Short delay limit of the delayed Duffing oscillator

Thomas Erneux
Université Libre de Bruxelles, Optique Nonlinéaire Théorique, Campus Plaine C.P. 231, 1050 Bruxelles, Belgium

Anton V. Kovalev and Evgeny A. Viktorov
ITMO University, Birzhevaya Liniya 14, 199034 Saint Petersburg, Russia

(Received 4 September 2023; accepted 8 November 2023; published 1 December 2023)

The delayed Duffing equation, $x'' + \varepsilon x' + x + x^3 + cx(t-\tau) = 0$, admits a Hopf bifurcation which becomes singular in the limit $\varepsilon \to 0$ and $\tau = O(\varepsilon) \to 0$. To resolve this singularity, we develop an asymptotic theory where $x(t-\tau)$ is Taylor expanded in powers of τ. We derive a minimal system of ordinary differential equations that captures the Hopf bifurcation branch of the original delay differential equation. An unexpected result of our analysis is the necessity of expanding $x(t-\tau)$ up to third order rather than first order. Our work is motivated by laser stability problems exhibiting the same bifurcation problem as the delayed Duffing oscillator [Kovalev *et al.*, Phys. Rev. E **103**, 042206 (2021)]. Here we substantiate our theory based on the short delay limit by showing the overlap (matching) between our solution and two different asymptotic solutions derived for arbitrary fixed delays.

DOI: 10.1103/PhysRevE.108.064201

I. INTRODUCTION

Several physical problems modeled by nonlinear delay differential equations (DDEs) admit Hopf bifurcation instabilities induced by a relatively short delay. Regenerative chatter in a drilling process, for example, is modeled by a DDE where the delay arises from the fact that the cutting tool passes over the metal surface repeatedly [1]. The delay is equal to the time period of one revolution of the workpiece. The short delay limit corresponds to high-speed drilling. Short time delays also appear in DDEs describing integrated photonic circuits subject to optical feedback [2,3]. While delayed feedback instabilities are well documented if the delay is sufficiently large [4,5], it was recently found that a Hopf bifurcation induced by a small delay is possible for some delayed feedback lasers [6,7].

For both the drilling and laser problems [6,8], it was noted that the weakly nonlinear analysis of a limit cycle emerging at the Hopf bifurcation [9,10] fails to provide the correct approximation in the limit of short delays. More precisely, the subsequent term in the expansion of the solution becomes larger than the previous one as the delay tends to zero. Numerically, we observe that the branch of limit cycles becomes nearly vertical as the delay decreases. Along the branch, the oscillations quickly change from nearly sinusoidal to pulsating as their amplitude increases.

A source of confusion in the mathematical literature is that sometimes small delays have no effects and sometimes cause dramatic changes; see Refs. [11–13], [[14], Sec. 4.4, p. 48], and [15,16]. A common feature of the drilling and laser equations is the fact that they can be reformulated as weakly damped conservative oscillators. We may reasonably expect that the destabilizing effect of a delayed feedback counterbalances the stabilizing effect of the physical damping. Both the drilling and laser problems depend on several parameters and require massive changes of variables before we may formulate a weakly perturbed conservative system of equations. This paper aims at clarifying the different steps needed for Hopf bifurcation analysis when both damping and delay are small. To this end, we consider a much simpler example of a delayed nearly conservative oscillator, the delayed Duffing oscillator, and explore the overlap (matching) of a short delay approximation with two different approximations derived for fixed delay.

The delayed Duffing equation [17] describes a damped nonlinear oscillator with a delayed restoring force [18]. The model has been used to model the regenerative effect in metal cutting [19] and the delayed control of a flexible beam [20]. Applications described in terms of the periodically driven delayed Duffing equation have also been investigated [21]. The delay differential equation is given by

$$x'' + \varepsilon x' + x + x^3 = -cx(t-\tau), \qquad (1)$$

where prime means differentiation with respect to time t. The parameter ε ($0 < \varepsilon << 1$) is the Duffing small damping coefficient and $c > 0$ is the amplitude of the delayed restoring force. The interest in the delayed Duffing equation also stems from the place Pyragas control method has taken in the physical and engineering sciences [22,23]. The control consists of replacing the right-hand side of Eq. (1) by $-c[x(t-\tau) - x]$. It is a noninvasive method since it doesn't change the steady states but may affect their stability properties. Because the feedback control is linear in $x(t-\tau)$, the conclusions of our analysis of Eq. (1) also apply for Duffing equation subject to Pyragas control.

Before we start our analysis of Eq. (1), it is worth stressing work on the delayed, undamped, Duffing oscillator ($\varepsilon = 0$). It has benefited from recent mathematical studies because of the existence of an infinite number of stable limit cycles. It was first suggested in a paper by Wahi and Chatterjee

[[24], Sec. 4.2], who considered the equation

$$x'' + x^3 = -\varepsilon x(t-1) \tag{2}$$

with $\varepsilon << 1$. By balancing harmonics, they determined a two-term expansion of the equation $x'' + x^3 = 0$ of the form $x = R(\sin(\omega t + \phi) + Rq\sin(3\omega t + 3\phi)$ where q is a constant. They then applied an averaging technique to formulate slow-time equations for R and ϕ. These equations predicted infinitely many limit cycles. Davidow *et al.* [25] looked for an approximate solution of

$$x'' + x^3 = -x(t-\tau) \tag{3}$$

of the the form $x = A\cos(\omega t)$ and came to the same conclusion. Mitra *et al.* [26] considered the complete Duffing equation (1) with $\varepsilon = 0$. They also looked for an approximation based on a single harmonic function and found an infinite number of limit cycles. The last two publications [25,26] noted that the amplitude of the periodic solutions grows to infinity as the delay $\tau \to 0$. Davidow *et al.* [25] and Mitra *et al.* [26] also analyzed their equations with damping ($\varepsilon > 0$) using again a single mode approximation. They observed that the number of limit cycles becomes finite and that their number progressively increases with the delay. Although the single mode approximation may provide a good numerical approximation of the solution of Eq. (2) [25], it is not an asymptotic approximation based on limiting values of a specific parameter. Sah *et al.* [27] analyzed Eq. (3) and demonstrated rigorously that the previously studied infinite sequence of stable limit cycles lose stability, eventually for delay τ such that $\tau^2 > 3\pi^2/2$. A similar result is obtained for Eq. (1) with $\varepsilon = 0$ in a more technical publication by Fiedler *et al.* [28]. To complete our review, we noted that the delayed bistable Duffing equation

$$x'' - x + x^3 = -cx(t-\tau) \tag{4}$$

has been analyzed by Cantisan *et al.* [29]. Bistability means that there are three fixed points: one unstable at the origin and two others corresponding to the bottom of the wells. Unlike Eq. (1) with $\varepsilon = 0$, Eq. (4) admits a finite number of stable limit-cycle solutions that are confined to one of the wells.

