


Griffiths singularity in quasi-one-dimensional restricted $\pm J$ Ising spin glass

Fateme Izadi  and Reza Sepehrinia ^{*}

Department of Physics, University of Tehran, Tehran 14395-547, Iran



(Received 7 June 2023; accepted 20 November 2023; published 11 December 2023)

We obtain the exact ground-state energy of the quasi-one-dimensional restricted $\pm J$ Ising spin glass in a uniform magnetic field using the transfer matrix method. Magnetic field dependence allows us to derive the magnetization as a function of concentration and magnetic field. It turns out that, in the limit of zero field, the magnetization tends to a nonzero value with a singular dependence on the magnetic field. We derive the explicit form of the singularity in thermodynamic quantities such as energy $E \simeq E_0 + m_0 h + E_1 e^{-h_0/h}$, which is an essential singularity known as Griffiths singularity. We confirm our analytical results using the numerical approach based on iterative equations for energy.

DOI: [10.1103/PhysRevE.108.064117](https://doi.org/10.1103/PhysRevE.108.064117)

I. INTRODUCTION

Spin glasses as the prototype of glassy systems are a subject of interest for many disciplines in science. From the fundamental point of view, the physics of these systems is not fully understood. The question of spin glass being a new magnetic phase of matter is somewhat open [1]. In addition to their fundamental importance, spin glass concepts, ideas, and mathematical tools were applied to problems in neural networks, combinatorial optimization, biological evolution, protein dynamics and folding, computer science, mathematics, and the social sciences [2–6].

Most of our theoretical knowledge on spin glasses is based on the mean-field theory or infinite-range Sherrington-Kirkpatrick model [7], whose thermodynamic behavior is identical to the short-range model in infinite dimensions. The theory of the short-range model in finite dimensions relies mainly on numerical simulations except in one dimension, where analytical proof exists. The current consensus is that there is no transition in dimensions less than 3 at a finite temperature. There is, however, evidence of true long-range spin glass order at zero temperature in two dimensions [8–11].

Therefore, the exact analytical results in low-dimensional systems can be helpful in understanding the physics of the spin glass phase, especially the one-dimensional and quasi-one-dimensional models, which are feasible for analytical treatment. The ground-state energy and magnetization of a linear chain with $\pm J$ [12–14] and continuous [15,16] distribution of couplings in the presence of a uniform magnetic field have been investigated. For a linear chain, the magnetization vanishes at the zero-field limit with a power-law dependence on the magnetic field for both discrete and continuous coupling distribution. For strips of a few coupled chains, it is also possible to calculate the ground-state energy analytically. Three coupled chains with a periodic boundary condition in the transverse direction and $\pm J$ distribution have been investigated by Derrida *et al.* [13]. The linear triangular chain has

been discussed in Refs. [17,18]. The method of Ref. [13] is generalized to calculate the ground-state energy of strips with larger widths in Ref. [19].

To obtain the magnetization, one needs the dependence of the energy on the magnetic field. Including the magnetic field makes the problem more complicated. For the cases in which the enumeration of the spin configurations is possible, the inclusion of the magnetic field can be done more easily. One such example, which is investigated in Ref. [20], is the Ising ladder with randomness only in transverse links with certain constraints on couplings. But in general, the enumeration of clusters that flip by increasing the magnetic field is not a simple problem. The transfer matrix method [13] is suitable as it allows the calculations without enumeration of spin configurations. This method is implemented numerically, and the magnetic field dependence of energy and magnetization has been investigated for strips of various widths by one of the authors and a collaborator [21].

In this paper, we study a simplified version of $\pm J$ Ising spin glass on a ladder structure in the presence of a magnetic field. We use the transfer matrix method to obtain the exact analytical expression for zero-temperature energy and magnetization. In contrast with the continuous distribution [21], where the magnetization tends to zero in zero field, in this case we find nonzero magnetization with essential singularity in the magnetic field dependence in the zero-field limit. This kind of singularity was first discovered by Griffiths [22] in diluted ferromagnets. Other studies suggest that such a singularity exists also in spin glasses [23,24].

II. MODEL AND TRANSFER MATRIX FORMALISM

The Hamiltonian of the $\pm J$ Ising spin glass on the ladder in a uniform magnetic field is given by

$$\mathcal{H} = - \sum_{i=1}^{L-1} (J_{i,1}^h \sigma_{i,1} \sigma_{i+1,1} + J_{i,2}^h \sigma_{i,2} \sigma_{i+1,2}) + \sum_{i=1}^L (J_i^v \sigma_{i,1} \sigma_{i,2}) - h \sum_{i=1}^L (\sigma_{i,1} + \sigma_{i,2}), \quad (1)$$

^{*}sepehrinia@ut.ac.ir

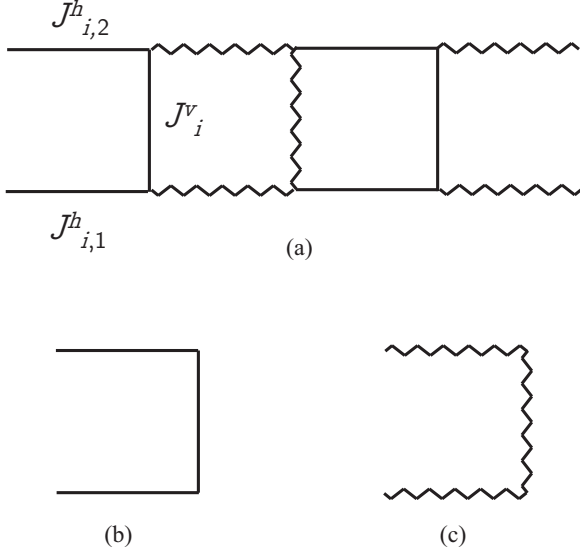


FIG. 1. (a) The Ising ladder with random $\pm J$ interactions. The wavy lines indicate antiferromagnetic interactions ($-J$). Unit cells with (b) all ferromagnetic couplings and (c) antiferromagnetic couplings.

where $\sigma_{ij} = \pm 1$, couplings J^h, J^v take $\pm J$ randomly, and h is a uniform magnetic field. As is well-known, the partition function of this quasi-one-dimensional (quasi-1D) system can be expressed in terms of the trace of the product of the transfer matrices $Z = \text{Tr}(\mathcal{M}_L)$, where $\mathcal{M}_L = \prod_{i=1}^{L-1} M_i$, and L is the length of the ladder. By increasing the length by one unit, three couplings will be added to the ladder. There are $2^3 = 8$ possibilities for these three couplings, and each of them can be represented by a transfer matrix M_i . Therefore, there will be eight matrices, and the total transfer matrix of the ladder will be a product of a random sequence of these matrices.

