Statistics of the number of renewals, occupation times, and correlation in ordinary, equilibrium, and aging alternating renewal processes

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The renewal process is a point process where an interevent time between successive renewals is an independent and identically distributed random variable. Alternating renewal process is a dichotomous process and a slight generalization of the renewal process, where the interevent time distribution alternates between two distributions. We investigate statistical properties of the number of renewals and occupation times for one of the two states in alternating renewal processes. When both means of the interevent times are finite, the alternating renewal process can reach an equilibrium. On the other hand, an alternating renewal process shows aging when one of the means diverges. We provide analytical calculations for the moments of the number of renewals, occupation time statistics, and the correlation function for several case studies in the interevent-time distributions. We show anomalous fluctuations for the number of renewals and occupation times when the second moment of interevent time diverges. When the mean interevent time diverges, distributional limit theorems for the number of events and occupation times are shown analytically. These are known as the Mittag-Leffler distribution and the generalized arcsine law in probability theory.

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I. INTRODUCTION

The renewal process is a point process where the duration times of a state are independent and identically distributed (IID) random variables. Theory of renewal processes can be applied to wide ranges of science such as physics, chemistry, biology, and seismology [1–5], as well as technology and industry such as queueing problem and replacing machines [6–10]. Occupation time statistics of renewal processes and the statistics of the number of renewals are extensively studied especially in physics and mathematics literatures because of its wide applications [1,6,7,11–18]. These theoretical results play an important role in fundamental theory as well as numerous applications to physical systems such as the mean magnetization in spin systems [1], the fluorescence of quantum dots [19–22], interface fluctuations in liquid crystals [23], α-percentile options in stock prices [24,25], and leads in sports games [26]. Moreover, the statistics of the number of changes of states play an important role in the continuous-time random walk and its generalization [27–29].

In biological and soft matter systems, diffusivity often changes randomly. This fluctuating diffusivity provides non-Gaussian fluctuations in the displacement distribution, anomalous diffusion, and ergodicity breaking [30–35]. Brownian motion with fluctuating diffusivity is a simple stochastic model of such systems [36–39]. These stochastic models show a non-Gaussian distribution in the displacement, anomalous diffusion, and nonergodic behaviors in the time-averaged diffusivity [36–41]. As a simple stochastic model of the Brownian motion with fluctuating diffusivity, Brownian motion with dichotomously fluctuating diffusivity is investigated [40,42], where low and high diffusivities change alternatively. In particular, such a fluctuating diffusivity is modeled by a dichotomous process.

An alternating renewal process is a slight modification of the renewal process, where duration times are IID random variables but the duration-time distribution alternates [6,43]. A dichotomous process is a typical stochastic process of the alternating renewal process. In dichotomous processes, duration-time PDFs for two states are not identical in general. Therefore, alternating renewal processes can be applied to a wider range of phenomena. In this paper, we aim to obtain occupation time statistics, the moments of the number of changes of states, and the correlation function in alternating renewal processes.

There are some universal fluctuations in stochastic processes. The central limit theorem provides one of the most universal fluctuations, i.e., the normal distribution [7], which states that the normalized sum of IID random variables converges in distribution to the normal distribution. Another classical theory of fluctuations is the arcsine law, which states that the occupation time distribution in the positive side in a random walk follows the arcsine distribution [7]. Its generalization is known as the generalized arcsine law [44]. The occupation time distribution depends on the domain. When the domain is finite, the occupation time distribution in the Brownian motion follows the half Gaussian and its generalization to general Markov processes provides the Mittag-Leffler distribution [45]. These universal fluctuations play a fundamental role in infinite ergodic theory [46–51].

This paper is organized as follows. In Sec. II, we describe an alternating renewal process and observables that we are interested in. In Sec. III, we derive the backward and the

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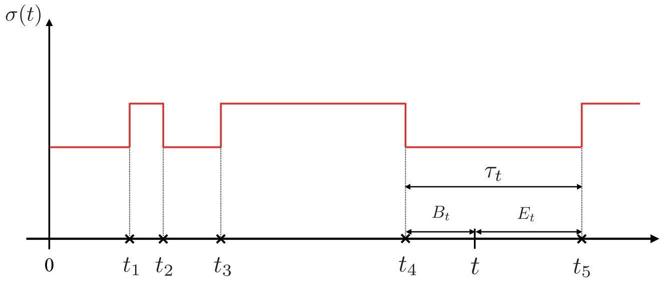

FIG. 1. Renewal time t_n, the forward recurrence time E_t, the backward recurrence time B_t, and the time interval straddling t, i.e., τ_t.

forward recurrence time distributions. In Sec. IV, we derive the moments of the number of renewals. In Sec. V, we show the occupation time statistics. In Sec. VI, we derive the correlation function of states. Section VII is devoted to the conclusion.

II. MODEL AND OBSERVABLES

The renewal process is a point process where an interevent time between successive renewals is an IID random variable. If there are two types of renewal processes, the interevent-time distributions for the renewal processes take different forms. In particular, if the interevent time distribution alternates between the two distributions, this process is called an alternating renewal process. We consider a dichotomous random signal $\sigma(t)$ described by an alternating renewal processes. The random signal $\sigma(t)$ alternates between $+$ and $-$ states, i.e., $\sigma(t) = +1$ or -1. Duration times for $+$ and $-$ states are random variables following different probability density functions (PDFs), $\rho_+(x)$ and $\rho_-(x)$ for $+$ and $-$ states, respectively. Here, we consider four cases for the PDFs of duration times. These cases are summarized by the Laplace transform of the PDFs as follows: (1) $\alpha_+ \leqslant \alpha_- < 1$, $\hat{\rho}_\pm(s) = 1 - a_\pm s^{\alpha_\pm} + o(s^{\alpha_\pm})$, (2) $1 < \alpha_+ < 2$ and $\alpha_+ \leqslant \alpha_-$, $\hat{\rho}_+(s) = 1 - \mu_+ s + a_+ s^{\alpha_+} + o(s^{\alpha_+})$ and $\hat{\rho}_-(s) = 1 - \mu_- s + o(s)$, (3) $2 < \alpha_\pm$, $\hat{\rho}_\pm(s) = 1 - \mu_\pm s + \frac{1}{2}(\sigma_\pm^2 + \mu_\pm^2)s^2 + o(s^2)$, and (4) $\alpha_+ < 1$ and $1 < \alpha_-$, $\hat{\rho}_+(s) = 1 - a_+ s^{\alpha_+} + o(s^{\alpha_+})$ and $\hat{\rho}_-(s) = 1 - \mu_- s + o(s)$, where $\hat{\rho}_\pm(s)$ is the Laplace transform of $\rho_\pm(t)$, $\mu_\pm$ is the mean duration time, and $\sigma_\pm^2$ is the variance. In what follows, we use $\alpha \equiv \min(\alpha_+, \alpha_-)$. In Case 1, both of the mean duration times diverge. In Case 2, both of the mean duration times are finite and both second moments of duration times diverge. In Case 3, both second moments of duration times are finite. In Cases 1, 2, and 3, the asymptotic behavior of the PDFs follow power-law distributions: $\rho_\pm(x) \propto x^{-1-\alpha_\pm}$ for $x \to \infty$. In Case 4, the asymptotic behavior of the PDF of duration times for the $+$ state follows power-law distributions: $\rho_+(x) \propto x^{-1-\alpha_+}$ for $x \to \infty$. Thus the mean duration time for the $+$ state diverges. Moreover, the asymptotic behavior of the PDF of duration times for the $-$ state follows a power-law distribution: $\rho_-(x) \propto x^{-1-\alpha_-}$ for $x \to \infty$.

We consider the following observables, which are illustrated in Fig. 1. t_n is the time at nth renewal, i.e., $t_n = \tau_1 + \tau_2 + \cdots + \tau_n$, where τ_k is the kth duration time. N_t is the number of renewals from 0 to t. E_t is the forward recurrence time defined by $E_t \equiv t_{N_t+1} - t$. B_t is the backward recurrence time defined by $B_t \equiv t - t_{N_t}$. τ_t is the time interval straddling t. $T_t^\pm$ is the occupation time for the $\pm$ state, which is represented by

$$T_t^+ = \frac{1}{2}\int_0^t [1 + \sigma(t')]dt'. \qquad (1)$$

When $\sigma(0) = +1$, $T_t^+ = \tau_1 + \tau_3 + \cdots + \tau_{2k+1}$ or $T_t^+ = \tau_1 + \tau_3 + \cdots + \tau_{2k-1} + B_t$ if $N_t = 2k+1$ or $N_t = 2k$, respectively. Moreover, when $\sigma(0) = -1$, $T_t^+ = \tau_2 + \tau_4 + \cdots + \tau_{2k} + B_t$ or $T_t^+ = \tau_2 + \tau_4 + \cdots + \tau_{2k}$ if $N_t = 2k+1$ or $N_t = 2k$, respectively. In what follows, we assume $\sigma(0) = +1$ except where specifically noted. It follows that the occupation time for the $-$ state with $\sigma(0) = \pm 1$, denoted by $T_t^-(\sigma_0 = \pm 1)$, can be represented by $T_t^-(\sigma_0 = +1) = T_t^+(\sigma_0 = -1)$ and $T_t^-(\sigma_0 = -1) = T_t^+(\sigma_0 = +1)$.

III. BACKWARD AND FORWARD RECURRENCE TIME DISTRIBUTIONS

A. Forward recurrence time distribution

We derive the forward recurrence time distribution. It is intuitively conjectured that the probability of finding the $+$ state, P_+, and $-$ state, P_-, for $t \to \infty$ can be represented by the means, i.e., $P_+ = \mu_+/(\mu_+ + \mu_-)$ and $P_- = \mu_-/(\mu_+ + \mu_-)$, respectively, if the means exist. Furthermore, the probabilities do not depend on the initial condition. This is rigorously proved in Ref. [6]. Here, we assume that the PDF of the duration time for the first renewal is the same as $\rho_+(\tau)$ or $\rho_-(\tau)$. This process is called an *ordinary alternating renewal process* [6]. We denote the PDF of forward recurrence time E_t by $f_E(E_t, t)$. As shown in Appendix A, for $s \to 0$, the double Laplace transforms $\hat{f}_E(u; s)$ of $f_E(E_t, t)$ for both initial conditions, i.e., $\sigma(0) = 1$ and -1, coincide and are given by

$$\hat{f}_E(u; s) \sim \frac{\hat{\rho}_-(u) - \hat{\rho}_-(s) + \hat{\rho}_+(u) - \hat{\rho}_+(s)}{(s-u)\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}}. \qquad (2)$$

Therefore, the PDF of the forward recurrence time does not depend on the initial state in the long-time limit ($t \to \infty$).

