Letter

Estimating the master stability function from the time series of one oscillator via reservoir computing

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The master stability function (MSF) yields the stability of the globally synchronized state of a network of identical oscillators in terms of the eigenvalues of the adjacency matrix. In order to compute the MSF, one must have an accurate model of an uncoupled oscillator, but often such a model does not exist. We present a reservoir computing technique for estimating the MSF given only the time series of a single, uncoupled oscillator. We demonstrate the generality of our technique by considering a variety of coupling configurations of networks consisting of Lorenz oscillators or Hénon maps.

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Synchronization is an emergent behavior in which a number of interacting oscillators do the same thing at the same time [1]. Synchronization is a crucial phenomenon in various systems, such as power grids [2], neuronal networks [3], cardiac tissue [4], and chemical [5], electronic [6], optoelectronic [7,8], and laser [9] systems. When the oscillators are discrete objects, the interactions can be described by a network, and so the synchronization of oscillator networks has long been a rich area of research [1,10,11].

The problem of global synchronization and its stability was solved by the master stability function (MSF) approach [12]. Once determined, the MSF gives the stability of the globally synchronized state of a network of identical oscillators for any network topology from the eigenvalues of the network Laplacian adjacency matrix.

In order to obtain the MSF, one must have either an accurate model of an uncoupled oscillator and the coupling function [12] or a set of three coupled oscillators with controllable coupling strengths [13]. In many situations, one may not have a model, and building such a tunable network may be difficult, yet one may still desire to know the stability properties of network synchronization. In this Letter, we demonstrate that reservoir computing can be used as a machine-learning technique for estimating the MSF given only the time series of a single, uncoupled oscillator. We also find that the trained reservoir computer can provide an accurate estimate of the MSF even when it is not successful at attractor reconstruction.

In our technique, we train a reservoir computer to reproduce the dynamics of a single oscillator for which we have the time series; this task is often called "attractor reconstruction," and can be performed with a variety of machine-learning modalities [14–18]. The trained reservoir computer now serves as a model for the uncoupled oscillator. If one knows the form of the coupling, one can compute the MSF of the learned model and use this as an estimate of the MSF of the true oscillator system. We find that the MSF of the reservoir computer can provide an excellent estimate of the MSF of the true oscillator system, for both a continuous-time system (the Lorenz system) and a discrete-time system (the Hénon map—see Supplemental Material [19]).

The trained reservoir computer is a high-dimensional dynamical system that displays not only the trained dynamical behaviors of the true oscillator, but also the untrained networked dynamical behaviors of the true oscillator near the synchronization manifold. While previous works have shown that an accurate estimate of the Lyapunov spectrum of a single oscillator can be obtained from a reservoir computer that achieves attractor reconstruction [14], in this Letter we demonstrate that a well-trained reservoir computer can provide an accurate estimate of the MSF (i.e., the largest transverse Lyapunov exponent of the master stability equation). These are distinct tasks: In the Supplemental Material, we show that successful attractor reconstruction is not necessary for obtaining an accurate estimate of the MSF [19].

In order for the MSF estimation to be accurate, the largest (of many) of the transverse Lyapunov exponents of the reservoir computer master stability equation must be approximately equal to the largest transverse Lyapunov exponent of the true master stability equation. We find that this behavior is more likely to occur when the spectral radius of the reservoir adjacency matrix is significantly less than one, a parameter regime in which reservoir computers are rarely operated, and we provide a reason why a small spectral radius is preferable.

While the dynamics of coupled reservoir computers have been considered previously [20,21], no correspondence has been shown between the reservoir computer coupling strength and the true oscillator coupling strength. In this Letter, we present a coupling scheme in which the coupling strength is proportional to the sampling time of the training data τ. We demonstrate that this factor of τ is essential for obtaining the correct MSF.

