Improving the Cramér-Rao bound with the detailed fluctuation theorem

Domingos S. P. Salazar
Unidade de Educação a Distância e Tecnologia, Universidade Federal Rural de Pernambuco, 52171-900 Recife, Pernambuco, Brazil

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In some nonequilibrium systems, the distribution of entropy production $p(\Sigma)$ satisfies the detailed fluctuation theorem (DFT) $p(\Sigma)/p(-\Sigma) = \exp(\Sigma)$. When the distribution $p(\Sigma)$ shows a time dependence, the celebrated Cramér-Rao (CR) bound asserts that the mean entropy production rate is upper bounded in terms of the variance of Σ and the Fisher information with respect to time. In this paper we employ the DFT to derive an upper bound for the mean entropy production rate that improves the CR bound. We show that this new bound serves as an accurate approximation for the entropy production rate in the heat exchange problem mediated by a weakly coupled bosonic mode. The bound is saturated for the same setup when mediated by a weakly coupled qubit.

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I. INTRODUCTION

In small-scale thermodynamics, the entropy production Σ is usually regarded as a fluctuating quantity [1–12]. Every time a thermodynamic process is repeated, quantities such as heat, work, and entropy production might output different random values. For that reason, it is natural to represent the randomness of the entropy production in a time-dependent probability distribution $p_t(\Sigma)$, where the subscript refers to time, $t \in [0, \infty)$.

Depending on the class of systems, the distribution $p_t(\Sigma)$ might display general properties. Of particular importance is the strong detailed fluctuation theorem (DFT) [2,13], which is a relation that constrains the asymmetry of $p_t(\Sigma)$,

$$\frac{p_t(\Sigma)}{p_t(-\Sigma)} = e^{\Sigma}, \tag{1}$$

forcing positive values of entropy production to be more likely to be observed. The strong DFT (1) arises, for instance, in stochastic thermodynamics with driving protocols that are symmetric under time reversal. In that case, the forward and backward probabilities of observing a stochastic trajectory Γ are the same $[P(\Gamma) = P_F(\Gamma) = P_B(\Gamma)]$ and we have $\Sigma(\Gamma) := \ln P_F(\Gamma)/P_B(\Gamma^\dagger) = \ln[P(\Gamma)/P(\Gamma^\dagger)]$, where $\Gamma^\dagger$ is the inverse trajectory and we considered $k_B = 1$. Defining $p(\Sigma) = \int P(\Gamma)\delta(\Sigma(\Gamma) - \Sigma)d\Gamma$, we obtain the relation (1). It also appears as the Evan-Searles fluctuation theorem [14,15], as the Gallavotti-Cohen relation [9], and in the exchange fluctuation framework [16–24]. The most known consequence of (1) is the integral fluctuation theorem $\langle e^{-\Sigma}\rangle_t = 1$, which results in the second law of thermodynamics, $\langle\Sigma\rangle_t \geqslant 0$ for all t, from Jensen's inequality, where $\langle f(\Sigma)\rangle_t := \sum_i f(\Sigma_i)p_t(\Sigma_i)$.

Understanding how the distribution $p_t(\Sigma)$ and the average entropy production $\langle\Sigma\rangle_t$ change over time is important, for instance, for devising optimal thermal machines that operate in finite time. In this context, the Fisher information with respect to time plays a relevant role,

$$I(t) := \left\langle \left(\frac{\partial}{\partial t} \ln p_t(\Sigma) \right)^2 \right\rangle_t, \tag{2}$$

as it was recently used in stochastic thermodynamics as the intrinsic speed of the system [25–27], in the context of thermodynamic length [28], and in connection with the thermodynamic uncertainty relation [29,30]. The authors in [25] noted that the rate of change of the average of any observable is bounded from above by its variance and the temporal Fisher information, evoking the famous Cramér-Rao (CR) bound [31] from estimation theory. Here we are interested in the entropy production as the stochastic quantity, for which the Cramér-Rao bound reads

$$\frac{d\langle\Sigma\rangle}{dt} \leqslant \sigma_\Sigma \sqrt{I(t)}, \tag{3}$$

where $\langle\Sigma\rangle := \langle\Sigma\rangle_t$ and $\sigma_\Sigma := \sqrt{\langle\Sigma^2\rangle_t - \langle\Sigma\rangle_t^2}$ are both functions of time. Finding upper bounds for the entropy production rate such as (3) is a relevant topic in stochastic [32–34] and quantum thermodynamics [35], as they are ultimately related to speed limits [36–42].