To make this model tractable, we simplify it by keeping only two of the matrices, one with probability x and the other with probability $1 - x$. We call the resulting model restricted $\pm J$ Ising spin glass. The schematic diagram of the system is shown in Fig. 1(a), which is composed of two units shown in Figs. 1(b) and 1(c). The corresponding transfer matrices of these units are given by

$$M_1 = \begin{pmatrix} z^{3+2\alpha} & z^{1+2\alpha} & z^{1+2\alpha} & z^{-1+2\alpha} \\ z^{-1} & z & z^{-3} & z^{-1} \\ z^{-1} & z^{-3} & z & z^{-1} \\ z^{-1-2\alpha} & z^{1-2\alpha} & z^{1-2\alpha} & z^{3-2\alpha} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} z^{-3+2\alpha} & z^{-1+2\alpha} & z^{-1+2\alpha} & z^{1+2\alpha} \\ z & z^{-1} & z^3 & z \\ z & z^3 & z^{-1} & z \\ z^{1-2\alpha} & z^{-1-2\alpha} & z^{-1-2\alpha} & z^{-3-2\alpha} \end{pmatrix}, \quad (2)$$

where $z = \exp(\beta J)$ and $\alpha = h/J$.

In selecting two matrices, we have made a point to preserve the main ingredients of the original model. As the resulting model contains a finite concentration of frustrated plaquettes,

and the enumeration of the overturned spins is not simple even for the zero magnetic field, we expect to capture the physics of the $\pm J$ spin glass.

Since these matrices have positive elements, the trace in the partition function can be replaced with any element of \mathcal{M}_L [13]. By considering the evolution of each element of \mathcal{M}_L under successive matrix products

$$\mathcal{M}_{L+1} = \mathcal{M}_L M_L, \quad (3)$$

the free energy per spin can be obtained directly.

III. ZERO-TEMPERATURE LIMIT

At the low-temperature limit, $z \rightarrow \infty$, we can keep only the leading term in z for each element of \mathcal{M}_L under successive matrix products. The leading terms in the first row of \mathcal{M}_L and \mathcal{M}_{L+1} can be considered as [13]

$$\mathcal{M}_L \simeq \begin{pmatrix} y_1 z^a & y_2 z^b & y_3 z^c & y_4 z^d \\ \ddots & & \ddots & \vdots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$\mathcal{M}_{L+1} \simeq \begin{pmatrix} Y_1 z^A & Y_2 z^B & Y_3 z^C & Y_4 z^D \\ \ddots & & \ddots & \vdots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \quad (4)$$

Using Eq. (3), one can obtain A, B, C, D knowing a, b, c, d and M_L . The equations of evolution of these exponents are given in Appendix A. The free energy can be expressed in terms of these exponents and their probabilities. It turns out that b and c remain equal along the ladder because the second and third columns of the transfer matrices M_1 and M_2 have the same set of values. Therefore, b and c remain equal as we multiply \mathcal{M}_L with any of the transfer matrices M_1 and M_2 . Also, we can see that $-2 \leq a - b \leq 2$ and $-4 \leq a - d \leq 4$. Using the equations given in Appendix A, by multiplication of M_1 , $A - B$ always remains less than 2, and by multiplication of M_2 , it will remain bigger than -2 . For example, in Eq. (A1) we can see that $A - B = 2$ or in the third set Eq. (A3) we have $A - B = a - d + 2 + 4\alpha$ and using $-4 \leq a - d \leq -4\alpha$ we have $A - B \leq 2$, and so on. Similarly, it can be shown that $A - D \leq 4$ for M_1 and $A - D \geq -4$ for M_2 .

The number of possible values of the exponents depends on α and can be very large, which makes the calculation tedious. To reduce the number of values of the exponents, for $\alpha < 1$, we assume $\alpha = 1/n$, where n only takes positive integer values. To obtain the possible values of the exponents, we start with an arbitrary set of values, and then by iteration of equations of evolution (Appendix A), we find a closed set of values. For $\alpha = \frac{1}{n}$, $n = 2, 3, \dots$ we have the following values for $(a - b, a - d)$:

$$(2\alpha n, 2\alpha i), \quad 0 \leq i \leq 2n \rightarrow p_i,$$

$$(-2\alpha, 2\alpha(i - 2)), \quad 0 \leq i \leq n - 1 \rightarrow q_i,$$

$$\begin{aligned}
 (2\alpha(n-i), -2\alpha i), \quad 0 \leq i \leq n-1 &\rightarrow r_i, \\
 (-2\alpha(i+1), -2\alpha(i+2)), \quad 0 \leq i \leq n-1 &\rightarrow s_i, \\
 (2\alpha, 0), &\rightarrow t_1, \\
 (2\alpha, 4\alpha), &\rightarrow t_2, \\
 (-2, -4\alpha), &\rightarrow t_3, \\
 (-2, 0), &\rightarrow t_4. \quad (5)
 \end{aligned}$$

The corresponding probabilities for each set of values are indicated in front of them.

