





Odd Cosserat elasticity in active materialsPiotr Surówka ^{1,2,3} Anton Souslov ⁴ Frank Jülicher ^{2,5} and Debarghya Banerjee ^{6,2,*}¹*Institute of Theoretical Physics, Wrocław University of Science and Technology, 50-370 Wrocław, Poland*²*Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany*³*Würzburg-Dresden Cluster of Excellence ct.qmat, 01187 Dresden, Germany*⁴*Department of Physics, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom*⁵*Cluster of Excellence Physics of Life, TU Dresden, 01062 Dresden, Germany*⁶*Institute for Theoretical Physics, University of Göttingen, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany*

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Stress-strain constitutive relations in solids with an internal angular degree of freedom can be modeled using Cosserat (also called micropolar) elasticity. In this paper, we explore Cosserat materials that include chiral active components and hence odd elasticity. We calculate static elastic properties and show that the static response to rotational stresses leads to strains that depend on both Cosserat and odd elasticity. We compute the dispersion relations in odd Cosserat materials in the overdamped regime and find the presence of exceptional points. These exceptional points create a sharp boundary between a Cosserat-dominated regime of complete wave attenuation and an odd-elasticity-dominated regime of propagating waves. We conclude by showing the effect of Cosserat and odd-elasticity terms on the polarization of Rayleigh surface waves.

DOI: [10.1103/PhysRevE.108.064609](https://doi.org/10.1103/PhysRevE.108.064609)**I. INTRODUCTION**

The elastic behavior of an isotropic solid at equilibrium can be characterized by two elastic constants, namely, the shear modulus and the bulk modulus [1]. This simple description of elastic properties using two coefficients is possible because of symmetries such as isotropy, parity, and time-reversal invariance. However, this simple definition does not apply to a variety of other systems, for example, nematic solids [2] and Cosserat (or micropolar) solids [3]. Typical elastic solids can be microscopically modeled by considering point masses connected by springs. By contrast, Cosserat elasticity is based on a more complex picture and includes an angle ϕ describing the microscopic orientational degree of freedom. Even for models consisting of point particles, Cosserat-like elasticity can emerge due to a geometry based on rotating elements [4]. Recent advances in additive manufacturing (or three-dimensional printing) have led to rapid developments in the design of metamaterials with Cosserat elasticity [5–11]. Cosserat elasticity can also emerge in disordered solids [12–17], elastic polymers [18,19], and biomembranes with viscoelastic responses [20–22]. The Cosserat filament model has been used to explore the effect of microrotations in biological filaments [23–27].

Active solids [28–33] are solids that are far from equilibrium due to forcing at the microscopic scales [34–36]. To consider the effect of activity and chirality, a situation that can emerge when activity is present in the form of an active torque, in an elastic Cosserat solid one must include the effect of odd elasticity. We illustrate some of these systems and situations where an odd-Cosserat-elasticity theory is relevant in Fig. 1. Analogous to the study of Cosserat-like terms in granular material [16] the presence of (rotational) activity would lead to odd Cosserat elasticity as indicated in Fig. 1(c); Fig. 1(d) indicates a chiral active biomembrane, which is another possible material that has odd Cosserat elasticity. The odd elasticity is connected to the breaking of two essential symmetries of classical solids, parity invariance and time-reversal invariance, and appears in the elasticity tensor as a term breaking the major symmetry of the fourth-rank elastic tensor, i.e., $\kappa_{ijkl}^o = -\kappa_{klij}^o$. Recent literature [37–52] has extensively studied the effects of odd elasticity and other forms of odd responses in solids and fluids and it is therefore worthwhile to study the effects of odd elasticity in solids with active torques.

In this paper we show how the simultaneous presence of both odd elasticity and the Cosserat term affects the static and dynamic elastic response of chiral active solids in the overdamped regime. We find that the static response to off-diagonal stresses is strongly dependent on both Cosserat and odd elasticity. We also find that dynamic modes have an exceptional point [53,54] in the dispersion relation due to the competition of Cosserat and odd elasticity. In the overdamped regime, this exceptional point is characterized by a transition from damped oscillations to diffusive (or attenuating) solutions. Furthermore, the edge of these solids exhibits edge modes [1,55,56] whose polarization is affected by the combination of odd and Cosserat elasticity. The

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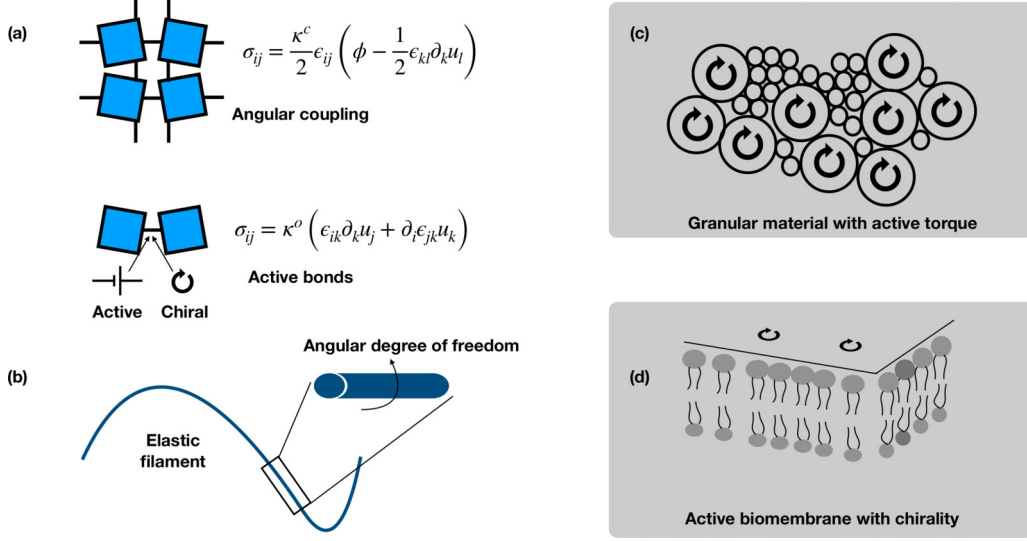


FIG. 1. Odd-Cosserat-elasticity materials. (a) The presence of extended objects at lattice points connected by springs necessitates an additional angle variable to define the strain. These microrotations are described by the so-called Cosserat term. In addition, the bonds can be chiral and active, which can be modeled (to leading order) using so-called odd elasticity. The active bonds illustrated above can be thought of as springs that inject energy and angular momentum, i.e., possess some form of active torque. (b) Rodlike elastic filaments (such as actin filaments in a cell) have an angular degree of freedom and the elastic properties of their networks have been modeled using Cosserat elasticity [25]. (c) Granular material with active torque and (d) active biomembranes with active chiral processes are two other examples of materials which could exhibit odd Cosserat elasticity.

Cosserat-elasticity coefficient results in a renormalization of the usual elastic terms for these edge waves, while the odd-elasticity coefficient mixes the longitudinal and transverse waves at the edge.