In this paper we investigate how can the DFT (1) be used to improve the Cramér-Rao upper bound (3) for the entropy production rate. The idea is that, since (3) was derived in the general setting of estimation theory, it might be further improved for the entropy production rate $d\langle\Sigma\rangle/dt$ in cases where $p_t(\Sigma)$ is constrained by the DFT (1). Such improvement would have direct impact on the estimation of the entropy production rate in physical systems arbitrarily far from equilibrium.

We show that, in situations where the DFT (1) is valid, the entropy production rate has an upper bound that improves the CR bound,

$$\frac{d\langle\Sigma\rangle}{dt} \leqslant \sigma_{h(\Sigma)}\sqrt{I(t)} \leqslant \sigma_\Sigma\sqrt{I(t)}, \tag{4}$$

which is our main result, where $\sigma_{h(\Sigma)} := \sqrt{\langle h(\Sigma)^2\rangle - \langle h(\Sigma)\rangle^2}$ and $h(\Sigma) := \Sigma \tanh(\Sigma/2)$. As applications, we also show how the bound acts as a good estimator for the entropy production rate for the heat exchange problem mediated by a bosonic mode with Lindblad's dynamics in comparison with the CR bound (3). We also show how the bound is saturated for the same problem when mediated by a qubit. We argue that the behavior of the bound in those cases is not accidental: The bound (4) is always saturated for a time-dependent maximal distribution [43], which was originally derived as the distribution that maximizes Shannon's entropy for a given mean, while satisfying the DFT (1). We show that the qubit case falls in the maximal distribution family and the bosonic case is very close to it.

II. FORMALISM

Let $\Sigma \in S = \{\Sigma_1, \Sigma_2, \ldots\}$ be a random variable with distribution $p_t(\Sigma)$ that depends on time $t \in [0, \infty)$ and satisfies the DFT (1). Let $\phi(\Sigma)$ be any odd function

$$\phi(-\Sigma) = -\phi(\Sigma). \tag{5}$$

The DFT imposes the following known property [17,19] on the average of odd functions:

$$\langle \phi(\Sigma) \rangle_t = \left\langle \phi(\Sigma) \tanh\left(\frac{\Sigma}{2}\right) \right\rangle_t. \tag{6}$$

The time derivative of (6) yields

$$\frac{d}{dt}\langle \phi(\Sigma) \rangle_t = \sum_i \phi(\Sigma_i) \tanh\left(\frac{\Sigma_i}{2}\right) \frac{\partial}{\partial t} p_t(\Sigma_i), \tag{7}$$

which can be written as

$$\frac{d}{dt}\langle \phi(\Sigma) \rangle_t = \sum_i \sqrt{p_t(\Sigma_i)}\left[\phi(\Sigma_i) \tanh\left(\frac{\Sigma_i}{2}\right) - c\right] \frac{\dot{p}_t(\Sigma_i)}{\sqrt{p_t(\Sigma_i)}}, \tag{8}$$

where c is any constant and $\dot{p}_t(\Sigma_i) := \partial p_t(\Sigma_i)/\partial t$. Now using the Cauchy-Schwarz inequality, we obtain from (8)

$$\left(\frac{d}{dt}\langle \phi(\Sigma) \rangle_t\right)^2 \leqslant \left\langle \left[\phi(\Sigma_i) \tanh\left(\frac{\Sigma_i}{2}\right) - c\right]^2 \right\rangle_t I(t), \tag{9}$$

with the Fisher information $I(t)$ given by (2). Finally, considering the special case $\phi(\Sigma) = \Sigma$ and setting $c = \langle \Sigma \tanh(\Sigma/2) \rangle_t = \langle h(\Sigma) \rangle_t$, we obtain the first inequality in (4),

$$\left| \frac{d\langle \Sigma \rangle}{dt} \right| \leqslant \sigma_{h(\Sigma)} \sqrt{I(t)}. \tag{10}$$

