Optimizing search processes with stochastic resetting on the pseudofractal scale-free web

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(Received 4 August 2023; accepted 15 November 2023; published 5 December 2023)

The pseudofractal scale-free web (PSFW) is a well-known model for a scale-free network with small-world characteristics. Understanding the dynamic properties of this network can provide valuable insights into dynamic processes occurring in general scale-free and small-world networks. In this study we investigate search processes using discrete-time random walks on the PSFW to reveal the impact of the resetting position on optimizing search efficiency, as measured by the mean first-passage time (MFPT). At each step the walker has two options: with a probability of $1-\gamma$, it moves to one of the neighboring sites, and with a probability of γ, it resets to the predefined resetting position. We explore various choices for the resetting position, present rigorous results for the MFPT to a given node of the network, determine the optimal resetting probability γ^* where the MFPT reaches its minimum, and evaluate the ratio of the minimum for MFPT to the MFPT without resetting for each case. Results show that, in large PSFWs, both the degree of the resetting position and the distance between the target and the resetting position significantly affect the search efficiency. A higher degree of the resetting position leads to a slower convergence of the walker to the target, while a greater distance between the target and the resetting position also results in a slower convergence. Additionally, we observe that resetting to a vertex randomly selected from the stationary distribution can significantly expedite the process of the walker reaching the target. The findings presented in this study shed light on optimizing stochastic search processes on large networks, offering valuable insights into improving search efficiency in real-world applications, where the target node's location is unknown.

DOI: 10.1103/PhysRevE.108.064109

I. INTRODUCTION

Stochastic searches occur in various domains, such as animal foraging [1,2], protein search phenomena in DNA [3–5], and algorithms seeking global optimal solutions [6]. These search processes can be modeled as random walks on underlining networks [7–10], and the mean first-passage time (MFPT), which measures the average time for a walker to reach a specific target site for the first time [11,12], serves as a key indicator of search efficiency of random search processes. Over the past several decades, researchers have extensively investigated the MFPT on various networks, employing different random walk strategies [13–16]. The results have consistently demonstrated that both the topology of the networks and the random walk strategies significantly influence the MFPT [17–20]. One can optimize the search efficiency by appropriately designing random walk strategies on the underlying networks [21–24].

Recently, "resetting" has been introduced in search processes, and it has been shown that resetting can improve search efficiency [25]. Consequently, search processes with stochastic resetting have garnered increasing attention [26–28]. In d-dimensional infinite space, a diffusion searcher faces the challenge of requiring an infinite mean time to reach a target for $d = 1$ and $d = 2$. Moreover, for $d > 2$, there is a nonzero probability that the searcher may not even reach the target. However, the introduction of resetting can effectively prevent particles from escaping in the system, thereby ensuring that the searcher can always find the target in finite time [29]. Specifically, when considering a diffusion process with Poissonian resetting, i.e., a diffusion process with a constant rate γ of resetting to a given site (i.e., the resetting position), the MFPT can be minimized at a nonzero optimal reset rate γ^* [25,29]. Moreover, for a diffusion process with a rate γ of resetting to a randomly drawn resetting position, the choice of resetting position or the probability distribution of the resetting position has a significant impact on the optimal MFPT [30]. Furthermore, optimization problems controlled by first-passage resetting have been introduced and investigated [31,32]. Resetting does not always lead to accelerated search processes in confined d-dimensional spaces, as reported in a study by Bonomo *et al.* [33]. The effectiveness of resetting in accelerating the search process relies on the distance between the target and the resetting position. The resetting position refers to the location where the walker restarts after resetting. The distance between these two points plays a crucial role in determining how efficiently resetting aids in speeding up the search process.

In the study of random walks on complex networks, various methods have been introduced to assess the survival probability and MFPT under different resetting strategies [34,35].

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Related strategies include a fixed probability of resetting the walker to a fixed site [34] or one of multiple sites [35–37] at each step and time-dependent probability [38] and node-dependent probability [39] of resetting the walker to a special site at each step. In these contexts, a relationship between the MFPT and the spectral properties of the transition matrix has been presented. Additionally, exact results for MFPT on small-scale networks have been obtained, indicating that resetting can expedite the search process. For the general case of first-passage progress under restart, the MFPT can be evaluated exactly using the probability generation functions of the first-passage time and the wait time between two successive resettings [33]. There are also work which focus on the effect of resetting position on the search efficiency [40] and work which evaluate the effect of stochastic resetting on the entropy rate for discrete-time Markovian processes [41].

However, to the best of our knowledge, a result of the MFPT for random walks with resetting on complex networks has just been obtained on one-dimensional lattices and some small-scale networks. The resetting positions in these studies are often chosen to be close to the target, and the impact of the resetting position on the optimization of MFPT on large-scale networks has not been explored. It should be noted that many real-world scenarios involving random searches can be represented as large-scale complex networks. Therefore, understanding the effect of resetting (or resetting position) on the MFPT for random walks on such large-scale complex networks becomes an intriguing problem. Addressing the question of how resetting affects the search process on large-scale complex networks would be highly challenging [42]. The inherent heterogeneity of complex networks adds complexity to the problem.

In this study we investigate the impact of resetting positions on the optimization of the MFPT for discrete-time random walks on the pseudofractal scale-free web (PSFW) [43]. Our study focuses on understanding how resetting affects the efficiency of the random search process. During each step of the random walk, the walker has two options: it can either move to one of the neighbors of the current site with a probability of $1-\gamma$ or reset to a specific resetting position with a probability of γ. To comprehensively explore the impact of different resetting positions, we evaluate analytically the MFPT to the hub, which is the vertex with the highest degree, for various choices of resetting positions. Accurate results for the MFPT to the hub of the networks and the optimal resetting probability γ^* where MFPT reaches the minimum are obtained. It is worth noting that scale-free and small-world properties are prevalent in many real-world networks, making them relevant for our investigation. The PSFW, being a well-known network model that exhibits both scale-free and small-world characteristics, serves as a suitable basis for our study.

II. NETWORK MODEL

The pseudofractal scale-free web (PSFW) is a deterministic network model designed to capture the characteristics of scale-free networks with small-world properties [43,44]. The PSFW, denoted as n, with $G(n)$ ($n \geqslant 0$), is constructed iteratively as follows: For the initial generation $n = 0$, $G(0)$ comprises only two vertices connected by an edge. For subsequent generations with $n > 0$, $G(n)$ is obtained from $G(n-1)$ by replacing each edge of $G(n-1)$ with a triangle. In other words, for every edge of $G(n-1)$, a new vertex is introduced, which connects to both endpoints of the edge. Figure 1 illustrates the constructions of PSFW $G(n)$ for generations $n = 0$, 1, and 2. Figure 2 depicts the construction of PSFW for generation $n = 3$.

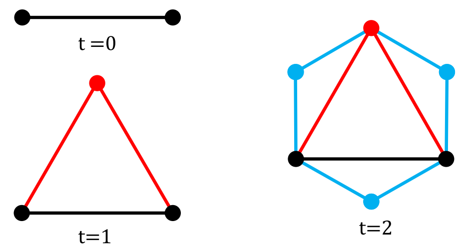

FIG. 1. Constructions of the PSFW $G(n)$ with generation $n = 0$, 1, 2.

As a result of this construction, the PSFW $G(n)$ with $n \geqslant 0$ exhibits certain properties. Specifically, the total number of edges E_n in $G(n)$ is equal to 3^n, while the total number of vertices N_n is given by $\frac{3^n+3}{2}$. During the growth from generation n to generation $n+1$, 3^n new vertices are introduced, and they are connected to the existing vertices. Consequently, the degree of each old vertex in $G(n)$ doubles. Furthermore, the three vertices present in $G(1)$ attain the highest degree in $G(n)$ ($n \geqslant 1$), and we refer to them as the hubs of PSFW $G(n)$. For convenience, in PSFW $G(n)$ with $n \geqslant 3$, we label the three hubs A, B, and C; the three new vertices in generation 2 D, E, and F; and, the nine new vertices in generation 3 G, H, I, J, K, L, M, N, and O, respectively. Figure 2 illustrates the labeled vertices for the PSFW with generation $n = 3$.

