Kappa distribution from particle correlations in nonequilibrium, steady-state plasmas

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(Received 18 May 2023; accepted 12 November 2023; published 8 December 2023)

Kappa-distributed velocities in plasmas are common in a wide variety of settings, from low-density to high-density plasmas. To date, they have been found mainly in space plasmas, but are recently being considered also in the modeling of laboratory plasmas. Despite being routinely employed, the origin of the kappa distribution remains, to this day, unclear. For instance, deviations from the Maxwell-Boltzmann distribution are sometimes regarded as a signature of the nonadditivity of the thermodynamic entropy, although there are alternative frameworks such as superstatistics where such an assumption is not needed. In this work we recover the kappa distribution for particle velocities from the formalism of nonequilibrium steady-states, assuming only a single requirement on the dependence between the kinetic energy of a test particle and that of its immediate environment. Our results go beyond the standard derivation based on superstatistics, as we do not require any assumption about the existence of temperature or its statistical distribution, instead obtaining them from the requirement on kinetic energies. All of this suggests that this family of distributions may be more common than usually assumed, widening its domain of application in particular to the description of plasmas from fusion experiments. Furthermore, we show that a description of kappa-distributed plasma is simpler in terms of features of the superstatistical inverse temperature distribution rather than the traditional parameters κ and the thermal velocity v_{th}.

DOI: 10.1103/PhysRevE.108.065207

I. INTRODUCTION

Modelling the velocity distribution of particles in a nonequilibrium, steady-state plasma is an interesting challenge from both theoretical and practical points of view [1–4]. In general, particles in a steady state plasma do not follow the typical Maxwell-Boltzmann distribution of velocities one would expect in an equilibrium system but, instead, their velocities are described by more general families of distributions. Among them, the kappa distribution, a power-law distribution with long tails describing highly energetic particles, appears predominantly in space plasmas [5,6]. These are weakly collisional plasmas in steady states found in the Earth's magnetosphere [7–10] and plasma sheet [8,11,12], as well as in the solar wind [13–15]. Beyond Earth, kappa distributions have also been found in other planetary atmospheres [16–19] and the interstellar medium [20–22].

From the point of view of laboratory plasmas, even when the energy distribution of suprathermal ions in fusion plasmas, such as the ones generated by Z-pinch discharges, has been known for decades to be well described by power laws [23–27], only recently kappa distributions have being proposed in this context as possible statistical models [28,29].

For the velocity $\boldsymbol{v}$ of a particle of mass m, the kappa distribution is commonly written in the form

$$P(\boldsymbol{v}|\kappa, v_{\text{th}}) = \frac{1}{\eta_\kappa(v_{\text{th}})}\left[1 + \frac{1}{\kappa - \frac{3}{2}}\frac{\boldsymbol{v}^2}{v_{\text{th}}^2}\right]^{-(\kappa+1)}, \tag{1}$$

where $\kappa \geqslant 0$ is a shape parameter, sometimes referred to as the spectral index, v_{th} is the thermal velocity, [30],

$$v_{\text{th}} := \sqrt{\frac{2k_BT}{m}}, \tag{2}$$

and $\eta_\kappa(v_{\text{th}})$ is a normalization constant given by

$$\eta_\kappa(v_{\text{th}}) := (\sqrt{\pi(\kappa - 3/2)}v_{\text{th}})^3\frac{\Gamma(\kappa - 1/2)}{\Gamma(\kappa + 1)}. \tag{3}$$

In the limit $\kappa \to \infty$, the kappa distribution in Eq. (1) reduces to the Maxwell-Boltzmann distribution,

$$P(\boldsymbol{v}|m, T) = \left(\sqrt{\frac{m}{2\pi k_BT}}\right)^3 \exp\left(-\frac{m\boldsymbol{v}^2}{2k_BT}\right), \tag{4}$$

precisely the distribution expected in equilibrium at temperature T. However, for finite κ the interpretation of the parameter T in Eq. (2) is not straightforward [31,32], mainly because there are multiple admissible definitions of temperature and not all of them agree with T.

Although the presence of kappa distributions in plasmas has been traditionally explained [33,34] by the use of nonextensive statistical mechanics, also known as Tsallis statistics

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[35], more recent frameworks such as superstatistics [36,37] can recover them in a direct manner. Moreover, recently we have shown [38] that superstatistics arises as a natural description for collisionless plasmas in nonequilibrium steady states, providing support to recent efforts [39–41] in establishing a foundational basis for steady-state distributions in plasmas using superstatistics as a starting point.

Despite these advances, superstatistics still requires the assumption of a gamma distribution for the inverse temperature $\beta := 1/(k_B T)$ in order to recover Tsallis statistics, and in particular, the kappa distributions. This particular choice of temperature distribution is referred to as χ^2 superstatistics. Several mechanisms aiming to explain the origin of this χ^2 family of superstatistics have been proposed in the literature since the theory was originally presented. For instance, by using the fact that a sum of squares of Gaussian random variables becomes gamma distributed as the number of such variables grows large [42], or by invoking Jaynes' maximum entropy principle on the distribution $P(\beta|S)$ under macroscopic constraints [43].