Note that we could write $|d\langle \Sigma \rangle/dt| = d\langle \Sigma \rangle/dt$ by construction, since the DFT (and the second law $\langle \Sigma \rangle_t \geqslant 0$) works for all time $t > 0$. The second inequality in (4) follows from

$$\tanh\left(\frac{\Sigma}{2}\right)^2 \leqslant 1 \rightarrow h(\Sigma)^2 \leqslant \Sigma^2, \tag{11}$$

which, upon taking the average of (11) over $p_t(\Sigma)$ and subtracting $\langle h(\Sigma) \rangle^2$, yields

$$\langle h(\Sigma)^2 \rangle_t - \langle h(\Sigma) \rangle_t^2 \leqslant \langle \Sigma^2 \rangle_t - \langle \Sigma \rangle_t^2, \tag{12}$$

where we use $\langle h(\Sigma) \rangle_t = \langle \Sigma \rangle_t$ from (6). Finally, we have, from (12),

$$\sigma_{h(\Sigma)} \sqrt{I(t)} \leqslant \sigma_\Sigma \sqrt{I(t)}, \tag{13}$$

which is the second inequality of our main result (4), showing that it improves the CR bound. In the examples below, we start with a dynamics that allows us to compute both $d\langle \Sigma \rangle/dt$ and $p_t(\Sigma)$ exactly. We check that $p_t(\Sigma)$ satisfies the DFT (1). Then we use $p_t(\Sigma)$ to find σ_Σ, $\sigma_{h(\Sigma)}$, and $I(t)$. Finally, we show the bounds (4) as a function of time in Figs. 1 and 2. Then we discuss why the bound is a surprisingly good approximation for $d\langle \Sigma \rangle/dt$ in both cases.

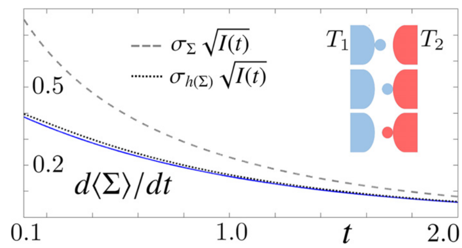

FIG. 1. Entropy production rate $d\langle \Sigma \rangle/dt$ (blue solid line) as a function of time for the heat exchange problem mediated by a bosonic mode ($\gamma = 1$, $\hbar\omega/k_B T_1 = 1$, and $T_2 = T_1/2$). The Cramér-Rao bound $\sigma_\Sigma \sqrt{I(t)}$ is depicted by a gray dashed line and $\sigma_{h(\Sigma)} \sqrt{I(t)}$ is shown by the black dotted line. In this case, the entropy production rate is very close to the bound but does not saturate it.

III. APPLICATION

A. Bosonic mode

We consider a bosonic mode with Hamiltonian $H = \hbar\omega(a^\dagger a + 1/2)$ weakly coupled to a thermal reservoir so that the system satisfies a Lindblad equation [44–46]

$$\partial_t \rho = \frac{-i}{\hbar}[H, \rho] + D(\rho) \tag{14}$$

for the dissipator given by

$$D(\rho) = \gamma(\overline{n}_2 + 1)\left[a\rho a^\dagger - \tfrac{1}{2}\{a^\dagger a, \rho\}\right] + \gamma \overline{n}_2\left[a^\dagger \rho a - \tfrac{1}{2}\{a a^\dagger, \rho\}\right], \tag{15}$$

where γ is a constant, $\overline{n}_i = [\exp(\hbar\omega/k_B T_i) - 1]^{-1}$, and $\beta_i = 1/k_B T_i$, $i \in \{1, 2\}$. The system is prepared in thermal equilibrium with the first reservoir (temperature T_1). At $t = 0$, an energy measurement is performed, resulting in $E_0 = \hbar\omega(n_0 + \frac{1}{2})$, $n_0 \in \{0, 1, 2, \ldots\}$. Then, for $t > 0$, the system is placed in thermal equilibrium with a second reservoir (temperature T_2) with dynamics (14). At a given $t > 0$, a second

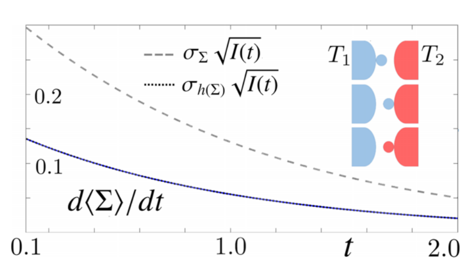

FIG. 2. Entropy production rate $d\langle \Sigma \rangle/dt$ (blue solid line) as a function of time for the heat exchange problem mediated by a qubit ($\gamma = 1$, $\hbar\omega/k_B T_1 = 1$, and $T_2 = T_1/2$). The Cramér-Rao upper bound $\sigma_\Sigma \sqrt{I(t)}$ is depicted by a gray dashed line and $\sigma_{h(\Sigma)} \sqrt{I(t)}$ is shown by the black dotted line. In this case, the entropy production rate matches the upper bound. Actually, $p_t(\Sigma)$ is a particular case of the maximal distribution, which always saturates the bound.

measurement is performed, resulting in $E_t = \hbar\omega(n_t + \frac{1}{2})$, where $n_t \in \{0, 1, 2, \ldots\}$, where $n_t := \mathrm{tr}(a^\dagger a\rho_t)$.