In this paper, we propose an asymptotic analysis of Eq. (1) valid in the double limit

$$\varepsilon \to 0 \text{ and } \tau = O(\varepsilon) \to 0 \tag{5}$$

but arbitrary feedback amplitude c. Our objective is to derive an ordinary differential equation (ODE) admitting the same bifurcation properties as the delay differential equation (DDE) in the limit of small delays. This particular limit was mathematically analyzed by Chicone [12], who determined conditions for which a DDE can be reduced to an ordinary differential equation (ODE) where τ is regarded as a small parameter [30]. As an example, he considered the delayed undamped Duffing equation [[12], p. 385]

$$x'' + \omega^2 x = -ax(t-\tau) - bx^3(t-\tau) \tag{6}$$

and formulated the following ODE problem:

$$x'' + \tau(a + 3bx^2)x' + (\omega^2 - a)x - bx^3 = 0. \tag{7}$$

Equation (7) is a form of van der Pol's oscillator and has a stable limit cycle for appropriate choices of its parameters. Chicone concluded that a small delay certainly does matter in this case. This is obvious since Eq. (7) with $\tau = 0$ is conservative and the dissipation of the oscillator, described by the coefficient multiplying x', is nonlinear. However, if $b = 0$, periodic solutions are possible only if $a = 0$, implying that the delay has no effects. The problem with $b = 0$ but $a \neq 0$ is precisely the problem we analyze in this paper. We show that a Hopf bifurcation is still possible for the weakly damped Duffing oscillator provided we expand $x(t-\tau)$ up to third order in τ.

The organization of this paper is as follows. In Sec. II we analyze Hopf perturbation solution of Eq. (1) and show that its expansion becomes nonuniform as $\tau \to 0$. The way it becomes nonuniform motivates a different expansion of the solution where τ is scaled with respect to ε. In Sec. III we find that the leading order solution fails to show how the branch of limit cycle solutions depends on the control parameter c. It then motivates a higher-order analysis which is developed in Sec. IV. We obtain the bifurcation equation relating the amplitude of the oscillations and c. Our result suggests the formulation of an ODE featuring the same bifurcation properties as the original DDE. Its numerical bifurcation diagram is compared to the numerical bifurcation of Eq. (1). Last, we show that solution of Eq. (1) for small delays is matching two different asymptotic solutions obtained for arbitrarily fixed delays. They contribute to an unified description of the Hopf bifurcation from small to arbitrary fixed values of the delay. Their derivations are long and tedious. They are relegated to the Appendix.

II. FAILURE OF HOPF LOCAL SOLUTION

We consider $c > 0$ as our bifurcation parameter. From the linearized equation, we determine the first Hopf bifurcation point at $c = c_0(\tau, \varepsilon)$ and analyze its limit as both $\tau \to 0$ and $\varepsilon \to 0$. In particular, we note that τ needs to be scaled by ε in order to keep c as an $O(1)$ quantity. We then determine the bifurcating solution near $c = c_0$ by using a standard perturbation analysis. We show that the expansion of the solution becomes nonuniform as $\tau \to 0$. However, the way it fails indicates how the deviation $|c - c_0|$ should be scaled with respect to ε.

From Eq. (1), we formulate the linearized equation for the zero solution $x = 0$. It is given by

$$x'' + \varepsilon x' + x = -cx(t-\tau). \tag{8}$$

Inserting $x = \exp(\lambda t)$ into (1), we determine the characteristic equation for the growth rate λ :

$$\lambda^2 + \varepsilon\lambda + 1 = -c\exp(-\lambda\tau). \tag{9}$$

There are no real λ if $c > 0$. But Hopf bifurcation instabilities are possible. We obtain the conditions by introducing $\lambda = i\omega$ into Eq. (9) and by separating the real and imaginary parts. We obtain

$$-\omega^2 + 1 = -c\cos(\omega\tau), \tag{10}$$

$$\varepsilon\omega = c\sin(\omega\tau). \tag{11}$$

An analytic solution is possible in parametric form. Introducing $z \equiv \omega\tau$, we formulate a quadratic equation for τ from

Eq. (10) and then an equation for c using Eq. (11). They are given by

$$\tau^2 + \varepsilon z \cot(z)\tau - z^2 = 0, \tag{12}$$

$$c = \varepsilon \frac{z}{\tau \sin(z)}. \tag{13}$$

By continuously increasing z from zero, we solve Eqs. (12) and (13), and determine the Hopf bifurcation lines in the (c, τ) parameter space (ε fixed). The first Hopf bifurcation point from $x = 0$ is denoted by $c = c_0$ and is obtained for the interval $0 < z < \pi/2$.