Networks of coupled oscillators. Consider a single oscillator with isolated (uncoupled) dynamics described by

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)), \tag{1}$$

where $\mathbf{x} \equiv [x^{(1)}, \ldots, x^{(\mathcal{D})}]$ is the state vector of a single node. One can couple a network of such oscillators together as

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follows:

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t)) - \epsilon L_{ij}\mathbf{H}(\mathbf{x}_j(t)), \tag{2}$$

where H is a function that describes the coupling and the sum over the subscript j is implied. In this Letter we are concerned with the stability of the globally synchronized state $\mathbf{x}_i(t) = \mathbf{x}_j(t)$ for all pairs i and j as $t \to \infty$. The existence of the synchronized state is guaranteed by the Laplacian adjacency matrix L [12]. The stability of the synchronized state is determined by the MSF [12].

Reservoir computing. In this Letter, we choose a reservoir computer [22,23] (also called an echo state network [24]) as our machine-learning modality. A reservoir computer is a type of recurrent neural network that is designed to be particularly easy to train and has been shown to be well suited to the performance of a variety of time-series tasks, including attractor reconstruction [14,16,25,26], Lyapunov exponent estimation [14,16], causal inference [27,28], and nonlinear control [29]. A reservoir computer has also been shown to have the ability to predict untrained bifurcations [25,26], to forecast network dynamics [30], and to synchronize with either the true chaotic system [31] or with identical copies of itself [20,21,32]. We use a reservoir computer for these reasons, but we note that MSF estimation and can be tried with other machine-learning modalities such as long short-term memory (LSTM) [17], nonlinear vector autoregression (NVAR) [18], and extreme learning machines [33].

Following Ref. [14], the reservoir computers used here consist of a randomly designed recurrent neural network of N discrete-time nodes. Let $\mathbf{r}[n]$ be an $N \times 1$ column vector that describes the state of the reservoir at time n. The reservoir computer is described in training mode by

$$\mathbf{r}[n+1] = \tanh\,(A\mathbf{r}[n] + W^{\text{in}}\mathbf{x}[n] + \mathbf{b}), \tag{3}$$

where $\mathbf{x}[n]$ is an $\mathcal{D} \times 1$ column vector describing the input signal to the reservoir, W^{in} is an $N \times \mathcal{D}$ matrix describing the input layer, A is an $N \times N$ matrix describing the internodal connections in the reservoir layer, and $\mathbf{b}$ is an $N \times 1$ bias vector, the purpose of which is to break the symmetry of the hyperbolic tangent function. In this Letter, we consider input signals that are created by sampling a continuous-time dynamical system described by the state vector $\mathbf{x}(t)$, such that $\mathbf{x}[n] \equiv \mathbf{x}(n\tau)$ where τ is the uniform sampling time.

In the training phase the reservoir is driven by the input signal $\mathbf{x}[n]$ and is trained to predict that signal at the next time step $\mathbf{x}[n+1]$; this prediction $\hat{\mathbf{x}}$ is obtained from the reservoir by

$$\hat{\mathbf{x}}[n] = W^{\text{out}}\mathbf{P}(\mathbf{r}[n]), \tag{4}$$

where $\mathbf{P}$ is an N_p-dimensional function of $\mathbf{r}$, and W^{out} is a $\mathcal{D} \times N_p$ matrix obtained by ridge regression [34]. Often, the choice $P(\mathbf{x}) = \mathbf{x}$ is made.

Once the training is complete, one can turn the reservoir into an autonomous dynamical system by feeding this prediction back into the reservoir according to

$$\mathbf{r}[n+1] = \tanh\,(A\mathbf{r}[n] + W^{\text{in}}\hat{\mathbf{x}}[n] + \mathbf{b}). \tag{5}$$

It is known that, when the training succeeds, the autonomous reservoir computer described by Eq. (5) can provide a stable reconstruction of the attractor of a dynamical system [14,16,25]. When the attractor is successfully reconstructed, short-term predictions are of course limited to a few Lyapunov times; however, the long-term climate of the true system is reproduced by the autonomous reservoir, and the Lyapunov exponents of Eq. (5) often agree with the Lyapunov exponents of the true dynamical system.