The time-dependent random variable $\Sigma := -(\beta_2 - \beta_1)(E_t - E_0)$ [18,24,47] is the entropy production in this case. Intuitively, we could see the bosonic mode (system) in equilibrium with the first reservoir, so the second reservoir transfers ΔE to the first reservoir in the form of heat. This heat exchange results in the entropy flux $-\beta_2\Delta E$ in the second reservoir and $\beta_1\Delta E$ in the first reservoir, which results in a total entropy flux $\Phi = -(\beta_2 - \beta_1)(E_t - E_0)$. The entropy production is given by $\Sigma = \Delta S_{\text{sys}} + \Phi$, but the system is eventually in thermal equilibrium with the first reservoir again (as in the initial state), which results in ($\Delta S_{\text{sys}} = 0$) and $\Sigma = \Phi = -(\beta_2 - \beta_1)(E_t - E_0)$.

For simplicity, let us consider $\hbar\omega/k_B T_1 = 1$ (hot) and $T_2 = T_1/2 < T_1$ (cold) such that $\Delta\beta\hbar\omega = 1$. The average entropy production (over all possible n_0 and n_t) is given by

$$\langle\Sigma\rangle_t = (\overline{n}_1 - \overline{n}_2)(1 - e^{-\gamma t}) \tag{16}$$

from the dynamics in (14) directly. The distribution $p_t(\Sigma)$ has a closed form [45,46] that satisfies the DFT (1),

$$p_t(\Sigma) = \frac{1}{A(\lambda_t)} \exp\left(\frac{\Sigma}{2} - \lambda_t \frac{|\Sigma|}{2}\right), \tag{17}$$

with support $\Sigma \in \{\pm m\}$, $m = 0, 1, 2, \ldots$, and the normalization constant reads $A(\lambda_t) := [1 - \exp(-1/2 - \lambda_t/2)]^{-1} + [1 - \exp(1/2 - \lambda_t/2)]^{-1} - 1$, where $\lambda_t > 1$. This situation corresponds to the heat exchange of ΔE from a cold (T_2) to a hot (T_1) reservoir mediated by a bosonic mode, so the second law $\langle\Sigma\rangle_t \geqslant 0 \to \langle\Delta E\rangle_t \leqslant 0$ tells us that the energy should flow from the hot to the cold reservoir on average, as expected. The value of λ_t is given implicitly by

$$(\overline{n}_1 - \overline{n}_2)(1 - e^{-\gamma t}) = \frac{e^{-1/2+\lambda_t/2}}{(e^{-1/2+\lambda_t/2} - 1)^2} - \frac{e^{-1/2-\lambda_t/2}}{(e^{-1/2-\lambda_t/2} - 1)^2}, \tag{18}$$

where the right-hand side comes from the distribution (17) and the left-hand side comes from (16). Using λ_t in (17), we calculate the Fisher information using (2). The values σ_Σ and $\sigma_{h(\Sigma)}$ are also given in terms of (17), using $\langle\Sigma^2\rangle_t = \sum_i \Sigma_i^2 p_t(\Sigma_i)$ and $\langle h(\Sigma_i)^2\rangle_t = \sum_i h(\Sigma_i)^2 p_t(\Sigma_i)$.

In Fig. 1 we plot the entropy production rate $d\langle\Sigma\rangle/dt$ from (16) as a function of time for $\hbar\omega/k_B T_1 = 1$, $T_2 = T_1/2 < T_1$, and $\gamma = 1$. We also plot the Cramér-Rao upper bound (3) and our result (4) for comparison, showing that the proposed bound is actually a good approximation to the entropy production rate when compared to the CR bound. In Sec. IV we will provide some insight into the reason for such good approximation.