II. EFFECTIVE THEORY WITH ODD ELASTICITY

We begin by considering the stress tensor in a two-dimensional odd-Cosserat-elasticity solid (Fig. 1). The minimal coupling of the angle and the strain can be obtained by considering the Cosserat part [3–6] of the stress, which has the form $\kappa^c/2\epsilon_{ij}(\phi - \frac{1}{2}\nabla \times \mathbf{u})$ in two dimensions, while the presence of activity and chirality leads to the presence of odd elasticity, as discussed in Ref. [57]. We can write the constitutive relation between stress σ_{ij} and strain u_{ij} in these materials as

$$\begin{aligned} \sigma_{ij} = & \mu u_{ij} + B \delta_{ij} u_{kk} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \nabla \times \mathbf{u} \right) \\ & + \kappa^o (\partial_i u_j^* + \partial_i^* u_j). \end{aligned} \quad (1)$$

Here the vector \mathbf{u} is the displacement field. The stress tensor σ_{ij} depends on the strain tensor u_{ij} , which is defined as the spatial gradient of the displacement $u_{ij} = 1/2(\partial_i u_j + \partial_j u_i)$, described by the elastic coefficients μ and B . The coefficient of Cosserat elasticity κ^c describes the coupling of the internal displacement gradients to an orientational degree of freedom with angle ϕ . The two-dimensional Levi-Civita symbol is denoted by ϵ_{ij} . The odd-elasticity coefficient is denoted by κ^o and we define $u_i^* = \epsilon_{ij} u_j$. The odd-Cosserat-elasticity model described in Eq. (1) respects rotational invariance.

The dynamics of solids can depend not just on the relation between the elastic stresses and strains, but also on viscous

stresses proportional to the strain rates, i.e., $\sigma_{ij}^{\text{vis}} = \eta_{ijkl} \dot{u}_{kl}$. A combination of viscous and elastic stresses in solids is described by the Kelvin-Voigt model of viscoelasticity, which in turn can be generalized to include odd-elasticity terms [39]. However, in this paper we focus on the elastic properties only and hence neglect for simplicity the viscous stress.¹ In addition to the constitutive relation given in Eq. (1), the equation of motion for the displacement field can be written as

$$\begin{aligned} \rho \partial_t^2 u_i + \Gamma \partial_t u_i = & \partial_j \sigma_{ij}, \\ \mathcal{I} \partial_t^2 \phi + \Gamma_\phi \partial_t \phi = & \alpha \nabla^2 \phi - \kappa^c \left(\phi - \frac{1}{2} \nabla \times \mathbf{u} \right) + \tau^a, \end{aligned} \quad (2)$$

where ρ is the mass density and \mathcal{I} is the moment of inertia density. The coefficients Γ and Γ_ϕ are friction coefficients that arise due to the damping of relative motion with respect to a substrate. The coefficient α is a diffusive coefficient. The term proportional to κ^c is required by angular momentum conservation.² The active torque is denoted by τ^a . Equations (2) do not take into account nonlinear terms.

So far we have discussed the presence of odd elasticity in the equation for u_{ij} . We now consider terms that constitute active contributions in the equation of motion for ϕ .

¹In the overdamped case, we consider a dominance of frictional damping due to the interaction with a substrate over the intrinsic viscous damping. In a more mathematical language we can say that the net damping is given by $(\Gamma + \eta k^2) \partial_t \mathbf{u}$, where Γ is the friction coefficient, η is the viscous coefficient, k is the wave number, and \mathbf{u} is the internal displacement. We consider the long-wavelength limit where $\Gamma \gg \eta k^2$.

²We can use $\nabla \times \mathbf{u}$ and $\nabla^* \cdot \mathbf{u}$ interchangeably without affecting any physics.

The Cosserat term can be derived from a free energy $\mathcal{F} = \int d\mathbf{x}(\kappa^c/2)(\phi - \frac{1}{2}\nabla \times \mathbf{u})^2$. Model A–type dynamics [58,59] using this free energy gives us the equations of motion for Cosserat materials (see Appendix C). In order for a term to qualify as an active contribution, the term has to be such that it cannot be derived from an equilibrium free energy. Therefore, we do not find a linear active term in the equation for ϕ . The leading-order active contribution arising in this equation is nonlinear and given by $\tau^a = \lambda|\nabla\phi|^2 + \dots$. Mathematically, these terms are similar to the type of active terms that arise in active binary mixtures [60–62].³ An important point to be noted here is that because of the positive-definite nature of the term, the sign of λ determines the sense of the active torque, making the system naturally chiral.

III. ELASTOSTATICS OF ODD-COSSERAT-ELASTICITY MATERIALS

Using the stress tensor in Eq. (1), we now discuss the static properties of odd-Cosserat-elasticity materials. In static equilibrium, solids balance external stresses by the elastic stresses, which are proportional to the strain. This force balance can be written as a set of linear equations with the applied stress on one side and the internal stress on the other side. If we now additionally neglect the higher-order spatial gradients, we then have a linear problem where we can choose a profile of external stress and from that obtain the strain in static equilibrium. Material properties like the Young modulus E , Poisson ratio ν , and odd ratio ν^o (defining the transverse tilt of the solid under uniaxial compression; see Ref. [57]) can be computed using this method. We find the emergence of auxetic properties (negative value of ν) in the limit of large odd elasticity $2(\kappa^o)^2 > \mu B$, which was previously reported for non-Cosserat odd-elasticity solids in Ref. [57]. Details of this computation are provided in Appendix C.

While the moduli E , ν , and ν^o remain largely unaffected under the application of a uniaxial pressure, we can obtain generic expressions of strain in the presence of applied stress. Let us consider a problem where we have only applied rotational and transverse stresses, i.e., only σ_{xy} and σ_{yx} are nonzero. Under such an external stress, components of the strain tensor have the form

$$\begin{aligned} \partial_x u_x &= \frac{-\kappa^o(v_1 + v_2)}{4\kappa^{o2} + \mu^2}, & \partial_y u_y &= \frac{\kappa^o(v_1 + v_2)}{4\kappa^{o2} + \mu^2}, \\ \partial_x u_y &= -\phi - \frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)}, \\ \partial_y u_x &= \phi + \frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)}, \end{aligned} \quad (3)$$

where $v_1 = \sigma_{xy}$ and $v_2 = \sigma_{yx}$.

³The ϕ here is a pseudoscalar, while the concentration field in the binary mixture problem is a scalar field, which implies that the coefficient λ is a pseudoscalar. Also, the references here have a nonequilibrium model B dynamics while the Cosserat has a model A dynamics.