Similarly, for $1 < \alpha < 2$ we obtain the possible values for $(a-b, a-d)$,

$$\begin{aligned}
 (2, 4) &\rightarrow u_1, \\
 (-2, -4) &\rightarrow u_2, \\
 (-2, 2-2\alpha) &\rightarrow u_3, \\
 (2, -4+4\alpha) &\rightarrow u_4. \quad (6)
 \end{aligned}$$

For $2 < \alpha < 3$, the possible values for $(a-b, a-d)$ are

$$\begin{aligned}
 (2, 4) &\rightarrow v_1, \\
 (-2, -4) &\rightarrow v_2, \\
 (-2, 2-2\alpha) &\rightarrow v_3. \quad (7)
 \end{aligned}$$

Finally, for $\alpha > 3$ we obtain

$$(a-b, a-d) = \begin{cases} (2, 4) &\rightarrow w_1, \\ (-2, -4) &\rightarrow w_2. \end{cases} \quad (8)$$

In the next section, we will derive the relations between probabilities and their solutions.

IV. INVARIANT MEASURE

For a long sequence of matrices, the probabilities of occurrence of different values of the exponents become stationary. According to the rules of evolution, presented in Appendix A, for $\alpha < 1$ the probabilities satisfy the following set of equations:

$$\begin{aligned}
 p_0 &= r_0, \\
 p_1 &= (1-x)(r_1 + q_1), \\
 p_2 &= (1-x)(p_0 + q_2 + t_1), \\
 p_4 &= (1-x)(p_2 + q_4 + t_2), \\
 p_i &= (1-x)(p_{i-2} + q_i), \quad 4 < i < n, \quad (9)
 \end{aligned}$$

$$p_i = (1-x)p_{i-2}, \quad n \leq i < 2n, \quad (10)$$

$$\begin{aligned}
 p_{2n} &= (1-x)(p_{2n-2} + p_{2n-1} + p_{2n}), \\
 q_i &= xr_i, \quad 0 \leq i \leq n-1, \quad (11)
 \end{aligned}$$

$$r_i = (1-x)(r_{i+2} + s_i), \quad 0 < i < n-2, \quad (12)$$

$$\begin{aligned}
 r_{n-1} &= (1-x)s_{n-1}, \\
 r_{n-2} &= (1-x)s_{n-2}, \\
 s_0 &= xr_0, \\
 s_i &= xp_i, \quad 0 < i < n-1, \quad (13) \\
 s_{n-1} &= x \sum_{i=n-1}^{2n} p_i, \\
 t_1 &= (1-x)t_3, \\
 t_2 &= (1-x)t_4, \\
 t_3 &= x(t_1 + t_2), \\
 t_4 &= x \left(\sum_{i=1}^{n-1} s_i + \sum_{i=0}^{n-1} q_i + t_3 + t_4 \right).
 \end{aligned}$$

This set of equations allows the following closed solution:

$$\begin{aligned}
 p_i &= \begin{cases} c_1\lambda_1^i + c_2\lambda_2^i, & 0 < \text{even } i < n, \\ c_3\lambda_1^i + c_4\lambda_2^i, & 0 < \text{odd } i < n, \\ c_5\lambda_3^i, & n \leq \text{even } i < 2n, \\ c_6\lambda_3^i, & n \leq \text{odd } i < 2n, \end{cases} \\
 r_i &= \begin{cases} c_7\lambda_1^i + c_8\lambda_2^i, & 0 < \text{even } i < n, \\ c_9\lambda_1^i + c_{10}\lambda_2^i, & 0 < \text{odd } i < n, \end{cases} \\
 q_i &= xr_i, \\
 s_i &= xp_i, \\
 t_1 &= \frac{x^3(1-x)^2}{1-x+x^2}, \\
 t_2 &= x^2(1-x), \\
 t_3 &= \frac{x^3(1-x)}{1-x+x^2}, \\
 t_4 &= x^2, \quad (14)
 \end{aligned}$$

where

$$\begin{aligned}
 \lambda_1 &= \frac{1}{\sqrt{1-x}} \left(1-x+x^3 - \frac{1}{2}x^4 - \frac{1}{2}x^{\frac{3}{2}}\sqrt{8-12x+4x^2+4x^3-4x^4+x^5} \right)^{\frac{1}{2}}, \\
 \lambda_2 &= \frac{1}{\sqrt{1-x}} \left(1-x+x^3 - \frac{1}{2}x^4 + \frac{1}{2}x^{\frac{3}{2}}\sqrt{8-12x+4x^2+4x^3-4x^4+x^5} \right)^{\frac{1}{2}}, \\
 \lambda_3 &= \sqrt{1-x}.
 \end{aligned}$$

λ_1 and λ_2 are the roots of the characteristic equation

$$((1-x) - \lambda^2)((1-x)\lambda^2 - 1) - x^2(1-x)^2\lambda^2 = 0, \quad (15)$$

and λ_3 is the root of

$$\lambda^2 = 1-x. \quad (16)$$

Equation (15) is obtained by replacing q_i from Eq. (11) in Eq. (9) and s_i from Eq. (13) in Eq. (12) and then using the ansatz $p_i \propto \lambda^i$ and $r_i \propto \lambda^i$ and setting the determinant of the coefficients of the resulting equations equal to zero. Equation (16) is obtained from Eq. (10) using the ansatz $p_i \propto \lambda^i$. Since the even and odd indices are separated in Eqs. (14), the roots with the negative signs will give the same solution.

The coefficients c_1, \dots, c_{10} should be determined in terms of x and n . Substituting the above solution in the corresponding equations, we obtain a set of equations for c_i 's which are given in Appendix B.