1. *Probabilities finding + and − states*

By Eqs. (A5) and (A6), we have the probabilities finding $+$ and $-$ states, P_+ and P_-, for $t \to \infty$. Taking limits of $s \to 0$ and $u = 0$ in Eqs. (A5) and (A6) yields the probabilities. For Cases 2 and 3 ($\alpha > 1$), the probabilities become

$$P_+ = \frac{\mu_+}{\mu_+ + \mu_-}, \quad P_- = \frac{\mu_-}{\mu_+ + \mu_-}. \qquad (3)$$

These results are consistent with the intuitive understanding. On the other hand, for $\alpha_+ < \alpha_- < 1$ (Case 1), the probability of finding the $-$ state decays to zero in the long-time limit. By Eq. (A6), the asymptotic behavior of the probability $P_-(t)$ of finding the $-$ state at time t for $\alpha_+ < \alpha_- < 1$ (Case 1) is given by

$$P_-(t) \sim \frac{a_-}{a_+\Gamma(1 + \alpha_+ - \alpha_-)} \frac{1}{t^{\alpha_- - \alpha_+}}. \qquad (4)$$

However, when $\alpha_+ = \alpha_- < 1$ (Case 1), the probabilities converge to a finite value:

$$P_+ = \frac{a_+}{a_+ + a_-}, \quad P_- = \frac{a_-}{a_+ + a_-}. \tag{5}$$

This is because the orders of divergences of the mean duration times for two states are exactly the same. As shown later, the ratio of the occupation time of one of two states to the total measurement time does not go to one or zero but becomes a random variable with a finite variance. Therefore, both the probabilities P_+ and P_- are finite. For $\alpha_+ < 1$ and $1 < \alpha_-$ (Case 4), the probability of finding a $-$ state also decays to zero in the long-time limit. By Eq. (A6), the asymptotic behavior is given by

$$P_-(t) \sim \frac{\mu_-}{a_+\Gamma(\alpha)}\frac{1}{t^{1-\alpha}}. \tag{6}$$

Therefore, the probability of finding a $-$ state is decreasing as time goes on. This is a mechanism of subdiffusion in the Langevin equation with dichotomously fluctuating diffusivity as will be shown later.

2. Asymptotic behavior of the forward recurrence time distribution

For Cases 1 and 4, the double Laplace transform of the forward recurrence time distribution for $s \ll 1$ and $u \ll 1$ with $s/u = O(1)$ becomes

$$\hat{f}_E(u;s) \sim \frac{u^\alpha - s^\alpha}{(u-s)s^\alpha}, \tag{7}$$

which is exactly the same as in the case of $\rho_+(x) = \rho_-(x)$ [1]. Using an inversion method of Ref. [1], we have the PDF of E_t/t ($E_t \gg 1$ and $t \gg 1$) for Cases 1 and 4:

$$\lim_{t\to\infty} f_{E_t/t}(x) = \frac{\sin\pi\alpha}{\pi}\frac{1}{x^\alpha(1+x)}, \tag{8}$$

which is the same as that in the ordinary renewal process. For Cases 2 and 3, i.e., $\hat{\rho}_\pm(s) = 1 - \mu_\pm s + o(s)$, in the long-time limit ($t \to \infty$), the Laplace transform of the PDF of E_t reads

$$\lim_{t\to\infty} \hat{f}_E(u;t) = \lim_{s\to 0} s\hat{f}_E(u;s) = \frac{2 - \hat{\rho}_+(u) - \hat{\rho}_-(u)}{(\mu_+ + \mu_-)u} \tag{9}$$

$$= P_+\hat{f}_{E,+}(u) + P_-\hat{f}_{E,-}(u), \tag{10}$$

where

$$\hat{f}_{E,+}(u) = \frac{1-\hat{\rho}_+(u)}{\mu_+ u}, \quad \hat{f}_{E,-}(u) = \frac{1-\hat{\rho}_-(u)}{\mu_- u}. \tag{11}$$

By the inverse Laplace transformation, we have the PDF of E_t in the long-time limit ($t \to \infty$) for Cases 2 and 3:

$$f_E(x) = \frac{1}{\mu_+ + \mu_-}\left[\int_x^\infty \rho_+(\tau)d\tau + \int_x^\infty \rho_-(\tau)d\tau\right]. \tag{12}$$

This is a simple extension of the forward recurrence time distribution in renewal processes, which is a weighted sum of the forward recurrence time distributions for $+$ and $-$ states. When the PDF of the duration time for the first renewal is given by Eq. (12), this process is called an *equilibrium alternating renewal process* [6]. In the equilibrium alternating renewal process, the probability of finding $\sigma(0) = 1$ is given by P_+.

The mean of E_t in the limit $t \to \infty$ diverges in Case 2, whereas the mean duration time is finite. Here, we calculate a long-time behavior of the mean of E_t. The Laplace transform of $\langle E_t\rangle$ with respect to t is given by

$$\mathcal{L}(\langle E_t\rangle) \equiv \int_0^\infty dt\, e^{-st}\langle E_t\rangle = -\left.\frac{\partial \hat{f}_E(u;s)}{\partial u}\right|_{u=0}, \tag{13}$$

which becomes

$$\mathcal{L}(\langle E_t\rangle) = \frac{2 - \rho_+(s) - \rho_-(s) - (\mu_+ + \mu_-)s}{s^2\{1 - \rho_+(s)\rho_-(s)\}}. \tag{14}$$

In the case of $1 < \alpha_+ = \alpha_- < 2$ (Case 2), we have

$$\int_0^\infty dt\, e^{-st}\langle E_t\rangle \sim \frac{a_+ + a_-}{\mu_+ + \mu_-}\frac{1}{s^{3-\alpha}} \quad (s \to 0). \tag{15}$$

By the inverse Laplace transform, the asymptotic behavior of $\langle E_t\rangle$ for $1 < \alpha_+ = \alpha_- < 2$ (Case 2) becomes

$$\langle E_t\rangle \sim \frac{a_+ + a_-}{(\mu_+ + \mu_-)\Gamma(3-\alpha)}t^{2-\alpha} \quad (t \to \infty). \tag{16}$$

B. Backward recurrence time distribution

Here we calculate the backward recurrence time distribution, which is almost the same as the calculation of the forward recurrence time distribution. For $\alpha > 1$, as shown in Appendix B, the backward recurrence time distribution is the same as the forward recurrence time distribution in the long-time limit ($t \to \infty$). For $\alpha < 1$ (Cases 1 and 4), the inverse Laplace transform of Eq. (B8) by the method in Ref. [1] yields

$$\lim_{t\to\infty} f_{B_t/t}(x) = \frac{\sin\pi\alpha}{\pi}x^{-\alpha}(1-x)^{\alpha-1}, \tag{17}$$

where $B_t \gg 1$ and $t \gg 1$. This result is the same as that in the ordinary renewal process.

C. Distribution of the time interval straddling *t*

Here we calculate the distribution of the time interval straddling t, i.e., τ_t [52]. Counterintuitively, as shown in Appendix C, this distribution is not the same as $P_+\rho_+(x) + P_-\rho_-(x)$. The inverse Laplace transform of Eq. (C6) by the inversion method in Ref. [1] yields

$$\lim_{t\to\infty} f_{\tau_t/t}(x) = \begin{cases} \frac{\sin\pi\alpha}{\pi x^{1+\alpha}}[1-(1-x)^{\alpha-1}] & (x < 1), \\ \frac{\sin\pi\alpha}{\pi x^{1+\alpha}}[1-2(x-1)^{\alpha-1}\cos\pi\alpha] & (x < 1). \end{cases} \tag{18}$$

This distribution plays an important role in obtaining the scaling function of the propagator in the subrecoil laser cooling [53] as well as the initial distribution of the diffusivity in the annealed transit time model [41].

IV. MOMENTS OF THE NUMBER OF RENEWALS

Here, we consider the moments of the number of renewals in the time interval $(0, t)$, i.e., N_t.

A. First moment

The renewal function, which is the mean of N_t, can be obtained as

$$H(t) \equiv \langle N_t \rangle = \sum_{r=0}^{\infty} r \Pr(N_t = r) = \sum_{r=1}^{\infty} \Pr(t_r < t). \quad (19)$$

Taking the Laplace transform with respect to t yields

$$\hat{H}(s) = \frac{1}{s} \sum_{r=1}^{\infty} \hat{\rho}_1(s) \cdots \hat{\rho}_r(s), \quad (20)$$

where $\hat{\rho}_r(s)$ is the Laplace transform for the rth duration-time PDF. In what follows, we consider three alternating renewal processes: equilibrium, ordinary, and aging alternating renewal processes. In the ordinary alternating renewal process, $\rho_1(x)$ is the same as $\rho_+(x)$, where we assume that the initial state is $\sigma(0) = 1$. In the equilibrium alternating renewal process, $\rho_1(x)$ is given by Eq. (12) and the probabilities finding $\pm$ at $t = 0$ are given by Eq. (3). We note that the equilibrium alternating renewal process exists only if $\alpha > 1$. For $\alpha \leqslant 1$, there is no equilibrium distribution in the forward recurrence time. In this case, the statistical properties in the time interval $[t_a, t_a + t]$ explicitly depend on the aging time t_a [1,54–57]. This process is called the *aging alternating renewal process*.

1. Ordinary alternating renewal process

The asymptotic behaviors of the renewal function can be obtained by the Tauberian theorem of the inverse Laplace transform. The asymptotic behavior of the Laplace transform of the renewal function is given in Appendix D. Using the Tauberian theorem, we have the asymptotic behaviors of the renewal function:

$$H(t) = \begin{cases} \frac{2t}{\mu_+ + \mu_-} + \frac{\sigma_+^2 + \sigma_-^2 - \mu_-^2 - \mu_+\mu_-}{(\mu_+ + \mu_-)^2} + o(1) & (2 < \alpha,\ \text{Case 3}), \\ \frac{2t}{\mu_+ + \mu_-} + \frac{a_+}{(\mu_+ + \mu_-)^2 \Gamma(3-\alpha)} t^{2-\alpha} + o(t^{2-\alpha}) & (1 < \alpha < 2,\ \text{Case 2}), \\ \frac{2}{(a_+ + a_-)\Gamma(\alpha+1)} t^{\alpha} + o(t^{\alpha}) & (\alpha_+ = \alpha_- < 1,\ \text{Case 1}), \\ \frac{2}{a_+ \Gamma(\alpha+1)} t^{\alpha} + o(t^{\alpha}) & (\alpha_+ < \alpha_- < 1,\ \text{Cases 1 and 4}). \end{cases} \quad (21)$$

When the mean duration time diverges, the renewal function increases sublinearly in the asymptotic behavior. This is a mechanism of subdiffusion in the continuous-time random walk (CTRW) because the mean square displacement (MSD) is proportional to the renewal function in the CTRW [58,59].

2. Equilibrium alternating renewal process

In the equilibrium renewal process ($\alpha > 1$), the PDF of the first renewal time PDF is given by Eq. (12). More precisely, if the initial state is $+$, the PDF of the first renewal time is given by

$$f_+(\tau) = \frac{1}{\mu_+} \int_{\tau}^{\infty} \rho_+(x) dx, \quad (22)$$

and

$$f_-(\tau) = \frac{1}{\mu_-} \int_{\tau}^{\infty} \rho_+(x) dx \quad (23)$$

otherwise. In the equilibrium process, the Laplace transform of the renewal function is exactly obtained as

$$\hat{H}(s) = \frac{2}{(\mu_+ + \mu_-)s^2}. \quad (24)$$

By the inverse Laplace transform, the renewal function for Cases 2 and 3 becomes

$$H(t) = \frac{2}{\mu_+ + \mu_-} t. \quad (25)$$

Therefore, the renewal function can be represented by the mean duration time $\mu = (\mu_+ + \mu_-)/2$, i.e., $H(t) = t/\mu$, which is consistent with the intuition.