IV. ELASTODYNAMICS OF ODD-COSSERAT-ELASTICITY MATERIALS

We now consider the dynamics of the system described by Eqs. (1) and (2). In the underdamped limit, i.e., $\Gamma/\rho \rightarrow 0$ and $\Gamma_\phi/\mathcal{I} \rightarrow 0$, and ignoring other viscous effects, we can obtain oscillatory solutions from the elastic terms, which represent the usual elastic waves [1]. In the overdamped limit, i.e., when $\Gamma/\rho \rightarrow \infty$ and $\Gamma_\phi/\mathcal{I} \rightarrow \infty$, a dominant odd elasticity gives odd-elasticity waves [57]. In this paper we focus on the overdamped limit in order to prevent any additional instability that can arise in the underdamped limit. We consider propagating waves of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, where \mathbf{x} is the position coordinate and \mathbf{k} is the wave vector which gives us wave number $k = |\mathbf{k}|$. The frequency is given by ω and t is time.

In typical elastic materials, such an overdamped limit gives rise to damped modes due to the effects of the shear modulus and bulk modulus. Odd elasticity gives rise to propagating waves, which are analogous to the Avron waves in odd-viscosity fluids [48,63,64]. However, for an odd solid these are waves in displacement and not in velocity. We now compute the dispersion relation for a Cosserat solid in the presence of odd elasticity. For the purpose of this calculation, we consider the limit where we can neglect the fluctuations in ϕ , i.e., $\phi = \phi_0$. We obtain a closed-form expression of the dispersion relation given by

$$\omega = i \frac{k^2}{8} [-8\mu - 4B - \kappa^c \pm \sqrt{(4B - \kappa^c)^2 - 64\kappa^{o2}}], \quad (4)$$

where we find an exceptional point at $4B - \kappa^c = 8\kappa^o$, where the eigenvalues are degenerate and have a square-root branch point [53,54]. At the exceptional point, two eigenvectors coalesce to a single one and the eigenvalues are degenerate. At this exceptional point, we find the transition from a diffusive solution with diffusivity proportional to $\sqrt{4B - \kappa^c}$ to a damped wave solution with speed proportional to $\sqrt{\kappa^o}$. This is the key differentiating feature of the odd Cosserat elasticity from both equilibrium elastic solids and active odd-elasticity solids. In the odd-Cosserat-elasticity solids we see the coexistence of the exceptional point due to the Cosserat elastic constant and the bulk elastic constant.

If we now do not ignore fluctuations in ϕ we obtain the eigenvalues as solutions to cubic equations (Appendix E). In Fig. 2 we show the real and imaginary parts of the dispersion relations showing the emergence of a nonzero real part for finite odd elasticity as a signature of the exceptional points. In odd-Cosserat-elasticity solids we find that there exists a third eigenmode which remains purely dissipative, as indicated in Fig. 2(a) by the zero real part of the eigenvalue.

V. RAYLEIGH WAVES

We consider the effect of both the Cosserat and the odd terms on a stress-free edge. We consider solutions of the form $\mathbf{u} = \mathbf{U}e^{i(kx - \omega t)}e^{ay}$ (and ignore fluctuations in ϕ) and a boundary on the line parallel to the x axis, i.e., the y direction is normal to the edge. Implementing a zero normal stress implies $\sigma_{yy} = \sigma_{xy} = 0$ (and $\phi = 0$). We find that while Cosserat elasticity renormalizes parameters, the polarization of these edge waves depends crucially on the presence and

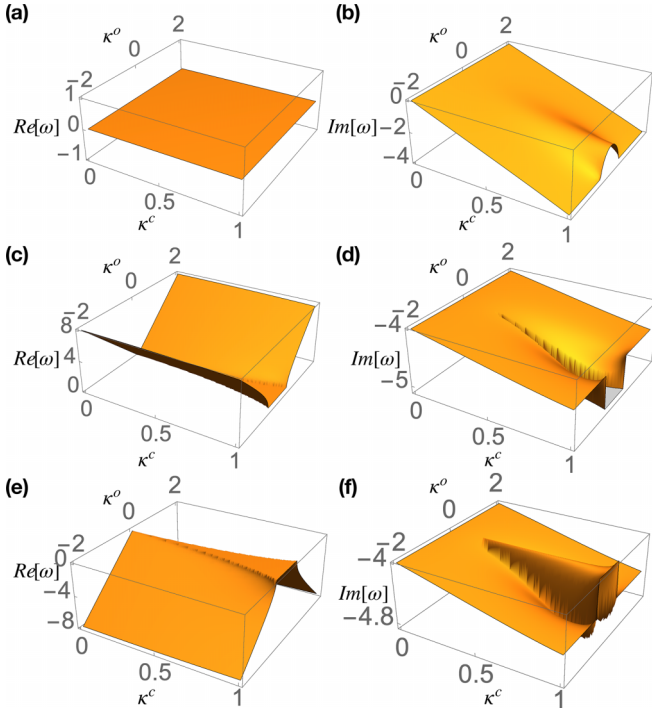


FIG. 2. Dependence of frequency on odd and Cosserat elasticity. Real and imaginary parts of the dispersion relation for an odd-Cosserat-elasticity solid are shown. The three solutions of ω (three eigenfrequencies) have both real and imaginary parts: (a) and (b) eigenfrequency corresponding to the first eigenmode, (c) and (d) eigenfrequency corresponding to the second eigenmode, and (e) and (f) eigenfrequency corresponding to the third eigenmode for the behavior of (a), (c), and (e) the real part and (b), (d), and (f) the imaginary part. The real part corresponds to oscillations and emerges only for finite odd elasticity in both the second and the third eigenmodes. For these plots, we have fixed the values of B , μ , and k at $B = 0$, $\mu = 1$, and $k = 1$.

nature of odd elasticity. We perform the detailed computation in Appendix F.

We now define two types of waves: one that is transverse and one that is longitudinal. The decay length in the y direction for the transverse and longitudinal waves are a_t and a_l , respectively. The amplitudes are given by U_t and U_l and the ratio of the amplitudes is given by ξ .

The amplitudes acquire phases proportional to the odd elasticity. Physically, this occurs because the odd-elasticity term mixes the longitudinal and the transverse waves at the edge while the Cosserat term has a much less drastic effect of simply renormalizing the equilibrium elastic constants. Therefore, in an odd-Cosserat-elasticity solid, we obtain surface waves that are in a mixed state of the usual longitudinal and transverse waves and the effective parameters are renormalized due to the presence of the Cosserat elasticity. In Fig. 3 we choose $\xi = \exp(-i\pi/2)$ and show that the edge waves appear for negative values of odd elasticity and the exact nature of the amplitudes depends on the magnitude of the Cosserat term. The complex length scale in the figures is located entirely in the negative κ^o half of the figures corresponding to the negative polarization between the transverse and longitudinal components, while the width of values of odd elasticity for