As is widely recognized, the PSFW exhibits both scale-free and small-world characteristics [43]. Notably, its diameter

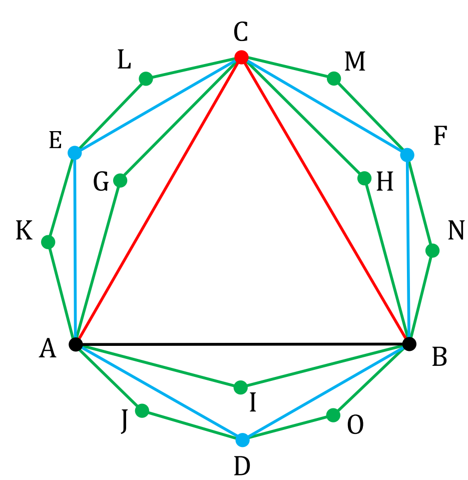

FIG. 2. Construction of PSFW $G(3)$, where the three hubs of $G(3)$ are labeled A, B, and C; the three new vertices in generation 2, indicated in blue, are labeled D, E, and F; and the nine new vertices in generation 3, indicated in green, are labeled G, H, I, J, K, L, M, N, and O respectively.

experiences logarithmic growth with the size of the network, denoted by N_n. Additionally, the number of vertices with a degree k follows a power-law distribution with a power exponent $\beta = 1 + \log 3/\log 2 \approx 2.585$ [43]. Although the PSFW displays a form of self-similarity, it has no finite fractal dimension, and it is not a fractal. Then it is aptly called as pseudofractal scale-free web. In 2007 a broader class of scale-free networks was introduced, known as (u, v)-flowers [45]. The specific network considered in this study corresponds to the $(1, 2)$-flowers configuration.

Note that the PSFW exhibits rich characteristics. Considerable effort has been dedicated to exploring its structure and dynamic properties, including diameter and average path length [46], spectral properties [47,48], number of spanning trees [49,50], first-passage properties [23,51,52], and percolation [53].

III. MFPT FOR GENERAL DISCRETE-TIME RANDOM WALKS UNDER RESETTING IN GENERAL NETWORKS

In this section we introduce a comprehensive method for determining the MFPT in a general discrete-time first-passage process under resetting. Subsequently, as an application of the general method, we apply this general method to derive a rigorous formula for calculating the MFPT in general discrete-time random walks with a fixed resetting probability on general networks. These findings will be utilized to compute the MFPT for random walks under resetting with a fixed probability on the PSFW in the subsequent sections.

A. MFPT for the discrete-time first-passage process under resetting

The first-passage process under resetting is characterized by restarting the process after some random time R has elapsed. Let T_R denote the first-passage time under resetting, and T represent the first-passage time without any interruptions. One can obtain [27]

$$T_R = \begin{cases} T, & T < R, \\ R + T'_R, & T \geqslant R, \end{cases} \tag{1}$$

where T'_R and T_R denote the independent and identically distributed random variables.

In the context of continuous-time first-passage processes under resetting, where R follows an exponentially distributed random variable, Ref. [27] provided formulas to calculate the first two moments of T_R utilizing the Laplace transform method, and Ref. [33] by using a probability generation function. However, a notable distinction arises in Ref. [33], in which $T_R = T$ if $T = R$, different from Eq. (1); this implies that the walker can reach the target precisely at the moment of a reset (e.g., $T = R$). In most cases, for a random walk with resetting, when the resetting time R is reached, the walker is reset to a given site, and there is no opportunity to reach the target at that specific instance. Therefore, we cannot directly apply the formulas from Ref. [33] to derive the MFPT for random walks under resetting on the PSFW.

Therefore, in this section we will investigate the relationship between the mean of T_R and the probability generation functions of T and R. We will then present the formulas to calculate the mean of T_R for discrete-time first-passage process under resetting, as shown in Eq. (1), by using the method presented in Ref. [33].

Let

$$I(T \geqslant R) = \begin{cases} 0, & T < R, \\ 1, & T \geqslant R. \end{cases} \tag{2}$$

Equation (1) can be rewritten as

$$T_R = \min(T, R) + I(T \geqslant R) \times T'_R. \tag{3}$$

Let $\langle \cdot \rangle$ denote the first moment of random variable "$\cdot$" and $\Pr(\cdot)$ represent the probability that event "$\cdot$" occurs. We have

$$\begin{aligned} \langle T_R \rangle &= \langle \min(T, R) \rangle + \langle I(T \geqslant R) T'_R \rangle \\ &= \langle \min(T, R) \rangle + \Pr(T \geqslant R) \langle T'_R \rangle \\ &= \langle \min(T, R) \rangle + [1 - \Pr(T < R)] \langle T'_R \rangle. \end{aligned} \tag{4}$$

Note that T_R and T'_R are independent and identically distributed random variables and $\langle T_R \rangle = \langle T'_R \rangle$. We get

$$\langle T_R \rangle = \frac{\langle \min(T, R) \rangle}{\Pr(T < R)}, \tag{5}$$

where

$$\begin{aligned} \langle \min(T, R) \rangle &= \sum_{m=0}^{\infty} \Pr(\min(T, R) > m) \\ &= \sum_{m=0}^{\infty} \left[\left(\sum_{k=m+1}^{\infty} \Pr(T = k) \right) \sum_{l=m+1}^{\infty} \Pr(R = l) \right] \end{aligned} \tag{6}$$

and

$$Pr(T < R) = \sum_{m=0}^{\infty} \left[\Pr(T = m) \sum_{l=m+1}^{\infty} \Pr(R = l) \right]. \tag{7}$$

B. MFPT for discrete-time random walks under resetting with fixed probability in general networks

Here we consider discrete-time random walks under resetting on general networks. At each step, there is a fixed probability γ that the walker is reset to the initial state (or site), and a probability $1 - \gamma$ that the walker jumps to one of the neighbors of the site currently occupied. Therefore, random variable R, the time it takes the walker to restart the random walk, follows a geometric distribution with parameter γ, reads as

$$\Pr(R = l) = (1 - \gamma)^{l-1} \gamma, \quad l \geqslant 1. \tag{8}$$

Therefore, for any $m \geqslant 0$,

$$\begin{aligned} \sum_{l=m+1}^{\infty} \Pr(R = l) &= \sum_{l=m+1}^{\infty} (1 - \gamma)^{l-1} \gamma \\ &= (1 - \gamma)^m. \end{aligned} \tag{9}$$

Substituting $\sum_{l=m+1}^{\infty} \Pr(R=l)$ from Eq. (9) in Eq. (6) and Eq. (7), we get

$$\begin{aligned}\langle \min(T,R)\rangle &= \sum_{m=0}^{\infty}\left[(1-\gamma)^m \sum_{k=m+1}^{\infty} \Pr(T=k)\right]\\ &= \sum_{k=1}^{\infty}\left[\Pr(T=k)\sum_{m=0}^{k-1}(1-\gamma)^m\right]\\ &= \sum_{k=1}^{\infty}\left[\Pr(T=k)\frac{1-(1-\gamma)^k}{\gamma}\right]\\ &= \frac{1}{\gamma}\left[1-\sum_{k=0}^{\infty}\left[\Pr(T=k)(1-\gamma)^k\right]\right]\\ &= \frac{1-\Phi_T(1-\gamma)}{\gamma} \end{aligned} \tag{10}$$

and

$$\begin{aligned}\Pr(T<R) &= \sum_{m=0}^{\infty}[\Pr(T=m)(1-\gamma)^m]\\ &= \Phi_T(1-\gamma),\end{aligned} \tag{11}$$

where $\Phi_T(z) = \sum_{k=0}^{\infty}[\Pr(T=k)z^k]$ is the probability-generating function of random variable T.

Inserting Eq. (10) and Eq. (11) into Eq. (5), we get

$$\langle T_R\rangle = \frac{1-\Phi_T(1-\gamma)}{\gamma\Phi_T(1-\gamma)}, \tag{12}$$

which is a function of resetting probability γ. Therefore, one can find the optimal γ^* where $\langle T_R\rangle$ obtained the minimum by solving the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\langle T_R\rangle = 0, \tag{13}$$

which is equivalent to

$$[\Phi_T(1-\gamma)]^2 - \Phi_T(1-\gamma) - \gamma\frac{\mathrm{d}\Phi_T(1-\gamma)}{\mathrm{d}\gamma} = 0. \tag{14}$$

If $\Phi_T(1-\gamma)$ is a rational function of γ, which can be expressed as

$$\Phi_T(1-\gamma) \equiv \frac{X_T(\gamma)}{Y_T(\gamma)}, \tag{15}$$

one can rewrite Eq. (14) as

$$\left(\frac{X_T(\gamma)}{Y_T(\gamma)}\right)^2 - \frac{X_T(\gamma)}{Y_T(\gamma)} - \gamma\frac{\frac{dX_T(\gamma)}{d\gamma}Y_T(\gamma) - \frac{dY_T(\gamma)}{d\gamma}X_T(\gamma)}{[Y_T(\gamma)]^2} = 0, \tag{16}$$

which is equivalent to

$$X_T^2(\gamma) - X_T(\gamma)Y_T(\gamma) - \gamma\left[\frac{dX_T(\gamma)}{d\gamma}Y_T(\gamma) - \frac{dY_T(\gamma)}{d\gamma}X_T(\gamma)\right] = 0. \tag{17}$$

It should be pointed out, for any discrete-time first-passage process under resetting, as introduced in Sec. III A, if random variable R follows a geometric distribution with parameter γ, results obtained here are also true.

