ up to time t times the distribution of a single jump $f(X)$. It is to be noted at this point that in the derivation of Eq. (6) we have nowhere explicitly used the fact that $f(x)$ is stretched-exponential and it holds for heavy-tailed distributions in general, including power laws, lognormal, and stretched exponentials. With the distribution of a single jump known, we only need to estimate $\langle N_t \rangle$. The mean number of jumps $\langle N_t \rangle$ attains a simple form in Laplace domain [27] $\langle \tilde{N}_s \rangle = \frac{\tilde{\psi}_s}{s(1-\tilde{\psi}_s)}$, where $\tilde{\psi}_s = \int_0^\infty dt \, e^{-st} \psi(t)$ is the Laplace transform of $\psi(\tau)$. The PDF $P(X, t)$ derived in Eq. (6) holds at arbitrary times, and is in excellent agreement with numerical simulations as seen from Fig. 2 for different classes of heavy tailed distributions: **(a)** power law, **(b)** log-normal, **(c)** generalized Gaussian forms, and **(d)** Cauchy distribution. Notwithstanding the fact that $P(X, t)$ for different $f(x)$ look similar, deciphering the behavior of the latter from the former is a difficult task from the perspective of experiments [65,93]. The average number of jumps $\langle N_t \rangle$ that appears in Eq. (6) was previously found to be an important quantity determining the tails of the PDF for Lévy walks [89,91,94].

Let us now consider the point $\beta = 1$. As this is a transition point on the β axis, separating the regions $\beta > 1$ and $\beta < 1$, it should be approached with caution. For example, for $\beta = 1$, $f(x) = \frac{a}{2} \exp(-a|x|)$, which implies that the large deviation form of the distribution of position X after N jumps is $P_N(X) \sim \exp[-NI_N(\frac{|X|}{N})]$ with the rate function $I_N(\frac{|X|}{N}) = -\frac{a|X|}{N} + \log(\frac{a|X|}{N})$. This implies that for large deviations, the distribution of the sum possesses exponentially decaying tails with logarithmic corrections. For $\psi(\tau)$ analytic near zero

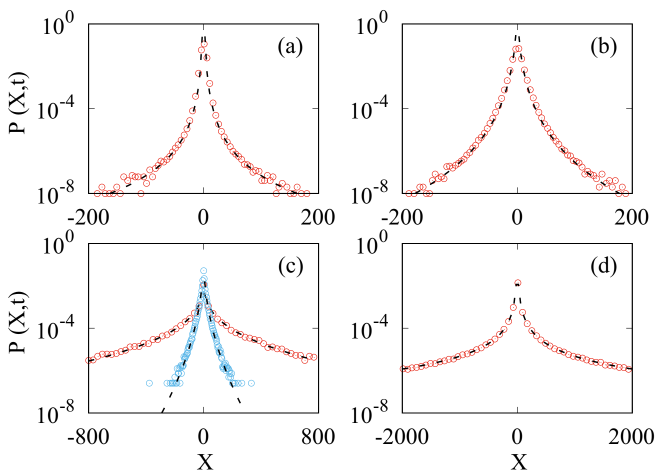

FIG. 2. Numerically estimated $P(X,t)$ for a CTRW with jump distribution $f(x)$ and waiting time distribution $\psi(\tau)$. (a) $f(x) = \frac{3}{2x^4}$ with $|x| \geqslant 1$, $\psi(\tau) = \tau^2 e^{-\tau}/2$ at $t = 15$. (b) $f(x) = \frac{1}{2\sqrt{2\pi}|x|}\exp(-(\ln|x|)^2/2)$, $\psi(\tau) = e^{-\tau}$ at $t = 6$. (c) $f(x) = \frac{\beta}{2\Gamma(1/\beta)}e^{-|x|^\beta}$ with $\beta = 1/3$, for $\psi(\tau) = \sqrt{2/\pi}e^{-\tau^2/2}$ (red) at $t = 0.4$ and $\beta = 1/2$ for $\psi(\tau) = \frac{1}{(1+\tau)^2}$ (blue) at $t = 1.5$. (d) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $\psi(\tau) = e^{-\tau}$ at $t = 15$. The circles represent numerically estimated $P(X,t)$ and the black dashed lines are the analytical form Eq. (6) with appropriate $f(x)$.