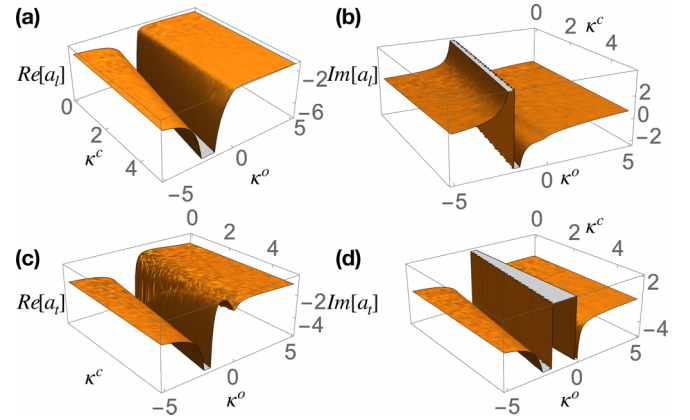


FIG. 3. Edge modes in odd-Cosserat-elasticity material. Here a_l and a_t are the decay rates of the edge modes as we move away from the edge. We use $\xi = \exp(-i\pi/2)$, which means we have left circular polarization in terms of the transverse and longitudinal modes. We find that the real and imaginary parts of a_l and a_t are 0 for positive odd elasticity. This means that, depending on the sign of odd elasticity, the edge modes have a sense of polarization. (a) Real part of a_l , (b) imaginary part of a_l , (c) real part of a_t , and (d) imaginary part of a_t .

which the edge waves occur depends on the the magnitude of Cosserat elasticity. In particular, we note in Fig. 3(d) a larger width of values of κ^o for which a finite imaginary part of the decay length is observed for greater values of κ^c . We consider this an important aspect of the Cosserat elasticity in odd-Cosserat-elasticity materials.

VI. CONCLUSION

We have studied in this paper the effects of the presence of odd elasticity in Cosserat solids. While studying the static response we found that strain at mechanical equilibrium acquires components that, generically, are dependent on both odd and Cosserat elasticity. The dispersion relations of odd-Cosserat-elasticity materials exhibit exceptional points where the solutions change from diffusive to propagating waves. However, unique to the Cosserat solids, we found there is always an eigenmode present that exhibits purely diffusive behavior which disappears on removing the fluctuation dynamics of the angle ϕ . Previous studies of odd-elasticity solids also observed exceptional points, but again, unique to odd-Cosserat-elasticity materials, the exceptional points that we have shown in this paper arise due to a competition between odd elasticity and Cosserat elasticity. A signature of these exceptional points appears in the surface waves by the generation of waves which are mixtures of transverse and longitudinal modes. The presence of Cosserat elasticity leads to an increased range of values of odd elasticity for which we have oscillatory transverse waves. We leave for future study consideration of the effect of various boundary conditions of the angle variable on the Rayleigh waves. We envision experimental verification of our results in robotic metamaterials, disordered solids, and active gels. It would also be interesting to see the effect of nonlinear terms [65] and hydrodynamic

interactions on the exceptional point in odd-Cosserat-elasticity material.

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APPENDIX A: COSSERAT AND ANGULAR MOMENTUM CONSERVATION (PASSIVE SOLIDS)

The underdamped equation of motion in two-dimensional space (neglecting other terms such as viscosity) in the presence of Cosserat stress is given by

$$\rho \partial_t^2 u_i = \partial_j \sigma_{ij}, \quad \mathcal{I} \partial_t^2 \phi = \alpha \nabla^2 \phi - \kappa^c (\phi - \frac{1}{2} \nabla \times \mathbf{u}), \quad (\text{A1})$$

where

$$\sigma_{ij} = \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \nabla \times \mathbf{u} \right). \quad (\text{A2})$$

These equations conserve angular momentum in spite of the presence of an antisymmetric stress [shown in Eq. (A2)] because of the term with coefficient κ^c in the equations of motion for local angular momentum. If one considers a closed-loop integral of the above stress, one can show that the net torque injected due to this stress is given by $\sigma_{xy} - \sigma_{yx}$, which is compensated by the κ^c term in the evolution equation for local angular momentum. This argument also works for micropolar fluids where the ϕ is replaced by the rate of

change of ϕ (we can call it Ω) and the displacement vector is replaced by the velocity vector.

In the overdamped case Eqs. (A1) and (A2) reach a state where the local angle relaxes to the curl of the strain, i.e., $2\phi = \nabla \times \mathbf{u}$, and one reaches a state where the antisymmetric stress vanishes and the local angular momentum evolution is entirely given by the evolution of the strain. For problems relating to micropolar fluids, similar arguments can be made and twice the local angular momentum relaxes to the vorticity of the fluid in a finite time. Thus, we find that even for passive solids the underdamped equation may have an important contribution from antisymmetric stress in the equation of motion.

APPENDIX B: IS SYMMETRIZATION OF THE STRESS TENSOR POSSIBLE IN PASSIVE COSSERAT SOLIDS?

In a passive fluid the antisymmetric part of the stress can be eliminated by a certain transformation of variables. To understand this, let us first write the conservation laws relevant in a fluid,

$$\partial_t g_i = -\partial_j \sigma_{ij}, \quad \partial_t \ell = \epsilon_{ij} \sigma_{ij} - \partial_j \chi_j, \quad (\text{B1})$$

where ℓ is the spin angular momentum density, χ_j is the flux of spin angular momentum, the linear momentum due to the strain rate is defined as $g_i = \rho \dot{u}_i$, and hence the conservation law is given by $\rho \ddot{u}_i = -\partial_j \sigma_{ij}$. For simplicity we consider a system with only the stress given in Eq. (A2). Let us consider the following transformations:

$$g'_i = g_i + \frac{1}{2} \partial_j \epsilon_{ij} \mathcal{I} \dot{\phi}, \quad \phi' = 0, \quad \sigma'_{ij} = 0. \quad (\text{B2})$$

In fluid systems this simply gives us the equations of motion in the primed variables. However, in solids this argument may not be simply used because the underdamped motion in solids does not give relaxation dynamics and hence antisymmetric modes remain intact and the overdamped equations are not conservation laws. This aspect of symmetrization of the stress tensor was reviewed for active fluids in Ref. [66] and discussion of liquid-crystal hydrodynamics is available in Ref. [67].