IV. MFPT ON THE PSFW FOR RANDOM WALKS UNDER RESETTING WITH A FIXED RESETTING POSITION

In this section we describe the analysis of the MFPT to the hub A for random walks under resetting on the PSFW. During each step the walker has a fixed probability γ of resetting to a predefined site, which we refer to as the resetting position. We explore different choices of the resetting position, namely, vertices B, D, I, F, and H, and the newest common neighbor of hub A and B, as shown in Fig. 2. For each resetting position, we determine the optimal resetting probability that results in the minimum MFPT. Subsequently, we discuss the influence of the resetting position on the optimal MFPT. It is important to note that in this analysis we assume that the initial position of the walker coincides with the chosen resetting position. However, the initial position of the walker affects only the steps leading up to the first reset occurrence, and its impact on the MFPT under resetting is inconsequential. The results presented in this section will shed light on how the choice of the resetting position affects the search efficiency, as measured by the MFPT.

A. MFPT under resetting for a random walk from hub B to hub A

In this subsection, we specifically focus on the case where the walker begins at the hub B (i.e., the neighbor of A with the highest degree). At each step, there is a constant probability γ that the walker will reset to its initial position. Let $T^{B\to A}(n)$, $T_R^{B\to A}(n)$ denote the first-passage time from node B to A, without and with resetting on the PSFW of generation n, respectively. We define $\Phi_T^{B\to A}(n,z)$ as the probability-generating function of $T^{B\to A}(n)$ and $\langle T_R^{B\to A}(n)\rangle$ as the mean of $T_R^{B\to A}(n)$.

Here we evaluate the MFPT from B to A under resetting, denoted as $\langle T_R^{B\to A}(n)\rangle$. Rigorous results for $\langle T_R^{B\to A}(n)\rangle$ are presented, along with the optimal value of $\gamma_B^*(n)$ where $\langle T_R^{B\to A}(n)\rangle$ reaches its minimum.

Recalling the result for the MFPT under resetting, as shown in Eq. (12), we aim to derive $\langle T_R^{B\to A}(n)\rangle$. To achieve this, our first step is to obtain the probability-generating function, denoted as $\Phi_T^{B\to A}(n,z)$, for the first-passage time $T^{B\to A}(n)$.

Previously obtained results have shown that [52,54], for any $n>1$, $\Phi_T^{B\to A}(n,z)$ satisfies

$$\Phi_T^{B\to A}(n,z) = \Phi_T^{B\to A}\big(1, \Phi_T^{B\to A}(n-1,z)\big), \tag{18}$$

with an initial condition $\Phi_T^{B\to A}(1,z) = \frac{z}{2-z}$. Solving the above recursive formula, we get

$$\Phi_T^{B\to A}(n,z) = \frac{z}{2^n-(2^n-1)z}. \tag{19}$$

Therefore

$$\Phi_T^{B\to A}(n,1-\gamma) = \frac{1-\gamma}{(2^n-1)\gamma+1}. \tag{20}$$

By replacing $\Phi_T(1-\gamma)$ with $\Phi_T^{B\to A}(n,1-\gamma)$ in Eq. (12), and using the expression for $\Phi_T^{B\to A}(n,1-\gamma)$ in Eq. (20), we can derive the MFPT from node B to A under resetting in this

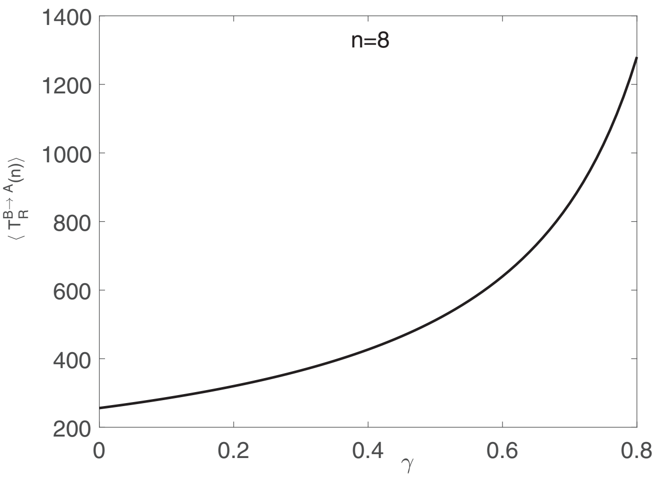

FIG. 3. Plot of $\langle T_R^{B\to A}(n)\rangle$ vs γ in the PSFW with generation $n = 8$. $\langle T_R^{B\to A}(n)\rangle$ increases monotonically in γ on interval [0, 1).

particular case:

$$\left\langle T_R^{B\to A}(n)\right\rangle = \frac{1-\Phi_T^{B\to A}(n,1-\gamma)}{\gamma\Phi_T^{B\to A}(n,1-\gamma)} = \frac{2^n}{1-\gamma}. \tag{21}$$

As shown in Fig. 3, $\langle T_R^{B\to A}(n)\rangle$ increases monotonically in γ on interval [0, 1) and reaches its minimum at $\gamma_B^*(n) = 0$. The minimum value of $\langle T_R^{B\to A}(n)\rangle$ is obtained when no resetting occurs. Based on the analysis and the results obtained, it can be concluded that resetting does not lead to a reduction in the MFPT or accelerate the search process in this particular case.

B. MFPT under resetting for a random walk from vertex *D* to hub *A*

In this subsection we focus on the scenario where the walker initiates from vertex D (i.e., the neighbor of A with the second highest degree). At each step, there is a fixed probability γ of resetting the walker back to its initial position. Our objective is to evaluate the MFPT from vertex D to A under resetting, denoted as $\langle T_R^{D\to A}(n)\rangle$. We will present the results for $\langle T_R^{D\to A}(n)\rangle$ and determine the optimal value of γ^* at which $\langle T_R^{D\to A}(n)\rangle$ reaches its minimum.

To derive $\langle T_R^{D\to A}(n)\rangle$, our first step is to derive the probability-generating function, denoted as $\Phi_T^{D\to A}(n,z)$, for the first-passage time from D to A. Similarly to the derivation of Eq. (18), we find [52,54] that for any $n>2$, $\Phi_T^{D\to A}(n,z)$ satisfies

$$\Phi_T^{D\to A}(n,z) = \Phi_T^{D\to A}\big(2, \Phi_T^{D\to A}(n-2,z)\big). \tag{22}$$

For $n=2$, we find $\Phi_T^{D\to A}(2,z) = \frac{z(z-2)}{3z-4}$, as shown in Appendix A. Solving the above recursive formula, we get

$$\Phi_T^{D\to A}(n,z) = \frac{(2^{n-1}-1)z^2 - 2^{n-1}z}{(5\times 2^{n-2} - 2^{2n-2} - 1)z^2 + (2^{2n-1} - 5\times 2^{n-2})z - 2^{2n-2}}. \tag{23}$$

Therefore,

$$\Phi_T^{D\to A}(n,1-\gamma) = \frac{(2^{n-1}-1)\gamma^2 + (2-2^{n-1})\gamma - 1}{(5\times 2^{n-2} - 2^{2n-2} - 1)\gamma^2 + (2 - 5\times 2^{n-2})\gamma - 1}. \tag{24}$$

Replacing $\Phi_T(1-\gamma)$ with $\Phi_T^{D\to A}(n,1-\gamma)$ in Eq. (12), and substituting $\Phi_T^{D\to A}(n,1-\gamma)$ from Eq. (24), we obtain

$$\left\langle T_R^{D\to A}(n)\right\rangle = \frac{(3\times 2^{n-2} - 2^{2n-2})\gamma - 3\times 2^{n-2}}{(2^{n-1}-1)\gamma^2 + (2-2^{n-1})\gamma - 1}. \tag{25}$$

By taking the first-order derivative with respect to γ on both sides of Eq. (25) and setting $\frac{\mathrm{d}}{\mathrm{d}\gamma}\langle T_R^{D\to A}(n)\rangle = 0$, we obtain

$$(2^{3n-3} + 3\times 2^{n-2} - 5\times 2^{2n-3})\gamma^2 + 3\times(2^{2n-2} - 2^{n-1})\gamma + 3\times 2^{n-2} - 2^{2n-3} = 0, \tag{26}$$

with root[1]

$$\gamma_D^*(n) = \frac{3(2^{n-1} - 2^{2n-2}) + \sqrt{2^{5n-4} - 2^{4n-3}}}{2^{3n-2} + 3\times 2^{n-1} - 5\times 2^{2n-2}} = \frac{2^{n-1}(3\times 2^{1-n} + \sqrt{2^n-2} - 3)}{2^{n-1}(2^n-5)+3}. \tag{27}$$

For any $n \geqslant 3$, $1 > \gamma_D^*(n) > 0$, and $\langle T_R^{D\to A}(n)\rangle$ reaches its minimum at $\gamma_D^*(n)$. Figure 4 shows the plot of $\langle T_R^{D\to A}(n)\rangle$ vs γ in the PSFW with generation $n=8$. $\langle T_R^{D\to A}(8)\rangle$ reaches its minimum at $\gamma_D^*(8) = 0.052$.