3. Aging alternating renewal process

For $\alpha < 1$ (Cases 1 and 4), there is no equilibrium distribution in the forward recurrence time. As a result, the forward recurrence time distribution explicitly depends on the elapsed time t_a (aging time) of the system, where the ordinary alternating renewal process is assumed at time $t = 0$. By Eq. (8), the asymptotic behavior of the forward recurrence time distribution for $t \gg 1$ becomes

$$f_{E_t}(x) \sim \frac{\sin \pi\alpha}{\pi} \frac{t^{\alpha}}{x^{\alpha}(t + x)}. \quad (26)$$

The asymptotic behavior of the double Laplace transform of the renewal function $H(t; t_a)$, which is the mean number of renewals in $[t_a, t_a + t]$, with respect to t and the aging time t_a ($t_a \leftrightarrow u$ and $t \leftrightarrow s$), is approximately given by

$$\hat{H}(s; u) \cong \frac{2\hat{f}_E(s; u)}{s[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)]}. \quad (27)$$

For $\alpha_+ < \alpha_- < 1$ (Cases 1 and 4), the leading order becomes

$$\hat{H}(s; u) \sim \begin{cases} \frac{2}{a_+ s^{1+\alpha} u} & (s \ll u), \\ \frac{2}{a_+ s^2 u^{\alpha}} & (s \gg u). \end{cases} \quad (28)$$

By the inverse Laplace transform, the asymptotic behavior of $H(t; t_a)$ for $\alpha_+ < \alpha_- < 1$ (Cases 1 and 4) becomes

$$H(t; t_a) \sim \begin{cases} \frac{2t^{\alpha}}{a_+ \Gamma(1+\alpha)} & (t \gg t_a), \\ \frac{2t t_a^{\alpha-1}}{a_+ \Gamma(\alpha)} & (t \ll t_a). \end{cases} \quad (29)$$

This explicit dependence of aging time t_a is a mechanism of the aging of the MSD in the CTRW [60,61].

B. Second moment

The second moment of N_t is also obtained as

$$H_2(t) \equiv \langle N_t^2 \rangle = \sum_{r=0}^{\infty} r^2 \Pr(N_t = r) \tag{30}$$

$$= \sum_{r=1}^{\infty} (2r-1) \Pr(t_r < t). \tag{31}$$

Taking the Laplace transform with respect to t yields

$$\hat{H}_2(s) = \frac{1}{s} \sum_{r=1}^{\infty} (2r-1) \hat{\rho}_1(s) \cdots \hat{\rho}_r(s). \tag{32}$$

1. Ordinary alternating renewal process

The asymptotic behavior of the Laplace transform of the second moment of N_t in the ordinary alternating renewal process is given in Appendix D. Using the Tauberian theorem, we have the asymptotic behaviors of the second moment of N_t:

$$H_2(t) = \begin{cases} \frac{4t^2}{(\mu_+ + \mu_-)^2} + \frac{8(\sigma_+^2 + \sigma_-^2) - 4\mu_+(\mu_+ + \mu_-)}{(\mu_+ + \mu_-)^3} t + o(t) & (2 < \alpha,\ \text{Case 3}), \\ \frac{4t^2}{(\mu_+ + \mu_-)^2} + \frac{16 a_+}{(\mu_+ + \mu_-)^3 \Gamma(4-\alpha)} t^{3-\alpha} + o(t^{3-\alpha}) & (1 < \alpha < 2,\ \text{Case 2}), \\ \frac{8}{a_+^2 \Gamma(2\alpha+1)} t^{2\alpha} & (\alpha_+ < \alpha_- < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{33}$$

It follows that the variance of N_t is given by

$$\mathrm{Var}(N_t) \sim \begin{cases} \frac{4(\sigma_+^2 + \sigma_-^2) - 4(\mu_+^2 - \mu_-^2)}{(\mu_+ + \mu_-)^3} t & (2 < \alpha,\ \text{Case 3}), \\ \frac{4 a_+ (1+\alpha)}{(\mu_+ + \mu_-)^3 \Gamma(4-\alpha)} t^{3-\alpha} & (1 < \alpha < 2,\ \text{Case 2}), \\ \frac{4[2\Gamma(\alpha+1)^2 - \Gamma(2\alpha+1)]}{a_+^2 \Gamma(2\alpha+1)\Gamma(\alpha+1)^2} t^{2\alpha} & (\alpha_+ < \alpha_- < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{34}$$

For $1 < \alpha < 2$, the variance of N_t increases as $t^{3-\alpha}$, where $3 - \alpha > 1$. Therefore, the variance grows faster than that for $\alpha > 2$. This is a mechanism of the field-induced superdiffusion because the variance of the displacement is proportional to the variance of N_t [62–64].

2. Equilibrium alternating renewal process

The asymptotic behavior of the Laplace transform of the second moment of N_t in the equilibrium alternating renewal process is given in Appendix D. Using the Tauberian theorem, we have the asymptotic behaviors of the second moment of N_t:

$$H_2(t) = \begin{cases} \frac{4t^2}{(\mu_+ + \mu_-)^2} + \frac{4(\sigma_+^2 + \sigma_-^2)}{(\mu_+ + \mu_-)^3} t + o(t) & (2 < \alpha,\ \text{Case 3}), \\ \frac{4t^2}{(\mu_+ + \mu_-)^2} + \frac{12 a_+}{(\mu_+ + \mu_-)^3 \Gamma(4-\alpha)} t^{3-\alpha} + o(t^{3-\alpha}) & (1 < \alpha < 2,\ \text{Case 2}). \end{cases} \tag{35}$$

It follows that the variance of N_t is given by

$$\mathrm{Var}(N_t) \sim \begin{cases} \frac{4(\sigma_+^2 + \sigma_-^2)}{(\mu_+ + \mu_-)^3} t & (2 < \alpha,\ \text{Case 3}), \\ \frac{12 a_+}{(\mu_+ + \mu_-)^3 \Gamma(4-\alpha)} t^{3-\alpha} & (2 < \alpha,\ \text{Case 2}). \end{cases} \tag{36}$$

The variance of N_t increases as $t^{3-\alpha}$ for $1 < \alpha < 2$, which is the same as that in the ordinary alternating renewal process. However, the coefficient of the variance is different from that in the ordinary alternating renewal process. This phenomena is also observed in the Lévy walk model of superdiffusion [16,65].

3. Aging alternating renewal process

The asymptotic behavior of the Laplace transform of the second moment of N_t in the aging alternating renewal process is given in Appendix D. The inverse Laplace transform yields

$$H_2(t; t_a) \sim \begin{cases} \frac{8 t^{2\alpha}}{a_+^2 \Gamma(1+2\alpha)} & (t \gg t_a \gg 1,\ \text{Cases 1 and 4}), \\ \frac{8 t^{1+\alpha} t_a^{\alpha-1}}{a_+^2 \Gamma(\alpha)\Gamma(2+\alpha)} & (1 \ll t \ll t_a,\ \text{Cases 1 and 4}). \end{cases} \tag{37}$$

When aging time t_a is much larger than time t, the second moment explicitly depends on aging time t_a, i.e., $t_a^{\alpha-1}$. We note that this dependence of t_a is the same as the renewal function in the aging alternating renewal process.

C. Asymptotic behaviors of higher moments

The nth moment of N_t is also obtained as

$$H_n(t) \equiv \langle N_t^n \rangle = \sum_{r=0}^{\infty} r^n \Pr(N_t = r)$$

$$= \sum_{r=1}^{\infty} \{r^n - (r-1)^n\} \Pr(t_r < t). \tag{38}$$

The asymptotic behavior of the Laplace transform with respect to t for $s \to 0$ becomes

$$\hat{H}_n(s) \sim \frac{n}{s}\sum_{r=1}^{\infty} r^{n-1}\hat{\rho}_1(s)\cdots\hat{\rho}_r(s). \tag{39}$$

1. Ordinary alternating renewal process

In the long-time limit, the nth moment of N_t becomes

$$H_n(t) = \begin{cases} \left(\frac{t}{\mu}\right)^n & (1 < \alpha,\ \text{Cases 1 and 4}), \\ \frac{2^n n!}{a_+^n \Gamma(n\alpha+1)} t^{n\alpha} & (\alpha_+ < \alpha_- < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{40}$$

2. Equilibrium alternating renewal process

In the equilibrium renewal process ($\alpha > 1$), the asymptotic behaviors of the higher moments of N_t are the same as Eq. (D9). Thus, for Cases 2 and 3, we have

$$H_n(t) \sim \left(\frac{t}{\mu}\right)^n. \tag{41}$$

3. Aging alternating renewal process

The asymptotic behaviors of the Laplace transform of the higher moments of N_t in the aging alternating renewal process are given in Appendix D. The inverse Laplace transform yields

$$H_n(t;t_a) \sim \begin{cases} \frac{2^n n! t^{n\alpha}}{a_+^n \Gamma(1+n\alpha)} & (t \gg t_a), \\ \frac{2^n n! t^{1+n\alpha} t_a^{\alpha-1}}{a_+^n \Gamma(\alpha)\Gamma(2+n\alpha)} & (t \ll t_a). \end{cases} \tag{42}$$

The dependences of all the higher moments on aging time t_a are given by $t_a^{\alpha-1}$ when $t \ll t_a$.

V. OCCUPATION TIME STATISTICS

Here, we consider the limit distribution of occupation times T_t^+ for $t \to \infty$ and the moments of T_t^+ as a function of time t. We denote the joint probability distribution of $T_t^+ = y$ and $N_t = n$ with $\sigma(0) = \pm 1$ by

$$g_n^{\pm}(y;t) = \langle \delta(y - T_t^+) I(t_n \leqslant t < t_{n+1})\rangle. \tag{43}$$

A. Fluctuations of T_t^+/t

1. Ordinary alternating renewal process

Here, we consider the distribution of T_t^+/t for an ordinary alternating renewal process. The double Laplace transform of the PDF of T_t is given by Eq. (E9), which shows that it does not depend on the initial condition. By the method of the inverse Laplace transform given in Appendix B in [1], the PDF $\tilde{g}(x)$ of $x = T_t^+/t$ for $\alpha = \alpha_+ = \alpha_- < 1$ (Case 1) in the long-time limit becomes

$$\lim_{t\to\infty} \tilde{g}(x) = \frac{a \sin \pi\alpha}{\pi} \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{a^2 x^{2\alpha} + 2a \cos \pi\alpha (1-x)^{\alpha} x^{\alpha} + (1-x)^{2\alpha}}, \tag{44}$$

where $a = a_-/a_+$. This is exactly the same as Lamperti's generalized arcsine law [66]. Counterintuitively, the ratio of occupation time in the positive side T_t^+/t does not converge to a constant but remains random for $\alpha < 1$ even in the long-time limit. For $\alpha > 1$ (Cases 2 and 3), on the other hand, T_t^+/t converges to $\mu_+/(\mu_+ + \mu_-)$ for $t \to \infty$.

2. Equilibrium alternating renewal process

We consider fluctuations of T_t^+ in an equilibrium alternating renewal process for $\alpha > 1$ (Cases 2 and 3). The Laplace transform of the PDF of T_t^+ is given by Eq. (E15). Thus, for $\alpha > 1$ (Cases 2 and 3), the Laplace transform of the PDF of T_t^+ in the asymptotic limit ($s < u \ll 1$) becomes

$$\hat{g}(u;s) \sim \frac{1}{1 - \hat{\rho}_+(s+u)\hat{\rho}_-(s)} \left[\frac{1 - \hat{\rho}_+(s+u)}{s+u} + \frac{1 - \hat{\rho}_-(s)}{s}\right], \tag{45}$$

which is the same as that for the ordinary alternating renewal process.