APPENDIX C: EQUILIBRIUM ELASTIC THEORY WITH AN ANGLE

In this Appendix we use the methods borrowed from the review in Ref. [66]. Let us begin by first deriving the Gibbs-Duhem relations and equilibrium stress for a Cosserat viscoelastic material. The free energy can be written as

$$F = \int d\mathbf{x} \left[\frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{1}{2} I \rho \dot{\phi}^2 + f_0(\partial_i u_j, \partial_i \phi, \phi) \right], \quad (\text{C1})$$

where we have used $\mathcal{I} \equiv I\rho$, with I a proportionality coefficient. This assumption states that the moment of inertia density is proportional to mass density. We can also define force densities as

$$\mu_\phi \equiv \frac{\partial f_0}{\partial \phi} - \partial_i \frac{\partial f_0}{\partial (\partial_i \phi)}, \quad \mu_j \equiv -\partial_i \frac{\partial f_0}{\partial (\partial_i u_j)}. \quad (\text{C2})$$

Under infinitesimal spatial translations the change in free energy is

$$\delta F = \int d\mathbf{x} \left(\rho \dot{u}_i \delta u_i + \frac{\dot{u}_i \dot{u}_i}{2} \delta \rho + I \rho \dot{\phi} \delta \dot{\phi} + \frac{I \dot{\phi}^2}{2} \delta \rho + \mu_\phi \delta \phi + \mu_i \delta u_i \right) + \oint dS_j \left(f \delta x_j + \frac{\partial f_0}{\partial (\partial_j \phi)} \delta \phi + \frac{\partial f_0}{\partial (\partial_j u_i)} \delta u_i \right), \quad (\text{C3})$$

where $f \equiv \frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{1}{2} I \rho \dot{\phi}^2 + f_0(\partial_i u_j, \partial_i \phi, \phi)$ and dS_j is the surface vector element, the integral of which encloses the volume. Now we use the following relations for infinitesimal transformations:

$$\delta \rho = -\delta x_j \partial_j \rho, \quad \delta \dot{u}_i = -\delta x_j \partial_j \dot{u}_i, \quad \delta \dot{\phi} = -\delta x_j \partial_j \dot{\phi}, \quad \delta \phi = -\delta x_j \partial_j \phi, \quad \delta u_i = -\delta x_j \partial_j u_i. \quad (\text{C4})$$

Using these expressions and the divergence theorem, we obtain

$$\delta F = \int d\mathbf{x} [(\partial_j \mu_\phi) \phi + (\partial_j \mu_i) u_i] \delta x_j + \oint dS_j \left((f_0 - \phi \mu_\phi - \mu_k u_k) \delta_{ij} \delta x_i - \frac{\partial f_0}{\partial (\partial_j \phi)} (\partial_i \phi) \delta x_i - \frac{\partial f_0}{\partial (\partial_j u_k)} (\partial_i u_k) \delta x_i \right). \quad (\text{C5})$$

From this expression for the free-energy change in an infinitesimal translation we obtain the equilibrium Ericksen-type stress as

$$\sigma_{ij}^e = (f_0 - \phi \mu_\phi - \mu_i u_i) \delta_{ij} - \frac{\partial f_0}{\partial (\partial_i \phi)} (\partial_j \phi) - \frac{\partial f_0}{\partial (\partial_i u_k)} (\partial_j u_k) \quad (\text{C6})$$

and the Gibbs-Duhem relation

$$-\partial_j \sigma_{ij}^e = (\partial_i \mu_\phi) \phi + (\partial_i \mu_k) u_k. \quad (\text{C7})$$

It is important to note that the divergence of the stress defined here goes to zero in the limit of a constant displacement ($u_i = \text{const}$). This is understood as the term μ_i goes to zero in this limit.

Irreversible thermodynamics and Cosserat viscoelasticity

For a nonequilibrium system the entropy evolution is given by

$$\partial_t s + \partial_\alpha J_\alpha^s = \theta, \quad (\text{C8})$$

where s is the entropy density, $J_\alpha^s = s v_\alpha + j_\alpha^s$ is the entropy flux, j_α^s is the relative entropy flux in the c.m. frame, and $\theta > 0$ is the entropy production rate per unit volume due to irreversible processes. Similarly, the free-energy density follows,

$$\partial_t f + \partial_\alpha J_\alpha^f = \theta_f, \quad (\text{C9})$$

where J_α^f is the free-energy flux and θ_f is the source of free energy. The free-energy density obeys the local thermodynamic relation $f = e - Ts$. The relative flux in the c.m. frame is j_α^f and we have the relation $J_\alpha^f = f v_\alpha + j_\alpha^f$. The total energy flux is the sum of free-energy transport and heat transport, i.e., $J_\alpha^e = (f + Ts) v_\alpha + j_\alpha^f + j_\alpha^Q$. We define $j_\alpha^Q = T j_\alpha^s$. Thus the free-energy flux $j_\alpha^f = j_\alpha^e - j_\alpha^Q$ is part of the relative energy flux that is not heat. In an isothermal system at temperature T the local reduction of free energy is directly related to entropy production: $T\theta = -\theta_f$.

Let us now derive the constitutive relations of Cosserat viscoelasticity from the principles of irreversible thermodynamics. We begin with a free energy given by

$$F = \int d\mathbf{x} \left[\frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{1}{2} I \rho \dot{\phi}^2 + f_0(\partial_i u_j, \partial_i \phi, \phi) \right], \quad f_0 \equiv \frac{\kappa^c}{2} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right)^2 + \kappa_{ijkl} u_k u_l u_j + \frac{\alpha}{2} (\partial_i \phi) (\partial_i \phi). \quad (\text{C10})$$

Therefore, we have

$$\frac{\partial f_0}{\partial \phi} = \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right), \quad -\partial_i \frac{\partial f_0}{\partial (\partial_i \phi)} = -\alpha \partial_i \partial_i \phi, \quad -\partial_i \frac{\partial f_0}{\partial (\partial_i u_j)} = \frac{\kappa^c}{2} \epsilon_{ij} \partial_i \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \partial_i \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \quad (\text{C11})$$

and the conservation laws given by

$$\partial_t (\rho \dot{u}_i) = -\partial_j \sigma_{ij}, \quad \partial_t \rho = -\partial_j (\rho \dot{u}_j), \quad \partial_t (I \rho \dot{\phi}) = \epsilon_{ij} \sigma_{ij} - \partial_j \chi_j, \quad (\text{C12})$$

where χ_j is the flux of angular momentum, which in our case can be simply a diffusive flux.

The free-energy evolution is given by

$$\begin{aligned} \dot{F} &= \int d\mathbf{x} \left[\frac{\dot{u}_i^2}{2} \dot{\rho} + \dot{u}_i \partial_t (\rho \dot{u}_i) + \frac{I}{2} \dot{\phi}^2 \dot{\rho} + \dot{\phi} \partial_t (I \rho \dot{\phi}) + \dot{\phi} \left(\frac{\partial f_0}{\partial \phi} - \partial_i \frac{\partial f_0}{\partial (\partial_i \phi)} \right) - \dot{u}_j \left(\frac{\partial f_0}{\partial (\partial_i u_j)} \right) \right] \\ &= \int d\mathbf{x} \left\{ \left(-\frac{\dot{u}_i \dot{u}_i}{2} - \frac{I \dot{\phi}^2}{2} \right) \partial_j (\rho \dot{u}_j) - \dot{u}_i \left[\partial_j \sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \partial_j \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \partial_j \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \right] \right. \\ &\quad \left. + \dot{\phi} \left[\epsilon_{ij} \sigma_{ij} - \partial_j \chi_j + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \alpha \partial_k \partial_k \phi \right] \right\}. \quad (\text{C13}) \end{aligned}$$

