Algebraic area of cubic lattice walks

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We obtain an explicit formula to enumerate closed random walks on a cubic lattice with a specified length and algebraic area. The algebraic area of a closed cubic lattice walk is defined as the sum of the algebraic areas obtained from the walk's projection onto the three Cartesian planes. This enumeration formula can be mapped onto the cluster coefficients of three types of particles that obey quantum exclusion statistics with statistical parameters $g = 1$, $g = 1$, and $g = 2$, respectively, subject to the constraint that the numbers of $g = 1$ (fermions) exclusion particles of two types are equal.

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I. INTRODUCTION

The algebraic area of a planar closed random walk is defined as the area swept by the walk, weighted by the winding number in each winding sector. The area is considered positive if the walk moves around the sector in a counterclockwise direction. In the continuous case, the probability distribution of the algebraic area A enclosed by closed Brownian curves after a time t is given by Lévy's stochastic area formula (also known as Lévy's law) [1]

$$P(A) = \frac{\pi}{2t} \frac{1}{\cosh^2(\pi A/t)}. \tag{1}$$

In the discrete case, a series of explicit algebraic area enumeration formulas [2–4] for closed random walks on various lattices have recently been obtained from the Kreft coefficients [5] encoding the Schrödinger equation of quantum Hofstadter-like models [6] that describe a charged particle hopping on planar lattices coupled to a perpendicular magnetic field. Essentially the enumeration amounts to calculating the trace of the power of the Hofstadter-like Hamiltonian and has an interpretation in terms of the statistical mechanics of particles that obey exclusion statistics with an integer exclusion parameter g ($g = 0$ for bosons, $g = 1$ for fermions, $g \geqslant 2$ for stronger exclusion than fermions). Figure 1 shows three examples of two-dimensional (2D) lattice random walks: the square lattice walk corresponds to the $g = 2$ exclusion, the Kreweras-like chiral walk on a triangular lattice corresponds to the $g = 3$ exclusion, and the honeycomb lattice walk corresponds to a mixture of the $g = 1$ and $g = 2$ exclusions, with an appropriate spectrum. Note that, in the context of Hofstadter-like model, the algebraic area can be expressed as $\frac{1}{2}\oint(\mathbf{r} \times d\mathbf{r}) \cdot \mathbf{B}$, where $\mathbf{B} = (0, 0, 1)$ and the integral is along the closed walk in the xy plane.

In this article, we extend the concept of algebraic area to closed cubic lattice walks by defining it as the sum of the algebraic areas of the walk projected onto the xy, yz, zx planes along the $-z$, $-x$, $-y$ directions. To count closed random walks on a cubic lattice with a given length and algebraic area, we begin by introducing three lattice hopping operators U, V, W along the x, y, z directions, as well as U^{-1}, V^{-1}, W^{-1} along the $-x, -y, -z$ directions. These operators satisfy the noncommutative three-tori algebra [7]

$$V\,U = \mathrm{Q}\,U\,V, \quad W\,V = \mathrm{Q}\,V\,W, \quad U\,W = \mathrm{Q}\,W\,U, \tag{2}$$

with Q a central element (that is, Q commutes with all operators). It amounts to saying that the planar walks that go around the unit lattice cell on the Cartesian planes in a counterclockwise direction enclose an algebraic area 1, i.e., $V^{-1}U^{-1}VU = \mathrm{Q}$, $W^{-1}V^{-1}WV = \mathrm{Q}$, and $U^{-1}W^{-1}UW = \mathrm{Q}$. The algebraic area A enclosed by a cubic lattice walk can thus be computed by reducing the corresponding hopping operators to Q^A using the commutation relations (2). See Fig. 2 for the closed six-step cubic lattice walk $UW^{-1}V^{-1}U^{-1}WV = \mathrm{Q}$ as an example. Another example involves enumerating closed four-step walks. By taking the trace of $(U + V + W + U^{-1} + V^{-1} + W^{-1})^4 = 6(11 + 2\mathrm{Q} + 2\mathrm{Q}^{-1}) + \cdots$, only terms with an equal number of U and U^{-1}, V and V^{-1}, W and W^{-1} survive, yielding the count of algebraic area[1]: 66 walks enclose an algebraic area $A = 0$, 12 walks enclose an algebraic area $A = 1$, and 12 walks enclose an algebraic area $A = -1$.