We are interested in exploring the limit of small delays. From expanding the exponential in Eq. (9) for small τ, we find that a Hopf bifurcation verifies the condition $\varepsilon - c\tau = 0$, in first approximation. Assuming $c = O(1)$, this implies that τ needs to be scaled by ε. But other scalings are possible for c and τ provided $c\tau = O(\varepsilon)$. By assuming $\tau = O(\varepsilon)$ and $z = O(\varepsilon)$, we obtain from Eq. (13) that

$$c_0 \to \frac{\varepsilon}{\tau} = O(1) \text{ as } \varepsilon \to 0. \tag{14}$$

From (14), we note that c_0 moves to infinity in the limit $\tau \to 0$, as we may expect since the zero solution is stable if $\tau = 0$. Finally, the Hopf bifurcation frequency $\omega = \omega_0$ is obtained from (10) with $z = \omega\tau = O(\varepsilon)$:

$$\omega_0 \to \sqrt{1 + c_0} \text{ as } \varepsilon \to 0. \tag{15}$$

A local analysis of the Hopf bifurcation is detailed in the Appendix and indicates that the bifurcation is supercritical ($c > c_0$). As $\tau \to 0$, we note that the amplitude of the oscillations approximates as

$$x \simeq 2\sqrt{\frac{2(c - c_0)}{\varepsilon\tau}} \cos(\omega_0 t). \tag{16}$$

In order to keep a fixed amplitude as $\tau \to 0$, $c - c_0$ needs to be scaled with respect to τ. We already indicate that τ needs to be an $O(\varepsilon)$ quantity. Consequently, the amplitude of the periodic solution (16) is fixed as $\varepsilon \to 0$ and $\tau = O(\varepsilon) \to 0$ provided that $c - c_0 = O(\varepsilon^2)$. These scalings will be useful for our asymptotic analysis, which we now detail.

III. SINGULARLY PERTURBED SOLUTION

We first Taylor expand the delayed variable in Eq. (1) up to third order and obtain

$$x'' + x + x^3 + \varepsilon x' + c\left[x - \tau x' + \frac{\tau^2}{2}x'' - \frac{\tau^3}{6}x''' + O(\tau^4)\right] = 0. \tag{17}$$

We next introduce

$$\tau = \varepsilon\tau_1, \tag{18}$$

where $\tau_1 = O(1)$, and seek a solution of the form

$$x = x_0(t) + \varepsilon x_1(t) + \cdots. \tag{19}$$

Inserting (18) and (19) into Eq. (17), we equate to zero the coefficients of each power of ε. The leading problem for x_0 is given by

$$x_0'' + (1 + c)x_0 + x_0^3 = 0 \tag{20}$$

and is an integrable Hamiltonian system with constant energy

$$E_0 = \frac{x_0'^2}{2} + (1 + c)\frac{x_0^2}{2} + \frac{x_0^4}{4}. \tag{21}$$

It admits a family $x_0 = x_0(t)$ of nested periodic orbits with amplitude A,

$$x(0) = A \text{ and } x'(0) = 0. \tag{22}$$

The frequency of the oscillations increases from $\omega = 1$ as the amplitude A increases from zero [27]. From now on, we denote by $x_0(t)$ the P-periodic solution of Eq. (20) where $P \equiv 2\pi/\omega$. To explore the contribution of the higher order terms, it will be useful to introduce the energy function

$$E(x, x') = \frac{x'^2}{2} + (1 + c)\frac{x^2}{2} + \frac{x^4}{4}, \tag{23}$$

which is suggested by (21). Differentiating (23) with respect to t and using (17) we determine an equation for E given by

$$E' = \varepsilon(-1 + c\tau_1)x'^2 + O(\varepsilon^2). \tag{24}$$

We note that the right-hand side of Eq. (24) is proportional to ε. This motivates us to solve this equation by introducing $E = E_0(t) + \varepsilon E_1(t) + \cdots$ into Eq. (24). The leading problem is $E_0' = 0$, which implies that E_0 is the constant given by (21). The next problem is $O(\varepsilon)$ and is given by $E_1' = (-1 + c\tau_1)x_0'^2$. In order to have a bounded solution for E_1 the right-hand side needs to satisfy the solvability condition

$$(-1 + c\tau_1)\int_0^P x_0'^2\, dt = 0, \tag{25}$$

which implies

$$c = \frac{1}{\tau_1}. \tag{26}$$

The expression (26) is the leading approximation of the Hopf bifurcation point since $c = 1/\tau_1 = \varepsilon/\tau$ [see (14)]. The Hopf bifurcation point moves to infinity as $\tau \to 0$, and the branch of P-periodic solutions becomes nearly vertical in the same limit. This information is valuable when we numerically investigate the bifurcation diagram of the periodic solutions. To find how the period and amplitude of each P-periodic solution change as we change c from $1/\tau_1$, a higher-order analysis is needed.

IV. HIGHER-ORDER ANALYSIS

It will be mathematically convenient to reformulate Eq. (17) by combining all terms multiplying the same derivative of x:

$$x''\left(1 + \frac{c\tau^2}{2}\right) + (1 + c)x + x^3 + (\varepsilon - c\tau)x' - c\frac{\tau^3}{6}x''' + O(\tau^4) = 0. \tag{27}$$

We next introduce $X_0(t)$ as the solution of the following equation:

$$X_0''\left(1 + \frac{c\tau^2}{2}\right) + (1 + c)X_0 + X_0^3 = 0. \tag{28}$$

Equation (28) is obtained by neglecting the last two terms in Eq. (27). It looks like Eq. (20) except for the coefficient of the second derivative of X_0. As we shall now demonstrate, Eq. (28) allows us to introduce a new energy function and derive a new solvability condition. Equation (28) admits a family of $\overline{P}$-periodic solutions with first integral

$$\overline{E}_0 = \frac{X_0'^2}{2}\left(1+\frac{c\tau^2}{2}\right) + (1+c)\frac{X_0^2}{2} + \frac{X_0^4}{4}. \quad (29)$$

As in the previous function, Eq. (29) suggests introducing the energy function

$$\overline{E}(x, x') = \frac{x'^2}{2}\left(1+\frac{c\tau^2}{2}\right) + (1+c)\frac{x^2}{2} + \frac{x^4}{4}. \quad (30)$$

We determine an equation for $\overline{E}$ by differentiating (30) and by simplifying using (27). We find

$$\overline{E}' = x'\left[-(\varepsilon - c\tau)x' + c\frac{\tau^3}{6}x'''\right]. \quad (31)$$

Recall that $\tau = \varepsilon\tau_1$. We wish that the two terms in the right-hand side of Eq. (31) were of the same order of magnitude with respect to ε. It then motivates us to introduce

$$c = \frac{1}{\tau_1} + \varepsilon^2 c_2, \quad (32)$$

where c_2 is our new control parameter. The ε^2 correction in (32) is also suggested by the expression of the Hopf bifurcation point [Eq. (A1) in the Appendix]. Inserting (18) and (32) into Eq (31) leads to

$$\overline{E}' = \varepsilon^3 x'\left(c_2\tau_1 x' + \frac{\tau_1^2}{6}x'''\right) + O(\varepsilon^5). \quad (33)$$