For other ranges of α , the equations and their solutions are as follows. For $1 < \alpha < 2$,

$$\begin{cases} u_1 = (1-x)(u_1 + u_3 + u_4), \\ u_2 = x(u_1 + u_4), \\ u_3 = x(u_2 + u_3), \\ u_4 = (1-x)u_2, \end{cases} \Rightarrow \begin{cases} u_1 = (1-x)(1-x+x^2), \\ u_2 = x(1-x), \\ u_3 = x^2, \\ u_4 = x(1-x)^2. \end{cases} \quad (17)$$

For $2 < \alpha < 3$,

$$\begin{cases} v_1 = (1-x)(v_1 + v_2 + v_3), \\ v_2 = xv_1, \\ v_3 = x(v_2 + v_3), \end{cases} \Rightarrow \begin{cases} v_1 = (1-x), \\ v_2 = x(1-x), \\ v_3 = x^2. \end{cases} \quad (18)$$

For $\alpha > 3$,

$$\begin{cases} w_1 = (1-x)(w_1 + w_2), \\ w_2 = x(w_1 + w_2), \end{cases} \Rightarrow \begin{cases} w_1 = 1-x, \\ w_2 = x. \end{cases} \quad (19)$$

V. GROUND-STATE ENERGY AND MAGNETIZATION

Now the ground-state energy can be calculated via [13]

$$\begin{aligned} E/J &= -\frac{1}{2} \lim_{L \rightarrow \infty} \langle a \rangle_L / L = -\frac{1}{2} \lim_{L \rightarrow \infty} (\langle a \rangle_{L+1} - \langle a \rangle_L), \\ &= -\frac{1}{2} \lim_{L \rightarrow \infty} \langle a_{L+1} - a_L \rangle, \end{aligned} \quad (20)$$

and differentiating energy with respect to the field gives magnetization $m = -\partial E / \partial h = -\partial(E/J) / \partial \alpha$. The factor $\frac{1}{2}$ is included to make the energy per spin.

The infinite length limit in Eq. (20) can be replaced with an average with respect to the stationary solution for probabilities given in Eqs. (6), (7), (8), and (14). For $\alpha < 1$, using the equations in Appendix A, we need to calculate the average of $a_{L+1} - a_L = A - a$. We should sum up different values of $A - a$ multiplied with corresponding probabilities. For instance, in the first set of equations (with probability $1 - x$) we can see that $A - a = 3 + 2\alpha$ in all cases. Therefore, from the

first set we only get the contribution $(3 + 2\alpha)(1 - x)$. Including the contributions from the second set (with probability x), we obtain

$$\begin{aligned} E/J &= -\frac{1}{2}(3 + 2\alpha)(1 - x) - \frac{1}{2}x \left[\sum_{i=n-1}^{2n} (-2\alpha n + 1)p_i \right. \\ &\quad + \sum_{i=0}^{n-2} [-2\alpha(i + 1) + 1]p_i + \sum_{i=1}^{n-1} [2\alpha(i - 1) + 1]r_i \\ &\quad + \sum_{i=0}^{n-1} [2\alpha(i + 1) + 1]s_i + \sum_{i=1}^{n-1} (2\alpha + 1)q_i \\ &\quad \left. + (1 - 2\alpha)(t_1 + t_2) + 3(t_3 + t_4) \right]. \end{aligned} \quad (21)$$

For $1 < \alpha < 3$, using Eqs. (6), (7), (17), and (18),

$$E/J = -\frac{1}{2}[3(2x^2 - 2x + 1) + 2\alpha(1 - x^2)], \quad (22)$$

and then by differentiating with respect to α , the magnetization is

$$m = 1 - x^2. \quad (23)$$

For $\alpha > 3$, using Eqs. (8) and (19), the energy is

$$E/J = -\frac{1}{2}[(3 + 2\alpha)(1 - x) + (-3 + 2\alpha)x], \quad (24)$$

and then by differentiating Eq. (24) with respect to α , one finds that the magnetization is saturated ($m = 1$), as expected.

VI. NUMERICAL CALCULATIONS

To confirm our analytical results, we repeat the calculations using a numerical method. We use the iterative approach, which is applied to the quasi-one-dimensional random Ising model and is based on the following set of coupled equations [21]:

$$\begin{aligned} E_l^\mu &= \min_v \left\{ E_{l-1}^v - \sum_j J_{l-1,j}^h \sigma_{l-1,j}^v \sigma_{l,j}^\mu \right\} \\ &\quad - \sum_j J_{l,j}^v \sigma_{l,j}^\mu \sigma_{l,j+1}^\mu - h \sum_j \sigma_{l,j}^\mu, \end{aligned} \quad (25)$$

where E_l^μ 's are the ground-state energies of the strip of length l for a given configuration, μ , of spins in the last column.

VII. RESULTS AND DISCUSSION

The ground-state energy per spin as a function of the concentration of antiferromagnetic bonds for different values of the uniform magnetic field is shown in Fig. 2. For zero magnetic field ($\alpha \rightarrow 0$), the curve is symmetric [19] under the exchange of x and $(1 - x)$, and it has a maximum at $x = \frac{1}{2}$. The minimum energy corresponds to $x = 0$ and 1 because there is no frustration in these cases. By applying a magnetic field, this symmetry breaks. At $x = 1$ the magnetic field does not split the energy because all bonds are antiferromagnetic and magnetization is zero for small magnetic fields, therefore the energy does not change by increasing the magnetic field up to a threshold value. In the presence of the magnetic field,

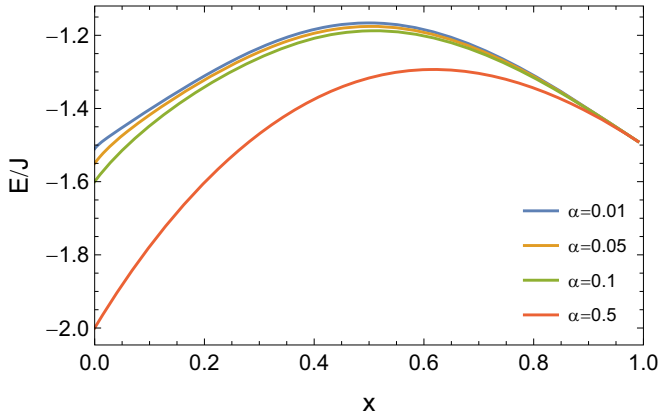


FIG. 2. Ground-state energy per spin as a function of the concentration for different values of magnetic field ($\alpha = h/J$).

the minimum energy corresponds to the ferromagnetic ladder ($x = 0$), and the concentration with maximum energy shifts to higher values of x .