By expressing the phase $\mathrm{Q} = \exp(2\pi \mathrm{i}\phi/\phi_0)$ in terms of the flux ϕ through the unit lattice cell on each of the three Cartesian planes in unit of the flux quantum ϕ_0, the Hermitian operator

$$H = U + V + W + U^{-1} + V^{-1} + W^{-1} \tag{3}$$

represents a Hamiltonian that describes a charged particle hopping on a cubic lattice coupled to a magnetic field $\mathbf{B} = (1, 1, 1)$, as indicated in the definition of the algebraic area for a cubic lattice walk. The energy spectrum with $\mathbf{B} = (1, 1, 1)$

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[1]By symmetry, we can focus on the walks that start with a step along the x direction (i.e., U). There are 11 walks that enclose an algebraic area 0: $U^{-1}UU^{-1}U$, $UU^{-1}U^{-1}U$, $V^{-1}VU^{-1}U$, $VV^{-1}U^{-1}U$, $W^{-1}WU^{-1}U$, $WW^{-1}U^{-1}U$, $U^{-1}U^{-1}UU$, $U^{-1}V^{-1}VU$, $U^{-1}VV^{-1}U$, $U^{-1}W^{-1}WU$, and $U^{-1}WW^{-1}U$, two walks that enclose an algebraic area 1: $V^{-1}U^{-1}VU$ and $WU^{-1}W^{-1}U$, and two walks that enclose an algebraic area -1: $VU^{-1}V^{-1}U$ and $W^{-1}U^{-1}WU$.

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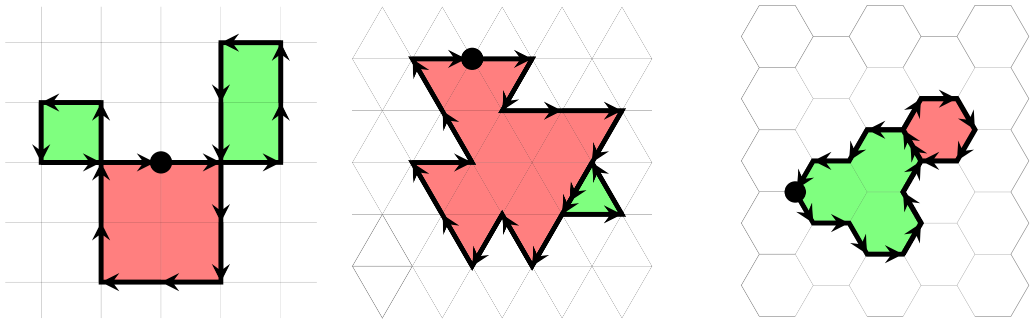

FIG. 1. A closed square lattice walk, chiral triangular lattice walk, and honeycomb lattice walk of length 18, starting and ending at the bullet point, with algebraic area -1, -14, and 2, respectively. The region inside the walk, i.e., the winding sector, is colored green if its area is positive, otherwise it is colored red. In the chiral triangular lattice walk, only three of the possible six directions are allowed at each step, namely, in directions with angles 0, $2\pi/3$ and $4\pi/3$ with respect to the horizontal axis.

on a cubic lattice was initially investigated in [8]. The three-dimensional (3D) Hofstadter model was studied earlier in [9], and the general case of the uniform magnetic field was explored in [10], with an experimental scheme proposed in [11]. Hofstadter models have also been studied on other 3D lattices, such as the tetragonal monoatomic and double-atomic lattice [12], and in four dimensions [13] as well.

As with the case of a planar lattice, the trace of H^{2n} provides the generating function for the number $C_{2n}(A)$ of closed random walks of length $2n$ (necessarily even) on a cubic lattice enclosing an algebraic area A. Specifically,

$$\sum_A C_{2n}(A)\mathrm{Q}^A = \mathbf{Tr}\, H^{2n}, \tag{4}$$

with the normalization $\mathbf{Tr}\, I = 1$, where I denotes the identity operator.

The article is organized as follows. Assuming that the flux is rational, we use the finite-dimensional representation of the algebra (2) to derive the trace of H^{2n}, establish its connection with quantum exclusion statistics ($g=1$, $g=1$, $g=2$), and provide a combinatorial interpretation based on the combinatorial coefficients $c_{1,1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j)$ labeled by the (1,1,2)-compositions. In Sec. III, we present the explicit formula for $C_{2n}(A)$, as well as its asymptotics as $n\to\infty$, and discuss potential generalizations and applications.