B. Qubit

We consider the same measurement scheme as before, the only difference being that the system mediating the heat exchange is a qubit with Hamiltonian $H = \hbar\omega\hat{\sigma}^\dagger\hat{\sigma}$, where $\hat{\sigma}^\dagger = |1\rangle\langle 0|$ and $\hat{\sigma} = |0\rangle\langle 1|$. The systems evolves with a Lindblad dynamics (14) with dissipator

$$D_2(\rho) = \gamma(1 - \overline{n}_2)\big(\hat{\sigma}\rho\hat{\sigma}^\dagger - \tfrac{1}{2}\{\hat{\sigma}^\dagger\hat{\sigma}, \rho\}\big) + \gamma\overline{n}_2\big(\hat{\sigma}^\dagger\rho\hat{\sigma} - \tfrac{1}{2}\{\hat{\sigma}\hat{\sigma}^\dagger, \rho\}\big), \tag{19}$$

where $\overline{n}_i = 1/(1 + e^{-\beta\omega})$ is the thermal occupation for this case. As in the previous example, the qubit is prepared in thermal equilibrium with the first reservoir T_1 and at $t = 0$ the first energy measurement takes place ($E_0 = \hbar\omega n_0$, $n_0 \in \{0, 1\}$); after that it is placed in thermal contact with the second reservoir T_2 for a time $t > 0$, modeled with the dynamics (14), when a second measurement takes place ($E_t = \hbar\omega n_t$, $n_t \in \{0, 1\}$). Using the same reasoning as before, the entropy production is given by $\Sigma = -\hbar\omega\Delta\beta(n_t - n_0)$, where n_0 and n_t are Bernoulli random variables. Again, for simplicity, let us consider $\hbar\omega/k_B T_1 = 1$ (hot) and $T_2 = T_1/2 < T_1$ (cold) such that $\Delta\beta\hbar\omega = 1$. The average entropy production over all possible n_0 and n_1 yields, from the dynamics,

$$\langle\Sigma\rangle_t = (\overline{n}_1 - \overline{n}_2)(1 - e^{-\gamma t}), \tag{20}$$

just as before (16), but now the occupation numbers $\overline{n}_i$ have different values, $\overline{n}_1 = 1/(1 + e)$ and $\overline{n}_2 = 1/(1 + e^2)$. For this setup, we have $P(n_0 = 1) = \overline{n}_1$ and $P(n_t = 1|n_0) = \overline{n}_2 + (n_0 - \overline{n_2})\exp(-\gamma t)$. Considering all possibilities of n_0 and n_t results in the distributions for $p_t(\Sigma)$,

$$p_t(0) = 1 - (1 - e^{-\gamma t})(\overline{n}_1 + \overline{n}_2 - 2\overline{n}_1\overline{n}_2) \tag{21}$$

for $\Sigma = 0$ and

$$p_t(\Sigma) = (1 - p_t(0))\frac{e^{\Sigma/2}}{e^{1/2} + e^{-1/2}} \tag{22}$$

for $\Sigma \in \{\pm 1\}$, which satisfy the DFT (1). Using the distribution (22), we compute $\langle\Sigma\rangle_t = \tanh(\frac{1}{2})[1 - p_t(0)]$, $\langle\Sigma^2\rangle_t = 1 - p_t(0)$, and $\langle h(\Sigma)^2\rangle_t = \tanh^2(\frac{1}{2})[1 - p_t(0)]$, which allows us to write σ_Σ and $\sigma_{h(\Sigma)}$ as functions of time. Finally, we can use (22) and (21) to find the Fisher information (2) also as a function of time. In this particular case, it yields $I(t) = [\partial p_t(0)/\partial t]^2/p_t(0)[1 - p_t(0)]$.

In Fig. 2, as in the previous example, we plot the entropy production rate, also given by $d\langle\Sigma\rangle/dt = (\overline{n}_1 - \overline{n}_2)e^{-\gamma t}$, as a function of time for $\hbar\omega/k_B T_1 = 1$, $T_2 = T_1/2 < T_1$, and $\gamma = 1$. We also plot the Cramér-Rao bound (3) and our result (4). In the case of the qubit, the upper bound is saturated while the CR bound is not. This fact led us to investigate what would be a sufficient condition for the saturation, as discussed in the next section.