3. Aging alternating renewal process

Here, we consider a particular case of $\alpha = \alpha_- = \alpha_+$ (Case 1) and occupation time of + state in $[t_a, t_a + t]$ denoted by $T_{t_a,t}^+$. Because there is no equilibrium distribution for the forward recurrence time in Case 1, the occupation time intrinsically depends on time t_a (*aging*). This aging extension of the generalized arcsine law was established in Ref. [67]. The PDF of $y = T_{t_a,t}^+$ denoted by $g(y;t,t_a)$ is given by

$$g(y;t,t_a) = P_+(t_a) g^+(y;t,t_a) + P_-(t_a) g^-(y;t,t_a). \tag{46}$$

The Laplace transform of $g(y;t,t_a)$ with respect to y, t, and t_a ($y \leftrightarrow u$, $t \leftrightarrow s$, and $t_a \leftrightarrow v$) is given by Eq. (E17). In the asymptotic limit ($u, s \ll 1$), we have

$$\hat{g}(u;s,v) \sim \frac{\hat{f}_{E,+}(s+u,v) + \hat{f}_{E,-}(s,v)}{1 - \hat{\rho}_+(s+u)\hat{\rho}_-(s)} \left[\frac{1 - \hat{\rho}_+(s+u)}{s+u} + \frac{1 - \hat{\rho}_-(s)}{s}\right]. \tag{47}$$

For $u, s \ll v$, we obtain $\hat{g}(u;s,v) \sim \hat{g}(u;s)/v$. Therefore, there is no explicit dependence of the distribution on t_a for $u, s \ll v$, i.e., $t_a \ll t, y$. On the other hand, there is an explicit dependence of the distribution on t_a for $u, s \gg v$, i.e., $t_a \gg t$.

B. First moment of T_t^+

Here, we consider the first moment of T_t^+.

1. Ordinary alternating renewal process

The Laplace transform of the first moment of T_t^+ with respect to t is obtained from Eq. (E6) and given by Eq. (F3). By the Tauberian theorem of the inverse Laplace transform, we have in the long-time limit

$$\langle T_t^+\rangle = \begin{cases} \frac{\mu_+}{\mu_++\mu_-}t - \frac{\sigma_+^2\mu_- - \sigma_-^2\mu_+ - \mu_+\mu_-(\mu_++\mu_-)}{2(\mu_++\mu_-)^2} & (2<\alpha,\ \text{Case 3}),\\ \frac{\mu_+}{\mu_++\mu_-}t - \frac{a_+\mu_-}{\Gamma(3-\alpha)(\mu_++\mu_-)^2}t^{2-\alpha} + o(t^{2-\alpha}) & (1<\alpha<2,\ \text{Case 2}),\\ t + o(t) & (\alpha<1,\ \text{Cases 1 and 4}). \end{cases} \tag{48}$$

2. Equilibrium alternating renewal process

The Laplace transform of the first moment of T_t^+ in an equilibrium alternating renewal process is obtained from Eq. (E15):

$$\hat{T}_1(s) = \frac{\mu_+}{(\mu_+ + \mu_-)s^2}. \tag{49}$$

Therefore, for Cases 2 and 3, we have in the equilibrium alternating renewal process

$$\langle T_t^+\rangle = \frac{\mu_+}{\mu_+ + \mu_-}t. \tag{50}$$

3. Aging alternating renewal process

The asymptotic behavior of $\hat{T}_1(s)$ in an aging alternating renewal process ($\alpha < 1$) is obtained from Eq. (47). For $u, s \ll v$, the result is the same as that for the ordinary alternating renewal process, i.e., $\langle T_t^+\rangle \sim t$. Moreover, for $u, s \gg v$, i.e., $t \ll t_a$, we have the same result, i.e., $\langle T_t^+\rangle \sim t$.

C. Second moment

Here, we consider the second moment of T_t^+.

1. Ordinary alternating renewal process

The Laplace transform of the second moment of T_t^+ is given by Eq. (F6). By the Tauberian theorem of the inverse Laplace transform, we have, in the long-time limit,

$$\langle (T_t^+)^2\rangle = \begin{cases} \left(\frac{\mu_+}{\mu_++\mu_-}t\right)^2 + \frac{\sigma_+^2\mu_-(\mu_--\mu_+)+2\sigma_-^2\mu_+^2+\mu_+^2\mu_-(\mu_++5\mu_-)}{(\mu_++\mu_-)^3}t & (2<\alpha,\ \text{Case 3}),\\ \left(\frac{\mu_+}{\mu_++\mu_-}t\right)^2 - \frac{4a_+\mu_+\mu_-}{\Gamma(4-\alpha)(\mu_++\mu_-)^3}t^{3-\alpha} + o(t^{3-\alpha}) & (1<\alpha<2,\ \text{Case 2}). \end{cases} \tag{51}$$

It follows that the relative standard deviation of T_t^+ is given by

$$\frac{\sqrt{\langle (T_t^+)^2\rangle - \langle T_t^+\rangle^2}}{\langle T_t^+\rangle} \sim \begin{cases} \sqrt{\frac{4\mu_-^2+\sigma_-^2}{\mu_++\mu_-}}t^{-\frac{1}{2}} & (2<\alpha,\ \text{Case 3}),\\ \sqrt{\frac{2a_+\tilde{\mu}^2}{\Gamma(4-\alpha)(\mu_++\mu_-)}}t^{-\frac{\alpha-1}{2}} & (1<\alpha<2,\ \text{Case 2}), \end{cases} \tag{52}$$

where $\tilde{\mu} = \mu_-/\mu_+$. For $1 < \alpha < 2$, the relaxation becomes slower than that for $\alpha > 2$. This slow relaxation is observed in diffusion of lipid molecules [68].

2. Equilibrium alternating renewal process

The Laplace transform of the second moment of T_t^+ is given by Eq. (F7). By the Tauberian theorem of the inverse Laplace transform, we have, in the long-time limit,

$$\langle (T_t^+)^2\rangle = \begin{cases} \left(\frac{\mu_+}{\mu_++\mu_-}t\right)^2 + \frac{4P_-\mu_+\mu_-(\sigma_+^2+\mu_+^2)+2\mu_+^2(\mu_+\mu_-+\sigma_-^2)}{(\mu_++\mu_-)^3}t & (2<\alpha,\ \text{Case 3}),\\ \left(\frac{\mu_+}{\mu_++\mu_-}t\right)^2 - \frac{a_+\mu_-^2}{\Gamma(3-\alpha)(\mu_++\mu_-)^3}t^{3-\alpha} + o(t^{3-\alpha}) & (1<\alpha<2,\ \text{Case 2}). \end{cases} \tag{53}$$

It follows that the relative standard deviation of T_t^+ is given by

$$\frac{\sqrt{\langle (T_t^+)^2\rangle - \langle T_t^+\rangle^2}}{\langle T_t^+\rangle} \sim \begin{cases} \sqrt{\frac{4P_-\tilde{\mu}(\sigma_+^2+\mu_+^2)+2(\mu_+\mu_-+\sigma_-^2)}{\mu_++\mu_-}}t^{-\frac{1}{2}} & (2<\alpha,\ \text{Case 3}),\\ \sqrt{\frac{a_+\tilde{\mu}^2}{\Gamma(3-\alpha)(\mu_++\mu_-)}}t^{-\frac{\alpha-1}{2}} & (1<\alpha<2,\ \text{Case 2}). \end{cases} \tag{54}$$

VI. ERGODIC PROPERTIES

Here, we discuss the ergodic properties for alternating renewal processes. Time average of an observable $f(x)$ in an alternating renewal process is defined by

$$\overline{f(t)} \equiv \frac{1}{t}\int_0^t f(\sigma(t'))dt'. \tag{55}$$

If the system is ergodic, time averages converge to the ensemble average in the long-time limit:

$$\overline{f(t)} \to \langle f(x)\rangle_{\rm eq} \quad \text{for } t \to \infty \tag{56}$$

for all trajectories $\sigma(t)$, where $\langle\cdot\rangle_{\rm eq}$ is the equilibrium ensemble average.

A. Number of renewals

We consider the time average of the number of renewals, i.e., N_t/t. If the system is ergodic,

$$\frac{N_t}{t} \to \lambda_{\rm eq} \quad \text{for } t \to \infty \tag{57}$$

for all paths N_t, where $\lambda_{\rm eq}$ is the equilibrium jump rate. For $\alpha > 1$ (Cases 2 and 3), by Eq. (21), we have $\langle N_t/t\rangle \to 1/\mu$ for $t \to \infty$. Moreover, by Eq. (34), the variance of N_t/t becomes zero in the long-time limit: $\langle (N_t/t)^2\rangle - \langle N_t/t\rangle^2 \to 0$ for $t \to \infty$. Therefore, alternating renewal processes with $\alpha > 1$ are ergodic in the sense that Eq. (57) holds with $\lambda_{\rm eq} = 1/\mu$.

For $\alpha < 1$ (Cases 1 and 4), the ergodic properties become different from those for $\alpha > 1$. By Eq. (21), we have

$$\langle N_t/t^\alpha\rangle \to \frac{2}{a_+\Gamma(1+\alpha)} \tag{58}$$

for $t \to \infty$. Moreover, by Eq. (34), the variance of N_t/t becomes zero in the long-time limit. However, the variance of the scaled time average, i.e., N_t/t^α, does not go to zero but converges to a finite value:

$$\langle (N_t/t^\alpha)^2\rangle - \langle N_t/t^\alpha\rangle^2 \to \frac{4[2\Gamma(\alpha+1)^2 - \Gamma(2\alpha+1)]}{a_+^2\Gamma(2\alpha+1)\Gamma(\alpha+1)^2} \tag{59}$$

for $t \to \infty$. Therefore, N_t/t^α does not converge to a constant even in the long-time limit but remains random, which means that alternating renewal processes with $\alpha < 1$ are not ergodic. Using the results for the higher moments, i.e., Eq. (40), we have the asymptotic behavior of the nth moment of $N_t/\langle N_t\rangle$. In particular, the nth moment of $N_t/\langle N_t\rangle$ converges to $n!\Gamma(1+\alpha)^n/\Gamma(n\alpha+1)$ for $t \to \infty$. This is the nth moment of the Mittag-Leffler distribution. In other words, $N_t/\langle N_t\rangle$ shows trajectory-to-trajectory fluctuations and the distribution of $N_t/\langle N_t\rangle$ converges to the Mittag-Leffler distribution. This distribution appears in stochastic processes [29,41,45,61] as well as the infinite ergodic theory [47,69].

B. Occupation times

We consider the time average of an occupation time, i.e., $f(x) = I_{(0,\infty)}(x)$, where $I_A(x)$ is the indicator function of A. If the system is ergodic,

$$\frac{1}{t}\int_0^t I_{(0,\infty)}(\sigma(t'))dt' \to \langle I_A(x)\rangle_{\rm eq} \quad \text{for } t \to \infty \tag{60}$$

for all trajectories $\sigma(t)$. The left-hand side is the ratio of the occupation time, i.e., T_t^+/t. For $\alpha > 1$ (Cases 2 and 3), by Eq. (48), we have $\langle T_t^+/t\rangle \to \mu_+/(\mu_+ + \mu_-)$ for $t \to \infty$. Moreover, by Eq. (51), the variance of T_t^+/t becomes zero in the long-time limit: $\langle (T_t^+/t)^2\rangle - \langle T_t^+/t\rangle^2 \to 0$ for $t \to \infty$. Therefore, alternating renewal processes with $\alpha > 1$ are ergodic in the sense that T_t^+/t converges to $\mu_+/(\mu_+ + \mu_-)$ in the long-time limit. Although alternating renewal processes with $\alpha > 1$ are ergodic, alternating renewal processes with $1 < \alpha < 2$ (Case 2) exhibits a slow relaxation, i.e., $\langle (T_t^+/t)^2\rangle - \langle T_t^+/t\rangle^2 \propto t^{-\frac{\alpha-1}{2}}$ for $t \to \infty$.