II. ALGEBRAIC AREA ENUMERATION OF CUBIC LATTICE WALKS

A. Hamiltonian

From now on, we assume that the magnetic flux on each Cartesian plane is rational, i.e., $\phi/\phi_0 = p/q$ with p and q being coprime, thus $\mathrm{Q} = \exp(2\pi \mathrm{i} p/q)$. To obtain the finite-dimensional representation of U, V, W, we introduce the $q\times q$ "clock" and "shift" matrices

$$u = \mathrm{e}^{\mathrm{i}k_x}\begin{pmatrix} \mathrm{Q} & 0 & 0 & \cdots & 0 & 0\\ 0 & \mathrm{Q}^2 & 0 & \cdots & 0 & 0\\ 0 & 0 & \mathrm{Q}^3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mathrm{Q}^{q-1} & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix},$$

$$v = \mathrm{e}^{\mathrm{i}k_y}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & 0 & \cdots & 0 & 0\end{pmatrix},$$

which satisfy $v\,u = \mathrm{Q}\,u\,v$ and contribute to the Hofstadter Hamiltonian $u + v + u^{-1} + v^{-1}$ for square lattice walks.

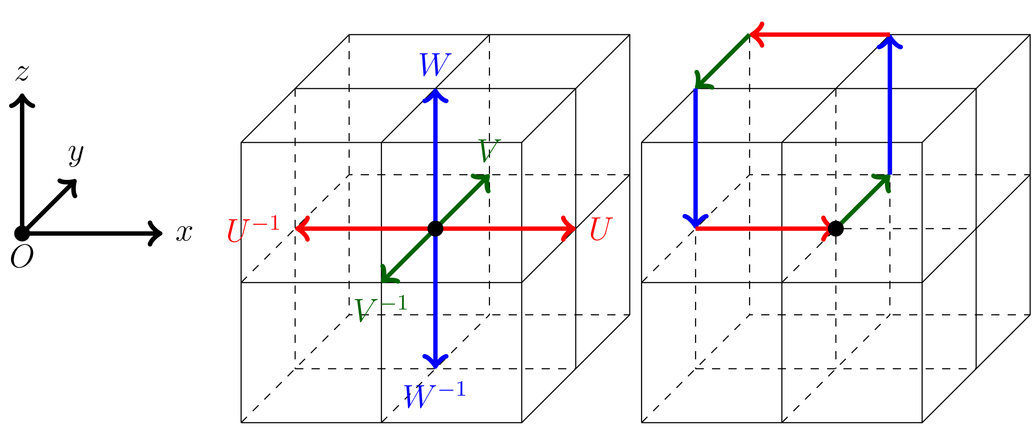

FIG. 2. (left) Three-dimensional Cartesian coordinate system; (middle) six lattice hopping operators in a cubic lattice; (right) closed 6-step cubic lattice walk $UW^{-1}V^{-1}U^{-1}WV$ whose algebraic area is given by $A = 1 + 1 + (-1) = 1$. Using the commutation relations (2), $UW^{-1}V^{-1}U^{-1}WV$ is simplified to Q^1 as expected.

Here, k_x and k_y denote the quasimomenta in the x and y directions. In the quantum trace, integration over k_x and k_y eliminates the unwanted terms containing u^q and v^q which correspond to open walks but can be closed by q periodicity. Another way to achieve this is by setting $k_x = k_y = 0$ and considering walks of length less than q.

Because of the open walk $UVW \neq I$, it is not possible to represent the operators U, V, W as u, v, $v^{-1}u^{-1}$, respectively, even though they satisfy the algebra (2). To address this, we introduce an additional vector space with dimension q', in which U and V act as identity operators, while W does not. Consequently, we obtain the representation of (2) as $qq' \times qq'$ matrices

$$U = u \otimes I_{q'}, \quad V = v \otimes I_{q'}, \quad W = (v^{-1}u^{-1}) \otimes u',$$

where u' is an arbitrary $q' \times q'$ matrix that is not proportional to $I_{q'}$. Again, the quasimomenta are set to be zero. The sought-after quantum trace of the $qq' \times qq'$ Hamiltonian matrix (3) reduces to the usual trace up to a normalization factor, that is,

$$\mathbf{Tr}\, H^{2n} = \frac{1}{qq'} \mathrm{tr}\, H^{2n}.$$

Let u' be diagonal and equal to $u|_{q\to q'}$ [therefore $\mathrm{Q} \to \mathrm{Q}' = \exp(2\pi i p/q')$ in u']. Performing the algebra-preserving transformation $u \to -u^{-1}v$, $v \to v^{-1}$, $u' \to -u'$ leads to the new Hamiltonian that describes walks on a deformed cubic lattice

$$H' = H_2 \otimes I_{q'} + u \otimes u' + u^{-1} \otimes u'^{-1},$$

where the Hofstadter Hamiltonian associated to the usual square lattice walks is

$$H_2 = -u^{-1}v - v^{-1}u + v + v^{-1} = \begin{pmatrix} 0 & \bar{f}_1 & 0 & \cdots & 0 & 0 \\ f_1 & 0 & \bar{f}_2 & \cdots & 0 & 0 \\ 0 & f_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \bar{f}_{q-1} \\ 0 & 0 & 0 & \cdots & f_{q-1} & 0 \end{pmatrix},$$