Motivated by this somewhat unsatisfactory assumption of gamma-distributed inverse temperatures, in this work we delve deeper into the formalism established in Ref. [38], by connecting it with more recent theoretical developments [44,45] on the structure of superstatistics. In particular, we show that the assumption of a gamma distribution for β can be replaced by a simpler, and perhaps more fundamental, assumption on the dependence between the kinetic energy of a test particle and that of its surrounding environment.

In the following section we provide a brief account of the superstatistical formalism and we connect it with a generalized definition of temperature for steady states [46,47], namely the fundamental inverse temperature function β_F.

II. NONEQUILIBRIUM STEADY STATES AND SUPERSTATISTICS

Steady states are a special kind of nonequilibrium states which are time independent, that is, where the nonequilibrium probability density of microstates $p(\boldsymbol{\Gamma}; t)$ at a time t reduces to $p(\boldsymbol{\Gamma})$. In particular, we will consider steady states where $p(\boldsymbol{\Gamma})$ depends on $\boldsymbol{\Gamma}$ only through the Hamiltonian $\mathcal{H}(\boldsymbol{\Gamma})$, and we will write their probability density as

$$P(\boldsymbol{\Gamma}|S) = \rho(\mathcal{H}(\boldsymbol{\Gamma})), \tag{5}$$

where ρ is the ensemble function, and S denotes the set of parameters that uniquely define the steady state.

Within this general framework, superstatistics is a natural extension of statistical mechanics to steady states in the form given by Eq. (5). Besides nonequilibrium plasmas, it has been successfully used in high-energy physics [48,49], anomalous diffusion [50,51], cosmology and gravitation [52], turbulence [53–55], seismicity [56], bioinformatics [57], as well as phenomena of interest in engineering such as the electrical fluctuations of power grids [58].

In superstatistics, the canonical ensemble

$$P(\boldsymbol{\Gamma}|\beta) = \frac{\exp(-\beta\mathcal{H}(\boldsymbol{\Gamma}))}{Z(\beta)} \tag{6}$$

is replaced by a superposition of canonical ensembles at different temperatures. The inverse temperature β is promoted from a constant to a random variable with probability density $P(\beta|S)$, such that its joint distribution with the microstates is given by

$$P(\boldsymbol{\Gamma}, \beta|S) = P(\boldsymbol{\Gamma}|\beta)P(\beta|S) = \left[\frac{\exp\left(-\beta\mathcal{H}(\boldsymbol{\Gamma})\right)}{Z(\beta)}\right]P(\beta|S). \tag{7}$$

By marginalization of β, the distribution of microstates becomes

$$P(\boldsymbol{\Gamma}|S) = \int_0^\infty d\beta P(\beta|S)\left[\frac{\exp(-\beta\mathcal{H}(\boldsymbol{\Gamma}))}{Z(\beta)}\right], \tag{8}$$

which has the form of Eq. (5) with an ensemble function,

$$\rho(E) = \int_0^\infty d\beta f(\beta)\exp(-\beta E), \tag{9}$$

that is the Laplace transform of the superstatistical weight function $f(\beta)$, defined by

$$f(\beta) := \frac{P(\beta|S)}{Z(\beta)}. \tag{10}$$

The distinction between $f(\beta)$ and $P(\beta|S)$ is an important one. The formulation of superstatistics as in Eq. (8) is known as type-B superstatistics, and is the standard version in use nowadays [37]. The original formulation [36], now known as type-A superstatistics, defines ρ as in Eq. (9) but $f(\beta)$ itself is taken as the probability density for β. This led to inconsistencies with the application of the sum and product rule of probability [59].

Among the possible families of distributions compatible with superstatistics, three universality classes have been shown to be especially relevant for nonequilibrium systems: the so-called χ^2 superstatistics where $f(\beta)$ has the form of a gamma distribution, log-normal superstatistics, and inverse-gamma superstatistics. Arguably the most predominant case is the χ^2 superstatistics, as it leads to the q-canonical ensemble of Tsallis statistics, and in particular to the kappa distribution. However, the log-normal superstatistics has been found in the context of turbulence [53,60], and in stellar systems [61], among several other contexts. On the other hand, the inverse-gamma superstatistics has been successfully employed to described the thermodynamics of small molecules [62] and the dynamics of protein diffusion [63].

Using the definition in Eq. (9) we can write $\rho(E) = \mathcal{L}\{f\}(E)$ and, conversely, $f(\beta) = \mathcal{L}^{-1}\{\rho\}(\beta)$. An important consequence of this is that ρ is completely determined by f and viceversa, and as the latter depends on both the inverse temperature distribution and the partition function, then both aspects together define the form of the statistical ensemble $P(\boldsymbol{\Gamma}|S)$.