We also observe that, for any $n>5$, $\gamma_D^*(n)$ decreases monotonically with increasing n, as shown in Fig. 5. Additionally, in the PSFW where the size is sufficiently large, $n\to\infty$,

$$\gamma_D^*(n) \approx \frac{1}{\sqrt{2^n}} \to 0. \tag{28}$$

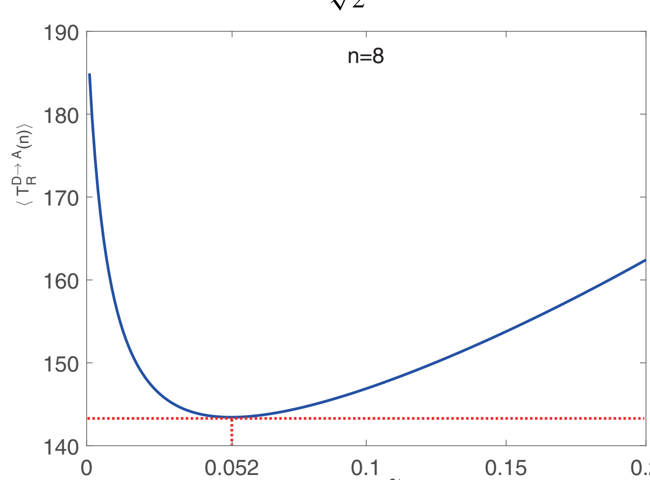

FIG. 4. Plot of $\langle T_R^{D\to A}(n)\rangle$ vs γ in the PSFW with generation $n = 8$. $\langle T_R^{D\to A}(8)\rangle$ reaches its minimum at $\gamma_D^*(8) = 0.052$.

[1]Another root is negative; it has thus been removed.

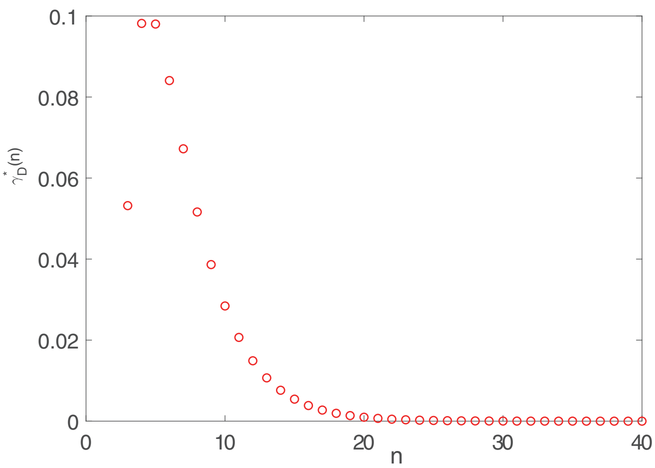

FIG. 5. Plot of the optimal resetting probability $\gamma_D^*(n)$ as a function of n. For any $n > 5$, $\gamma_D^*(n)$ decreases monotonically with increasing n, and $\gamma_D^*(n) \to 0$, as $n \to \infty$.

Replacing γ in Eq. (25) from Eq. (27), we get the MFPT from vertex D to A under resetting with $\gamma = \gamma^*(D)$,

$$\left\langle T_R^{D\to A}(n)\right\rangle_{\gamma=\gamma^*(D)} \approx \frac{3\sqrt{2^n} - 2^n\sqrt{2^n} - 3\times 2^n}{2 - \frac{4}{2^n} + \frac{8}{\sqrt{2^n}} - 2\sqrt{2^n} - 4}. \tag{29}$$

Letting $\gamma = 0$ in Eq. (25), we can obtain the MFPT without resetting,

$$\langle T^{D\to A}(n)\rangle = \frac{3\times 2^n}{4}. \tag{30}$$

By calculating the ratio of $\langle T_R^{D\to A}(n)\rangle_{\gamma=\gamma^*(D)}$ to $\langle T^{D\to A}(n)\rangle$, we obtain, in the PSFW where the size is sufficiently large, i.e. $n \to \infty$,

$$\frac{\left\langle T_R^{D\to A}(n)\right\rangle_{\gamma=\gamma^*(D)}}{\langle T^{D\to A}(n)\rangle} \approx \lim_{n\to\infty} \frac{3\sqrt{2^n} - 2^n\sqrt{2^n} - 3\times 2^n}{2 - \frac{4}{2^n} + \frac{8}{\sqrt{2^n}} - 2\sqrt{2^n} - 4} \times \frac{4}{3\times 2^n} \to \frac{2}{3}. \tag{31}$$

This result indicates that, although the optimal resetting probability, where $\langle T_R^{D\to A}(n)\rangle$ reaches its minimum, is approximately equal to 0 in the PSFW with a large enough size, resetting can always expedite the search process in this case.

C. MFPT under resetting for a random walk from vertex *I* to hub *A*

In this subsection we consider the case where the walker initiates from vertex I (i.e., the neighbor of A with the third highest degree). At each step, there is a fixed probability γ of resetting the walker back to its initial position. Similarly, we calculate the MFPT from vertex I to A under resetting, denoted as $\langle T_R^{I\to A}(n)\rangle$, and find the optimal value of $\gamma_I^*(n)$ at which $\langle T_R^{I\to A}(n)\rangle$ reaches its minimum.

First, we derive the probability-generating function, $\Phi_T^{I\to A}(n, z)$, of the first-passage time from I to A. Similar to Eq. (18), we find [54] that for any $n > 3$,

$$\Phi_T^{I\to A}(n, z) = \Phi_T^{I\to A}\big(3, \Phi_T^{I\to A}(n-3, z)\big). \tag{32}$$

For $n = 3$, $\Phi_T^{I\to A}(3, z) = \frac{z(3z-4)}{7z-8}$, as shown in Appendix A. Solving the aforementioned recursive formula, we obtain

$$\Phi_T^{I\to A}(n, z) = \frac{(2^{n-1}-1)z^2 - 2^{n-1}z}{(9\times 2^{n-3} - 2^{2n-3} - 1)z^2 + (2^{2n-2} - 9\times 2^{n-3})z - 2^{2n-3}}.$$

Therefore,

$$\Phi_T^{I\to A}(n, 1-\gamma) = \frac{(2^{n-1}-1)\gamma^2 + (2 - 2^{n-1})\gamma - 1}{(9\times 2^{n-3} - 2^{2n-3} - 1)\gamma^2 + (2 - 9\times 2^{n-3})\gamma - 1}. \tag{33}$$

Then, by replacing $\Phi_T(1-\gamma)$ with $\Phi_T^{I\to A}(n, 1-\gamma)$ in Eq. (12), and substituting $\Phi_T^{I\to A}(n, 1-\gamma)$ from Eq. (33), we obtain

$$\left\langle T_R^{I\to A}(n)\right\rangle = \frac{(5\times 2^{n-3} - 2^{2n-3})\gamma - 5\times 2^{n-3}}{(2^{n-1}-1)\gamma^2 + (2 - 2^{n-1})\gamma - 1}. \tag{34}$$

By taking the first-order derivative with respect to γ on both sides of Eq. (34) and setting $\frac{d}{d\gamma}\langle T_R^{I\to A}(n)\rangle = 0$, we obtain

$$(2^{3n-4} + 5\times 2^{n-3} - 7\times 2^{2n-4})\gamma^2 + 5\times(2^{2n-3} - 2^{n-2})\gamma + 5\times 2^{n-3} - 3\times 2^{2n-4} = 0, \tag{35}$$

with a root[2]

$$\gamma_I^*(n) = \frac{5*(2^{n-2} - 2^{2n-3}) + \sqrt{3\times(2^{5n-6} - 2^{4n-5})}}{2^{3n-3} + 5\times 2^{n-2} - 7\times 2^{2n-3}} = \frac{2^{n-1}(5\times 2^{1-n} + \sqrt{3\times 2^n - 6} - 5)}{2^{n-1}(2^n - 7) + 5}. \tag{36}$$

For any $n \geqslant 4$, $1 > \gamma_I^*(n) > 0$, and $\langle T_R^{I\to A}(n)\rangle$ reaches its minimum at $\gamma_I^*(n)$. Figure 6(a) depicts the plot of $\langle T_R^{I\to A}(n)\rangle$ vs γ in the PSFW with generation $n = 8$. $\langle T_R^{I\to A}(8)\rangle$ reaches its minimum at $\gamma_I^*(8) = 0.091$. Similar to $\gamma_D^*(n)$, for any $n > 5$, $\gamma_I^*(n)$ decreases monotonically with increasing n, as

[2]Another root is negative; it has thus been removed.