We now proceed as in the previous section. A bounded energy requires the solvability condition

$$\int_0^{\overline{P}} X_0'\left(c_2\tau_1 X_0' + \frac{\tau_1^2}{6}X_0'''\right)dt = 0. \quad (34)$$

Differentiating (28) once gives

$$X_0''' = -\frac{1}{\left(1+\frac{c\tau^2}{2}\right)}\left[(1+c)X_0' + 3X_0^2X_0'\right]. \quad (35)$$

Introducing (35) into (34) and taking the limit $\varepsilon \to 0$ leads to

$$\left[c_2 - \frac{\tau_1}{6}\left(1+\frac{1}{\tau_1}\right)\right]\int_0^{\overline{P}} X_0'^2\, dt - \frac{\tau_1}{2}\int_0^{\overline{P}} X_0^2 X_0'^2\, dt = 0, \quad (36)$$

where $X_0(t)$ is now redefined as the $\overline{P}$-periodic solution of Eq. (28) evaluated at $\varepsilon = 0$:

$$X_0'' + \left(1+\frac{1}{\tau_1}\right)X_0 + X_0^3 = 0. \quad (37)$$

Equation (36) is the bifurcation equation relating the amplitude of the $\overline{P}$-periodic solution $X_0(t)$ to the bifurcation parameter c_2. The definite integrals can be computed since Eq. (37) admits an exact solution in terms of Jacobian elliptic functions [31].

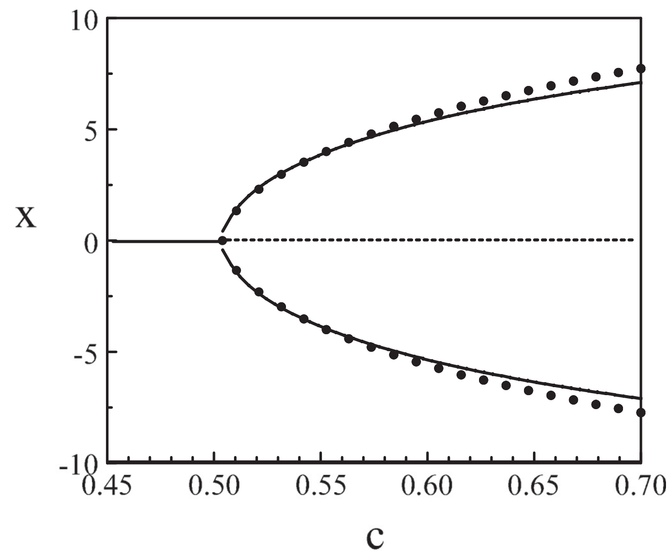

FIG. 1. Numerical bifurcation diagram. The full lines are the extrema of x determined from solving numerically Eq. (40) with $\varepsilon = 0.1$ and $\tau = 0.2$. The Hopf bifurcation is located at $c_0 \simeq \varepsilon/\tau = 0.5$. The dots are the extrema of x obtained by solving the original DDE (1).

We wish to compare the bifurcation diagram of the periodic solutions using Eq. (36) and the numerical bifurcation diagram obtained from the original DDE. Instead of computing the elliptic functions and then the integrals in (36), we choose a more direct method. Specifically, our bifurcation analysis suggests an ODE obtained by Taylor expanding $x(t-\tau)$ up to third order. It is given by Eq. (27), which can be further simplified by introducing $\tau = \varepsilon\tau_1$. We determine an expression for x''' (1) by differentiating Eq. (27) once, (2) by introducing (32), and (3) by simplifying the coefficient of x''. We obtain

$$x'''\left(1+\frac{\varepsilon^2\tau_1}{2}\right) + \left(1+\frac{1}{\tau_1}+\varepsilon^2 c_2\right)x' + 3x^2x' - \varepsilon^3\tau_1 c_2 x'' + \frac{\varepsilon^3\tau_1^3}{6}\left(\frac{1}{\tau_1}+\varepsilon^2 c_2\right) + O(\varepsilon^4) = 0. \quad (38)$$

Equation (38) implies that x''' is given by

$$x''' = -\left(1+\frac{1}{\tau_1}\right)x' - 3x^2x' + O(\varepsilon^2) \quad (39)$$

as $\varepsilon \to 0$. We note that the correction term is $O(\varepsilon^2)$. Substituting (39) into Eq. (27), we finally obtain the desired minimal second-order ODE valid up to $O(\varepsilon^4)$ corrections

$$x''\left(1+\frac{c\tau^2}{2}\right) + (1+c)x + x^3 + (\varepsilon - c\tau)x' + \frac{c\tau^3}{6}(1+\tau_1^{-1}+3x^2)x' + O(\varepsilon^4) = 0. \quad (40)$$

We compare in Fig. 1 the numerical bifurcation diagrams of the ODE (40) and the DDE (1) and verify their agreement.

In summary, we numerically showed that the periodic solution of a minimal ODE obtained by Taylor expanding the delayed variable up to third order correctly matches the periodic solution of the DDE in the limit of small delays. In the Appendix, we derive two different asymptotic solutions valid for arbitrary fixed delays. The first solution is the Hopf classical small-amplitude solution of the Duffing equation (1). We show that the small-amplitude limit of the bifurcation equation (36) correctly matches the Hopf bifurcation equation for small-amplitude solutions, evaluated for small delays. The second solution is based on the limit of small gains of the Duffing equation (1). By looking for the small delay limit of its bifurcation equation, we verify the analytical matching with Eq. (36).

V. DISCUSSION

The field of dynamical systems characterized by the presence of time delays is rapidly developing. New concepts have been emerging in recent years together with an increasing number of applications and tools to analyze real data [32,33]. While instabilities caused by a large delay have led to systematic asymptotic theories [34], the case of a small delay inducing a Hopf bifurcation instability remains poorly documented. Our asymptotic analysis of the weakly damped Duffing equation allowed us to mathematically substantiate our analysis of the laser rate equations when the delay is short [6]. Specifically, we determined an asymptotic solution of Duffing equation in the short delay and weak damping limits and then showed its overlap (matching) with other asymptotic solutions derived for arbitrary fixed delays. In this way, an unified picture of the Hopf bifurcation from small to moderated delays is possible. Physically, we emphasized the gradual change of the Hopf bifurcation branch as the delay approaches zero: the Hopf bifurcation point moves to infinity, while its branch of solutions becomes vertical and generates strongly nonlinear pulsating oscillations.