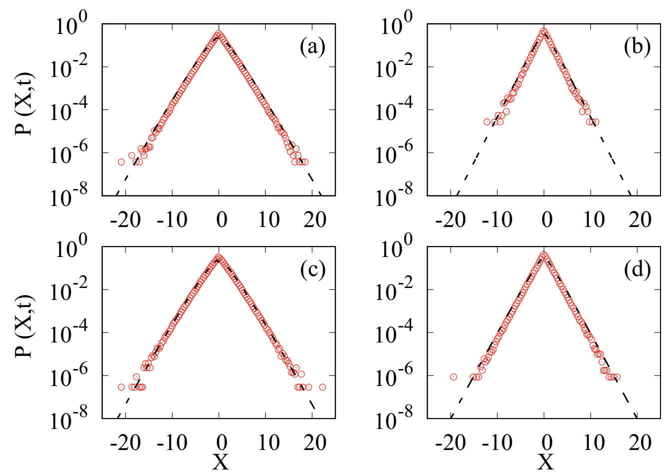

FIG. 3. Comparison of numerically estimated $P(X,t)$ (red circles) against the solution given in Eq. (7) (black dashed line). The waiting time distributions are the following: (a) exponential mixture $\psi(\tau) = p_1 r_1 e^{-r_1\tau} + p_2 r_2 e^{-r_2\tau}$ with $r_1 = 1/4, r_2 = 5/2, p_1 = 1/4, p_2 = 3/4$ at $t = 0.7$; (b) gamma distribution $\psi(\tau) = \tau^3 e^{-\tau}/6$ at $t = 0.8$; (c) half-Gaussian distribution $\psi(\tau) = \sqrt{2/\pi}e^{-\tau^2/2}$ at $t = 1.5$; (d) power-law distribution $\psi(\tau) = 1/(1+\tau)^2$ at $t = 0.4$.

$\psi(\tau) \overset{\tau\to 0}{\sim} C_A\tau^A + C_{A+1}\tau^{A+1} + C_{A+2}\tau^{A+2} + \cdots$ [40], where A is a non-negative integer, $Q_t(N)$ admits a large deviation form $Q_t(N) \overset{N\to\infty}{\sim} \exp[-NI_N(t)]$ with a universal rate function [40,95] $I_N(t) = -\frac{C_{A+1}}{C_A}\frac{t}{N} - (A+1)[1 + \log\{\frac{(C_A\Gamma(A+1))^{\frac{1}{A+1}}}{A+1}\frac{t}{N}\}]$. Using $P_N(X)$ from above and $Q_t(N)$ in Eq. (1) we have for large $|X|/t$ [87]

$$P(X,t) \underset{|X|/t\to\infty}{\sim} \sqrt{\frac{2\pi}{A+2}\mu(a|X|)^{\frac{1}{A+2}}} \times \exp\left[-t\left\{C + \frac{a|X|}{t} - \left(C_A\Gamma(A+2)\frac{a|X|}{t}\right)^{\frac{1}{A+2}}\right\}\right]. \tag{7}$$

From Fig. 3 we see that the large deviation form of $P(X,t)$ evaluated in Eq. (7) agrees with numerically estimated $P(X,t)$ for different waiting time distributions. In other words, $P(X,t)$ possesses exponentially decaying tails in the limit of large $|X|/t$ when the distribution of jumps is Laplace distributed.

The results in Eqs. (6) and (7), in conjunction with the large deviation description in Ref. [40], show that two distinct behaviors of $P(X,t)$ exist: (I) universal exponential decay when $f(x)$ decays as $\exp(-x/a)$ or faster and (II) very specific decay dictated by the form of $f(x)$, i.e., Eq. (6), when the jump size distribution decays slower than exponential. The intricate differences between $\beta = 1$ and $\beta > 1$ (see the SM) imply that the PDF of a CTRW critically changes at $\beta = 1$, the value for which the order parameter $F_\beta(X)$ shows a critical transition. The evaluation of $P(X,t)$ further corroborates our assertion of a universal to nonuniversal transition as seen from the analysis of $F_\beta(X)$ (see Fig. 1). The fact that $\lim_{|X|\to\infty}F_\beta(X) = 0$ for $\beta \geqslant 1$ is analogous to saying that $P(X,t) \sim \exp[-tI(|X|/t)]$ exists with a nontrivial rate function $I(|X|/t)$ for every $\beta \geqslant 1$. This rate function $I(z)$ attains a linear growth for large z and, therefore, the universal exponential decay of the PDF. On the other hand, for $\beta < 1$, $\lim_{|X|\to\infty}F_\beta(X) = 1$ and Eq. (6) further shows that for $\beta < 1$, the rate function $I(|X|/t)$ is trivially zero. The decay of $P(X,t)$ can attain any mathematical form that decays slower than the exponential, like power-law, stretched-exponential or log normal. Notice that the transition from universal to non-universal behavior, as determined by $F_\beta(X)$, takes place at finite times but for large values of X. In contrast to the Gaussian to α stable Lévy transition, which is characterized by $\phi_\alpha(t)$, occurs at large times but for finite values of X. Furthermore, the diffusion-to-subdiffusion transition is accompanied by divergences of the mean trapping time, the critical transition reported in the present study is free from such divergences.

Hopping dynamics which is an intrinsic feature of CTRW, has been ubiquitously observed in polymer melts [96], colloidal suspensions [97], rodlike particles through smectic layers [98,99], polymer glasses [100], binary mixtures [101], in one, two, and three spatial dimensions [102], to mention a few. A characteristic feature of motion in glassy materials [38,39,43,103] and at the liquid-solid interface [37,104], where hopping dynamics is observed, has been the exponential decay of the tails of the positional PDF. In sharp contrast there are a class of systems which are intrinsically out of equilibrium and are either active like dynamics of cytoskeleton [44,45], swimmers [46], Janus particles [47], chromatin dynamics [53], intracellular reactions [54] or involving multiple players like market fluctuations [49–51,59,62], income distributions [58], movement of vehicles [48] or individuals moving in a sporting arena [60], etc., where the jump sizes typically exhibit a slower than exponential decay. In the present work we show that such out of equilibrium systems belong to a totally different class for which the rare fluctuations are singular and not characterized by an accumulation of many events.

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