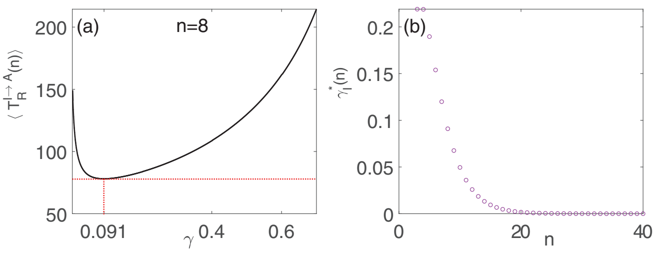

FIG. 6. (a) Plot of $\langle T_R^{I\to A}(n)\rangle$ vsγ in the PSFW with generation $n=8$. $\langle T_R^{I\to A}(8)\rangle$ reaches its minimum at $\gamma_I^*(8)=0.091$. (b) The plot of the optimal resetting probability $\gamma_I^*(n)$ as a function of n. For any $n>5$, $\gamma_I^*(n)$ decreases monotonically with increasing n, and $\gamma_I^*(n)\to 0$, as $n\to\infty$.

shown in Fig. 6(b). Furthermore, in the PSFW with a sufficiently large size, $n\to\infty$,

$$\gamma_I^*(n)\approx\sqrt{\frac{3}{2^n}}\to 0. \tag{37}$$

Replacing γ from Eq. (36) in Eq. (34), we obtain the MFPT under resetting at the optimal $\gamma=\gamma_I^*(n)$,

$$\langle T_R^{I\to A}(n)\rangle_{\gamma=\gamma_I^*(n)}=\frac{5\sqrt{3\times 2^n}-2^n\sqrt{3\times 2^n}-5\times 2^n}{12-\frac{24}{2^n}+16\sqrt{\frac{3}{2^n}}-4\sqrt{3\times 2^n}-8}.$$

Let $\gamma=0$ in Eq. (34), we obtain the MFPT without resetting

$$\langle T^{I\to A}(n)\rangle=\frac{5\times 2^n}{8}. \tag{38}$$

Calculating the ratio of $\langle T_R^{I\to A}(n)\rangle_{\gamma=\gamma_I^*(n)}$ to $\langle T^{I\to A}(n)\rangle$, we get

$$\frac{\langle T_R^{I\to A}(n)\rangle_{\gamma=\gamma_I^*(n)}}{\langle T^{I\to A}(n)\rangle}\approx\lim_{n\to\infty}\frac{5\sqrt{3\times 2^n}-2^n\sqrt{3\times 2^n}-5\times 2^n}{12-\frac{24}{2^n}+16\sqrt{\frac{3}{2^n}}-4\sqrt{3\times 2^n}-8}\times\frac{8}{5\times 2^n}\to\frac{2}{5}, \tag{39}$$

this indicates that resetting can always hasten the search process, although the optimal resetting probability where $\langle T_R^{I\to A}(n)\rangle$ reaches its minimum is approximately equal to 0 in the PSFW, which is sufficiently large.

D. MFPT under resetting for a random walk from the newest common neighbor of hub *A* and *B* to hub *A*

Let ζ_n denote the newest common neighbor of hub A and B in the PSFW with generation n, and $T^{\zeta_n\to A}(n)$ represent the first-passage time from ζ_n to hub A in the PSFW with generation n. In this subsection we will evaluate the MFPT under resetting for a random walk from ζ_n to hub A and find the optimal value of $\gamma_{\zeta_n}^*(n)$ at which $\langle T_R^{\zeta_n\to A}(n)\rangle$ reaches its minimum.

First, we derive the probability-generating function, $\Phi_T^{\zeta_n\to A}(n,z)$, of the first-passage time from ζ_n to A. It is clear that $\Pr(T^{\zeta_n\to A}(n)=0)=0$, $\Pr(T^{\zeta_n\to A}(n)=1)=\frac{1}{2}$, and for any $k>1$,

$$\Pr(T^{\zeta_n\to A}(n)=k)=\tfrac{1}{2}\Pr(T^{B\to A}(n)=k-1).$$

Therefore,

$$\begin{aligned}\Phi_T^{\zeta_n\to A}(n,z)&=\sum_{k=0}^{\infty}z^k\Pr(T^{\zeta_n\to A}(n)=k)\\&=z\frac{1}{2}+\sum_{k=2}^{\infty}z^t\frac{1}{2}\Pr(T^{B\to A}(n)=k-1)\\&=\frac{z}{2}+\frac{z}{2}\sum_{t=2}^{\infty}z^{t-1}\Pr(T^{B\to A}(n)=k-1)\\&=\frac{z}{2}\big[1+\Phi_T^{B\to A}(n,z)\big]\\&=z\left[\frac{2^{n-1}-(2^{n-1}-1)z}{2^n-(2^n-1)z}\right],\end{aligned} \tag{40}$$

where the last line of Eq. (40) is obtained by replacing $\Phi_T^{B\to A}(n,z)$ from Eq. (19). Thus

$$\Phi_T^{\zeta_n\to A}(n,1-\gamma)=\frac{-(2^{n-1}-1)\gamma^2+(2^{n-1}-2)\gamma+1}{(2^n-1)\gamma+1}. \tag{41}$$

Then, replacing $\Phi_T(1-\gamma)$ with $\Phi_T^{\zeta_n\to A}(n,1-\gamma)$ in Eq. (12), and substituting $\Phi_T^{\zeta_n\to A}(n,1-\gamma)$ from Eq. (41), we obtain the MFPT under resetting as follows:

$$\langle T_R^{\zeta_n\to A}(n)\rangle=\frac{(2^{n-1}-1)\gamma+2^n-2^{n-1}+1}{(1-2^{n-1})\gamma^2+(2^{n-1}-2)\gamma+1}. \tag{42}$$

By taking the first-order derivative with respect to γ on both sides of Eq. (42) and setting $\frac{d}{d\gamma}\langle T_R^{\zeta_n\to A}(n)\rangle=0$, we obtain

$$(2^{2n-2}-2^n+1)\gamma^2+(2^{2n-1}-2)\gamma+(-2^{2n-2}+2^n+1)=0, \tag{43}$$

with a root[3]

$$\begin{aligned}\gamma_{\zeta_n}^*(n)&=\frac{2-2^{2n-1}+\sqrt{2^{4n-1}-2^{3n+1}+2^{2n+1}}}{2^{2n-1}-2^{n+1}+2}\\&=\frac{2^n[\sqrt{(2^{n-1}-2)2^n+2}-2^{n-1}]+2}{2^n(2^{n-1}-2)+2}.\end{aligned} \tag{44}$$

For any $n\geqslant 3$, $1>\gamma_{\zeta_n}^*(n)>0$, and $\langle T_R^{\zeta_n\to A}(n)\rangle$ reaches its minimum at $\gamma_{\zeta_n}^*(n)$. Figure 7 illustrates the plot of $\langle T_R^{\zeta_n\to A}(n)\rangle$ vs γ in the PSFW with generation $n=8$. $\langle T_R^{\zeta_n\to A}(8)\rangle$ reaches its minimum at $\gamma_I^*(8)=0.41$.

As shown in Fig. 8, $\gamma_{\zeta_n}^*(n)$ increases monotonically with increasing n, and in the PSFW with a sufficiently large size, $n\to\infty$,

$$\gamma_{\zeta_n}^*(n)\to\sqrt{2}-1\approx 0.4142. \tag{45}$$

Similar to the derivation of Eqs. (31) and (41), calculating the ratio of the MFPT with optimal resetting probability to the MFPT without resetting, we find, in the PSFW with a

[3] Another root is negative; it has thus been removed.