In Fig. 3 we have plotted the magnetization as a function of the concentration of antiferromagnetic bonds. At small magnetic fields, magnetization has a nonmonotonic dependence on concentration and exhibits a local minimum, but at higher magnetic fields the behavior is different and the minimum disappears. Magnetization tends to zero by increasing the concentration of antiferromagnetic bonds due to antiferromagnetic order at higher values of x . It is interesting to compare Fig. 3 with the same result for the single chain (Fig. 1 of Ref. [13]). We see that nonmonotonic behavior is peculiar to the ladder. To see whether the nonmonotonicity persists by changing the randomness or not, we calculated the magnetization with all eight matrices, and it turns out that it disappears at least for the magnetic field of the order $h \sim 0.01$.

Figure 4 shows the numerical results for magnetization as a function of the magnetic field for different values of the concentration. The steplike behavior is observed as in the single chain, although the jumps occur at different sets of magnetic fields. However, the magnetization behaves differently in the zero magnetic field limit as it tends to a nonzero

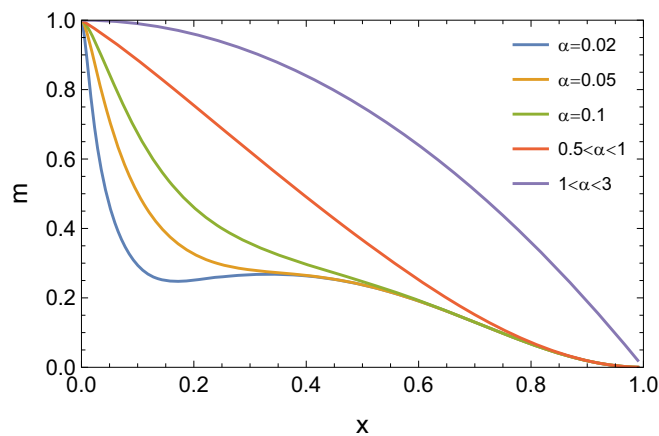


FIG. 3. Magnetization per spin as a function of the concentration for different values of magnetic field ($\alpha = h/J$).

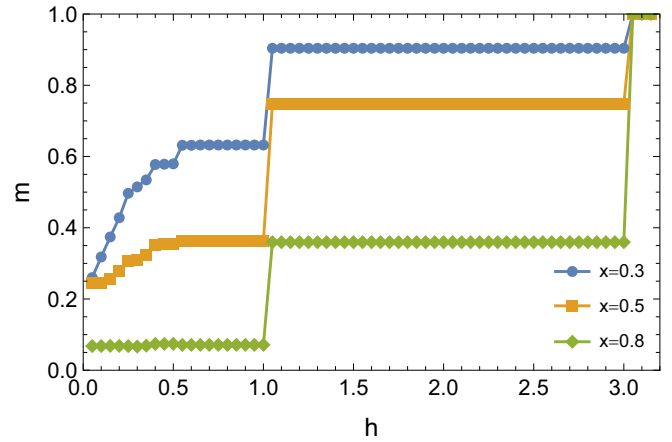


FIG. 4. Numerical results for magnetization as a function of the magnetic field for different values of concentration.

value depending on the concentration. We have plotted the analytical result for small magnetic fields in Fig. 5. It can be seen that the limiting values for $h \rightarrow 0$ agree in Figs. 4 and 5. The precise functional form of magnetization in the zero-field limit can be revealed using our analytical result. As we explained in Sec. IV, we obtain the probabilities in terms of x and n . Then we insert them in Eq. (20), and by doing the summations and keeping the leading terms as $n \rightarrow \infty$, we find that the energy has an essential singularity known as Griffiths singularity,

$$E \simeq E_0 + m_0 h + E_1 e^{-h_0/h}, \quad h \rightarrow 0, \quad (26)$$

from which we obtain

$$m \simeq m_0 + m_1 \frac{e^{-h_0/h}}{h^2}, \quad h \rightarrow 0, \quad (27)$$

where $m_1 = E_1 h_0$, and m_0 is the zero-field magnetization. Similar singularity is known to exist in a diluted Ising ferromagnet [25,26] and the higher-dimensional random temperature Ginzburg-Landau Hamiltonian [27]. The existence of the Griffiths phase in the spin glass is also suggested by studying the dynamics [23] and distribution of Lee-Yang zeros in equilibrium [24]. It is argued in Refs. [23,24] that a common

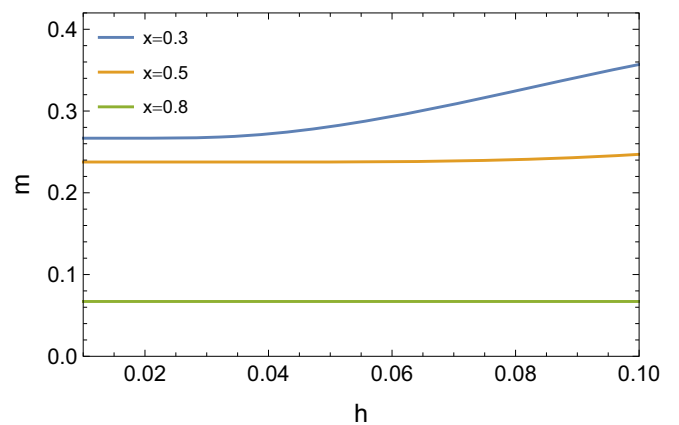


FIG. 5. Magnetization as a function of the magnetic field for different values of concentration.

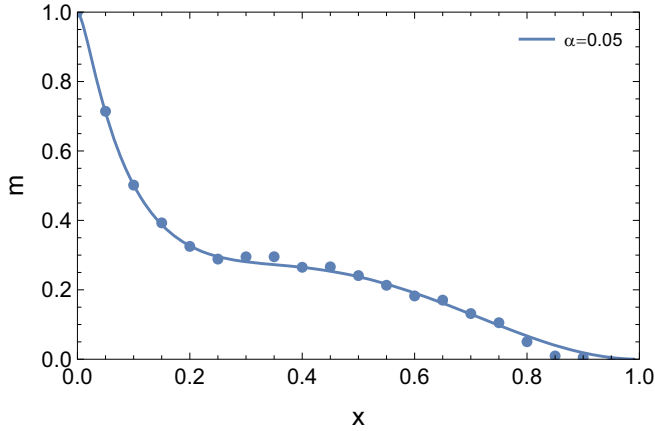


FIG. 6. Magnetization as a function of the concentration for $\alpha = 0.05$. The solid curve is the analytical result, and circles show the numerical result.

physics underlies the singularity as in the diluted ferromagnets, namely the large-sized connected clusters, although the connected clusters are not trivially defined in the spin glass. We believe the singularity that we have found here has the same cause. In our case, the magnetic field dependence of the magnetization is governed by the large ferromagnetic clusters that occur with an exponentially small probability. Our model allows an estimation of this effect as follows. The probability of the ferromagnetic cluster of length ℓ is of order $(1 - x)^\ell$. Such a cluster will flip when the magnetic field is such that $\ell h \sim J$ so the change in magnetization will be proportional to $(1 - x)^{J/h}$. This is, of course, a crude estimate that only shows how the essential singularity arises, but our result in Eq. (27) shows the true nature of the singularity.