IV. DISCUSSION

We note that the bound (4) was verified in both applications, as expected from the DFT (1). However, the bound also worked as a good estimator of the entropy production rate in both cases, matching the exact value for the qubit. Now we investigate the intuition behind it. First, we consider the following a general distribution $p_t(\Sigma)$ of the exponential family [25,28] that satisfies the DFT (1),

$$p_t(\Sigma) = \frac{1}{Z(\lambda_t)} \exp\left(\frac{\Sigma}{2} - \frac{\lambda_t}{2} f(\Sigma)\right), \tag{23}$$

where f is even, $f(\Sigma) = f(-\Sigma)$, and $Z(\lambda_t) := \int \exp[\Sigma/2 - (\lambda_t/2) f(\Sigma)]$ is a normalization constant. We also have $-\partial \ln Z(\lambda_t)/\partial\lambda_t = \frac{1}{2}\langle f(\Sigma)\rangle_t$ and

$$\frac{d\langle\Sigma\rangle}{dt} = \frac{\dot{\lambda}_t}{2}[\langle\Sigma\rangle_t \langle f(\Sigma)\rangle_t - \langle\Sigma f(\Sigma)\rangle_t]. \tag{24}$$

Now using the property (6) in (24) for the odd functions $\phi(\Sigma) = \Sigma$ and $\phi(\Sigma) = \Sigma f(\Sigma)$, we have $\langle\Sigma\rangle_t = \langle h(\Sigma)\rangle_t$ and $\langle\Sigma f(\Sigma)\rangle_t = \langle h(\Sigma) f(\Sigma)\rangle_t$, which results in

$$\frac{d\langle\Sigma\rangle}{dt} = \frac{\dot{\lambda}_t}{2}[\langle h(\Sigma)\rangle_t \langle f(\Sigma)\rangle_t - \langle h(\Sigma) f(\Sigma)\rangle_t]. \tag{25}$$

The Fisher information (2) from (23) is given by

$$I(t) = \langle [f(\Sigma) - \langle f(\Sigma)\rangle_t]^2\rangle_t \frac{\dot{\lambda}_t^2}{4} = \sigma^2_{f(\Sigma)} \frac{\dot{\lambda}_t^2}{4}. \tag{26}$$

Finally, using (25) and (26) in (10), we obtain, for $\dot{\lambda}_t \neq 0$,

$$|\langle h(\Sigma) f(\Sigma)\rangle_t - \langle h(\Sigma)\rangle_t \langle f(\Sigma)\rangle_t| \leqslant \sigma_{h(\Sigma)}\sigma_{f(\Sigma)}, \tag{27}$$

which can be rearranged as

$$|r_{h(\Sigma),f(\Sigma)}| := \frac{|\mathrm{cov}_{h(\Sigma),f(\Sigma)}|}{\sigma_{h(\Sigma)}\sigma_{f(\Sigma)}} \leqslant 1. \tag{28}$$

The left-hand side of (28) is the absolute value of the Pearson correlation coefficient $r_{h(\Sigma),f(\Sigma)}$ between the random variables $h(\Sigma)$ and $f(\Sigma)$. The saturation of the bound is thus obtained for $|r_{h(\Sigma),f(\Sigma)}| = 1$, resulting from the identity $f(\Sigma) = h(\Sigma)$. In this case, the probability density function (23) reads

$$p_t(\Sigma) = \frac{1}{Z(\lambda_t)} \exp\left[\frac{\Sigma}{2} - \lambda_t \frac{\Sigma}{2} \tanh\left(\frac{\Sigma}{2}\right)\right], \tag{29}$$

which is the maximal distribution [43], originally derived as the distribution that maximizes Shannon's entropy for a given mean with the DFT (1) as a constraint.

Comparing the maximal distribution (29) with the qubit example (22) shows that it is indeed a member of this family (for a specific support $\Sigma \in \{-\hbar\omega\Delta\beta, 0, \hbar\omega\Delta\beta\}$). For that reason, the entropy production rate actually matches the upper bound in Fig. 2. Alternatively, the bosonic case (17) does not saturate the bound in Fig. 1, but it is very close. Using the notation in (23), the bosonic case has $f(\Sigma) = |\Sigma| \approx h(\Sigma)$, which is a close approximation to the maximal distribution. In general, for any given system in the exponential family, the upper bound will serve as a good approximation whenever $|r_{f(\Sigma),h(\Sigma)}| \approx 1$.

V. CONCLUSION

We used the DFT (1) improve the Cramér-Rao upper bound for the entropy production rate. We checked the behavior of the bound in the heat exchange problem mediated by two relevant physical systems in the weak-coupling approximation: a bosonic mode and a qubit. We found that the bound is very close to the entropy production rate as a function of time for the bosonic case and it saturates for the qubit. Finally, in a more general setting, we showed that the bound is actually saturated for a maximal distribution, which contains the qubit example as a particular case and it approximates the bosonic case. Due to the recent developments of the DFT outside stochastic thermodynamics, especially in quantum correlated systems [48], we believe this result will have an impact on the understanding of the limiting behavior of open quantum systems.

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