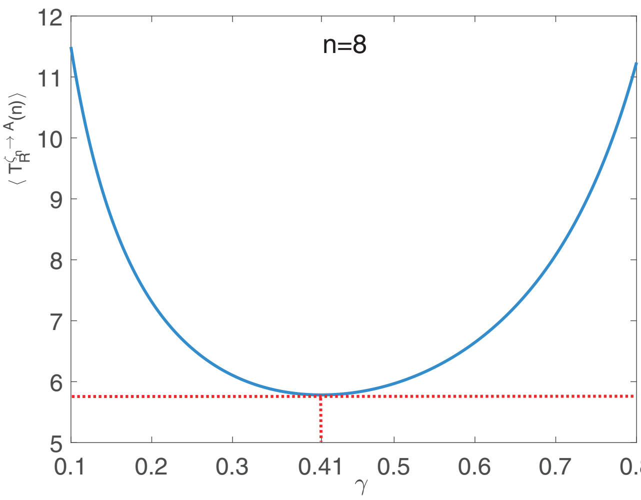

FIG. 7. Plot of $\langle T_R^{\zeta_n \to A}(n)\rangle$ vs γ in the PSFW with generation $n = 8$. $\langle T_R^{\zeta_n \to A}(8)\rangle$ reaches its minimum at $\gamma_{\zeta_n}^*(8) = 0.41$.

sufficiently large size,

$$\frac{\langle T_R^{\zeta_n \to A}(n)\rangle_{\gamma=\gamma_{\zeta_n}^*(n)}}{\langle T^{\zeta_n \to A}(n)\rangle} \to 0. \tag{46}$$

Therefore, in this case, resetting can significantly expedite the search process in a large PSFW, and the optimal resetting probability where the MFPT reaches its minimum is approximately equal to 0.4142.

Recalling the results presented in Secs. IV A–IV C, we observe significant differences in the optimal resetting probabilities and the ratio of the MFPT with optimal resetting probability to the MFPT without resetting. With these observations, we arrive at the following conclusion. For random walks on the PSFW, the degree of the resetting position has a tremendous influence on the MFPT to hub A. In a large PSFW, if at each step the walker has a fixed probability of resetting to vertex B, which is the neighbor of hub A with the highest degree, resetting would slow the search process to reach target A; if at each step the walker has a fixed probability of resetting to vertex D (or I), which are neighbors of hub A with the second highest, or third highest, degrees, resetting can always expedite the search process to reach target A to a certain extent, but the optimal resetting probability, where the MFPT reaches its minimum, is almost equal to 0. If the walker has a fixed probability of resetting to vertex ζ_n, which is a common neighbor of hub A and hub B with the lowest degree, resetting can substantially shorten the MFPT to hub A and accelerate the search process. Even though the distances between these resetting positions (i.e., B, D, I, ζ_n) and hub A are all equal to 1, the effect of resetting on the MFPT to hub A is remarkably different. The reason lies in the varying degrees of these resetting positions. The higher the degree of the resetting position, the more difficult it is for the walker to reach hub A from that resetting position.

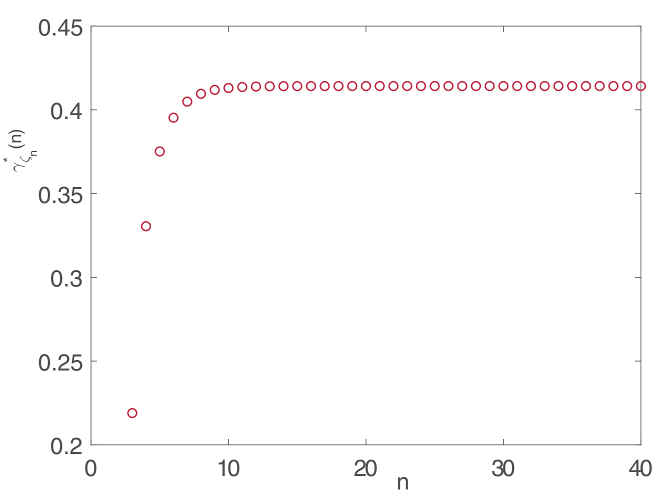

FIG. 8. Plot of optimal resetting probability $\gamma_{\zeta_n}^*(n)$ as a function of n. For all $n > 3$, $\gamma_{\zeta_n}^*(n)$ increases monotonically with increasing n, and $\gamma_{\zeta_n}^*(n) \to 0.4142$, as $n \to \infty$.

E. MFPT under resetting for a random walk from vertex *F* (or *H*) to hub *A*

In this subsection we consider the case where the walker initiates from a vertex [e.g., F (or H), as shown in Fig. 2], which is not a neighbor of hub A. Furthermore, at each step, there is a fixed probability γ of resetting the walker back to the initial position. Similarly, we calculate the MFPT from vertex F (or H to A under resetting), referred to as $\langle T_R^{F\to A}(n)\rangle$ [or $\langle T_R^{H\to A}(n)\rangle$] and find the optimal $\gamma_F^*(n)$ [or $\gamma_H^*(n)$] where $\langle T_R^{F\to A}(n)\rangle$ [or $\langle T_R^{H\to A}(n)\rangle$] reaches its minimum.

First, we derive the probability-generating function, $\Phi_T^{F\to A}(n,z)$ [or $\Phi_T^{H\to A}(n,z)$], of the first-passage time from F (or H) to A. Similar to Eq. (18), we find [54], for any $n > 2$,

$$\Phi_T^{F\to A}(n,z) = \Phi_T^{F\to A}(2, \Phi_T^{F\to A}(n-2,z)), \tag{47}$$

with $\Phi_T^{F\to A}(2,z) = \frac{z^2}{4z-3}$; and for any $n > 3$,

$$\Phi_T^{H\to A}(n,z) = \Phi_T^{F\to A}(3, \Phi_T^{F\to A}(n-3,z)), \tag{48}$$

with $\Phi_T^{H\to A}(3,z) = \frac{z^2}{8z-7}$.

Solving Eqs. (47) and (48), and replacing z with $1-\gamma$, we obtain

$$\Phi_T^{F\to A}(n, 1-\gamma) = \frac{\gamma^2 - 2\gamma + 1}{[2^{2n-2} - 5\times 2^{n-2} + 1]\gamma^2 + [5\times 2^{n-2} - 2]\gamma + 1} \tag{49}$$

and

$$\Phi_T^{H\to A}(n, 1-\gamma) = \frac{\gamma^2 - 2\gamma + 1}{[2^{2n-3} - 9\times 2^{n-3} + 1]\gamma^2 + [9\times 2^{n-3} - 2]\gamma + 1}. \tag{50}$$

Therefore,

$$\langle T_R^{F\to A}(n)\rangle = \frac{1-\Phi_T^{F\to A}(n,1-\gamma)}{\gamma\Phi_T^{F\to A}(n,1-\gamma)} = \frac{(2^{2n-2} - 5\times 2^{n-2})\gamma + 5\times 2^{n-2}}{(\gamma-1)^2} \tag{51}$$

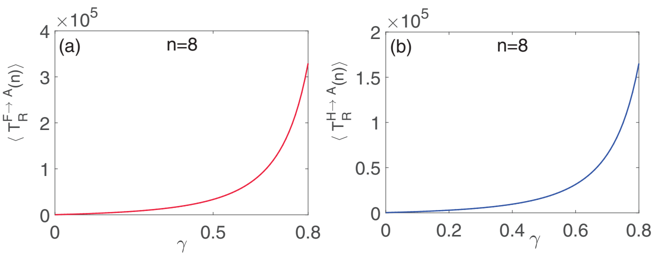

FIG. 9. Result of a random walk by the searcher in the eight-generation (1, 2)-flowers. (a) Plot of the MFPT of the searcher from the initial node F with respect to the reset probability. (b) Plot of the MFPT of the searcher from the initial node H with respect to the reset probability.

and

$$\langle T_R^{H\to A}(n)\rangle = \frac{1-\Phi_T^{H\to A}(n,1-\gamma)}{\gamma\Phi_T^{H\to A}(n,1-\gamma)} = \frac{(2^{2n-3}-9\times 2^{n-3})\gamma+9\times 2^{n-3}}{(\gamma-1)^2}. \quad (52)$$

As shown in Fig. 9, both $\langle T_R^{F\to A}(n)\rangle$ and $\langle T_R^{H\to A}(n)\rangle$ increase monotonically in γ on interval [0,1), and the minimum of the MFPT to hub A is obtained at $\gamma = 0$, where no resetting occurs in the entire process.

Note that the distance between F (or H) and A is greater than that between B (or D, I, and ζ_n) and A. By comparing the results obtained in this subsection with those obtained in Secs. IV A–IV D, we confirm that the distance between the resetting position and the target (i.e., A) has a tremendous effect on the MFPT to the target. The farther the distance, the smaller the effect of resetting on the optimization of MFPT.