For a comparison of analytical results with the results of numerical computations, we have plotted magnetization as a function of concentration in Fig. 6.

In conclusion, we presented analytical results on the zero-temperature properties of the restricted $\pm J$ Ising spin glass ladder in the presence of a uniform magnetic field. We found that, in contrast to the single chain model, magnetization of the frustrated ladder tends to a nonzero value by decreasing the magnetic field. Another notable feature of magnetization of the ladder is that depending on the randomness and magnetic field, the magnetization can be a nonmonotonic function of concentration. We also found that thermodynamic quantities are not analytic functions of magnetic field as $h \rightarrow 0$, which is known as Griffiths singularity. We derived the explicit functional form of the singularity. The accuracy of the analytical results was demonstrated by performing numerical calculations.

As we mentioned earlier, we expect to see similar behavior in the original $\pm J$ model because of the following argument. The singularity results from certain large sequences of matrices in the ladder. Adding other matrices in the ladder does not eliminate those sequences. They can still occur, but with different probability. Therefore, we expect the singularity to be generated by those sequences, although with different strength. The other feature of the magnetization is its zero-field limit, m_0 . We expect that this also remains nonzero in

the case of the original eight-matrix model because there will be a finite concentration of frustrated plaquettes, therefore the ground state will be degenerate and a small magnetic field induces a finite magnetization. We support this prediction with a numerical calculation of the magnetization of the eight-matrix model, which shows the same features that we obtained in the two-matrix model.

Our solution paves the way to studying the models with more realistic disorders and higher widths, and it can also be used to study the accuracy of optimization algorithms to find the spin glass ground state.

ACKNOWLEDGMENTS

We would like to acknowledge financial support from the research council of University of Tehran. This work is based upon research funded by Iran National Science Foundation (INSF) under Project No. 4005950.

APPENDIX A

In this Appendix, we derive the equations of evolution for the exponents of the two-matrix model with M_1 and M_2 , given in Eq. (2), with probabilities $1 - x$ and x , respectively, and magnetic fields $\alpha = 1/n$ for $n = 2, 3, \dots$ ($\alpha = h/J$).

By keeping the leading term in the elements of the first row of \mathcal{M}_L after being multiplied by one of the matrices, using Eqs. (3) and (4), the new exponents can be expressed in terms of the old ones as follows. After being multiplied by M_1 ,

$$\begin{aligned} A &= \max(a + 3 + 2\alpha; b - 1; c - 1; d - 1 - 2\alpha), \\ B &= \max(a + 1 + 2\alpha; b + 1; c - 3; d + 1 - 2\alpha), \\ C &= \max(a + 1 + 2\alpha; b - 3; c + 1; d + 1 - 2\alpha), \\ D &= \max(a - 1 + 2\alpha; b - 1; c - 1; d + 3 - 2\alpha), \end{aligned}$$

and after being multiplied by M_2 ,

$$\begin{aligned} A &= \max(a - 3 + 2\alpha; b + 1; c + 1; d + 1 - 2\alpha), \\ B &= \max(a - 1 + 2\alpha; b - 1; c + 3; d - 1 - 2\alpha), \\ C &= \max(a - 1 + 2\alpha; b + 3; c - 1; d - 1 - 2\alpha), \\ D &= \max(a + 1 + 2\alpha; b + 1; c + 1; d - 3 - 2\alpha). \end{aligned}$$

Then using $b = c$ (as we explained in the text), we find with probability $(1 - x)$,

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ 4 - 4\alpha \leq a - d \leq 4 \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = a + 1 + 2\alpha \\ D = a - 1 + 2\alpha \end{cases} \quad (\text{A1})$$

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ -4\alpha \leq a - d \leq 4 - 4\alpha \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = a + 1 + 2\alpha \\ D = d + 3 - 2\alpha \end{cases} \quad (\text{A2})$$

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ -4 \leq a - d \leq -4\alpha \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = d + 1 - 2\alpha \\ D = d + 3 - 2\alpha \end{cases} \quad (\text{A3})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ 4 - 4\alpha \leq a - d \leq 4 \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = b + 1 \\ D = b - 1 \end{cases} \quad (\text{A4})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ -4\alpha \leq a - d \leq 4 - 4\alpha \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = b + 1 \\ D = d + 3 - 2\alpha \end{cases} \quad (\text{A5})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ -4 \leq a - d \leq -4\alpha \\ -2\alpha \leq b - d \leq 2 \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = b + 1 \\ D = d + 3 - 2\alpha \end{cases} \quad (\text{A6})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ -4 \leq a - d \leq -4\alpha \\ -2 \leq b - d \leq -2\alpha \end{cases} \Rightarrow \begin{cases} A = a + 3 + 2\alpha \\ B = d + 1 - 2\alpha \\ D = d + 3 - 2\alpha \end{cases} \quad (\text{A7})$$

and with probability x , we have

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ 4 - 4\alpha \leq a - d \leq 4 \end{cases} \Rightarrow \begin{cases} A = b + 1 \\ B = b + 3 \\ D = a + 1 + 2\alpha \end{cases} \quad (\text{A8})$$