We now consider the integrand and look at it part by part. The first part is given by

$$-\frac{1}{2} (\dot{u}_i \dot{u}_i + I \dot{\phi}^2) \partial_j (\rho \dot{u}_j) = -\frac{1}{2} \partial_j [\dot{u}_i \dot{u}_i \rho \dot{u}_j + I \dot{\phi}^2 \rho \dot{u}_j] + \frac{1}{2} \rho \dot{u}_j \partial_j (\dot{u}_i \dot{u}_i + I \dot{\phi}^2). \quad (\text{C14})$$

This gives us the integral

$$\int d\mathbf{x} (\rho \dot{u}_i \dot{u}_j \partial_j \dot{u}_i + I \rho \dot{\phi} \dot{u}_j \partial_j \dot{\phi}) + (\text{surface terms}). \quad (\text{C15})$$

The second part is given by

$$\begin{aligned}
 & -\dot{u}_i \left[\partial_j \sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \partial_j \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \partial_j \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \right] \\
 & = -\partial_j \left\{ \dot{u}_i \left[\sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \right] \right\} \\
 & \quad + \partial_j \dot{u}_i \left[\sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \right], \tag{C16}
 \end{aligned}$$

which gives the integral

$$\int d\mathbf{x} \partial_j \dot{u}_i \left[\sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \right] + (\text{surface terms}). \tag{C17}$$

Finally, the third part is

$$\begin{aligned}
 & \dot{\phi} \left[\epsilon_{ij} \sigma_{ij} - \partial_j \chi_j + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \alpha \partial_k \partial_k \phi \right], \\
 & \dot{\phi} \left[\epsilon_{ij} \sigma_{ij} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) \right] - \partial_k \left[\dot{\phi} (\chi_k + \alpha \partial_k \phi) \right] + (\chi_k + \alpha \partial_k \phi) \partial_k \dot{\phi}. \tag{C18}
 \end{aligned}$$

This gives the integral

$$\int d\mathbf{x} \dot{\phi} \left[\epsilon_{ij} \sigma_{ij} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) \right] + (\chi_k + \alpha \partial_k \phi) \partial_k \dot{\phi} + (\text{surface terms}). \tag{C19}$$

Thus we get the final form of the integral excluding the surface terms,

$$\begin{aligned}
 \dot{F} = & \int d\mathbf{x} \partial_j \dot{u}_i \left[\rho \dot{u}_i \dot{u}_j + \sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \right] + \partial_j \dot{\phi} (I \rho \dot{\phi} \dot{u}_j + \chi_j + \alpha \partial_j \phi) \\
 & + \dot{\phi} \left[\epsilon_{ij} \sigma_{ij} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) \right], \tag{C20}
 \end{aligned}$$

$$\text{flux} \leftrightarrow \text{force}, \quad \rho \dot{u}_i \dot{u}_j + \sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) \leftrightarrow \partial_j \dot{u}_i,$$

$$I \rho \dot{\phi} \dot{u}_j + \chi_j + \alpha \partial_j \phi \leftrightarrow \partial_j \dot{\phi}, \quad \epsilon_{ij} \sigma_{ij} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) \leftrightarrow \dot{\phi}. \tag{C21}$$

Therefore, we obtain the phenomenological equations

$$\begin{aligned}
 \rho \dot{u}_i \dot{u}_j + \sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) &= \eta \partial_j \dot{u}_i + \lambda_1 \partial_i \partial_j \dot{\phi}, \\
 I \rho \dot{\phi} \dot{u}_j + \chi_j + \alpha \partial_j \phi &= \lambda_2 \partial_i \partial_j \dot{u}_i + \lambda_3 \partial_j \dot{\phi}, \\
 \epsilon_{ij} \sigma_{ij} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) &= \lambda_4 \dot{\phi} + \lambda_5 \partial_j \partial_j \dot{\phi} + \lambda_6 \partial_i \partial_j \partial_j \dot{u}_i. \tag{C22}
 \end{aligned}$$

If we now consider the part of force and flux that is even under time-reversal symmetry we get

$$\begin{aligned}
 \rho \dot{u}_i \dot{u}_j + \sigma_{ij}^{\text{reactive}} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) &= 0, \\
 I \rho \dot{\phi} \dot{u}_j + \chi_j^{\text{reactive}} + \alpha \partial_j \phi &= 0, \\
 \epsilon_{ij} \sigma_{ij}^{\text{reactive}} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) &= 0. \tag{C23}
 \end{aligned}$$

Similarly, if we consider the part of flux and force that is odd under time-reversal symmetry we get

$$\sigma_{ij}^{\text{dissipative}} = \eta \partial_j \dot{u}_i + \lambda_1 \partial_i \partial_j \dot{\phi}, \quad \chi_j^{\text{dissipative}} = \lambda_2 \partial_i \partial_j \dot{u}_i + \lambda_3 \partial_j \dot{\phi}, \quad \epsilon_{ij} \sigma_{ij}^{\text{dissipative}} = \lambda_4 \dot{\phi} + \lambda_5 \partial_j \partial_j \dot{\phi} + \lambda_6 \partial_i \partial_j \partial_j \dot{u}_i. \tag{C24}$$

In the equilibrium limit of a solid, all explicit time derivatives are zero and we obtain the equilibrium limit

$$\sigma_{ij} + \frac{\kappa^c}{2} \epsilon_{ij} \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) - \left(\kappa_{ijkl} \frac{1}{2} (\partial_k u_l + \partial_l u_k) \right) = 0, \quad \chi_j + \alpha \partial_j \phi = 0, \quad \epsilon_{ij} \sigma_{ij} + \kappa^c \left(\phi - \frac{1}{2} \epsilon_{kl} \partial_k u_l \right) = 0. \tag{C25}$$

Thus, we obtain the equilibrium theory of Cosserat elastic solids. We have assumed that all the coefficients are even under time reversal in the above discussion and we find a Kelvin-Voigt-type theory of viscoelastic solids.