For $\alpha_+ = \alpha_- < 1$ (Case 1), the ergodic properties are completely different from those for $\alpha > 1$. As shown in Sec. V A 1, T_t^+/t does not converge to a constant but exhibits trajectory-to-trajectory fluctuations. The distribution of T_t^+/t converges to the generalized arcsine distribution in the long-time limit. Therefore, ergodicity of the alternating renewal process breaks down. We note $T_t^+/t \to 1$ for $t \to \infty$ for $\alpha_+ < \alpha_- < 1$ (Case 1) and $\alpha < 1, \alpha_- > 1$ (Case 4).

VII. CORRELATION FUNCTION

We consider the correlation function of $D(t)$, which is defined by $D(t) = D_+$ or D_- when $\sigma(t) = +1$ or $\sigma(t) = +1$, respectively. If there exists an equilibrium distribution for $D(t)$, the correlation function is defined by

$$C(t) \equiv \langle D(0)D(t)\rangle - \langle D(0)\rangle\langle D\rangle_{\rm eq}, \tag{61}$$

where $\langle D\rangle_{\rm eq}$ is the equilibrium ensemble average of D, which is given by

$$\langle D\rangle_{\rm eq} = \frac{D_+\mu_+ + D_-\mu_-}{\mu_+ + \mu_-}. \tag{62}$$

In this section, we consider the correlation function for the ordinary and the equilibrium alternating process and assume that there exists an equilibrium ensemble average, i.e., $\langle D\rangle_{\rm eq}$.

1. Ordinary alternating renewal process

In the ordinary alternating renewal process, we assume $D(0) = D_+$. If there exists an equilibrium distribution, i.e., $\alpha > 1$, the correlation function is represented as

$$C(t) = D_+^2 W_{++}(t) + D_+D_-W_{+-}(t) - D_+\langle D\rangle_{\rm eq}, \tag{63}$$

where $W_{++}(t) = \Pr\{D(t) = D_+|D(0) = D_+\}$ and $W_{+-}(t) = \Pr\{D(t) = D_-|D(0) = D_+\}$ are the transition probabilities from $D(0) = D_+$ to $D(t) = D_+$ and D_-, respectively. In the long-time limit, the transition probabilities become $W_{++}(t) \to P_+$ and $W_{+-}(t) \to P_-$ for $t \to \infty$.

The Laplace transform of the correlation function is given by Eq. (G8). When both of the duration-time PDFs follow exponential distributions, we have

$$\hat{C}(s) = \frac{D_+(D_+ - D_-)\mu_-}{\mu_+ + \mu_-}\frac{1}{(\mu_+ + \mu_-)/(\mu_+\mu_-) + s}. \tag{64}$$

The inverse Laplace transform yields

$$C(t) = \frac{D_+(D_+ - D_-)\mu_-}{\mu_+ + \mu_-}\exp\left(-\frac{\mu_+ + \mu_-}{\mu_+\mu_-}t\right). \tag{65}$$

For $1 < \alpha_+ < \alpha_- < 2$ (Case 2), we have

$$\hat{C}(s) \sim \frac{a_- D_+ (D_+ - D_-)}{(\mu_+ + \mu_-)^2 s^{2-\alpha_-}} \tag{66}$$

for $s \to 0$. The inverse Laplace transform reads

$$C(t) \sim \frac{a_- D_+ (D_+ - D_-)}{\Gamma(2-\alpha_-)(\mu_+ + \mu_-)^2} t^{-(\alpha_- - 1)} \quad (t \to \infty). \tag{67}$$

The correlation function exhibits a power-law decay and the power-law exponent is given by α_-.

2. Equilibrium alternating renewal process

In the equilibrium alternating renewal process, the correlation function is represented as

$$C(t) = D_+^2 P_+ W_{++}(t) + D_+ D_- P_+ W_{+-}(t) + D_- D_+ P_- W_{-+}(t) + D_-^2 P_- W_{--}(t) - \langle D \rangle_{\rm eq}^2, \tag{68}$$

where $W_{-+}(t) = \Pr\{D(t) = D_+ | D(0) = D_-\}$ and $W_{--}(t) = \Pr\{D(t) = D_- | D(0) = D_-\}$.

The Laplace transform of the correlation function is given by Eq. (G14). When both of the duration-time PDFs follow exponential distributions, we have

$$\hat{C}(s) = \frac{(D_+ - D_-)^2 \mu_+ \mu_-}{(\mu_+ + \mu_-)^2} \frac{1}{(\mu_+ + \mu_-)/(\mu_+ \mu_-) + s}. \tag{69}$$

The inverse Laplace transform yields

$$C(t) = \frac{(D_+ - D_-)^2 \mu_+ \mu_-}{(\mu_+ + \mu_-)^2} \exp\left(-\frac{\mu_+ + \mu_-}{\mu_+ \mu_-} t\right). \tag{70}$$

For $\alpha > 2$ (Case 3), we have

$$\hat{C}(s) = \frac{(D_+ - D_-)^2 (\mu_+ \mu_-)^2}{2(\mu_+ + \mu_-)^3} \left(\frac{\sigma_+^2}{\mu_+^2} + \frac{\sigma_-^2}{\mu_-^2}\right) \tag{71}$$

in the small $s \ll 1$. For $1 < \alpha < 2$ (Case 2), we have

$$\hat{C}(s) \sim \frac{(D_+ - D_-)^2}{(\mu_+ + \mu_-)^2} a_+ \mu_- s^{\alpha - 2} \tag{72}$$

for $s \to 0$. The inverse Laplace transform reads

$$C(t) \sim \frac{a_+ \mu_- (D_+ - D_-)^2}{\Gamma(2-\alpha)(\mu_+ + \mu_-)^2} t^{-(\alpha - 1)} \quad (t \to \infty). \tag{73}$$

The correlation function exhibits a power-law decay in the equilibrium alternating renewal process. Unlike the ordinary alternating renewal process, the power-law exponent is given by $\alpha = \alpha_+$.

VIII. APPLICATION TO LANGEVIN EQUATION WITH ALTERNATELY FLUCTUATING DIFFUSIVITY

Here, we discuss how our results can be applied to experiments and stochastic processes such as the Langevin equation with fluctuating diffusivity. The CTRW is described by the Langevin equation with alternating fluctuating diffusivity [70]. In particular, instantaneous diffusivity $D(t)$ is given by $D(t) = 0$ or 1, which correspond to $\sigma(t) = 1$ or -1, respectively. In this model, the MSD becomes

$$\langle x(t)^2 \rangle = 2 \int_0^t \langle D(t') \rangle dt', \tag{74}$$

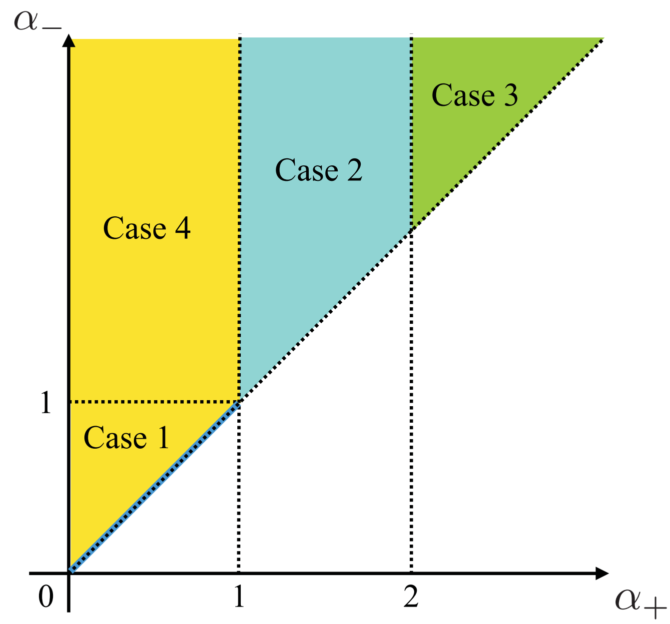

FIG. 2. Phase diagram of the Langevin equation with alternately fluctuating diffusivity. The MSD shows anomalous diffusion and the time-averaged SD is nonergodic in Case 1 with $\alpha_+ = \alpha_-$. The MSD shows normal diffusion and the time-averaged SD is ergodic in Case 2. Moreover, the correlation function and the relative standard deviation of the time-averaged SD exhibits a slow relaxation, i.e., a power-law decay, in Case 2. The MSD shows normal diffusion and the time-averaged SD is ergodic in Case 3. The MSD shows anomalous diffusion and the time-averaged SD is ergodic in Case 1 with $\alpha_+ \neq \alpha_-$ and Case 4.

where $x(t)$ is the displacement with $x(0) = 0$. When the mean duration time of state $D(t) = 1$ is finite, the MSD can be described by the number of changes of states: $\langle x(t)^2 \rangle \sim \mu_- H(t)$. Therefore, the MSD becomes $\langle x(t)^2 \rangle \sim 2\langle D \rangle_{\rm eq}\, t$ in Cases 2 and 3 and exhibits anomalous diffusion: $\langle x(t)^2 \rangle \propto t^\alpha$ in Cases 1 and 4, where $\langle D \rangle_{\rm eq}$ is the ensemble average of D in equilibrium and defined by Eq. (62). In Cases 2 and 3, $\langle D \rangle_{\rm eq}$ becomes $\langle D \rangle_{\rm eq} = \mu_-/(\mu_+ + \mu_-)$. Therefore, some information of μ_+ and μ_- can be obtained from the MSD.

We discuss the ergodic properties of the squared displacement (SD) in the Langevin equation with alternating fluctuating diffusivity, i.e., $D(t) = D_+$ or D_- [40,42,71]. The time-averaged SD can be defined by

$$\overline{\delta^2(\Delta; t)} = \frac{1}{t - \Delta} \int_0^{t-\Delta} dt' [(x(t' + \Delta) - x(t')]^2. \tag{75}$$

If the system is ergodic, time-averaged SD $\overline{\delta^2(\Delta; t)}$ converges to MSD $\langle x(\Delta)^2 \rangle_{\rm eq}$ for all lag times Δ in the long-time limit $(t \to \infty)$, where the equilibrium ensemble average implies the equilibrium alternating renewal process. The time-averaged SD can be represented by occupation time T_t^+:

$$\overline{\delta^2(\Delta; t)} \sim 2\left(D_- + \frac{(D_+ - D_-)T_t^+}{t}\right)\Delta. \tag{76}$$

Therefore, our results in occupation time statistics of T_t^+ can be applied to the time-averaged SD. For Cases 2 and 3, the ensemble average of $\overline{\delta^2(\Delta; t)}$ converges to $2\langle D \rangle_{\rm eq}\, \Delta$. Moreover, the variance of $\overline{\delta^2(\Delta;t)}$ becomes zero for $t \to \infty$ because of the results Eqs. (52) and (54). Therefore, the system is ergodic in the sense that $\overline{\delta^2(\Delta;t)} \to 2\langle D\rangle_{\rm eq}\,\Delta$ for $t \to \infty$. Furthermore, the relative standard deviation of the time-averaged SD or the diffusion coefficient is related to the occupation time statistics and the correlation function. In particular, in Case 2, the relative standard deviation exhibits a power-law decay and the power-law decay is related to the correlation function. For Case 1 with $\alpha_+ = \alpha_-$, T_t^+/t remains random in the long-time limit, which means that the time-averaged SD is also random. Therefore, the system is not ergodic in Case 1 with $\alpha_+ = \alpha_-$. On the other hand, T_t^+/t converges to 1 in the long-time limit for Case 1 with $\alpha_+ \neq \alpha_-$ and 4. Thus the time-averaged SD converges to a constant in the long-time limit, i.e., $\overline{\delta^2(\Delta;t)} \to 2D_+\Delta$ for $t \to \infty$. In this sense, the system is ergodic in Case 1 with $\alpha_+ \neq \alpha_-$ and 4. Phase diagram of the Langevin equation with alternately fluctuating diffusivity based on the diffusivity and the ergodic properties is summarized in Fig. 2.