Let us now consider a composite system, divided into subsystems A and B such that $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma}_A, \boldsymbol{\Gamma}_B)$, and where the Hamiltonian of the entire system is of the form

$$\mathcal{H}(\boldsymbol{\Gamma}_A, \boldsymbol{\Gamma}_B) = \mathcal{H}_A(\boldsymbol{\Gamma}_A) + \mathcal{H}_B(\boldsymbol{\Gamma}_B). \tag{11}$$

Please note that, because we are considering a superstatistical ensemble function $\rho(E)$ of the form in Eq. (8), it is no longer true that additive subsystems have a joint distribution that is the product of their marginal distributions. That is, in general,

$$\rho(E|S) \neq \rho(E_A|S)\rho(E_B|S). \tag{12}$$

The statistical independence of subsystems only remains true for the canonical ensemble, where

$$\rho(E_A + E_B|\beta_0) = \rho(E_A|\beta_0)\rho(E_B|\beta_0).$$

However, it is easy to show in the general case that $P(\beta|S)$ is a universal property of the entire system and its parts, that is, the same $P(\beta|S)$ function is involved when expressing the ensemble function of an arbitrary subsystem ν as in Eq. (8),

$$P(\boldsymbol{\Gamma}_\nu|S) = \int_0^\infty d\beta P(\beta|S)\left[\frac{\exp(-\beta\mathcal{H}_\nu(\boldsymbol{\Gamma}_\nu))}{Z_\nu(\beta)}\right], \tag{13}$$

noting that $P(\beta|S)$, unlike $\mathcal{H}_\nu$ and Z_ν, does not carry the subindex ν. It follows that $f_\nu(\beta)$, being the ratio between $P(\beta|S)$ and the partition function $Z_\nu(\beta)$, will in fact be dependent on the details of the subsystem. We can show that Eq. (13) holds as follows. Let the composite system be described by an inverse temperature distribution $P(\beta|S)$. Then we have

$$P(\boldsymbol{\Gamma}_A, \boldsymbol{\Gamma}_B|S) = \int_0^\infty d\beta P(\beta|S)\left[\frac{\exp(-\beta(\mathcal{H}_A + \mathcal{H}_B))}{Z_{AB}(\beta)}\right], \tag{14}$$

and the marginal distribution of $\boldsymbol{\Gamma}_A$ is given by

$$\begin{aligned} P(\boldsymbol{\Gamma}_A|S) &= \int d\boldsymbol{\Gamma}_B \int_0^\infty d\beta P(\beta|S)\left[\frac{\exp(-\beta(\mathcal{H}_A + \mathcal{H}_B))}{Z_{AB}(\beta)}\right] \\ &= \int_0^\infty d\beta P(\beta|S)\exp(-\beta\mathcal{H}_A(\boldsymbol{\Gamma}_A)) \\ &\quad\times \int d\boldsymbol{\Gamma}_B\left[\frac{\exp(-\beta\mathcal{H}_B(\boldsymbol{\Gamma}_B))}{Z_{AB}(\beta)}\right] \\ &= \int_0^\infty d\beta P(\beta|S)\frac{Z_B(\beta)}{Z_{AB}(\beta)}\exp(-\beta\mathcal{H}_A(\boldsymbol{\Gamma}_A)), \end{aligned} \tag{15}$$

that is,

$$P(\boldsymbol{\Gamma}_A|S) = \int_0^\infty d\beta P(\beta|S)\left[\frac{\exp(-\beta\mathcal{H}_A(\boldsymbol{\Gamma}_A))}{Z_A(\beta)}\right], \tag{16}$$

where we have used the well-known factorization of the partition function for additive systems, $Z_{AB}(\beta) = Z_A(\beta)Z_B(\beta)$. We see from Eq. (16) and Eq. (14) that the subsystem $\boldsymbol{\Gamma}_A$ is governed by the same inverse temperature distribution $P(\beta|S)$ as the composite system $(\boldsymbol{\Gamma}_A, \boldsymbol{\Gamma}_B)$ and, because the choice of A and B is arbitrary, it follows that any possible subsystem is governed by the same $P(\beta|S)$. In the following we will use this fact to recover subsystem-independent parameters for the kappa distribution describing the velocity of a single particle.

We will now define the fundamental inverse temperature function β_F, motivated by the conditional distribution of β given a fixed energy E. First, note that the distribution of energy in an steady state given by Eq. (5) is

$$\begin{aligned} P(E|S) = \langle\delta(E - \mathcal{H})\rangle_S &= \int d\boldsymbol{\Gamma}\rho(\mathcal{H}(\boldsymbol{\Gamma}))\delta(E - \mathcal{H}(\boldsymbol{\Gamma})) \\ &= \rho(E)\Omega(E), \end{aligned} \tag{17}$$

where $\Omega(E) := \int d\boldsymbol{\Gamma}\delta(E - \mathcal{H}(\boldsymbol{\Gamma}))$ is the density of states associated to $\mathcal{H}$. Now, from Bayes' theorem [64,65] we obtain

$$P(\beta|E, S) = \frac{P(\beta|S)P(E|\beta, S)}{P(E|S)} \tag{18}$$

and, because exact knowledge of β supersedes the state of knowledge S, we can replace $P(E|\beta, S)$ in the numerator with the usual canonical distribution of energy,

$$P(E|\beta) = \frac{\exp(-\beta E)\Omega(E)}{Z(\beta)}, \tag{19}$$

a particular case of Eq. (17) with $\rho(E) = \exp(-\beta E)/Z(\beta)$. Therefore, replacing Eq. (19) and Eq. (17) into Eq. (18) and canceling the factor $\Omega(E)$, we have

$$P(\beta|E, S) = \frac{f(\beta)\exp(-\beta E)}{\rho(E)}, \tag{20}$$

and we immediately see that Eq. (9) ensures that the left-hand side is a properly normalized distribution. The fluctuation-dissipation theorem [66] associated to $P(\beta|E, S)$ is