Since an autonomous reservoir computer is just a dynamical system, a set of identical autonomous reservoir computers can be coupled together in a network. Let $\mathbf{r}_i$ represent the $\mathcal{N} \times 1$ state vector of the ith reservoir. We hypothesize that it will be interesting and useful to couple together M autonomous reservoir computers in the following way,

$$\mathbf{r}_i[n+1] = \tanh(A\mathbf{r}_i[n] + W^{\text{in}}\{\hat{\mathbf{x}}_i[n] - \epsilon\tau L_{ij}\mathbf{H}(\hat{\mathbf{x}}_j[n])\} + \mathbf{b}), \tag{6}$$

where the sum over j is implied, L is the $M \times M$ Laplacian coupling matrix, ϵ is an overall coupling strength, and $\mathbf{H}$ is a coupling function that describes how and which elements of the output of reservoir j couple into which input elements of reservoir i. We hypothesize that this is analogous to Eq. (2). While coupled reservoir computers have been considered previously [20,21] using different coupling schemes, in the coupling scheme used here the coupling strength is proportional to the sampling time τ of the training data. Some justification for this is presented in the Supplemental Material [19]. As we show, this results in the equivalence of the coupling strength ϵ in the ordinary differential equation (2) and in the discrete-time reservoir equation (6) when $\tau \ll 1/\epsilon\lambda_k$, where λ_k are the eigenvalues of L.

We assume that the form of the coupling $\mathbf{H}(\mathbf{x})$ is known. In cases where it is not known, there are a variety of techniques for estimating the coupling function [35].

Master stability function for coupled reservoir computers. To compute the MSF, one needs both the dynamics of the synchronized state [Eq. (1)] and the Jacobian of the variational equation corresponding to Eq. (2) in the basis in which L is diagonal. The former is provided by the trained reservoir Eq. (5). Alternatively, it could be provided the measured dynamics of the nonlinear oscillator, as we demonstrate in the Supplemental Material [19].

To estimate the latter we linearize Eq. (6) about the synchronous flow given by Eq. (5), where we use a subscript s to indicate the synchronized state:

$$\begin{aligned}\delta\mathbf{r}_i[n+1] &= \text{sech}^2(A\mathbf{r}_s[n] + W^{\text{in}}\hat{\mathbf{x}}_s[n] + \mathbf{b})\\ &\quad\times\{A\delta\mathbf{r}_i[n] + W^{\text{in}}\delta\hat{\mathbf{x}}_i[n]\},\\ \delta\hat{\mathbf{x}}_i[n] &= W^{\text{out}}d\mathbf{P}(\mathbf{r}_s[n])\delta\mathbf{r}_i\\ &\quad - \epsilon\tau L_{ij}d\mathbf{H}(\hat{\mathbf{x}}_s[n])W^{\text{out}}d\mathbf{P}(\mathbf{r}_s[n])\delta\mathbf{r}_j[n].\end{aligned} \tag{7}$$

We want to consider only variations that are transverse to the synchronization manifold. The first term in the curly brackets is block diagonal with $M \times M$ blocks. The second term is treated by performing a linear transformation to a basis in which L is diagonal. This transformation does not affect the first term. This results in the master stability equation

$$\begin{aligned}\delta\mathbf{q}_k[n+1] &= \text{sech}^2(A\mathbf{r}_s[n] + W^{\text{in}}\hat{\mathbf{x}}_s[n] + \mathbf{b})D,\\ D &= \{A + W^{\text{in}}W^{\text{out}}d\mathbf{P}(\mathbf{r}_s[n])\\ &\quad - \epsilon\tau\lambda_k W^{\text{in}}d\mathbf{H}(\hat{\mathbf{x}}_s[n])W^{\text{out}}d\mathbf{P}(\mathbf{r}_s[n])\}\delta\mathbf{q}_k[n],\end{aligned} \tag{8}$$

where we define $K \equiv \epsilon\lambda$ and λ_k are the M eigenvalues of L. For $K = \epsilon\lambda_k = 0$, Eq. (8) is the variational equation for the synchronization manifold. All other eigenvalues correspond to transverse eigenvectors and therefore determine the stability of the globally synchronized state.