IX. CONCLUSION

We have investigated the statistics of the number of renewals and the occupation time statistics in alternating renewal processes. We analytically obtain the recurrence time distributions for the ordinary alternating renewal process and show that there is an equilibrium distribution when the mean duration time exists. The equilibrium distribution in the alternating renewal process is a simple extension of that in the normal renewal process. On the other hand, when the mean duration time diverges, there is no equilibrium distribution for the recurrence time distribution and the system exhibits aging, which is the same as in the normal renewal process. In other words, the recurrence time distribution explicitly depends on the elapsed time of the system, i.e., aging time t_a. Therefore, we have considered the ordinary, equilibrium, and aging alternating renewal processes.

Here, we summarize the results of the statistics of the number of renewals. When both of the duration-time PDFs have finite variance (Case 3), the renewal function and the variance of the number of renewals increase linearly with time for ordinary, equilibrium, and aging alternating renewal processes. When one of the duration-time PDFs follows a power-law distribution with time divergent second moment (Case 2), the renewal function increases linearly with time but the variance of the number of renewals exhibits a power law increasing for the ordinary and equilibrium alternating renewal processes. Moreover, the coefficients of the variances for the ordinary and equilibrium alternating renewal processes do not coincide in Case 2. When the means of duration times diverge (Cases 1 and 4), the renewal function increases sublinearly with time and the distribution of the number of renewals converges to the Mittag-Leffer distribution in the long-time limit for the ordinary alternating renewal process. In aging alternating renewal processes (Cases 1 and 4), the renewal function depends explicitly on the aging time t_a. All the results of the statistics of the number of renewals are basically the same as in the normal renewal processes where a power-law exponent of the duration-time PDF is α.

We compare the results for the occupation time statistics in alternating renewal processes with those in normal renewal processes. When one of the duration-time PDFs in the alternating renewal process follows a power-law distribution with time divergent second moment (Case 2), the relative standard deviation of occupation times as well as the correlation function exhibit a power-law decay. This power-law decay is also observed for the normal renewal process where a power-law exponent of the duration-time PDF is α. When the means of duration times diverge (Cases 1 and 4), the distribution of the ratio of occupation time follows the asymmetric generalized arcsine distribution in the alternating renewal process. On the other hand, the distribution of the ratio of occupation time in the normal renewal process follows the symmetric generalized arcsine distribution.

Finally, we discuss the ergodic properties in the alternating renewal process and apply our theory of the alternating renewal process to the Langevin equation with alternately fluctuating diffusivity. As a result, we obtain the MSD and the time-averaged MSD in the Langevin equation with alternately fluctuating diffusivity. Furthermore, the relative standard deviation of the diffusion coefficient of the time-averaged MSD is also obtained analytically.

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APPENDIX A: LAPLACE TRANSFORM OF FORWARD RECURRENCE TIME DISTRIBUTION

The joint PDF of E_t and $N_t = 2n$ $(n = 0, 1, \ldots)$ for fixed t is represented by

$$f_{E,2n}(E_t;t) = \langle \delta(E_t - t_{2n+1} + t) I(t_{2n} < t < t_{2n+1})\rangle, \quad \text{(A1)}$$

where $\langle\cdot\rangle$ is the expectation value and $I(t_{2n} < t < t_{2n+1})$ is the indicator function, i.e., $I(t_{2n} < t < t_{2n+1}) = 1$ if t satisfies $t_{2n} < t < t_{2n+1}$ and $I(t_{2n} < t < t_{2n+1}) = 0$ otherwise. The double Laplace transform of $f_{2n}(E_t;t)$ with respect to E_t and t is defined by

$$\hat{f}_{E,2n}(u;s) \equiv \left\langle \int_0^\infty dE_t e^{-uE_t} \int_0^\infty dt\, e^{-st} \delta(E_t - t_{2n+1} + t) \times I(t_{2n} < t < t_{2n+1})\right\rangle. \quad \text{(A2)}$$

A simple calculation yields

$$\hat{f}_{E,2n}(u;s) = \{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^n \frac{\hat{\rho}_+(u) - \hat{\rho}_+(s)}{s-u}. \quad \text{(A3)}$$

In the same way, we have the double Laplace transform of $f_{E,2n+1}(E_t;t)$ $(n = 0, 1, \ldots)$:

$$\hat{f}_{E,2n+1}(u;s) = \{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^n \hat{\rho}_+(s) \frac{\hat{\rho}_-(u) - \hat{\rho}_-(s)}{s-u}. \quad \text{(A4)}$$

The double Laplace transforms of the PDFs of E_t with final states being $+$ and $-$, denoted by $\hat{f}_{E,+}(u;s)$ and $\hat{f}_{E,-}(u;s)$, are given by $\hat{f}_{E,+}(u;s) = \sum_{n=0}^\infty \hat{f}_{E,2n}(u;s)$ and

$\hat{f}_{E,-}(u;s) = \sum_{n=0}^{\infty} \hat{f}_{E,2n+1}(u;s)$, respectively. Note that we assumed $\sigma(0) = +1$. We obtain

$$\hat{f}_{E,+}(u;s) = \frac{\hat{\rho}_+(u) - \hat{\rho}_+(s)}{(s-u)\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}} \tag{A5}$$

and

$$\hat{f}_{E,-}(u;s) = \frac{\hat{\rho}_+(s)\{\hat{\rho}_-(u) - \hat{\rho}_-(s)\}}{(s-u)\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}}. \tag{A6}$$

The double Laplace transform of the forward recurrence time distribution becomes

$$\hat{f}_E(u;s) = \sum_{n=0}^{\infty} \hat{f}_{E,2n}(u;s) + \sum_{n=0}^{\infty} \hat{f}_{E,2n+1}(u;s) \tag{A7}$$

$$= \frac{\hat{\rho}_+(u) - \hat{\rho}_+(s) + \hat{\rho}_+(s)\{\hat{\rho}_-(u) - \hat{\rho}_-(s)\}}{(s-u)\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}}. \tag{A8}$$

We note that the result with $\rho_+(x) = \rho_-(x)$ is consistent with Ref. [1]. In the same way, we have the double Laplace transform of the forward recurrence time distribution in the case of $\sigma(0) = -1$:

$$\hat{f}_E(u;s) = \frac{\hat{\rho}_-(u) - \hat{\rho}_-(s) + \hat{\rho}_-(s)\{\hat{\rho}_+(u) - \hat{\rho}_+(s)\}}{(s-u)\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}}. \tag{A9}$$

APPENDIX B: LAPLACE TRANSFORM OF FORWARD RECURRENCE TIME DISTRIBUTION

The joint PDF of B_t and $N_t = 2n$ is given by

$$f_{B,2n}(B_t;t) = \langle \delta(B_t - t + t_{2n}) I(t_{2n} < t < t_{2n+1}) \rangle. \tag{B1}$$

The double Laplace transforms of $f_{B,2n}(B_t;t)$ and $f_{B,2n+1}(B_t;t)$ with respect to B_t and t are given by

$$\hat{f}_{B,2n}(u;s) = \{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^n \frac{1 - \hat{\rho}_+(s+u)}{s+u} \tag{B2}$$

and

$$\hat{f}_{B,2n+1}(u;s) = \{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^n \rho_+(s) \frac{1 - \hat{\rho}_-(s+u)}{s+u}. \tag{B3}$$

It follows that the double Laplace transform of the PDF of B_t is given by

$$\hat{f}_B(u;s) = \sum_{n=0}^{\infty} \hat{f}_{B,2n}(u;s) + \sum_{n=0}^{\infty} \hat{f}_{B,2n+1}(u;s) \tag{B4}$$

$$= \frac{1 - \hat{\rho}_+(s+u) + \hat{\rho}_+(s)\{1 - \hat{\rho}_-(s+u)\}}{(s+u)\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}}. \tag{B5}$$

For $\alpha > 1$ (Cases 2 and 3), in the long-time limit ($t \to \infty$), the Laplace transform of the PDF of B_t reads

$$\lim_{t\to\infty} \hat{f}_B(u;t) = \lim_{s\to 0} s\hat{f}_B(u;s) = \frac{2 - \hat{\rho}_+(u) - \hat{\rho}_-(u)}{(\mu_+ + \mu_-)u} \tag{B6}$$

$$= P_+ \hat{f}_{E,+}(u) + P_- \hat{f}_{E,-}(u). \tag{B7}$$

Therefore, the backward recurrence time distribution is the same as the forward recurrence time distribution when $\alpha > 1$. On the other hand, for $\alpha < 1$ (Cases 1 and 4), the Laplace transform for $s \ll 1$ and $u \ll 1$ with $s/u = O(1)$ becomes

$$\hat{f}_B(u;s) \cong s^{-\alpha}(s+u)^{\alpha-1}, \tag{B8}$$

which is exactly the same as in the case of $\rho_+(x) = \rho_-(x)$.

APPENDIX C: LAPLACE TRANSFORM OF DISTRIBUTION OF THE TIME INTERVAL STRADDLING t

The joint PDF of τ_t and $N_t = 2n$ is given by

$$f_{2n}(\tau_t;t) = \langle \delta(\tau_t - t_{2n+1} + t_{2n}) I(t_{2n} < t < t_{2n+1}) \rangle. \tag{C1}$$

The double Laplace transforms of $f_{2n}(\tau_t;t)$ and $f_{2n+1}(\tau_t;t)$ with respect to τ_t and t are given by

$$\hat{f}_{2n}(u;s) = \{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^n \frac{\hat{\rho}_+(u) - \hat{\rho}_+(s+u)}{s} \tag{C2}$$

and

$$\hat{f}_{2n+1}(u;s) = \{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^n \hat{\rho}_+(s) \frac{\hat{\rho}_-(u) - \hat{\rho}_-(s+u)}{s}. \tag{C3}$$

It follows that the double Laplace transform of the PDF of τ_t is given by

$$\hat{f}(u;s) = \frac{\hat{\rho}_+(u) - \hat{\rho}_+(s+u) + \hat{\rho}_+(s)\{\hat{\rho}_-(u) - \hat{\rho}_-(s+u)\}}{s\{1 - \hat{\rho}_+(s)\hat{\rho}_-(s)\}}. \tag{C4}$$

For $\alpha > 1$ (Cases 2 and 3), in the long-time limit ($t \to \infty$), the Laplace transform of the PDF of τ_t reads

$$\lim_{t\to\infty} \hat{f}(u;t) = \lim_{s\to 0} s\hat{f}(u;s) = -\frac{\hat{\rho}_+'(u) + \hat{\rho}_-'(u)}{\mu_+ + \mu_-}. \tag{C5}$$

Therefore, the distribution of the time interval straddling t is not the same as $P_+\rho_+(x) + P_-\rho_-(x)$. For $\alpha < 1$ (Cases 1 and 4), the Laplace transform for $s \ll 1$ and $u \ll 1$ with $s/u = O(1)$ becomes

$$\hat{f}(u;s) \cong \frac{(s+u)^\alpha - s^\alpha}{s^{1+\alpha}}, \tag{C6}$$

which is the same as in the case of $\rho_+(x) = \rho_-(x)$.