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ -4\alpha \leq a - d \leq 4 - 4\alpha \\ -2\alpha \leq b - d \leq 2 \end{cases} \Rightarrow \begin{cases} A = b + 1 \\ B = b + 3 \\ D = a + 1 + 2\alpha \end{cases} \quad (\text{A9})$$

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ -4\alpha \leq a - d \leq 4 - 4\alpha \\ -2 \leq b - d \leq -2\alpha \end{cases} \Rightarrow \begin{cases} A = d + 1 - 2\alpha \\ B = b + 3 \\ D = a + 1 + 2\alpha \end{cases} \quad (\text{A10})$$

$$\text{if } \begin{cases} -2\alpha \leq a - b \leq 2 \\ -4 \leq a - d \leq -4\alpha \end{cases} \Rightarrow \begin{cases} A = d + 1 - 2\alpha \\ B = b + 3 \\ D = a + 1 + 2\alpha \end{cases} \quad (\text{A11})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ -4\alpha \leq a - d \leq 4 \end{cases} \Rightarrow \begin{cases} A = b + 1 \\ B = b + 3 \\ D = b + 1 \end{cases} \quad (\text{A12})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ -4 \leq a - d \leq -4\alpha \\ -2\alpha \leq b - d \leq 2 \end{cases} \Rightarrow \begin{cases} A = b + 1 \\ B = b + 3 \\ D = b + 1 \end{cases} \quad (\text{A13})$$

$$\text{if } \begin{cases} -2 \leq a - b \leq -2\alpha \\ -4 \leq a - d \leq -4\alpha \\ -2 \leq b - d \leq -2\alpha \end{cases} \Rightarrow \begin{cases} A = d + 1 - 2\alpha \\ B = b + 3 \\ D = b + 1 \end{cases} \quad (\text{A14})$$

APPENDIX B

In this Appendix, we shall give the equations for coefficients c_1, \dots, c_{10} ,

$$c_1 = \frac{x(1-x)\lambda_1^2}{\lambda_1^2 - (1-x)} c_7, \quad (\text{B1})$$

$$c_2 = \frac{x(1-x)\lambda_2^2}{\lambda_2^2 - (1-x)} c_8, \quad (\text{B2})$$

$$c_3 = \frac{x(1-x)\lambda_1^2}{\lambda_1^2 - (1-x)} c_9, \quad (\text{B3})$$

$$c_4 = \frac{x(1-x)\lambda_2^2}{\lambda_2^2 - (1-x)} c_{10}, \quad (\text{B4})$$

$$c_5 = \begin{cases} \left(\frac{\lambda_1}{\lambda_3}\right)^{n-2} c_1 + \left(\frac{\lambda_2}{\lambda_3}\right)^{n-2} c_2, & \text{even } n, \\ \left(\frac{\lambda_1}{\lambda_3}\right)^{n-1} c_1 + \left(\frac{\lambda_2}{\lambda_3}\right)^{n-1} c_2, & \text{odd } n \end{cases} \quad (\text{B5})$$

$$c_6 = \begin{cases} \left(\frac{\lambda_1}{\lambda_3}\right)^{n-1} c_3 + \left(\frac{\lambda_2}{\lambda_3}\right)^{n-1} c_4, & \text{even } n, \\ \left(\frac{\lambda_1}{\lambda_3}\right)^{n-2} c_3 + \left(\frac{\lambda_2}{\lambda_3}\right)^{n-2} c_4, & \text{odd } n. \end{cases} \quad (\text{B6})$$

For even n ,

$$c_1 = \left(\frac{\lambda_2}{\lambda_1}\right)^{n-4} \left(\frac{(1-x) - \lambda_2^2[1-x^2(1-x)^2]}{[1-x^2(1-x)^2]\lambda_1^2 - (1-x)}\right) c_2. \quad (\text{B7})$$

For odd n ,

$$c_3 = \left(\frac{\lambda_2}{\lambda_1}\right)^{n-4} \left(\frac{(1-x) - \lambda_2^2[1-x^2(1-x)^2]}{[1-x^2(1-x)^2]\lambda_1^2 - (1-x)}\right) c_4, \quad (\text{B8})$$

$$c_3\lambda_1 + c_4\lambda_2 = (1-x^2)(c_9\lambda_1 + c_{10}\lambda_2). \quad (\text{B9})$$

For even n ,

$$c_9\lambda_1^{n-1} + c_{10}\lambda_2^{n-1} = x(1-x) \left(c_5 \sum_{k=\frac{n}{2}}^{n-2} \lambda_3^{2k} + c_6 \sum_{k=\frac{n}{2}}^{n-1} \lambda_3^{2k-1} \right) + (1-x)(c_5\lambda_3^{2n-2} + c_6\lambda_3^{2n-1}). \quad (\text{B10})$$

For odd n ,

$$c_7\lambda_1^{n-1} + c_8\lambda_2^{n-1} = x(1-x) \left(c_5 \sum_{k=\frac{n-1}{2}}^{n-2} \lambda_3^{2k} + c_6 \sum_{k=\frac{n-1}{2}}^{n-1} \lambda_3^{2k-1} \right) + (1-x)(c_5\lambda_3^{2n-2} + c_6\lambda_3^{2n-1}). \quad (\text{B11})$$

Normalization of the probabilities,

$$\sum_{i=1}^{2n} p_i + \sum_{i=0}^{n-1} (q_i + r_i) + \sum_{i=1}^{n-1} s_i + t_1 + t_2 + t_3 + t_4 = 1, \quad (\text{B12})$$