It may be argued that the destabilizing effect of a delayed feedback could be expected in the case of weakly damped conservative oscillators but would not appear for Hopf bifurcations of first-order DDEs. Consider the nonlinear first-order DDE

$$\tau^{-1}x' = -x - \lambda x(s-1) + x^3(s-1), \tag{41}$$

where $s \equiv t/\tau$ is the dimensionless time, prime means the derivative of x with respect to s, and λ is the bifurcation parameter. It is the simplest example of the so-called delay recruitment equations: $x' = -x + f[x(t-\tau)]$ where $f(x)$ is the production function. From the linear stability analysis of the zero solution, we find that the first Hopf bifurcation approaches the limit

$$\lambda_H \to \frac{\pi}{2\tau} \text{ as } \tau \to 0. \tag{42}$$

As for the Duffing problem, the singularity of the Hopf bifurcation as $\tau \to 0$ can be removed by a new expansion of the solution ($x = \tau^{-1/2}x_1(s) + \cdots$ and $\lambda = \tau^{-1}p$). Both for the Duffing equation (1) and Eq. (41), a small (but nonzero) delay is essential for inducing a Hopf bifurcation provided the delayed feedback is sufficiently strong.

Although we have determined periodic solutions of arbitrary amplitudes, our theory remains local and is valid only in the vicinity of the first Hopf bifurcation point $c = c_0 = O(1)$. The bifurcation branch is nearly vertical, but the transient evolution to the sustained oscillations is slow. From the linearized theory we note that the growth rate from the unstable steady state is proportional to $|\varepsilon - c\tau| \sim \varepsilon$ since $\tau = O(\varepsilon)$ and $c = O(1)$. The next primary Hopf bifurcation points of the delayed Duffing equation (1) appear at much higher values of the bifurcation parameter c in the limit of small delays.

We have simulated numerically Eq. (1) for higher values of the feedback amplitude c but found no secondary bifurcations. Higher order bifurcations require large delays as shown in [35–37].[1]

ACKNOWLEDGMENTS

The work of A.V.K. and E.A.V. was supported by the Ministry of Science and Higher Education of the Russian Federation, research Project No. 2019-1442 (Project Reference No. FSER-2020-0013).

APPENDIX: ASYMPTOTIC SOLUTIONS FOR ARBITRARY DELAYS

In this Appendix we determine two asymptotic solutions of the Duffing equation (1) valid for arbitrary fixed delays. The first analysis concentrates on the Hopf small-amplitude solution close to its bifurcation point, and the second study considers the limit of small feedback gains. Our goal is to demonstrate that both analytical solutions overlap the solution derived in the main text in the limit of short delays.

1. Hopf weakly nonlinear analysis

The first Hopf bifurcation responsible for the change of stability of $x = 0$ appears at $c = c_0$ with a frequency $\omega = \omega_0$. The solution in parametric form is provided by Eqs. (12) and (13) with $z \equiv \omega_0\tau$ as the parameter ($0 < z < \pi/2$). It will be useful to determine the first two terms of the expansions of c_0 and ω_0 in the limit $\varepsilon \to 0$, with $\tau = \varepsilon\tau_1$. We find

$$c_0 = \frac{1}{\tau_1} + \varepsilon^2\frac{\tau_1 + 1}{6} + \cdots, \tag{A1}$$

$$\omega_0 = \sqrt{1 + \frac{1}{\tau_1}}\left(1 - \varepsilon^2\frac{\tau_1}{6} + \cdots\right). \tag{A2}$$

We next apply the Poincaré-Lindstedt method [9] and seek a 2π-periodic solution of the form

$$x(s,\delta) = \delta x_1(s) + \delta^2 x_2(s) + \delta^3 x_3(s) + \cdots, \tag{A3}$$

where all functions are 2π-periodic in $s \equiv \sigma t$ where σ is an unknown frequency to be determined. The small parameter δ is defined as

$$\delta \equiv \frac{1}{2\pi}\int_0^{2\pi} x(s,\delta)\exp(-is)\,ds. \tag{A4}$$

[1]In Ref. [36], a high linear frequency is equivalent to a large delay after rescaling time.

Inserting (A3) into (A4) leads to normalization conditions for the unknown functions $x_1, x_2, \ldots$:

$$\frac{1}{2\pi}\int_0^{2\pi} x_1(s)\exp(-is)\,ds = 1,$$
$$\int_0^{2\pi} x_j(s)\exp(-is)\,ds = 0\ (j = 2, 3, \ldots). \qquad \text{(A5)}$$

Introducing $s \equiv \sigma t$ into Eq. (1), we obtain

$$\sigma^2 x'' + x + \varepsilon\sigma x' + x^3 + cx(s - \sigma\tau) = 0, \qquad \text{(A6)}$$

where the prime now means differentiation with respect to time s. We next expand frequency σ and bifurcation parameter c in power series of δ^2:

$$\sigma = \sigma_0 + \delta^2\sigma_2 + \cdots, \qquad \text{(A7)}$$

$$c = d_0 + \delta^2 d_2 + \cdots. \qquad \text{(A8)}$$

Inserting (A3), (A7), and (A8) into Eq. (A6) and equating to zero the coefficients of each power of δ lead to the following problems for x_1, x_2, and x_3:

$$O(\delta) : Lx_1 \equiv \sigma_0^2 x_1'' + x_1 + \varepsilon\sigma_0 x_1' + d_0 x_1(s - \sigma_0\tau) = 0, \qquad \text{(A9)}$$

$$O(\delta^2) : Lx_2 = 0, \qquad \text{(A10)}$$

$$O(\delta^3) : Lx_3 = -\begin{bmatrix} 2\sigma_0\sigma_2 x_1'' + \varepsilon\sigma_2 x_1' + x_1^3 \\ +d_2 x_1(s - \sigma_0\tau) \\ -d_0(\sigma_2\tau)x_1'(s - \sigma_0\tau) \end{bmatrix}. \qquad \text{(A11)}$$