APPENDIX D: ASYMPTOTIC BEHAVIORS OF THE LAPLACE TRANSFORMS OF THE MOMENTS OF THE NUMBER OF RENEWALS

1. First moment

a. Ordinary alternating renewal process

In the ordinary alternating renewal process, we have

$$\hat{H}(s) = \frac{\hat{\rho}_+(s)[1 + \hat{\rho}_-(s)]}{s[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)]}. \tag{D1}$$

The leading order is given by

$$\hat{H}(s) = \begin{cases} \frac{2}{(\mu_+ + \mu_-)s^2} + \frac{\sigma_+^2 + \sigma_-^2 - \mu_-^2 - \mu_+\mu_-}{(\mu_+ + \mu_-)^2 s} + o(s^{-1}) & (2 < \alpha,\ \text{Case 3}),\\ \frac{2}{(\mu_+ + \mu_-)s^2} + \frac{a_+}{(\mu_+ + \mu_-)^2 s^{3-\alpha}} + o(s^{-3-\alpha}) & (1 < \alpha < 2,\ \text{Case 2}),\\ \frac{2}{(a_+ + a_-)s^{\alpha+1}} + o(s^{-1-\alpha}) & (\alpha_+ = \alpha_- < 1,\ \text{Case 1}),\\ \frac{2}{a_+ s^{\alpha+1}} + o(s^{-1-\alpha}) & (\alpha_+ < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{D2}$$

2. Second moment

a. Ordinary alternating renewal process

In the ordinary renewal process, we have

$$\hat{H}_2(s) = \frac{\hat{\rho}_+(s)[1 + 3\hat{\rho}_-(s) + 3\hat{\rho}_-(s)\hat{\rho}_+(s) + \hat{\rho}_-(s)^2\hat{\rho}_+(s)]}{s[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)]^2}. \tag{D3}$$

The asymptotic behaviors are given by

$$\hat{H}_2(s) = \begin{cases} \frac{8}{(\mu_+ + \mu_-)^2 s^3} + \frac{8(\sigma_+^2 + \sigma_-^2) - 4\mu_+(\mu_+ + \mu_-)}{(\mu_+ + \mu_-)^3 s^2} + o(s^{-2}) & (2 < \alpha,\ \text{Case 3}),\\ \frac{8}{(\mu_+ + \mu_-)^2 s^3} + \frac{16 a_+}{(\mu_+ + \mu_-)^3 s^{4-\alpha}} + o(s^{\alpha-4}) & (1 < \alpha < 2,\ \text{Case 2}),\\ \frac{8}{a_+^2 s^{2\alpha+1}} + o(s^{-1-\alpha}) & (\alpha_+ < \alpha_- < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{D4}$$

b. Equilibrium alternating renewal process

In equilibrium renewal process ($\alpha > 1$), the Laplace transform of $H_2(t)$ with respect to t yields

$$\hat{H}_2(s) = \frac{2[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)^2 - \hat{\rho}_+(s)^2\hat{\rho}_-(s) + \hat{\rho}_+(s) + \hat{\rho}_-(s) - \hat{\rho}_+(s)^2\hat{\rho}_-(s)^2]}{(\mu_+ + \mu_-)[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)]^2 s^2}. \tag{D5}$$

The asymptotic behaviors are given by

$$\hat{H}_2(s) = \begin{cases} \frac{8}{(\mu_+ + \mu_-)^2 s^3} + \frac{4(\sigma_+^2 + \sigma_-^2)}{(\mu_+ + \mu_-)^3 s^2} + o(s^{-2}) & (2 < \alpha,\ \text{Case 3}),\\ \frac{8}{(\mu_+ + \mu_-)^2 s^3} + \frac{12 a_+}{(\mu_+ + \mu_-)^3 s^{4-\alpha}} + o(s^{\alpha-4}) & (1 < \alpha < 2,\ \text{Case 2}). \end{cases} \tag{D6}$$

c. Aging alternating renewal process

By a similar calculation of the first moment in the aging alternating renewal process, the asymptotic behavior of the double Laplace transform of the second moment of the number of renewals in $[t_a, t_a + t]$, i.e., $N_{t+t_a} - N_{t_a}$, with respect to t and the aging time t_a, is approximately given by

$$\hat{H}_2(s; u) \cong \frac{8\hat{f}_E(s; u)}{s[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)]^2}. \tag{D7}$$

The leading order for Cases 1 and 4 becomes

$$\hat{H}_2(s; u) \sim \begin{cases} \frac{8}{a_+^2 s^{1+2\alpha} u} & (s \ll u),\\ \frac{8}{a_+^2 s^{2+\alpha} u^{\alpha}} & (s \gg u). \end{cases} \tag{D8}$$

3. Higher moments

a. Ordinary alternating renewal process

In the ordinary renewal process, we have

$$\hat{H}_n(s) \sim \frac{2^n n}{s} \sum_{r=1}^{\infty} r^{n-1}\{\hat{\rho}_+(s)\hat{\rho}_-(s)\}^r \sim \frac{2^n n!}{s[1 - \hat{\rho}_+(s)\hat{\rho}_-(s)]^n}. \tag{D9}$$

The asymptotic behaviors are given by

$$\hat{H}_n(s) = \begin{cases} \frac{2^n n!}{(\mu_+ + \mu_-)^n s^{n+1}} + o(s^{-n-1}) & (1 < \alpha,\ \text{Cases 2 and 3}),\\ \frac{2^n n!}{a_+^n s^{n\alpha+1}} + o(s^{-n\alpha-1}) & (\alpha_+ < \alpha_- < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{D10}$$

b. Aging alternating renewal process

For Cases 1 and 4, the asymptotic behavior of the double Laplace transform of the number of renewals in $[t_a, t_a + t]$, i.e., $N_{t+t_a} - N_{t_a}$, with respect to t and the aging time t_a is approximately given by

$$\hat{H}_n(s;u) \cong \frac{2^n n! \hat{f}_E(s;u)}{s[1-\hat{\rho}_+(s)\hat{\rho}_-(s)]^n}. \tag{D11}$$

The leading order becomes

$$\hat{H}_n(s;u) \sim \begin{cases} \frac{2^n n!}{a_+^n s^{1+n\alpha} u} & (s \ll u), \\ \frac{2^n n!}{a_+^n s^{2+n\alpha} u^{\alpha}} & (s \gg u). \end{cases} \tag{D12}$$

APPENDIX E: LAPLACE TRANSFORM OF DISTRIBUTION OF T_t/t

The double Laplace transform of $g_n^{\pm}(y;t)$ with respect to y and t is given by

$$\hat{g}_n^{\pm}(u;s) = \int_0^\infty dt\, e^{-st} \int_0^\infty dy\, e^{-uy} \langle \delta(y - T_t^+) I(t_n \leqslant t < t_{n+1}) \rangle = \left\langle \int_{t_n}^{t_{n+1}} dt\, e^{-st} e^{-uT_t^+} \right\rangle. \tag{E1}$$

1. Ordinary alternating renewal process

For $\sigma(0) = +1$, we have

$$\hat{g}_{2k+1}^+(u;s) = \frac{1-\hat{\rho}_-(s)}{s} \hat{\rho}_-^k(s) \hat{\rho}_+^k(s+u) \hat{\rho}_1(s+u) \tag{E2}$$

and

$$\hat{g}_{2k}^+(u;s) = \begin{cases} \frac{1-\hat{\rho}_+(s+u)}{s+u} \hat{\rho}_-^k(s) \hat{\rho}_+^{k-1}(s+u) \hat{\rho}_1(s+u) & (k \geqslant 1), \\ \frac{1-\hat{\rho}_1(s+u)}{s+u} & (k = 0). \end{cases} \tag{E3}$$

For $\sigma_0 = -1$, we have

$$\hat{g}_{2k}^-(u;s) = \begin{cases} \frac{1-\hat{\rho}_-(s)}{s} \hat{\rho}_1(s) \hat{\rho}_-^{k-1}(s) \hat{\rho}_+^k(s+u) & (k \geqslant 1), \\ \frac{1-\hat{\rho}_1(s)}{s} & (k = 0) \end{cases} \tag{E4}$$

and

$$\hat{g}_{2k+1}^-(u;s) = \frac{1-\hat{\rho}_+(s+u)}{s+u} \hat{\rho}_+^k(s+u) \hat{\rho}_-^k(s) \hat{\rho}_1(s). \tag{E5}$$

Using Eqs. (E2)–(E5), we have the Laplace transform of the PDF of T_t^+:

$$\hat{g}^+(u;s) = \sum_{n=0}^{\infty} \{\hat{g}_{2n}^+(u;s) + \hat{g}_{2n+1}^+(u;s)\} = \frac{1}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)} \left\{ \frac{1-\hat{\rho}_-(s)}{s} \hat{\rho}_+(s+u) + \frac{1-\hat{\rho}_+(s+u)}{s+u} \right\} \tag{E6}$$

and

$$\hat{g}^-(u;s) = \sum_{n=0}^{\infty} \{\hat{g}_{2n}^-(u;s) + \hat{g}_{2n+1}^-(u;s)\} = \frac{1}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)} \left\{ \frac{1-\hat{\rho}_+(s+u)}{s+u} \hat{\rho}_-(s) + \frac{1-\hat{\rho}_-(s)}{s} \right\}. \tag{E7}$$

In the small s and u limit,

$$\hat{g}^+(u;s) \sim \hat{g}^-(u;s) \sim \frac{1}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)} \left\{ \frac{1-\hat{\rho}_-(s)}{s} + \frac{1-\hat{\rho}_+(s+u)}{s+u} \right\}. \tag{E8}$$

For $\alpha = \alpha_+ = \alpha_- < 1$ (Case 1), fluctuations of $x = T_t^+/t$ are intrinsic even in the long-time limit. The double Laplace transform for $\hat{g}^{\pm}(u;s)$ becomes

$$\hat{g}^+(u;s) \sim \hat{g}^-(u;s) \sim \frac{a_+(s+u)^{\alpha-1} + a_- s^{\alpha-1}}{a_+(s+u)^{\alpha} + a_- s^{\alpha}} = \frac{1}{s} \frac{a_+(1+u/s)^{\alpha-1} + a_-}{a_+(1+u/s)^{\alpha} + a_-}. \tag{E9}$$

2. Equilibrium alternating renewal process

For $N_t = 2k+1$ $(k = 0, 1, \ldots)$ with $\sigma_0 = +1$, we have

$$\hat{g}_{2k+1}^+(u;s) = \frac{1-\hat{\rho}_-(s)}{s} \hat{\rho}_-^k(s) \hat{\rho}_+^k(s+u) \hat{f}_+(s+u), \tag{E10}$$

where $\hat{f}_+(s)$ is the Laplace transform of $f_+(x)$. For $N_t = 2k$ $(k = 0, 1, \ldots)$ with $\sigma_0 = +1$, we have

$$\hat{g}^+_{2k}(u;s) = \begin{cases} \frac{1-\hat{\rho}_+(s+u)}{s+u}\hat{\rho}^k_-(s)\hat{\rho}^{k-1}_+(s+u)\hat{f}_+(s+u) & (k \geqslant 1), \\ \frac{1-\hat{f}_+(s+u)}{s+u} & (k = 0). \end{cases} \tag{E11}$$