$$\frac{\partial}{\partial E}\langle\omega\rangle_{E,S} = \left\langle\frac{\partial\omega}{\partial E}\right\rangle_{E,S} + \left\langle\omega\frac{\partial}{\partial E}\ln P(\beta|E, S)\right\rangle_{E,S} \tag{21}$$

which, by replacing Eq. (20), becomes

$$\frac{\partial}{\partial E}\langle\omega\rangle_{E,S} = \left\langle\frac{\partial\omega}{\partial E}\right\rangle_{E,S} + \langle\omega(\beta_F - \beta)\rangle_{E,S}, \tag{22}$$

where we have defined the fundamental inverse temperature function $\beta_F(E)$ by

$$\beta_F(E) := -\frac{\partial}{\partial E}\ln\rho(E). \tag{23}$$

Two consequences of the fluctuation-dissipation relation in Eq. (22) are straightforward to obtain. First, by using $\omega = 1$ and recalling that $\langle f\rangle_{E,S} = f(E)$ for any function $f(E)$ of the energy, we immediately see that

$$\beta_F(E) = \langle\beta\rangle_{E,S}, \tag{24}$$

which then gives meaning to the fundamental inverse temperature in superstatistics: it is the conditional expectation of the superstatistical inverse temperature given the energy of the system. Second, by taking expectation of Eq. (24) under S on both sides, we obtain

$$\langle\beta_F\rangle_S = \langle\beta\rangle_S, \tag{25}$$

that is, the expectation values of β_F and β coincide, and we can use this common value to define the inverse temperature β_S of the ensemble S without ambiguity as

$$\beta_S := \langle\beta_F\rangle_S. \tag{26}$$

In the following sections, we will recover the kappa distribution for the single-particle velocity from superstatistics plus just one additional assumption. Furthermore, we will show how a superstatistical approximation produces a distribution

$P(\beta|S)$ as the thermodynamic limit of the distribution of the inverse fundamental temperature, $P(\beta_F|S)$, thus proving a deeper connection between the superstatistical parameter β and the function β_F.

III. THE KAPPA DISTRIBUTION IN STEADY STATE PLASMAS

The total energy of a system of N classical, nonrelativistic interacting particles forming a plasma in a steady state can be written as

$$E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_N) = \sum_{i=1}^{N} \frac{m_i \boldsymbol{v}_i^2}{2} + \Phi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N), \tag{27}$$

in such a way that the details of the interaction with the (self-consistent) electromagnetic fields are contained inside the potential energy function Φ. This energy function E is different from the Hamiltonian $\mathcal{H}$, as the latter should be written in terms of momenta instead of velocities. However, in a steady state the joint probability of positions and velocities actually depends only on the energy function E (as we have shown earlier [38]), that is, is of the form

$$P(\boldsymbol{R}, \boldsymbol{V}|S) = \rho(E(\boldsymbol{R}, \boldsymbol{V}); S), \tag{28}$$

where we have introduced the shortcut notation $\boldsymbol{R} := (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ and $\boldsymbol{V} := (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$. The joint distribution of velocities can be obtained by marginalization of the particle positions,

$$\begin{aligned} P(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N|S) &= \int d\boldsymbol{R}\, \rho(E(\boldsymbol{R}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_N); S) \\ &= p_N\left(\sum_{i=1}^{N} \frac{m_i \boldsymbol{v}_i^2}{2}\right), \end{aligned} \tag{29}$$

where this relation defines the N-particle ensemble function of velocities p_N. Moreover, the single-particle velocity distribution, which is our main target in this work, is given by marginalization in $P(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N|S)$ of the remaining $N-1$ particle velocities,

$$P(\boldsymbol{v}_1|S) = \int d\boldsymbol{v}_2 \ldots d\boldsymbol{v}_N P(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N|S) = p_1\left(\frac{m_1 \boldsymbol{v}_1^2}{2}\right). \tag{30}$$

Here it is important to note that Eq. (28) together with the form of the energy function in Eq. (27) will only lead to isotropic velocity distributions because then $P(\boldsymbol{v}_1|S)$ depends on $\boldsymbol{v}_1$ through its magnitude, according to Eq. (30). By comparing $P(\boldsymbol{v}_1|S)$ with the kappa distribution in Eq. (1), we see that our single-particle ensemble function p_1 must be given by

$$p_1(k_1) = \frac{1}{\eta_\kappa(v_{\text{th}})}\left[1 + \frac{2k_1}{m_1 v_{\text{th}}^2\left(\kappa - \frac{3}{2}\right)}\right]^{-(\kappa+1)}, \tag{31}$$

where $k_1 := m_1 \boldsymbol{v}_1^2/2$ is the kinetic energy of the particle with $i = 1$. In the next section, we will arrive at the kappa form for $p_1(k_1)$ using a single requirement on the dependence between the kinetic energy k_1 of a particle and the kinetic energy K of its surrounding environment.