gives, for even n ,

$$\begin{aligned} & \sum_{k=2}^{\frac{n}{2}-1} (\lambda_1^{2k} c_1 + \lambda_2^{2k} c_2 + \lambda_1^{2k-1} c_3) \\ & + c_4 \sum_{k=2}^{\frac{n}{2}-1} \lambda_2^{2k-1} + c_5 \sum_{k=\frac{n}{2}}^{n-2} \lambda_3^{2k} + c_6 \sum_{k=\frac{n}{2}}^{n-1} \lambda_3^{2k-1} \\ & + \frac{\lambda_3^{2n-2}}{x} (c_5 + \lambda_3 c_6) + \left(\frac{2-x}{(1-x)^2} - x^2 + x \right) (c_1 \lambda_1^4 + c_2 \lambda_2^4) \\ & + \sum_{k=2}^{\frac{n}{2}-1} (\lambda_1^{2k} c_7 + \lambda_2^{2k} c_8) + \sum_{k=2}^{\frac{n}{2}} (\lambda_1^{2k-1} c_9 + \lambda_2^{2k-1} c_{10}) \\ & + \left(\frac{x}{x-1} + (1-x)^2(1-x^2) - x \right) (c_7 \lambda_1^4 + c_8 \lambda_2^4) \\ & + \left(\frac{(2-x^2)(1-x)}{1-x(1-x)(1-x^2)} \right) (c_9 \lambda_1^3 + c_{10} \lambda_2^3) \\ & - \left(1 + x(1-x) + \frac{1}{(1-x)} + x^2(1-x) \right) x^2(1-x) \\ & - \frac{x^3(1-x)^2}{1-x+x^2} = \frac{(1-x)^2}{1-x+x^2}, \end{aligned} \tag{B13}$$

and for odd n ,

$$\begin{aligned} & \sum_{k=2}^{\frac{n-3}{2}} (\lambda_1^{2k} c_1 + \lambda_2^{2k} c_2) + \sum_{k=2}^{\frac{n-1}{2}} (\lambda_1^{2k-1} c_3 + \lambda_2^{2k-1} c_4) \\ & + c_5 \sum_{k=\frac{n-1}{2}}^{n-2} \lambda_3^{2k} + c_6 \sum_{k=\frac{n+1}{2}}^{n-1} \lambda_3^{2k-1} + \frac{1}{x} c_5 \lambda_3^{2n-2} \\ & + \frac{1}{x} c_6 \lambda_3^{2n-1} + \left(\frac{1}{(1-x)^2} + \frac{1}{(1-x)} - x^2 + x \right) (c_1 \lambda_1^4 + c_2 \lambda_2^4) \\ & + \sum_{k=2}^{\frac{n-1}{2}} (\lambda_1^{2k} c_7 + \lambda_2^{2k} c_8 + \lambda_1^{2k-1} c_9 + \lambda_2^{2k-1} c_{10}) \\ & + \left(\frac{x}{x-1} + (1-x)^2(1-x^2) - x \right) (c_7 \lambda_1^4 + c_8 \lambda_2^4) \\ & + \left(\frac{(2-x^2)(1-x)}{1-x(1-x)(1-x^2)} \right) (c_9 \lambda_1^3 + c_{10} \lambda_2^3) \\ & - \left(1 + x(1-x) + \frac{1}{(1-x)} + x^2(1-x) \right) x^2(1-x) \\ & - \frac{x^3(1-x)^2}{1-x+x^2} = \frac{(1-x)^2}{1-x+x^2}. \end{aligned} \tag{B14}$$

[1] P. Charbonneau, E. Marinari, G. Parisi, F. Ricci-Tersenghi, G. Sicuro, and F. Zamponi, *Spin Glass Theory and Far Beyond-Replica Symmetry Breaking After 40 Years* (World Scientific, Singapore, 2023).

[2] D. L. Stein and C. M. Newman, *Spin Glasses and Complexity* (Princeton University Press, Princeton, NJ, 2013), Vol. 4.

[3] J. A. Hertz, *Introduction to the Theory of Neural Computation* (CRC, Boca Raton, FL, 1991).

[4] M. Mezard and A. Montanari, *Information, Physics, and Computation* (Oxford University Press, Oxford, UK, 2009).

[5] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Clarendon, Oxford, UK, 2001), p. 111.

[6] H. Huang, *Statistical Mechanics of Neural Networks* (Springer, Berlin, 2022).

[7] D. Sherrington and S. Kirkpatrick, *Phys. Rev. Lett.* **35**, 1792 (1975).

[8] T. Jörg, J. Lukic, E. Marinari, and O. C. Martin, *Phys. Rev. Lett.* **96**, 237205 (2006).

[9] F. Romá, S. Risau-Gusman, A. J. Ramirez-Pastor, F. Nieto, and E. E. Vogel, *Phys. Rev. B* **82**, 214401 (2010).

[10] A. K. Hartmann and A. P. Young, *Phys. Rev. E* **108**, 024142 (2023).

[11] A. A. Middleton, *Phys. Rev. Lett.* **83**, 1672 (1999).

[12] A. Vilenkin, *Phys. Rev. B* **18**, 1474 (1978).

[13] B. Derrida, J. Vannimenus, and Y. Pomeau, *J. Phys. C* **11**, 4749 (1978).

[14] S. B. S. Ardebili and R. Sepehrinia, *J. Stat. Phys.* **163**, 568 (2016).

[15] H.-H. Chen and S.-k. Ma, *J. Stat. Phys.* **29**, 717 (1982).

[16] E. Gardner and B. Derrida, *J. Stat. Phys.* **39**, 367 (1985).

[17] T. Morita and T. Horiguchi, *J. Phys. C* **13**, 6483 (1980).

[18] B. Fechner and M. Blaszyk, *J. Phys. C* **19**, 2785 (1986).

[19] T. Kadowaki, Y. Nonomura, and H. Nishimori, *J. Phys. Soc. Jpn.* **65**, 1609 (1996).

[20] P. Timonin, *Phys. Solid State* **45**, 1935 (2003).

[21] R. Sepehrinia and F. Chalangari, *Phys. Rev. B* **97**, 104201 (2018).

[22] R. B. Griffiths, *Phys. Rev. Lett.* **23**, 17 (1969).

[23] M. Randeria, J. P. Sethna, and R. G. Palmer, *Phys. Rev. Lett.* **54**, 1321 (1985).

[24] Y. Matsuda, H. Nishimori, and K. Hukushima, *J. Phys. A* **41**, 324012 (2008).

[25] M. Wortis, *Phys. Rev. B* **10**, 4665 (1974).

[26] Y. Imry, *Phys. Rev. B* **15**, 4448 (1977).

[27] V. Dotsenko, *J. Stat. Phys.* **122**, 197 (2006).