The MSF is the largest Lyapunov exponent of Eq. (8) as a function of the complex number K. Once computed, the sign of the MSF gives the stability of global synchronization of any reservoir computer network described by any Laplacian coupling matrix L in terms of the eigenvalues of L.

Results. Is the stability of synchronization of a network of coupled, trained reservoirs predictive of the stability of synchronization of a network of the true oscillators? We investigate this question by training a reservoir computer on the Lorenz system with linear, one-component coupling [36] (e.g., for $x \to x$ coupling, $[\mathbf{H}(\mathbf{x})]_{ij} = x$ for $i = j = 1$ and zero otherwise), and comparing the MSF of the coupled reservoir system with the MSF of the true Lorenz system. In the Supplemental Material [19], we also demonstrate that this technique can be used to estimate the MSF of the Rössler oscillator and the discrete-time Hénon map.

We use the same type of reservoir as in Ref. [14]. The 300-node reservoir was set up as follows. To form the adjacency matrix, a sparse random Erdös-Rényi network with average degree 6 was created, with each nonzero element of A drawn independently and uniformly from the range $[-1, 1]$. All elements were then scaled such that the spectral radius is ρ. The input matrix W^{in} was size 300×3. The first 100 elements of the first column, the second 100 elements of the second column, and the final 100 elements of the third column were nonzero; the remaining elements were zero. The nonzero elements of W^{in} were randomly drawn independently and uniformly from $[-\sigma, \sigma]$. No bias was used ($\mathbf{b} = \mathbf{0}$).

Following Ref. [14], the reservoir output $\hat{\mathbf{x}} = [\hat{x}, \hat{y}, \hat{z}]$ is given by

$$\begin{bmatrix} \hat{x}[n] \\ \hat{y}[n] \\ \hat{z}[n] \end{bmatrix} = W^{\text{out}} \begin{bmatrix} \mathbf{r}[n] \\ \mathbf{r}[n] \\ \tilde{\mathbf{r}}[n] \end{bmatrix}, \tag{9}$$

where $\tilde{\mathbf{r}}$ is defined such that the first half of its elements are the same as that of $\mathbf{r}$, while $\tilde{\mathbf{r}} = r^2$ for the remaining half of the reservoir nodes. The W^{out} matrix is trained by ridge regression.

The input Lorenz signal used was obtained by integrating the Lorenz equations [37] using a second-order Runge-Kutta method with time step $dt = 0.001$, then downsampling the signal to have a sampling time $\tau = 0.02$. Dynamical noise was by modeled by adding white Gaussian noise of standard deviation $0.01\sqrt{dt}$ at each time step according to the method described in Ref. [38]; the presence of dynamical noise seems to be beneficial, consistent with previous work on network reconstruction using reservoir computing [27,28]. We found that a ridge parameter of 0 worked best, perhaps because the noise was sufficient to prevent overfitting [39,40]. The Lyapunov exponents of Eq. (8) were computed using the QR method [41].

We trained 1000 different Erdös-Rényi reservoirs with spectral radius ρ drawn from a uniform distribution with limits [0,0.3] and σ from a uniform distribution with limits [0,0.2] on the same Lorenz time series. We note that these values of ρ are much smaller than that typically advised in the literature; one reason for why small values of ρ may be better in this case is provided in the Supplemental Material [19]. We used 40 000 training time steps. We then computed the MSF of the 100 reservoirs that gave the best prediction error over seven Lyapunov times. Each MSF was computed over 50 000 time steps. MSFs that were extreme outliers relative to the population were discarded; the rest were retained. In all cases, more than 90 MSFs were retained. The results shown are the mean MSF of the retained reservoirs, and the error bars are the standard deviation. The exclusion of outlier MSFs had little effect on the mean, but did reduce the standard deviation in some cases.