IV. DERIVATION OF THE KAPPA DISTRIBUTION

In the following analysis, we will be considering a group of $n \leqslant N$ particles as a subsystem, regarding only their kinetic energy. Without loss of generality we can take the first particle as a test particle with kinetic energy k_1, and the remaining $n-1$ particles as its environment with kinetic energy

$$K := \sum_{i=2}^{n} \frac{m_i \boldsymbol{v}_i^2}{2}. \tag{32}$$

Then, the energy $\mathcal{K}$ of the subsystem is directly $\mathcal{K} := k_1 + K$. Recalling that the density of states of kinetic energy for a group of n particles is given by

$$\Omega_n(\mathcal{K}) := \int d\boldsymbol{v}_1 \ldots d\boldsymbol{v}_n \delta\left(\mathcal{K} - \sum_{i=1}^{n} \frac{m_i \boldsymbol{v}_i^2}{2}\right) = W_n\, \mathcal{K}^{\frac{3n}{2}-1}, \tag{33}$$

where we have defined the constants

$$W_n := \frac{(2\pi)^{\frac{3n}{2}} M^{-\frac{3}{2}}}{\Gamma\left(\frac{3n}{2}\right)} \tag{34}$$

and $M := \prod_{i=1}^{n} m_i$, the partition function associated to Ω_n is its Laplace transform,

$$\begin{aligned} Z_n(\beta; M) &= \int_0^\infty d\mathcal{K}\Omega_n(\mathcal{K}) \exp(-\beta\mathcal{K}) = W_n \beta^{-\frac{3n}{2}} \Gamma\left(\frac{3n}{2}\right) \\ &= (2\pi)^{\frac{3n}{2}} M^{-\frac{3}{2}} \beta^{-\frac{3n}{2}}, \end{aligned} \tag{35}$$

which contains the single-particle partition function $Z_1(\beta; m)$ as a particular case with $n = 1$ and $M = m$,

$$Z_1(\beta; m) = \left(\sqrt{\frac{2\pi}{m}}\right)^3 \beta^{-\frac{3}{2}}. \tag{36}$$

Now we will show that only one condition is sufficient to obtain the kappa distribution for a single particle in a plasma, namely that the most probable kinetic energy k^* of the test particle given the kinetic energy K of its $(n-1)$-particle environment is linear in K. In more precise terms, we require that

$$k^* := \text{argmax}_{k_1} P(k_1|K, S) = \gamma_n + \alpha_n K, \tag{37}$$

where the parameters γ_n and α_n are functions of n. In order to show that Eq. (37) leads to the kappa distribution, let us first compute the joint distribution $P(k_1, K|S)$ of test particle plus environment, which is given by

$$\begin{aligned} P(k_1, K|S) &= \left\langle \delta\left(k_1 - \frac{m_1 \boldsymbol{v}_1^2}{2}\right) \delta\left(K - \sum_{i=2}^{n} \frac{m_i \boldsymbol{v}_i^2}{2}\right) \right\rangle_S \\ &= \int d\boldsymbol{v}_1 \ldots d\boldsymbol{v}_n p_n\left(\sum_{i=1}^{n} \frac{m_i \boldsymbol{v}_i^2}{2}\right) \delta\left(k_1 - \frac{m_1 \boldsymbol{v}_1^2}{2}\right) \\ &\quad \times \delta\left(K - \sum_{i=2}^{n} \frac{m_i \boldsymbol{v}_i^2}{2}\right) \end{aligned}$$

$$= p_n(k_1+K)\left[\int d\boldsymbol{v}_1\delta\left(k_1-\frac{m_1\boldsymbol{v}_1^2}{2}\right)\right]$$
$$\times\left[\int d\boldsymbol{v}_2\ldots d\boldsymbol{v}_n\delta\left(K-\sum_{i=2}^{n}\frac{m_i\boldsymbol{v}_i^2}{2}\right)\right], \quad (38)$$

and that by using the definition of Ω_n in Eq. (33) becomes

$$P(k_1,K|S)=p_n(k_1+K)\Omega_1(k_1)\Omega_{n-1}(K). \quad (39)$$

The conditional distribution $P(k_1|K,S)$ appearing in Eq. (37) can then be obtained as

$$P(k_1|K,S)=\frac{P(k_1,K|S)}{P(K|S)}=\frac{p_n(k_1+K)\Omega_1(k_1)}{p_{n-1}(K)}, \quad (40)$$

where a factor $\Omega_{n-1}(K)$ has been canceled, and the single-particle density of states $\Omega_1(k_1)$ is readily obtained from Eq. (33) with $n=1$,

$$\Omega_1(k_1)=\frac{2}{\sqrt{\pi}}\left(\frac{2\pi}{m}\right)^{3/2}\sqrt{k_1}. \quad (41)$$

Now, because k^* is the argument of the maximum of $P(k_1|K,S)$ according to Eq. (37), it follows that k^* is the solution of the extremum equation

$$0=\left[\frac{\partial}{\partial k_1}\ln P(k_1|K,S)\right]_{k_1=k^*}, \quad (42)$$

and by replacing Eq. (40) and Eq. (41) we obtain

$$\beta_F^{(n)}(k^*+K)=\frac{1}{2k^*}, \quad (43)$$

where $\beta_F^{(n)}$ is the fundamental inverse temperature of the group of n particles, defined by

$$\beta_F^{(n)}(\mathcal{K}):=-\frac{\partial}{\partial\mathcal{K}}\ln p_n(\mathcal{K}). \quad (44)$$