APPENDIX D: ELASTOSTATICS CALCULATION

The linear problem of elastostatics can be defined by

$$\begin{pmatrix} B + \mu & B & \kappa^o & \kappa^o & 0 \\ B & B + \mu & -\kappa^o & -\kappa^o & 0 \\ -\kappa^o & \kappa^o & \frac{\mu}{2} - \frac{\kappa^c}{4} & \frac{\mu}{2} + \frac{\kappa^c}{4} & \frac{\kappa^c}{2} \\ -\kappa^o & \kappa^o & \frac{\mu}{2} + \frac{\kappa^c}{4} & \frac{\mu}{2} - \frac{\kappa^c}{4} & -\frac{\kappa^c}{2} \\ 0 & 0 & \frac{\kappa^c}{2} & -\frac{\kappa^c}{2} & -\kappa^c - \varepsilon \end{pmatrix} \begin{pmatrix} \partial_x u_x \\ \partial_y u_y \\ \partial_x u_y \\ \partial_y u_x \\ \phi \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ v_1 \\ v_2 \\ \theta \end{pmatrix}. \quad (\text{D1})$$

The general forms of the solutions are

$$\begin{aligned} \partial_x u_x &= \frac{2\kappa^{o2}(p_1 + p_2) - \kappa^o(v_1 + v_2)(\mu + 2B) + \mu[\mu p_1 + B(p_1 - p_2)]}{(4\kappa^{o2} + \mu^2)(\mu + 2B)}, \\ \partial_y u_y &= \frac{2\kappa^{o2}(p_1 + p_2) + \kappa^o(v_1 + v_2)(\mu + 2B) + \mu[\mu p_2 + B(p_2 - p_1)]}{(4\kappa^{o2} + \mu^2)(\mu + 2B)}, \\ \partial_x u_y &= \frac{\kappa^o(p_1 - p_2)}{4\kappa^{o2} + \mu^2} - \phi + \frac{v_2 - v_1}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)}, \\ \partial_y u_x &= \frac{\kappa^o(p_1 - p_2)}{4\kappa^{o2} + \mu^2} + \phi + \frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)}, \end{aligned} \quad (\text{D2})$$

where we have used

$$\phi = -\frac{\theta + v_1 - v_2}{\varepsilon}. \quad (\text{D3})$$

From the above description we can extract the Young modulus E , Poisson ratio ν , and odd ratio ν^o by setting $p_1 = v_1 = v_2 = \theta = 0$ and $p_2 = p$ (uniaxial stress in the y direction). We note that writing the tensorial equations of Cosserat elasticity in the matrix form renders the matrix of the coefficients noninvertible and results form an nonphysical splitting of the angle ϕ and the asymmetric part of the strain tensor $\frac{1}{2}\epsilon_{kl}\partial_k u_l$. Therefore, in order to invert the matrix, we introduce a regulator ε . It will disappear from the final result once we reintroduce the combination $\phi - \frac{1}{2}\epsilon_{kl}\partial_k u_l$. We obtain

$$E \equiv \frac{p}{\partial_y u_y} = \frac{(4\kappa^{o2} + \mu^2)(\mu + 2B)}{2\kappa^{o2} + \mu^2 + \mu B}, \quad \nu \equiv -\frac{\partial_x u_x}{\partial_y u_y} = \frac{-2\kappa^{o2} + \mu B}{2\kappa^{o2} + \mu^2 + \mu B}, \quad \nu^o \equiv -\frac{\partial_y u_x}{2\partial_y u_y} = \frac{\kappa^o(\mu/2 + B)}{2\kappa^{o2} + \mu^2 + \mu B}. \quad (\text{D4})$$

This expression is in agreement with the inversion of tensorial equations in Cosserat elasticity [68], upon setting the coefficient of odd elasticity to zero. For $p_1 = p_2 = 0$ we get

$$\begin{aligned} \partial_x u_x &= \frac{-\kappa^o(v_1 + v_2)}{(4\kappa^{o2} + \mu^2)}, \quad \partial_y u_y = \frac{\kappa^o(v_1 + v_2)}{(4\kappa^{o2} + \mu^2)}, \quad \partial_x u_y = -\phi - \frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)}, \\ \partial_y u_x &= \phi + \frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)}. \end{aligned} \quad (\text{D5})$$

The odd ratio is then given by

$$\nu^o \equiv -\frac{\partial_y u_x}{2\partial_y u_y} = -\frac{1}{2} \left(\frac{(4\kappa^{o2} + \mu^2)}{\kappa^o(v_1 + v_2)} \right) \left(\phi + \frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)} \right). \quad (\text{D6})$$

Alternatively, for a Cosserat solid one can define the odd ratio in a different way to avoid the appearance of the angle on the right-hand side as

$$\nu^o \equiv -\frac{\partial_y u_x - \phi}{2\partial_y u_y} = -\frac{1}{2} \left(\frac{(4\kappa^{o2} + \mu^2)}{\kappa^o(v_1 + v_2)} \right) \left(\frac{v_1 - v_2}{\kappa^c} + \frac{\mu(v_1 + v_2)}{2(4\kappa^{o2} + \mu^2)} \right). \quad (\text{D7})$$

This modified definition removes the ϕ dependence on the right-hand side of ν^o for stresses given above but introduces angular dependence when stresses are along the diagonal components of the stress tensor.

APPENDIX E: DISPERSION RELATION FROM THE CUBIC EQUATION

The dispersion relation (note that frequency $\omega = i\Lambda$) of an odd-Cosserat-elasticity solid can be obtained as a solution to the cubic equation

$$\Lambda^3 + b\Lambda^2 + c\Lambda + d = 0, \quad (\text{E1})$$

where

$$\begin{aligned} b &= 4\kappa^c + \kappa^c k^2 + 8\mu k^2 + 4Bk^2, \\ c &= 32\mu\kappa^c k^2 + 16B\kappa^c k^2 + 4(4\mu^2 + \mu\kappa^c + B\kappa^c + 4B\mu + 4\kappa^{o2})k^4, \\ d &= 64\kappa^c(\kappa^{o2} + \mu^2 + B\mu)k^4. \end{aligned} \quad (\text{E2})$$

The general solutions of the equation is given by

$$\begin{aligned} \Lambda &= -\frac{b}{3} - \frac{2^{1/3}(-b^2 + 3c)}{3\Theta} + \frac{\Theta}{3 \times 2^{1/3}} - \frac{b}{3} + \frac{(1 + i\sqrt{3})(-b^2 + 3c)}{3 \times 2^{2/3}\Theta} - \frac{(1 - i\sqrt{3})\Theta}{6 \times 2^{1/3}} \\ &\quad - \frac{b}{3} + \frac{(1 - i\sqrt{3})(-b^2 + 3c)}{3 \times 2^{2/3}\Theta} - \frac{(1 + i\sqrt{3})\Theta}{6 \times 2^{1/3}}, \end{aligned} \quad (\text{E3})$$

where

$$\Theta = (-2b^3 + 9bc - 27d + 3\sqrt{3}\sqrt{-b^2c^2 + 4c^3 + 4b^2d - 18bcd + 27d^2})^{1/3}. \quad (\text{E4})$$

In the case of $\kappa^o = 0$, the above relations greatly simplify and we obtain

$$\Lambda = -k^2(B + \mu) = \frac{1}{8}\{-4\kappa^c - \kappa^c k^2 - 4\mu k^2 \pm \sqrt{-64k^2\kappa^c\mu + [(4 + k^2)\kappa^c + 4k^2\mu]^2}\}. \quad (\text{E5})$$