Equation (A9) is the linearized equation evaluated at the Hopf bifurcation point. Using the first condition in (A5), it admits the solution

$$x_1 = \exp(is) + \text{c.c.}, \qquad \text{(A12)}$$

where c.c. means complex conjugate. $\sigma_0 = \omega_0$ and $d_0 = c_0$ verify the condition for the first Hopf bifurcation point ($0 < \omega_0\tau < \pi/2$) given by

$$-\omega_0^2 + 1 + i\varepsilon\omega_0 + c_0\exp(-i\omega_0\tau) = 0. \qquad \text{(A13)}$$

The solution of Eq. (A10) simply is

$$x_2 = 0 \qquad \text{(A14)}$$

because of the second condition in (A5) with $j = 2$. The solvability condition for Eq. (A11) with $\sigma_0 = \omega_0$ and $d_0 = c_0$ requires no terms $\exp(\pm is)$ on the right-hand side. This condition is

$$-2\omega_0\sigma_2 + \varepsilon i\sigma_2 + 3 + d_2\exp(-i\omega_0\tau)$$
$$- ic_0(\sigma_2\tau)\exp(-i\omega_0\tau) = 0. \qquad \text{(A15)}$$

We eliminate $\exp(-i\omega_0\tau)$ in Eq. (A15) by using (A13) and obtain

$$-2\omega_0\sigma_2 + \varepsilon i\sigma_2 + 3 + \frac{d_2}{c_0}\left(\omega_0^2 - 1 - \varepsilon i\omega_0\right)$$
$$- i(\sigma_2\tau)\left(\omega_0^2 - 1 - \varepsilon i\omega_0\right) = 0. \qquad \text{(A16)}$$

From the real and imaginary parts, we determine two equations for σ_2 and d_2 given by

$$-\omega_0\sigma_2(2 + \varepsilon\tau) + 3 + \frac{d_2}{c_0}\left(\omega_0^2 - 1\right) = 0, \qquad \text{(A17)}$$

$$\sigma_2\left[\varepsilon - \tau\left(\omega_0^2 - 1\right)\right] - \frac{d_2}{c_0}\varepsilon\omega_0 = 0. \qquad \text{(A18)}$$

The solution of Eqs. (A17) and (A18) is

$$d_2 = 3c_0\frac{\left[\varepsilon + \tau\left(-\omega_0^2 + 1\right)\right]}{\varepsilon\omega_0^2(2 + \varepsilon\tau) + \left(-\omega_0^2 + 1\right)\left[\varepsilon + \tau\left(-\omega_0^2 + 1\right)\right]}, \qquad \text{(A19)}$$

$$\sigma_2 = \frac{3\varepsilon\omega_0}{\varepsilon\omega_0^2(2 + \varepsilon\tau) + \left(-\omega_0^2 + 1\right)\left[\varepsilon + \tau\left(-\omega_0^2 + 1\right)\right]}. \qquad \text{(A20)}$$

The numerical evaluation of Eq. (A19) with $\varepsilon = 0.1$ indicates that $d_2(\tau) > 0$ meaning that the bifurcation is supercritical ($c > c_0$).

We are interested in evaluating (A19) and (A20) in the double limit $\varepsilon \to 0$ and $\tau = \varepsilon\tau_1 \to 0$. To this end, we use the expression of c_0 and ω_0 given by (A1) and (A2), respectively. The numerator and denominator in (A19) simplify as

$$N = \frac{3\varepsilon^3\tau_1}{2}\left(1 + \frac{1}{\tau_1}\right) + \cdots, \qquad \text{(A21)}$$

$$D = 2\varepsilon\left(1 + \frac{1}{\tau_1}\right) + \cdots \qquad \text{(A22)}$$

as $\varepsilon \to 0$ and lead to the following expression for d_2:

$$d_2 = \frac{\varepsilon^2\tau_1}{2}. \qquad \text{(A23)}$$

The evaluation of (A20) as $\varepsilon \to 0$ is simpler. We find

$$\sigma_2 = \frac{3}{2\sqrt{1 + \frac{1}{\tau_1}}}. \qquad \text{(A24)}$$

Using (A3), (A8) with $d_0 = c_0$, and (A12), the leading approximation of the bifurcating solution is

$$x \simeq \sqrt{\frac{c - c_0}{d_2}}\,2\cos(\omega_0 t). \qquad \text{(A25)}$$

We note from (A23) that $d_2 \to 0$ as $\varepsilon^2 \to 0$, which then requires that $c - c_0 = O(\varepsilon^2)$ in (A25) to keep the amplitude fixed. Physically, the Hopf bifurcation branch of periodic solutions becomes vertical in the limit $\varepsilon \to 0$.

The bifurcation equation (A8) relates the bifurcation parameter c and the amplitude δ of the solution. Using (A1) for $d_0 = c_0$ and (A23) for d_2, it reduces to

$$c = \frac{1}{\tau_1} + \frac{\varepsilon^2}{6}(\tau_1 + 1) + \delta^2\frac{\varepsilon^2\tau_1}{2} + O(\varepsilon^4) \text{ as } \varepsilon \to 0. \qquad \text{(A26)}$$

We next show that (A26) correctly matches the limit of small-amplitude solutions of the bifurcation equation (36). The latter was obtained by considering the limit $\varepsilon \to 0$ and $\tau = \varepsilon\tau_1 \to 0$ of Eq. (1) but for solutions of arbitrary amplitude. By again using the Poincaré-Lindstedt method, we find that the small amplitude solution of Eq. (37) for X_0 is

$$X_0 = \delta\exp(i\nu t) + \text{c.c.} + O(\delta^3), \qquad \text{(A27)}$$

where

$$\nu = \sqrt{1 + \frac{1}{\tau_1}} + O(\delta^2). \quad \text{(A28)}$$

The period is $\overline{P} \equiv 2\pi/\nu$. Using (A27), we compute the leading approximations of the two integrals in (36). After simplifications, we obtain

$$c_2 - \frac{\tau_1}{6}\left(1 + \frac{1}{\tau_1}\right) - \frac{\tau_1}{2}\delta^2 = 0. \quad \text{(A29)}$$

Finally, using (32), we find that c is related to the amplitude δ as

$$c = \frac{1}{\tau_1} + \varepsilon^2\left[\frac{1}{6}(\tau_1 + 1) + \frac{\tau_1}{2}\delta^2\right], \quad \text{(A30)}$$

which is identical to (A26).