We can replace k^* in Eq. (43) in terms of K using Eq. (37) and, after some algebra, obtain

$$\beta_F^{(n)}(\mathcal{K})=\frac{\alpha_n+1}{2(\gamma_n+\alpha_n\mathcal{K})}, \quad (45)$$

from which we can recover the n-particle ensemble function p_n by integration,

$$p_n(\mathcal{K})=p_n(0)\exp\left(-\frac{\alpha_n+1}{2}\int_0^{\mathcal{K}}\frac{d\epsilon}{\gamma_n+\alpha_n\epsilon}\right)$$
$$=p_n(0)\left[1+\left(\frac{\alpha_n}{\gamma_n}\right)\mathcal{K}\right]^{-\frac{1}{2\alpha_n}-\frac{1}{2}}, \quad (46)$$

where $p_n(0)$ is a normalization constant to be determined. By marginalizing K in Eq. (39) and using Eq. (17) as

$$P(k_1|S)=p_1(k_1)\Omega_1(k_1), \quad (47)$$

we see that

$$p_1(k_1)=\int_0^\infty dK\,p_n(k_1+K)\Omega_{n-1}(K). \quad (48)$$

Now, making use of the definite integral

$$\int_0^\infty dy\,y^m[1+r(x+y)]^{-c}$$
$$=r^{-m-1}B(c-m-1,m+1)\cdot[1+rx]^{m+1-c} \quad (49)$$

for $x>0$, $r>0$, $m>-1$, and $c>m+1$ with $B(a,b):=\int_0^1 dt\,t^{a-1}(1-t)^{b-1}$ the Beta function, we finally arrive at

$$p_1(k_1)=p_1(0)\left[1+\left(\frac{\alpha_n}{\gamma_n}\right)k_1\right]^{\frac{3n}{2}-\frac{1}{2\alpha_n}-2}. \quad (50)$$

By comparing Eq. (50) and Eq. (31) we see that we have recovered the kappa distribution for the test particle. However, the dependence of α_n and γ_n with n is not yet known. Because superstatistics imposes, through Eq. (9), that

$$p_1(k_1)=\left(\sqrt{\frac{m}{2\pi}}\right)^3\int_0^\infty d\beta P(\beta|S)\exp(-\beta k_1)\beta^{\frac{3}{2}}, \quad (51)$$

and we have already shown that $P(\beta|S)$ is size independent, then $p_1(k_1)$ must also be size independent, even when α_n and γ_n are functions of n. This allows us to define new size-independent parameters u and β_S such that

$$\frac{1}{u}:=\frac{1}{2}-\frac{3n}{2}+\frac{1}{2\alpha_n}, \quad (52a)$$
$$\beta_S:=\frac{\alpha_n}{u\,\gamma_n}, \quad (52b)$$

and whose meaning will be revealed shortly. In terms of these parameters, we can rewrite Eq. (50) as

$$p_1(k_1)=p_1(0)[1+(u\beta_S)k_1]^{-(\frac{1}{u}+\frac{3}{2})}. \quad (53)$$

Comparison with Eq. (31) gives the usual parameters κ and v_{th} of the kappa distribution for a single particle in terms of u and β_S as

$$\kappa=\frac{1}{u}+\frac{1}{2}, \quad (54a)$$
$$\frac{mv_{\text{th}}^2}{2}=\frac{1}{(1-u)\beta_S}, \quad (54b)$$

and we can use these new parameters u and β_S to rewrite the fundamental inverse temperature $\beta_F^{(n)}(\mathcal{K})$ in Eq. (45) as

$$\beta_F^{(n)}(\mathcal{K})=\left(1+\frac{3nu}{2}\right)\left[\frac{\beta_S}{1+u\beta_S\mathcal{K}}\right]. \quad (55)$$

We see that $u\to 0$, that is, $\kappa\to\infty$ reduces $\beta_F^{(n)}(\mathcal{K})$ to the constant function equal to β_S for all $\mathcal{K}$, thus recovering the canonical ensemble. Replacing Eq. (53) and Eq. (41) into Eq. (47) we obtain the single-particle energy distribution, which, after normalization, yields

$$P(k_1|u,\beta_S)=\frac{(\sqrt{u\beta_S})^3}{B\left(\frac{3}{2},\frac{1}{u}\right)}[1+(u\beta_S)k_1]^{-(\frac{1}{u}+\frac{3}{2})}\sqrt{k_1}; \quad (56)$$

result that fixes the normalization constant $p_1(0)$ to be

$$p_1(0)=\left(\sqrt{\frac{mu\beta_S}{2\pi}}\right)^3\frac{\Gamma\left(\frac{1}{u}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{u}\right)}, \quad (57)$$

in full agreement with $p_1(0) = \eta_\kappa^{-1}$ as it appears in Eq. (3). The mean and relative variance of $P(k_1|u, \beta_S)$ in Eq. (56) are given by

$$\langle k_1\rangle_{u,\beta_S} = \frac{3}{2\beta_S(1-u)}, \tag{58a}$$

$$\frac{\langle(\delta k_1)^2\rangle_{u,\beta_S}}{\langle k_1\rangle^2_{u,\beta_S}} = \frac{2+u}{3(1-2u)}, \tag{58b}$$

and from these two equations we can, in principle, determine u and β_S from the observed statistics of k_1. Note that the relative variance in Eq. (58b) increases monotonically with u from its value of $2/3$ for $u = 0$. Additionally, we see that in order to keep $\langle(\delta k_1)^2\rangle_{u,\beta_S}$ a nonnegative quantity, it is required that $u < 1/2$, that is, the spectral index κ must be larger than $5/2$. Again, in the limit $u \to 0$ we can confirm, using