APPENDIX F: RAYLEIGH EDGE MODES

Similar to the bulk waves discussed above, one can derive surface waves in the above system of equations. We consider the equation

$$\dot{\mathbf{u}} = \mu\nabla^2\mathbf{u} + B\nabla(\nabla \cdot \mathbf{u}) + \kappa^o\nabla^2\mathbf{u}^* - \frac{\kappa^c}{4}\nabla^*(\nabla \times \mathbf{u}) \quad (\text{F1})$$

and an ansatz $\mathbf{u} = \mathbf{U}e^{i(kx - \omega t)}e^{ay}$ in a plane semi-infinite in the y direction with $y < 0$. This gives a dispersion relation

$$\omega = \frac{i}{8}(a^2 - k^2)\{8\mu + \kappa^c + 4B \pm [(4B - \kappa^c)^2 - 64\kappa^{o2}]^{1/2}\}. \quad (\text{F2})$$

Let us now consider zero normal stress boundary conditions ($\sigma_{yy} = 0$ and $\sigma_{xy} = 0$). We obtain the following boundary conditions at $y = 0$:

$$\begin{aligned} \sigma_{yy} = 0 &= B\partial_x u_x + (\mu + B)\partial_y u_y - \kappa^o\partial_x u_y - \kappa^o\partial_y u_x, \\ \sigma_{xy} = 0 &= \kappa^o\partial_y u_y - \kappa^o\partial_x u_x + \frac{\kappa^c}{2}\phi + \left(\frac{\mu}{2} - \frac{\kappa^c}{2}\right)\partial_x u_y + \left(\frac{\mu}{2} + \frac{\kappa^c}{2}\right)\partial_y u_x. \end{aligned} \quad (\text{F3})$$

For our purpose, here we will set $\phi = 0$ as a boundary condition on ϕ at $y = 0$. Now the field \mathbf{u} will have a transverse and a longitudinal component such that $\nabla \cdot \mathbf{u}^l = 0$ and $\nabla \times \mathbf{u}^t = 0$, where the superscripts l and t denote longitudinal and transverse, respectively. We will also use a_l and a_t to denote the decay length of the longitudinal and transverse waves, respectively, and U_l and U_t for amplitudes of the longitudinal and transverse waves. Therefore, we obtain

$$\begin{aligned} u_x^t &= a_t U_t \exp(ikx + a_t y - i\omega t), \\ u_y^t &= -ik U_t \exp(ikx + a_t y - i\omega t), \\ u_x^l &= k U_l \exp(ikx + a_l y - i\omega t), \\ u_y^l &= -ia_l U_l \exp(ikx + a_l y - i\omega t). \end{aligned} \quad (\text{F4})$$

Now we can write the x and y components in terms of the longitudinal and transverse components. Therefore, we get

$$\begin{aligned} u_x &= (a_t U_t e^{a_t y} + k U_l e^{a_l y}) \exp[i(kx - \omega t)], \\ u_y &= -i(k U_t e^{a_t y} + a_l U_l e^{a_l y}) \exp[i(kx - \omega t)]. \end{aligned} \quad (\text{F5})$$

Using the above forms, we can compute the spatial derivatives of \mathbf{u} :

$$\begin{aligned} \partial_x u_x &= ik(a_t U_t e^{a_t y} + k U_l e^{a_l y}) \exp[i(kx - \omega t)], \\ \partial_x u_y &= k(k U_t e^{a_t y} + a_l U_l e^{a_l y}) \exp[i(kx - \omega t)], \\ \partial_y u_x &= (a_t^2 U_t e^{a_t y} + ka_l U_l e^{a_l y}) \exp[i(kx - \omega t)], \\ \partial_x u_y &= -i(ka_t U_t e^{a_t y} + a_l^2 U_l e^{a_l y}) \exp[i(kx - \omega t)]. \end{aligned} \quad (\text{F6})$$

At $y = 0$ we get

$$\begin{aligned}\partial_x u_x &= ik(a_t U_t + k U_l) \exp[i(kx - \omega t)], \\ \partial_x u_y &= k(k U_t + a_l U_l) \exp[i(kx - \omega t)], \\ \partial_y u_x &= (a_t^2 U_t + k a_l U_l) \exp[i(kx - \omega t)], \\ \partial_y u_y &= -i(k a_k U_t + a_l^2 U_l) \exp[i(kx - \omega t)].\end{aligned}\quad (\text{F7})$$

Returning to the condition on normal stress, we obtain

$$\begin{aligned}U_t \left(-i2k\kappa^o a_t + \frac{\mu}{2}(k^2 + a_t^2) - \frac{\kappa^c}{4}(k^2 - a_t^2) \right) + U_l [-i\kappa^o(k^2 + a_t^2) + \mu k a_l] &= 0, \\ U_t [-i\mu k a_t - \kappa^o(k^2 + a_t^2)] + U_l [iB(k^2 - a_t^2) - i\mu a_t^2 - 2\kappa^o k a_l] &= 0.\end{aligned}\quad (\text{F8})$$

Now we divide the expressions by U_l and μ and obtain

$$\begin{aligned}\xi \left(-i2k\kappa^o a_t + \frac{1}{2}(k^2 + a_t^2) - \frac{\kappa^c}{4}(k^2 - a_t^2) \right) + [-i\kappa^o(k^2 + a_t^2) + k a_l] &= 0, \\ \xi [k a_t - i\kappa^o(k^2 + a_t^2)] + [-B(k^2 - a_t^2) + a_t^2 - i2\kappa^o k a_l] &= 0,\end{aligned}\quad (\text{F9})$$

where the elastic coefficients are all scaled by μ (i.e., B is B/μ , κ^c is κ^c/μ , and κ^o is κ^o/μ). In the limit of $\kappa^o \rightarrow 0$ we get the expressions for a_l and a_t ,

$$a_l = \frac{1}{2\xi + \kappa^c \xi} (-2k \pm \sqrt{4k^2 - 4k^2 \xi^2 + k^2 \kappa^c \xi^2}) \quad (\text{F10})$$

and

$$\begin{aligned}a_t &= \frac{1}{2\xi^2 + \kappa^c \xi^2} \{ 2k\xi - k\kappa^c \xi + 4kB\xi - 8k/(2\xi + \kappa^c \xi) - 8kB/(2\xi + \kappa^c \xi) \} \\ &\pm [4\sqrt{k^2(4 - 4\xi^2 + \kappa^c \xi^2)} / (2\xi + \kappa^c \xi) \pm [4B\sqrt{k^2(4 - 4\xi^2 + \kappa^c \xi^2)} / (2\xi + \kappa^c \xi)]].\end{aligned}\quad (\text{F11})$$

The above equations allow the presence of real solutions of a_l and a_t for real ξ and $\xi < 1$. However, for κ^o not equal to zero we obtain polarized light and complex solutions of a_l and a_t . While Cosserat elasticity has the effect of rescaling parameters in the Rayleigh waves, the presence of odd elasticity introduces an additional phase in the amplitudes, indicating that the Rayleigh surface waves are circularly polarized and the sense of polarization given by the sign of odd elasticity.

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