V. MFPT ON THE PSFW FOR RANDOM WALKS UNDER RESETTING WITH THE RESETTING POSITION SELECTED RANDOMLY

In this section we examine the scenario where the walker starts from a vertex randomly selected according to a stationary distribution (i.e., $\Pi = (\frac{d_1}{\sum_k d_k}, \frac{d_2}{\sum_k d_k}, \dots, \frac{d_N}{\sum_k d_k})$, with d_k being the degree of vertex k, and N being the total number of vertexes of the networks), and at each step, there is a fixed probability γ of resetting to another vertex randomly drawn from the same probability distribution Π. Similarity, we calculate the MFPT under resetting from a randomly drawn vertex to hub A, referred to as $\langle T_R^{\Pi\to A}(n)\rangle$, and aim to find the optimal value of $\gamma_\Pi^*(n)$ at which $\langle T_R^{\Pi\to A}(n)\rangle$ reaches its minimum.

First, we derive the probability-generating function, $\Phi_T^{\Pi\to A}(n,z)$, of the first-passage time from a randomly drawn vertex to hub A.

As derived in Appendix B,

$$\Phi_T^{\Pi\to A}(n,z) = \frac{2^{n-1}}{3^n}\times\frac{z(1+z)}{\prod_{k=1}^n \chi(k,z)}, \quad (53)$$

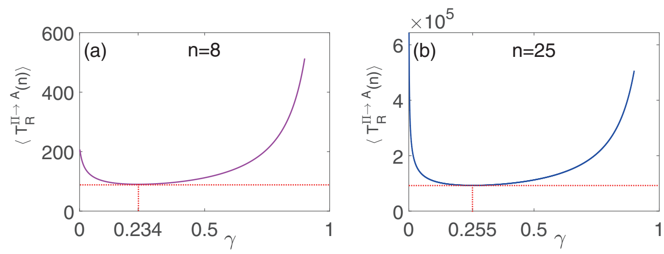

FIG. 10. (a) $\langle T_R^{\Pi\to A}(n)\rangle$ as a function of γ for $n = 8$. The minimum value of $\langle T_R^{\Pi\to A}(n)\rangle$ occurs at $\gamma_\Pi^*(8) = 0.234$. (b) $\langle T_R^{\Pi\to A}(n)\rangle$ as a function of γ for $n = 25$. The minimum value of $\langle T_R^{\Pi\to A}(n)\rangle$ occurs at $\gamma_\Pi^*(25) \approx 0.255$.

with

$$\chi(k,z) = \frac{[2^k+(4-2^k)z][2^k+(1-2^k)z]}{[2^k+(3-2^k)z][2^k+(2-2^k)z]}, \quad k = 1,2,\dots,n.$$

Thus,

$$\langle T_R^{\Pi\to A}(n)\rangle = \frac{1-\Phi_T^{\Pi\to A}(n,1-\gamma)}{\gamma\Phi_T^{\Pi\to A}(n,1-\gamma)} = \frac{1-\frac{2^{n-1}(1-\gamma)(2-\gamma)}{3^n\times\prod_{k=1}^n\chi(k,1-\gamma)}}{\gamma\frac{2^{n-1}(1-\gamma)(2-\gamma)}{3^n\times\prod_{k=1}^n\chi(k,1-\gamma)}}$$
$$= \frac{3^n\prod_{k=1}^n\chi(k,1-\gamma)-2^{n-1}(1-\gamma)(2-\gamma)}{2^{n-1}\gamma(1-\gamma)(2-\gamma)}$$
$$= \frac{3^n\phi(n,\gamma)-2^{n-1}(1-\gamma)(2-\gamma)}{2^{n-1}\gamma(1-\gamma)(2-\gamma)}, \quad (54)$$

where $\phi(n,\gamma) = \prod_{k=1}^n \chi(k,1-\gamma)$.

The expression for $\langle T_R^{\Pi\to A}(n)\rangle$ may appear lengthy and complex. However, by plotting $\langle T_R^{\Pi\to A}(n)\rangle$ as a function of γ, one can find the optimal $\gamma_\Pi^*(n)$ where $\langle T_R^{\Pi\to A}(n)\rangle$ reaches its minimum. Figure 10 shows the plot of $\langle T_R^{\Pi\to A}(n)\rangle$ vs γ for $n = 8$ and $n = 25$.

Furthermore, as shown in Fig. 11, in the PSFW with a large size, $n\to\infty$, we find that

$$\phi(n,\gamma) = \prod_{k=1}^n \chi(k,1-\gamma)\approx[1-(\gamma-1)^2]^{\frac{299}{500}}. \quad (55)$$

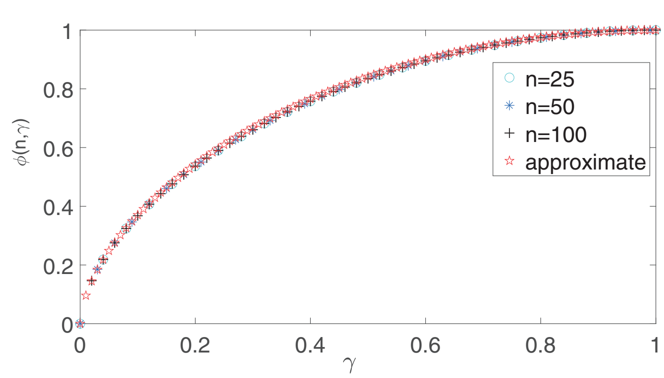

FIG. 11. Comparison of plots $\phi(n,\gamma)$ vs γ with different choices of n ($n = 25$, $n = 50$, and $n = 100$) and the approximation obtained from $[1-(\gamma-1)^2]^{\frac{299}{500}}$. The four curves closely overlap, confirming the accuracy of Eq. (57).

Replacing $\phi(n,\gamma)$ from Eq. (55) in Eq. (54), we obtain

$$\langle T_R^{\Pi\to A}(n)\rangle \approx \frac{3^n[1-(\gamma-1)^2]^{\frac{299}{500}} - 2^{n-1}(1-\gamma)(2-\gamma)}{2^{n-1}\gamma(1-\gamma)(2-\gamma)}. \tag{56}$$

By taking the first-order derivative with respect to γ on both sides of Eq. (56) and setting $\frac{d}{d\gamma}\langle T_R^{\Pi\to A}(n)\rangle = 0$, we obtain

$$\frac{299}{250}3^n 2^{n-1}[2\gamma-\gamma^2]^{\frac{299}{500}}(1-\gamma)^2 - 3^n 2^{n-1}[2\gamma-\gamma^2]^{\frac{299}{500}} \times(3\gamma^2-6\gamma+2) + [2^{n-1}(1-\gamma)(2-\gamma)]^2 = 0, \tag{57}$$

which can be simplified as

$$\gamma(\gamma-2)(451\gamma^2-902\gamma+201)=0, \tag{58}$$

with a root[4]

$$\gamma_\Pi^*(n) = 1-\sqrt{\frac{250}{451}} \approx 0.255, \tag{59}$$

and $\langle T_R^{\Pi\to A}(n)\rangle$ reaches its minimum at $\gamma_\Pi^*(n)\approx 0.255$ while n is big enough.

By taking the first-order derivative with respect to γ on both sides of Eq. (53) and setting $z=1$, we obtain the MFPT without resetting:

$$\begin{aligned}
&\langle T^{\Pi\to A}(n)\rangle \\
&= \left.\frac{\partial\Phi_T^{\Pi\to A}(n,z)}{\partial z}\right|_{z=1} \\
&= \frac{2^{n-1}\times 3^{n+1}\times\phi(n,1) - 3^n\times\left[\left.\frac{\partial\phi(n,z)}{\partial z}\right|_{z=1}\right]\times 2^n}{[3^n\times\phi(n,1)]^2} \\
&= \frac{2^{n-1}\times 3^{n+1}\times\frac{2^n}{3^n} - 3^n\times\frac{2^{n-1}}{3^{n-1}}\left[-\frac{5}{9}(2^n-1)\right]\times 2^n}{\left[3^n\times\frac{2^n}{3^n}\right]^2} \\
&= \frac{(5\times 2^n+4)}{6}.
\end{aligned} \tag{60}$$

Calculating the ratio of the MFPT at the optimal resetting probability to the MFPT without resetting, in the PSFW with a sufficiently large size, $n\to\infty$, we find

$$\frac{\langle T_R^{\Pi\to A}(n)\rangle_{\gamma=\gamma_\Pi^*(n)}}{\langle T^{\Pi\to A}(n)\rangle} \to 0. \tag{61}$$

Therefore, in a large PSFW, resetting to a vertex randomly drawn from the stationary distribution proves to be a favorable strategy for expediting the search process. In scenarios where the location of the target is unknown, the findings from this study provide valuable insights into optimizing stochastic search in the PSFW and other large networks.