$$\lim_{u\to 0}[1 + (u\beta_S)k_1]^{-(\frac{1}{u}+\frac{3}{2})} = \exp(-\beta_S k_1),$$

that $P(k_1|u, \beta_S)$ in Eq. (56) reduces to the Maxwell-Boltzmann distribution of single-particle energies,

$$P(k_1|\beta) = \left(\frac{2}{\sqrt{\pi}}\right)\beta^{\frac{3}{2}}\exp(-\beta k_1)\sqrt{k_1} \tag{59}$$

with $\beta = \beta_S$. Similarly, using Eq. (17) as $P(\mathcal{K}|u, \beta_S, n) = p_n(\mathcal{K})\Omega_n(\mathcal{K})$ we obtain the energy distribution for the group of n particles as

$$P(\mathcal{K}|u, \beta_S, n) = \frac{(\sqrt{u\beta_S})^{3n}}{B\left(\frac{3n}{2}, \frac{1}{u}\right)}[1 + u\beta_S\mathcal{K}]^{-\left(\frac{1}{u}+\frac{3n}{2}\right)}\mathcal{K}^{\frac{3n}{2}-1}, \tag{60}$$

and we can verify that

$$\langle\mathcal{K}\rangle_{u,\beta_S} = \frac{3n}{2\beta_S(1-u)} = n\langle k_1\rangle_{u,\beta_S}, \tag{61}$$

hence the mean kinetic energy is an extensive quantity for all $n > 1$ and for all u. By simple inspection we can also confirm that Eq. (60) includes Eq. (56) as a particular case with $n = 1$ and $\mathcal{K} \to k_1$.

We can gain further insight on the relationship between k^* and K if we write our original requirement in Eq. (37) in terms of u, β_S, and n as

$$k^*(K) = \frac{1 + u\beta_S K}{\beta_S([3n-1]u + 2)}. \tag{62}$$

We readily see that the only case where k^* is independent of K corresponds to $u = 0$, that is, to the canonical ensemble with

$$\beta_S = \frac{1}{2k^*}, \tag{63}$$

while for $u > 0$ in the thermodynamic limit we have

$$\lim_{n\to\infty} k^*(K) = \lim_{n\to\infty}\frac{K}{3(n-1)} = \frac{k}{3}, \tag{64}$$

where we have defined $k := \lim_{n\to\infty} K/(n-1)$ as the average kinetic energy of the environment. This is in agreement with the mode and mean of the Maxwell-Boltzmann distribution of energies in Eq. (59), namely

$$k^*(\beta) = \frac{1}{2\beta} = \frac{1}{3}\langle k_1\rangle_\beta. \tag{65}$$

On the other hand, the joint distribution $P(k_1, K|u, \beta_S)$ in Eq. (39) yields the covariance between k_1 and K as

$$\langle\delta k_1\delta K\rangle_{u,\beta_S} = \frac{9u(n-1)}{4\beta_S^2(1-u)^2(1-2u)} \geqslant 0, \tag{66}$$

with equality only for $u = 0$. We can check that this covariance increases monotonically with u, and that k_1 and K are statistically independent if and only if $u = 0$.

V. STATISTICAL DISTRIBUTION OF INVERSE TEMPERATURES

The superstatistical distribution of the inverse temperature β, namely $P(\beta|u, \beta_S)$, can now be determined by using Eq. (10) in the form

$$P(\beta|u, \beta_S) = f_1(\beta)Z_1(\beta), \tag{67}$$

with $f_1 = \mathcal{L}^{-1}\{p_1\}$ the inverse Laplace transform of the single-particle ensemble function p_1 in Eq. (53). Because the inverse Laplace transform is unique if it exists, and recalling the Euler integral

$$\int_0^\infty d\beta\exp(-\beta A)\beta^{R-1} = \Gamma(R)A^{-R}, \tag{68}$$

we obtain for $A = k_1 + 1/(u\beta_S)$ and $R = 1/u + 3/2$, that

$$f_1(\beta) = \frac{p_1(0)}{u\beta_S\Gamma\left(\frac{3}{2}+\frac{1}{u}\right)}\exp\left(-\frac{\beta}{u\beta_S}\right)\left(\frac{\beta}{u\beta_S}\right)^{\frac{1}{u}+\frac{1}{2}}. \tag{69}$$

After multiplying by $Z_1(\beta)$ in Eq. (36) and replacing Eq. (57), we obtain the properly normalized probability distribution for β as

$$P(\beta|u, \beta_S) = \frac{1}{u\beta_S\,\Gamma(1/u)}\exp\left(-\frac{\beta}{u\beta_S}\right)\left(\frac{\beta}{u\beta_S}\right)^{\frac{1}{u}-1}, \tag{70}$$

which is a gamma distribution with mean and variance given by

$$\langle\beta\rangle_{u,\beta_S} = \beta_S, \tag{71a}$$

$$\langle(\delta\beta)^2\rangle_{u,\beta_S} = u(\beta_S)^2. \tag{71b}$$

Here we see that β_S is directly the mean superstatistical inverse temperature, in agreement with Eq. (26) and Eq. (25), while u is the relative variance of β, thus together with $u < 1/2$ we see that we must have $0 \leqslant u < 1/2$. The most probable inverse temperature is given by