We first tested whether our reservoirs could reproduce the Lyapunov exponents of the true Lorenz system [that is, we computed the Lyapunov exponents of Eq. (8) with $K = 0$]. An autonomous reservoir has many Lyapunov exponents: For the three largest (averaged over all 100 retained reservoirs), we obtained 0.906, 0.000, −14.40. The true Lorenz Lyapunov exponents are 0.906, 0.000, −14.6. It is known that reservoir computers often struggle to replicate the negative Lyapunov exponent of the Lorenz system [14], though it is interesting to note that our reservoir computer (with its small spectral radius) comes closer than the large spectral radius reservoir computer of Ref. [14].

We then computed the MSF for real K of the true Lorenz system and of the coupled reservoir system Eq. (8) for all possible single-variable linear coupling schemes. The results are shown in Fig. 1. The true MSF appears to agree excellently with that computed in Ref. [36]. For the reservoir computer MSF, the results (error bars) shown are the average (standard deviation) over the retained reservoirs. The agreement between the reservoir computer MSF and the true MSF is excellent everywhere for cross-coupling (e.g., $x \to y$ and $y \to x$). For self-coupling ($x \to x$, etc.) the agreement is good though not perfect. It it not clear why the results are not as good for the case of self-coupling. We also note that, in general, the accuracy of the MSF estimation seems to be better for smaller values of K. The reason is that our scheme for coupling the reservoir computers is equivalent to the true coupling scheme only when $\tau \ll 1/K$; this is explained further in the Supplemental Material [19].

We have investigated the accuracy of the estimation of the MSF by reservoir computing on networks of coupled Hénon maps in the Supplemental Material [19]. We find good agreement between the true and estimated MSFs when ρ is small, and provide analysis that demonstrates that the disagreement at very negative values of the MSF is due to the nonzero spectral radius of the reservoir. We further demonstrate the generality of our MSF estimation technique by using the measured dynamics of the synchronized state $\mathbf{x}_s$ instead of $\hat{\mathbf{x}}_s$ in Eq. (8) on a reservoir computer trained on the Rössler system (for which it has been found that obtaining a reliable attractor reconstruction using reservoir computing is more difficult than for the Lorenz system [42]) in the Supplemental Material [19]. Driving the reservoir with $\mathbf{x}_s$ is particularly useful when attractor reconstruction fails but one still requires an estimate of the MSF. These results confirm that successful