Moreover, we have

$$\hat{g}^-_{2k+1}(u;s) = \frac{1-\hat{\rho}_+(s+u)}{s+u}\hat{f}_-(s)\hat{\rho}^k_+(s+u)\hat{\rho}^k_-(s) \tag{E12}$$

and

$$\hat{g}^-_{2k}(u;s) = \begin{cases} \frac{1-\hat{\rho}_-(s)}{s}\hat{f}_-(s)\hat{\rho}^k_+(s+u)\hat{\rho}^{k-1}_-(s) & (k \geqslant 1), \\ \frac{1-\hat{f}_-(s)}{s} & (k = 0), \end{cases} \tag{E13}$$

where $\hat{f}_-(s)$ is the Laplace transform of $f_-(x)$. It follows that the Laplace transform of the PDF of T_t^+ is given by

$$\hat{g}(u;s) = P_+\hat{g}^+(u;s) + P_-\hat{g}^-(u;s) \tag{E14}$$

$$= P_+\left\{\frac{1-\hat{f}_+(s+u)}{s+u} + \frac{\hat{f}_+(s+u)}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)}\left[\frac{1-\hat{\rho}_+(s+u)}{s+u}\hat{\rho}_-(s) + \frac{1-\hat{\rho}_-(s)}{s}\right]\right\}$$
$$+ P_-\left\{\frac{1-\hat{f}_-(s)}{s} + \frac{\hat{f}_-(s)}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)}\left[\frac{1-\hat{\rho}_+(s+u)}{s+u} + \frac{1-\hat{\rho}_-(s)}{s}\hat{\rho}_+(s+u)\right]\right\}. \tag{E15}$$

3. Aging alternating renewal process

The Laplace transform of $g(y;t,t_a)$ with respect to y, t, and t_a can be calculated as

$$\hat{g}(u;s,v) = \int_0^\infty dt\, e^{-st}\int_0^\infty dt_a e^{-vt_a}\int_0^\infty dy\, e^{-uy}\left\langle\delta\left(y - T^+_{t_a,t}\right)\right\rangle \tag{E16}$$

$$= \frac{\hat{f}_{E,+}(0,v) - \hat{f}_{E,+}(s+u,v)}{s+u} + \frac{\hat{f}_{E,+}(s+u,v)}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)}\left[\frac{1-\hat{\rho}_+(s+u)}{s+u}\hat{\rho}_-(s) + \frac{1-\hat{\rho}_-(s)}{s}\right]$$
$$+ \frac{\hat{f}_{E,+}(0,v) - \hat{f}_{E,-}(s,v)}{s} + \frac{\hat{f}_{E,-}(s,v)}{1-\hat{\rho}_+(s+u)\hat{\rho}_-(s)}\left[\frac{1-\hat{\rho}_+(s+u)}{s+u} + \frac{1-\hat{\rho}_-(s)}{s}\hat{\rho}_+(s+u)\right]. \tag{E17}$$

APPENDIX F: LAPLACE TRANSFORM OF MOMENTS OF T_t

1. First moment of T_t^+

The Laplace transform of the first moment of T_t^+ with respect to t is defined as

$$\hat{T}_1(s) \equiv \int_0^\infty e^{-st}\langle T_t^+\rangle dt. \tag{F1}$$

The Laplace transform of the first moment of T_t^+ with respect to t is obtained from Eq. (E6):

$$\hat{T}_1(s) = -\left.\frac{\partial\hat{g}^+(u;s)}{\partial u}\right|_{u=0}. \tag{F2}$$

a. Ordinary alternating renewal process

The asymptotic behaviors of $\hat{T}_1(s)$ in the ordinary alternating renewal process are given by

$$\hat{T}_1(s) = \begin{cases} \frac{\mu_+}{(\mu_++\mu_-)s^2} - \frac{\sigma_+^2\mu_- - \sigma_-^2\mu_+ - \mu_+\mu_-(\mu_++\mu_-)}{2(\mu_++\mu_-)^2 s} + o(s^{-1}) & (2 < \alpha,\ \text{Case 3}), \\ \frac{\mu_+}{(\mu_++\mu_-)s^2} - \frac{a_+\mu_-}{(\mu_++\mu_-)^2 s^{3-\alpha}} + o(s^{\alpha-3}) & (1 < \alpha < 2,\ \text{Case 2}), \\ \frac{1}{s^2} + o(s^{-2}) & (\alpha < 1,\ \text{Cases 1 and 4}). \end{cases} \tag{F3}$$

The Laplace transform of the second moment of T_t^+ with respect to t is defined as

$$\hat{T}_2(s) \equiv \int_0^\infty e^{-st}\langle (T_t^+)^2\rangle dt. \tag{F4}$$

2. Second moment of T_t^+

The Laplace transform of the second moment of T_t^+ with respect to t is obtained from

$$\hat{T}_2(s) = \left.\frac{\partial^2 \hat{g}^+(u;s)}{\partial u^2}\right|_{u=0}. \tag{F5}$$

a. Ordinary alternating renewal process

The asymptotic behaviors of $\hat{T}_2(s)$ in the ordinary alternating renewal process are given by

$$\hat{T}_2(s) = \begin{cases} \frac{2\mu_+^2}{(\mu_+ + \mu_-)^2 s^3} + \frac{\sigma_+^2\mu_-(\mu_- - \mu_+) + 2\sigma_-^2\mu_+^2 + \mu_+^2\mu_-(\mu_+ + 5\mu_-)}{(\mu_+ + \mu_-)^3 s^2} + o(s^{-1}) & (2 < \alpha,\ \text{Case 3}), \\ \frac{2\mu_+^2}{(\mu_+ + \mu_-)^2 s^3} - \frac{2a_+\mu_-[(3-\alpha)\mu_+ - (\alpha-1)\mu_-]}{(\mu_+ + \mu_-)^3 s^{4-\alpha}} + o(s^{\alpha-4}) & (1 < \alpha < 2,\ \text{Case 2}). \end{cases} \tag{F6}$$

b. Equilibrium alternating renewal process

The asymptotic behaviors of $\hat{T}_2(s)$ are given by

$$\hat{T}_2(s) = \begin{cases} \frac{2\mu_+^2}{(\mu_+ + \mu_-)^2 s^3} + \frac{4P_-\mu_+\mu_-(\sigma_+^2 + \mu_+^2) + 2\mu_+^2(\mu_+\mu_- + \sigma_-^2)}{(\mu_+ + \mu_-)^3 s^2} + o(s^{-1}) & (2 < \alpha,\ \text{Case 3}), \\ \frac{2\mu_+^2}{(\mu_+ + \mu_-)^2 s^3} - \frac{(3-\alpha)a_+\mu_-^2}{(\mu_+ + \mu_-)^3 s^{4-\alpha}} + o(s^{\alpha-4}) & (1 < \alpha < 2,\ \text{Case 2}). \end{cases} \tag{F7}$$

APPENDIX G: LAPLACE TRANSFORM OF THE CORRELATION FUNCTION

1. Ordinary alternating renewal process

The transition probabilities are written as

$$W_{++}(t) = \sum_{n=0}^{\infty} \Pr(N_t = 2n) = \sum_{n=1}^{\infty} \{\Pr(N_t < 2n+1) - \Pr(N_t < 2n)\} + \Pr(N_t = 0) \tag{G1}$$

$$= \sum_{n=1}^{\infty} \{\Pr(S_{2n} < t) - \Pr(S_{2n+1} < t)\} + \Pr(N_t = 0) \tag{G2}$$

and

$$W_{+-}(t) = \sum_{n=0}^{\infty} \Pr(N_t = 2n+1) = \sum_{n=0}^{\infty} \{\Pr(N_t < 2n+2) - \Pr(N_t < 2n+1)\} \tag{G3}$$

$$= \sum_{n=0}^{\infty} \{\Pr(S_{2n+1} < t) - \Pr(S_{2n+2} < t)\}. \tag{G4}$$

The Laplace transforms of $W_{--}(t)$ and $W_{-+}(t)$ in the ordinary alternating renewal process are given by

$$\hat{W}_{++}(s) = \frac{1}{s}\sum_{n=1}^{\infty} \hat{\rho}_+(s)\{\hat{\rho}_-(s)\hat{\rho}_+(s)\}^{n-1}\hat{\rho}_-(s) - \frac{1}{s}\sum_{n=1}^{\infty} \hat{\rho}_+(s)\{\hat{\rho}_-(s)\hat{\rho}_+(s)\}^{n} + \frac{1-\hat{\rho}_+(s)}{s} \tag{G5}$$

$$= \frac{1}{s} - \frac{\hat{\rho}_+(s)}{s}\frac{1-\hat{\rho}_-(s)}{1-\hat{\rho}_+(s)\hat{\rho}_-(s)} \tag{G6}$$

and

$$\hat{W}_{+-}(s) = \frac{\hat{\rho}_+(s)}{s}\frac{1-\hat{\rho}_-(s)}{1-\hat{\rho}_+(s)\hat{\rho}_-(s)}. \tag{G7}$$

It follows that the Laplace transform of $C(t)$ is given by

$$\hat{C}(s) = \frac{\mu_- D_+(D_+ - D_-)}{(\mu_+ + \mu_-)s} - \frac{D_+(D_+ - D_-)\hat{\rho}_+(s)\{1-\hat{\rho}_-(s)\}}{\{1-\hat{\rho}_+(s)\hat{\rho}_-(s)\}s}. \tag{G8}$$

2. Equilibrium alternating renewal process

The Laplace transforms of $W_{--}(t)$, $W_{-+}(t)$, $W_{--}(t)$, and $W_{-+}(t)$ for the equilibrium alternating renewal process are given by

$$\hat{W}_{++}(s) = \frac{1}{s}\sum_{n=1}^{\infty} \hat{f}_{E,+}(s)\{\hat{\rho}_-(s)\hat{\rho}_+(s)\}^{n-1}\hat{\rho}_-(s) - \frac{1}{s}\sum_{n=1}^{\infty} \hat{f}_{E,+}(s)\{\hat{\rho}_-(s)\hat{\rho}_+(s)\}^{n} + \frac{1-\hat{f}_{E,+}(s)}{s} \tag{G9}$$

$$= \frac{1}{s} - \frac{\hat{f}_{E,+}(s)}{s}\frac{1-\hat{\rho}_-(s)}{1-\hat{\rho}_+(s)\hat{\rho}_-(s)}, \tag{G10}$$

$$\hat{W}_{+-}(s) = \frac{\hat{f}_{E,+}(s)}{s}\frac{1-\hat{\rho}_-(s)}{1-\hat{\rho}_+(s)\hat{\rho}_-(s)}, \tag{G11}$$

$$\hat{W}_{--}(s) = \frac{1}{s} - \frac{\hat{f}_{E,-}(s)}{s}\frac{1-\hat{\rho}_+(s)}{1-\hat{\rho}_+(s)\hat{\rho}_-(s)}, \tag{G12}$$

and

$$\hat{W}_{-+}(s) = \frac{\hat{f}_{E,-}(s)}{s}\frac{1-\hat{\rho}_+(s)}{1-\hat{\rho}_+(s)\hat{\rho}_-(s)}. \tag{G13}$$

It follows that the Laplace transform of $C(t)$ is given by

$$\hat{C}(s) = \left(\frac{D_+ - D_-}{\mu_+ + \mu_-}\right)^2 \frac{\mu_+\mu_-}{s} - \frac{(D_+ - D_-)^2}{\mu_+ + \mu_-}\frac{\{1-\hat{\rho}_+(s)\}\{1-\hat{\rho}_-(s)\}}{\{1-\hat{\rho}_+(s)\hat{\rho}_-(s)\}s^2}. \tag{G14}$$

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