$$\beta_S^* := \beta_S(1-u), \tag{72}$$

and it is clear that $u \to 0$ recovers the canonical ensemble, because

$$\langle(\delta\beta)^2\rangle_{u,\beta_S} \to 0, \tag{73}$$

$$\beta_S^* \to \beta_S, \tag{74}$$

which together imply $P(\beta|u, \beta_S) \to \delta(\beta - \beta_S)$, in agreement with the limit $\kappa \to \infty$ of the kappa distribution, i.e.,, the Maxwell-Boltzmann distribution. Furthermore, using Eq. (72) and letting $k_B T_S^* := 1/\beta_S^*$, we can rewrite Eq. (54b) as

$$v_{\text{th}} = \sqrt{\frac{2k_B T_S^*}{m}}, \tag{75}$$

which agrees with Eq. (2) if we interpret the parameter T appearing in the kappa distribution as T_S^* of the superstatistical description. The conditional distribution of inverse temperature given K follows from Bayes' theorem as

$$P(\beta|K,u,\beta_S,n) = \frac{P(\beta|u,\beta_S)P(K|\beta)}{P(K|u,\beta_S,n)} = \frac{P(\beta|u,\beta_S)\exp(-\beta K)}{Z_{n-1}(\beta)\,p_{n-1}(K)}, \tag{76}$$

where we have canceled a factor $\Omega_{n-1}(K)$. This is also a gamma distribution, written explicitly as

$$P(\beta|K,u,\beta_S,n) = \frac{[1+u\beta_S K]^{\frac{1}{u}+\frac{3(n-1)}{2}}}{u\beta_S\Gamma\left(\frac{1}{u}+\frac{3(n-1)}{2}\right)} \times \exp\left(-\frac{\beta}{u\beta_S}[1+u\beta_S K]\right)\left(\frac{\beta}{u\beta_S}\right)^{\frac{1}{u}+\frac{3(n-1)}{2}-1}, \tag{77}$$

but, unlike $P(\beta|u,\beta_S)$ in Eq. (70), this distribution is explicitly dependent on the size n. The mean inverse temperature given K is

$$\langle\beta\rangle_{K,u,\beta_S,n} = \left(1+\frac{3(n-1)u}{2}\right)\left[\frac{\beta_S}{1+u\beta_S K}\right], \tag{78}$$

and, by comparing with Eq. (55), we can verify that Eq. (24) holds in the form

$$\langle\beta\rangle_{K,u,\beta_S,n} = \beta_F^{(n-1)}(K). \tag{79}$$

This means $\langle\beta\rangle_{K,u,\beta_S,n}$ also reduces to β_S in the limit $u \to 0$ with finite n, becoming independent of K. In the thermodynamic limit, that is, when $n \to \infty$, we have that

$$\lim_{n\to\infty}\langle\beta\rangle_{K,u,\beta_S,n} = \lim_{n\to\infty}\frac{3(n-1)}{2K} = \frac{3}{2k} \tag{80}$$

for $u > 0$. The relative variance of $P(\beta|K,u,\beta_S,n)$ is

$$\frac{\langle(\delta\beta)^2\rangle_{K,u,\beta_S,n}}{\langle\beta\rangle^2_{K,u,\beta_S,n}} = \frac{2u}{2+3(n-1)u}, \tag{81}$$

and vanishes both in the limit $u \to 0$ and in the thermodynamic limit with $u > 0$, unlike the relative variance of $P(\beta|u,\beta_S)$ which is independent of n. This last result, combined with Eq. (80), implies that

$$\lim_{n\to\infty} P(\beta|K,u,\beta_S,n) = \delta\left(\beta - \frac{3}{2k}\right). \tag{82}$$

We can interpret this result as the following statement: in the thermodynamic limit, the kinetic energy of a group of particles uniquely fixes its superstatistical temperature, and this temperature becomes exactly the fundamental temperature.

VI. SUMMARY AND DISCUSSION

We have shown that the kappa distribution for particle velocities in a plasma can be recovered from superstatistics plus a single assumption, namely Eq. (37), which imposes linearity of the most probable kinetic energy k^* of a test particle as a function of the kinetic energy K of its environment. Our results do not rely on the concept of entropy or its maximization, nonadditivity, or any such concept, and do not assume any particular distribution of temperature *a priori*. Nevertheless, in such a plasma the inverse temperature β does have a well-defined distribution, namely the gamma distribution $P(\beta|u,\beta_S)$ in Eq. (70).

Our result shows that the kappa distribution can arise whenever there are kinetic energy correlations, suggesting that it may be realized in more diverse experimental conditions than are currently considered. Relevant new scenarios to be explored may include laser-produced plasmas [67], Z-pinches [68], and, in particular, plasma focus devices [69–71] where a rich phenomenology has been observed, including dense plasma [72], plasma shocks [73], plasma filaments [74], and supersonic plasma jets [69–72]. The recently postulated relationship between the mechanism of magnetic reconnection and kappa distributions [75,76] suggests that this distribution may also describe the emission of plasma foci, as magnetic reconnection may also be a relevant process in those devices [77].

An open question, left for future studies, is the possibility that a similar mechanism of constraining the correlations between observables may lead to the other two universality classes in superstatistics, namely log-normal and inverse gamma forms for $f(\beta)$.

ACKNOWLEDGMENT

S.D. gratefully acknowledges funding from Grant ANID FONDECYT 1220651.

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