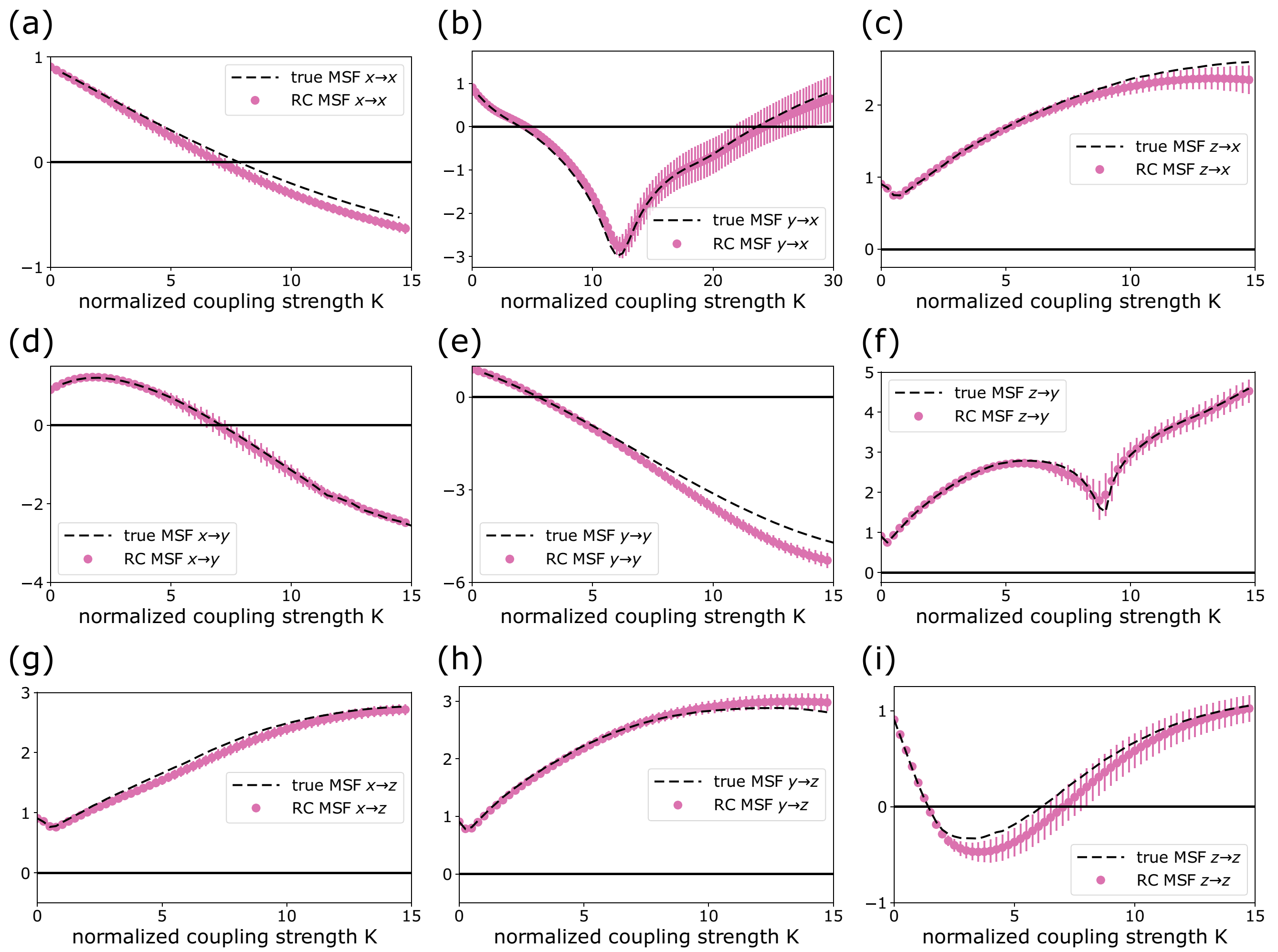

FIG. 1. Master stability function of the true Lorenz system (black dashed line) and the trained reservoir computer (pink dots). A solid black line is shown at 0 to indicate the stability boundary.

attractor reconstruction is not necessary for accurate MSF (or Lyapunov exponent) estimation.

Conclusions. We showed that the master stability function of a reservoir computer trained on a time series from a single, uncoupled nonlinear oscillator can provide an accurate estimate of the master stability function of the true oscillator. Therefore, a reservoir computer trained on a time series from a single nonlinear oscillator can be used to determine the stability of any network of coupled oscillators of the same type. We demonstrated this on a variety of coupling configurations of the chaotic Lorenz oscillator and the chaotic Hénon map (see Supplemental Material [19]), demonstrating that this technique can work for both continuous- and discrete-time oscillators.

The technique presented here offers the potential to solve the problem of global network synchronization by a simple measurement of an uncoupled oscillator. More generally, this result demonstrates that machine-learned systems, when coupled together, can quantitatively replicate the dynamical behavior of a true oscillator network, indicating a deep relationship between the trained reservoir computer and true oscillator system. The machine-learned model is a high-dimensional dynamical system that displays not only the trained dynamical behaviors of the true oscillator (i.e., attractor reconstruction), but also the untrained networked dynamical behaviors of the true oscillator.

Future work includes extending the network stability estimation technique for synchronization on networks with delays [3,43], nonlinear coupling (known or unknown) [35], time-varying connections [44,45] and cluster synchronization [46,47], and for stability metrics other than the largest Lyapunov exponent [12,48]. The MSF formalism only tells about local stability; it would also be interesting to test whether the trained reservoir could be used to predict the basin stability [49] as well. For cluster synchronization and basin stability, the reservoir would need to be able to model dynamics away from the attractor on which it was trained. Recent developments [25,26] may enable this.

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