2. Weak feedback limit

The asymptotic method for the limit of small feedback gains but arbitrary delays has been developed for the laser rate equations in Refs. [35–37], and we summarize the main details for Duffing equation (1). Specifically, we assume that

$$c = \varepsilon\alpha, \quad \text{(A31)}$$

where $\alpha = O(1)$. We then seek a solution of the form

$$x = x_0(t) + \varepsilon x_1(t) + \cdots. \quad \text{(A32)}$$

Inserting (A31) and (A32) into Eq. (1) and equating to zero the coefficients of each power of ε lead to the following problems for x_0 and x_1:

$$x_0'' + x_0 + x_0^3 = 0, \quad \text{(A33)}$$

$$x_1'' + x_1 + 3x_0^2 x_1 = -x_0' - \alpha x_0(t - \tau). \quad \text{(A34)}$$

Equation (A33) admits a one-parameter family of periodic solutions with energy

$$E_A = \frac{x_0'^2}{2} + \frac{x_0^2}{2} + \frac{x_0^4}{4}. \quad \text{(A35)}$$

The energy of the periodic solution $x_0(t)$ can be evaluated at $t = 0$, where $x_0(0) = A$ and $x_0'(0) = 0$, to be

$$E_A = \frac{A^2}{2} + \frac{A^4}{4}. \quad \text{(A36)}$$

Solving (A35) for $x' = dx/dt$, and separating variables, determines the period as

$$P_A = 4\int_0^A \frac{dx_0}{\sqrt{2E - x_0^2 - x_0^4/2}}. \quad \text{(A37)}$$

We next consider Eq. (A34). Its left-hand side admits the periodic solution $x_1 = x_0'$. Consequently, its right-hand side needs to satisfy a solvability condition given by

$$\int_0^{P_A} x_0'^2\,dt + \alpha\int_0^{P_A} x_0(t - \tau)x_0'\,dt = 0. \quad \text{(A38)}$$

We now want to check if Eq. (A38) correctly matches the bifurcation equations (36) with (37). More precisely, we need to verify if the limit $\tau_1 \to \infty$ of Eqs. (36) and (37) is identical to the limit $\tau \to 0$ of Eqs. (A38).

From Eqs. (36) and (37), we find

$$\left[c_2 - \frac{\tau_1}{6}\right]\int_0^{\overline{P}} X_0'^2\,dt - \frac{\tau_1}{2}\int_0^{\overline{P}} X_0^2 X_0'^2\,dt = 0, \quad \text{(A39)}$$

$$X_0'' + X_0 + X_0^3 = 0 \quad \text{(A40)}$$

in the limit $\tau_1 \to \infty$.

From Eq. (A38), we determine the small τ limit by expanding the delayed variable up to third order

$$\left[\begin{array}{c}\int_0^{P_A} x_0'^2 dt + \alpha\int_0^{P_A} x_0 x_0'\,dt - \alpha\tau\int_0^{P_A} x_0'^2\,dt + \alpha\frac{\tau^2}{2}\int_0^{P_A} x_0'' x_0'\,dt \\ -\alpha\frac{\tau^3}{6}\int_0^{P_A} x_0''' x_0'\,dt + O(\tau^4)\end{array}\right] = 0. \quad \text{(A41)}$$

We note that the integrals

$$\int_0^{P_A} x_0 x_0'\,dt = \oint x_0\,dx_0 = 0, \quad \int_0^{P_A} x_0'' x_0'\,dt = \oint x_0'\,dx_0' = 0. \quad \text{(A42)}$$

Taking the derivative of Eq. (A38) once provides x_0''' as

$$x_0''' = -x_0' - 3x_0^2 x_0'. \quad \text{(A43)}$$

Inserting (A43) into the last integral in Eq. (A41) and using (A42), we find

$$\left(1 - \alpha\tau + \alpha\frac{\tau^3}{6}\right)\int_0^{P_A} x_0'^2\,dt + \alpha\frac{\tau^3}{2}\int_0^{P_A} x_0^2 x_0'^2\,dt + O(\alpha\tau^4) = 0. \quad \text{(A44)}$$

We next multiply Eq. (A44) by ε and recall that $c = \varepsilon\alpha$. Eq. (A44) becomes

$$\left(\varepsilon - c\tau + c\frac{\tau^3}{6}\right)\int_0^{P_A} x_0'^2\,dt + c\frac{\tau^3}{2}\int_0^{P_A} x_0^2 x_0'^2\,dt + O(c\tau^4) = 0. \quad \text{(A45)}$$

We introduce the expansion (32) for c and $\tau = \varepsilon\tau_1$. From (A45), we find

$$O(\varepsilon^3): \left(-c_2 + \frac{\tau_1}{6}\right)\int_0^{P} x_0'^2\,dt + \frac{\tau_1}{2}\int_0^{P} x_0^2 x_0'^2\,dt = 0. \quad \text{(A46)}$$

Equation (A46) is identical to (A39) since the equations for X_0 and x_0 are identical and $P_A = \overline{P}$.

[1] S. A. Campbell, E. Stone, and T. Erneux, Delay induced canards in high speed machining, Dyn. Syst. **24**, 373 (2009).