VI. CONCLUSION

In this work we explored discrete-time random walks under resetting on the PSFW. During each step of the random walks, the walker either moves to one of the neighbors of the current site with a probability of $1-\gamma$, or resets to the resetting position with a probability of γ. We have considered various choices for the resetting position and rigorously derived results for the MFPT to the hub (i.e., vertex with the highest degree) of the networks. Additionally, we have identified the optimal resetting probability γ^* at which MFPT reaches its minimum and evaluated the ratio of the MFPT with optimal resetting probability and the MFPT without resetting. Our findings demonstrate that, in a large PSFW, if the resetting position is set to be the one of the neighbors with the highest degree, or to be a common neighbor of B and C, the resetting process slows the walker's progress towards the target A; if the resetting position is set to be the one of the neighbors with the second highest, or third highest, degree of the target A, resetting can always expedite the search process to reach target A to a certain extent, but the optimal resetting probability γ^* at which MFPT to target A reaches its minimum is approximately 0. If the resetting position is set to be one of the neighbors with the lowest degree of node A, resetting can significantly expedite the process of the walker reaching the target node A. In this case the optimal resetting probability γ^*, where the MFPT to target A reaches its minimum, is approximately 0.4142. We have also observed that if the resetting position is a vertex randomly drawn from the stationary distribution, resetting can also significantly speed the process of the walker reaching the target node A in the large PSFW. The optimal resetting probability γ^*, at which the MFPT to target A reaches its minimum, is approximately 0.255. Note that, in the general case, where the target node's location is unknown, it becomes challenging to select a suitable resetting position close to the target with a small degree to expedite the search process. Therefore, in such scenarios, our results suggest that resetting to a vertex randomly drawn from the stationary distribution in the large PSFW can be an acceptable strategy to expedite the search process. To further our understanding, interesting questions remain: What is the effect of the resetting position on the MFPT in other network types? Can resetting to a vertex randomly drawn from the stationary distribution expedite the search process in other networks as well? These questions warrant further investigation and exploration.

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ACKNOWLEDGMENT

The work is supported by the Guangdong Basic and Applied Basic Research Foundation (Grant No. 2022B1515120060).

APPENDIX A: METHOD FOR DERIVING THE PROBABILITY GENERATION FUNCTION FOR THE FIRST-PASSAGE TIME AND RETURN TIME ON A SMALL-SIZED PSFW

For a small PSFW, the probability generation function for the first-passage time and return time can be directly obtained by using the symbolic toolbox of MATLAB. First, we present the general method.

Let

$$\Pi = (P_{ij})$$

[4]All the other roots do not belong to (0, 1) and have thus been removed.

be the transfer probability matrix for a random walk on the PSFW, where

$$P_{ij} = \begin{cases} \frac{1}{d_i} & \text{if } i \sim j, \text{ and } i \text{ is not a trap}, \\ 0, & \text{others}, \end{cases} \tag{A1}$$

with "$i \sim j$" indicating that a link exists between i and j. Therefore, the probability-generating function for the passage time and the return time can be obtained using the following equation [55]:

$$\Phi(z) = \sum_{n=0}^{\infty} (z\Pi)^n = (I - z\Pi)^{-1}, \tag{A2}$$

where $\Phi(z) = [\phi_{ij}(z)]$, and $\phi_{ij}(z)$ represents the probability-generating function of the time for a walker, starting from site i, to be found at site j. If site j is a trap, $\phi_{ij}(z)$ is simply the probability-generating function for the first-passage time from site i to j. On the other hand, if there are no traps in the network, $\phi_{ii}(z)$ is simply the probability-generating function for the return time for a random walk starting from site i, i.e., the time for a walker, starting from site i, to be found at site i.

Then we present some examples to demonstrate the calculation of the probability-generating function for the first-passage time in the small PSFW. In the PSFW with generation 1, let vertex 1 (i.e., vertex A) be a trap:

$$\Pi = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{A3}$$

Substituting Π from Eq. (A3) in Eq. (A2) and calculating the matrix inverse using the MATLAB symbol toolbox, we obtain

$$\Phi_T^{B\to A}(1, z) = \phi_{21}(1, z) = \frac{z}{2 - z}. \tag{A4}$$

In the PSFW with generation 2, let vertex 1 (i.e., vertex A) be a trap:

$$\Pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{A5}$$

Substituting Π from Eq. (A5) in Eq. (A2) and calculating the matrix inverse using the MATLAB symbol toolbox, we obtain

$$\Phi_T^{D\to A}(2, z) = \phi_{41}(2, z) = \frac{z(z - 2)}{3z - 4} \tag{A6}$$

and

$$\Phi_T^{F\to A}(2, z) = \phi_{61}(2, z) = \frac{z^2}{3z - 4}. \tag{A7}$$

Similarly, in the PSFW with generation 3, we obtain

$$\Phi_T^{I\to A}(3, z) = \phi_{91}(3, z) = \frac{z(3z - 4)}{7z - 8} \tag{A8}$$

and

$$\Phi_T^{H\to A}(3, z) = \phi_{81}(3, z) = \frac{z^2}{7z - 8}. \tag{A9}$$

Finally, we present some examples demonstrating the calculation of the probability-generating function for the return time in the small PSFW. In the PSFW with generation 0, let

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A10}$$

Substituting Π from Eq. (A10) in Eq. (A2) and calculating the matrix inverse using the MATLAB symbol toolbox, we obtain the probability-generating function for the return time for vertex A in the PSFW with generation 0,

$$\Phi_{\text{RT}}^{A}(0, z) = \phi_{11}(0, z) = \frac{1}{(1 + z)(1 - z)}. \tag{A11}$$

Similarly, in the PSFW with generation 1, let

$$\Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{A12}$$

and calculate the matrix inverse using the MATLAB symbol toolbox, we get the probability-generating function for the return time for vertex A in the PSFW with generation 1,

$$\Phi_{\text{RT}}^{A}(1, z) = \phi_{11}(1, z) = \frac{2 - z}{(2 + z)(1 - z)}. \tag{A13}$$

APPENDIX B: DERIVATION OF EQ. (55)

In this Appendix we present a detailed derivation of the probability-generating function, $\Phi_T^{\Pi\to A}(n, z)$, for the first-passage time from a randomly drawn vertex to hub A in the PSFW with generation n ($n > 2$).

It is known that [54],

$$\Phi_T^{\Pi\to A}(n, z) = \frac{2^{n-1} z}{(1 - z)3^n} \times \frac{1}{\Phi_{\text{RT}}^{A}(n, z)}, \tag{B1}$$

where $\Phi_{\text{RT}}^{A}(n, z)$ is the probability-generating function of the return time for a random walk starting from vertex A on the PSFW for generation n. It is defined such that $\Phi_{\text{RT}}^{A}(n, z)(n, z)$ satisfies the following equation:

$$\Phi_{\text{RT}}^{A}(n, z) = \frac{1}{\Psi(\Phi_T^{B\to A}(n - 1, z))} \times \Phi_{\text{RT}}^{A}(n - 1, z), \tag{B2}$$

with $\Psi(z) = \frac{\Phi_{\text{RT}}^{A}(0,z)}{\Phi_{\text{RT}}^{A}(1,z)}$.

Using the method presented in Appendix A, we obtain

$$\Phi_{\text{RT}}^{A}(0, z) = \frac{1}{(1 + z)(1 - z)} \tag{B3}$$

and

$$\Phi^{A}_{\mathrm{RT}}(1,z) = \frac{2-z}{(2+z)(1-z)}. \tag{B4}$$

Therefore,

$$\begin{aligned}\Phi^{A}_{\mathrm{RT}}(n,z) &= \frac{[2^n+(4-2^n)z][2^n+(1-2^n)z]}{[2^n+(3-2^n)z][2^n+(2-2^n)z]}\Phi^{A}_{\mathrm{RT}}(n-1,z)\\ &\equiv \chi(n,z)\Phi^{A}_{\mathrm{RT}}(n-1,z),\end{aligned} \tag{B5}$$

where $\chi(n,z) \equiv \frac{[2^n+(4-2^n)z][2^n+(1-2^n)z]}{[2^n+(3-2^n)z][2^n+(2-2^n)z]}$.

Using Eq. (B5) recursively, we obtain

$$\Phi^{A}_{\mathrm{RT}}(n,z) = \frac{\prod_{k=1}^{n}\chi(k,z)}{(1+z)(1-z)}. \tag{B6}$$

Substituting $\Phi^{A}_{\mathrm{RT}}(n,z)$ from Eq. (B6) into Eq. (B1), we obtain

$$\Phi^{\Pi\to A}_{T}(n,z) = \frac{2^{n-1}}{3^n} \times \frac{z(1+z)}{\prod_{k=1}^{n}\chi(k,z)}. \tag{B7}$$

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