[2] S. Bauer, O. Brox, J. Kreissl, B. Sartorius, M. Radziunas, J. Sieber, H. J. Wünsche, and F. Henneberger, Nonlinear

dynamics of semiconductor lasers with active optical feedback, Phys. Rev. E **69**, 016206 (2004).
[3] J. P. Toomey, D. M. Kane, C. McMahon, A. Argyris, and D. Syvridis, Integrated semiconductor laser with optical feedback: Transition from short to long cavity regime, Opt. Express **23**, 18754 (2015).
[4] Y. K. Chembo, D. Brunner, M. Jacquot, and L. Larger, Optoelectronic oscillators with time-delayed feedback, Rev. Mod. Phys. **91**, 035006 (2019).
[5] M. C. Soriano, J. García-Ojalvo, C. R. Mirasso, and I. Fischer, Complex photonics: Dynamics and applications of delay coupled semiconductors lasers, Rev. Mod. Phys. **85**, 421 (2013).
[6] A. V. Kovalev, E. A. Viktorov, and T. Erneux, Short-delayed-feedback semiconductor lasers, Phys. Rev. E **103**, 042206 (2021).
[7] M. Guionie, M. Romanelli, A. Thorette, A. Carré, E. Pinsard, L. Lablonde, B. Cadier, M. Alouini, M. Vallet, and M. Brunel, Delay-induced instability in phase-locked dual-polarization distributed-feedback fiber laser, Phys. Rev. A **101**, 043843 (2020).
[8] E. Stone and S. Campbell, Stability and bifurcation analysis of a nonlinear DDE model for drilling, J. Nonlinear Sci. **14**, 27 (2004).
[9] P. G. Drazin, *Nonlinear Systems*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 1992).
[10] G. Iooss and D. D. Joseph, *Elementary Stability and Bifurcation Theory*, 2nd ed., Undergraduate Texts in Mathematics (Springer, New York, 1990).
[11] J. K. Hale and S. M. Verduyn Lunel, Effects of small delays on stability and control, *Operator Theory: Advances and Applications* (Birhäuser Verlag Basel/Switzerland, 2001), Vol. 122, pp. 275–301.
[12] C. Chicone, Inertial and slow manifolds for delay equations with small delays, J. Diff. Equ. **190**, 364 (2003).
[13] T. Vyhlídal, W. Michiels, P. Zíteka, and P. McGahan, Stability impact of small delays in proportional-derivative state feedback, Control Eng. Practice **17**, 382 (2009).
[14] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Texts in Applied Mathematics Vol. 57 (Springer Science & Business Media, New York, 2011).
[15] Á. Fehér, L. Márton, and M. Pituk, Approximation of a linear autonomous differential equation with small delay, Symmetry **11**, 1299 (2019).
[16] T. Insperger, On the approximation of delayed systems by Taylor series expansion, J. Comput. Nonlinear Dyn. **10**, 024503 (2015).
[17] *The Duffing Equation: Nonlinear Oscillators and Their Behaviour*, edited by I. Kovacic and M. J. Brennan (Wiley, Hoboken, New Jersey, 2011).
[18] J. Cao, R. Yuan, H. Jiang, and J. Song, Hopf bifurcation and multiple periodic solutions in a damped harmonic oscillator with delayed feedback, J. Comput. Appl. Math. **263**, 14 (2014).
[19] R. Rusinek, A. Mitura, and J. Warminski, Time delay Duffing's systems: Chaos and chatter control, Meccanica **49**, 1869 (2014).
[20] M. Hamdi and M. Belhaq, Self-excited vibration control for axially fast excited beam by a time delay state feedback, Chaos Solitons Fractals **41**, 521 (2009).
[21] R. Rusinek, A. Weremczuk, K. Kecik, and J. Warminski, Dynamics of a time delayed Duffing oscillator, Int. J. Non Linear Mech. **65**, 98 (2014).
[22] K. Pyragas, Continuous control of chaos by self-controlling feedback, Phys. Lett. A **170**, 421 (1992).
[23] K. Pyragas, *A Twenty-Year Review of Time-Delay Feedback Control and Recent Developments*, IEICE Proceeding Series Vol. 1 (2012), pp. 683–686, www.proceeding.ieice.org.
[24] P. Wahi and A. Chatterjee, Averaging oscillations with small fractional damping and delayed terms, Nonlinear Dyn. **38**, 3 (2004).
[25] M. Davidow, B. Shayak, and R. H. Rand, Analysis of a remarkable singularity in a nonlinear DDE, Nonlinear Dyn. **90**, 317 (2017).
[26] R. K. Mitra, S. Chatterjee, and A. K. Banok, Limit cycle oscillation and multiple entrainment phenomena in a Duffing oscillator under time-delayed displacement feedback, J. Vibration Control **23**, 2742 (2017).
[27] S. M. Sah, B. Fiedler, B. Shayak, and R. H. Rand, Unbounded sequences of stable limit cycles in the delayed Duffing equation: An exact analysis, Nonlinear Dyn. **103**, 503 (2021).
[28] B. Fiedler, A. Lopez Nieto, R. H. Rand, S. M. Sah, I. Schneider, and B. de Wolff, Coexistence of infinitely many large, stable, rapidly oscillating periodic solutions in time-delayed Duffing oscillators, J. Diff. Equ. **268**, 5969 (2020).
[29] J. Cantisan, M. Coccolo, J. M. Seoane, and M. A. F. Sanjuan, Delay-induced resonance in the time-delayed Duffing oscillator, Intl. J. Bifurcation Chaos **30**, 2030007 (2020).
[30] D. Hill and D. Shafer, Asymptotics and stability of the delayed Duffing equation, J. Diff. Equ. **265**, 33 (2018).
[31] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd ed. (Springer-Verlag, New York, 1971).
[32] T. Erneux, J. Javaloyes, M. Wolfrum, and S. Yanchuk, Introduction to focus issue: Time-delay dynamics, Chaos **27**, 114201 (2017).
[33] J. Milton and T. Ohira, *Mathematics as a Laboratory Tool: Dynamics, Delays and Noise*, 2nd ed. (Springer Nature, Switzerland, AG, 2021).
[34] M. Lichtner, M. Wolfrum, and S. Yanchuk, The spectrum of delay differential equations with large delay, SIAM J. Math. Anal. **43**, 788 (2011).
[35] T. Erneux, *Applied Delay Differential Equations* (Springer, 2009).
[36] M. Hamdi and M. Belhaq, Control of bistability in a delayed Duffing oscillator, Adv. Acoustics Vibration **2012**, 872498 (2012).
[37] A. V. Kovalev, M. Shariful Islam, A. Locquet, D. S. Citrin, E. A. Viktorov, and T. Erneux, Resonances between fundamental frequencies for lasers with large delayed feedbacks, Phys. Rev. E **99**, 062219 (2019).