with $f_k = 1 - \mathrm{Q}^k$. Note that H_2 is a $g = 2$ matrix in the sense that its secular determinant $\det(I_q - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor}(-1)^n Z_n z^{2n}$ captures the Kreft coefficient [5]

$$Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n},$$
$$Z_0 = 1,$$

as a trigonometric multiple nested sum with $+2$ shifts among the spectral functions $s_k := f_k \bar{f}_k = 4\sin^2(k\pi p/q)$. In statistical mechanics, Z_n can be interpreted as the n-body partition function for n particles in a one-body spectrum ϵ_k $(k = 1, 2, \ldots, q-1)$ with Boltzmann factor $\mathrm{e}^{-\beta\epsilon_k} = s_k$. The $+2$ shifts indicate that these particles obey $g = 2$ exclusion statistics, i.e., no two particles can occupy adjacent quantum states. By introducing cluster coefficients b_n via $\log\left(\sum_{n=0}^{\lfloor q/2 \rfloor} Z_n x^n\right) = \sum_{n=1}^{\infty} b_n x^n$ with fugacity $x = -z^2$, and using the identity $\log\det(I_q - zH_2) = \mathrm{tr}\,\log(I_q - zH_2) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\, H_2^n$ we establish a connection between the generating function for algebraic area enumeration of square lattice walks and the cluster coefficients with $g = 2$ exclusion statistics, that is,

$$\mathbf{Tr}\, H_2^{2n} = \frac{1}{q}\mathrm{tr}\, H_2^{2n} = 2n(-1)^{n+1}\frac{1}{q} b_n = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \ldots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}, \tag{5}$$

where $c_2(l_1, l_2, \ldots, l_j) = \frac{1}{l_1}\prod_{i=2}^{j}\binom{l_{i-1}+l_i-1}{l_i}$. As we will see in Sec. II B, the algebraic area enumeration for cubic lattice walks can also be mapped onto cluster coefficients with appropriate exclusion parameters and spectral functions.

Now come back to the Hamiltonian H'. Introduce $\tilde{s}_{k,k'} = \mathrm{Q}^k \mathrm{Q}'^{k'} + \mathrm{Q}^{-k}\mathrm{Q}'^{-k'}$ and $q' \times q'$ diagonal matrices $\tilde{\mathbf{s}}_k = \mathrm{diag}(\tilde{s}_{k,1}, \tilde{s}_{k,2}, \ldots, \tilde{s}_{k,q'})$, $\mathbf{f}_k = f_k I_{q'}$, $\bar{\mathbf{f}}_k = \bar{f}_k I_{q'}$, $\mathbf{0} = 0 I_{q'}$. H' can be expressed as a $qq' \times qq'$ block tridiagonal matrix

$$H' = \begin{pmatrix} \tilde{\mathbf{s}}_1 & \bar{\mathbf{f}}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{f}_1 & \tilde{\mathbf{s}}_2 & \bar{\mathbf{f}}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{f}_2 & \tilde{\mathbf{s}}_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{s}}_{q-1} & \bar{\mathbf{f}}_{q-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{f}_{q-1} & \tilde{\mathbf{s}}_q \end{pmatrix}.$$

Applying the trace computation techniques described in [14] we obtain

$$\frac{1}{qq'}\mathrm{tr}\, H'^{2n} = 2n \sum_{\substack{\tilde{l}_1, \ldots, \tilde{l}_{j+1}; l_1, \ldots, l_j \\ (1,2)\text{-composition of } 2n}} c_{1,2}(\tilde{l}_1, \ldots, \tilde{l}_{j+1}; l_1, \ldots, l_j) \times \frac{1}{q}\sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j} \times \frac{1}{q'}\sum_{k'=1}^{q'} \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \cdots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}}, \tag{6}$$

with

$$c_{1,2}(\tilde{l}_1, \ldots, \tilde{l}_{j+1}; l_1, \ldots, l_j) = \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, l_k}.$$

By convention $l_k = 0$ for $k > j$. We define the sequence of integers $\tilde{l}_1, \ldots, \tilde{l}_{j+1}; l_1, \ldots, l_j$, $j \geqslant 0$, as a (1,2)-composition of $2n$ if they satisfy the conditions

$$2n = (\tilde{l}_1 + \cdots + \tilde{l}_{j+1}) + 2(l_1 + \cdots + l_j), \tilde{l}_i \geqslant 0, l_i > 0,$$

i.e., l_i's are the usual compositions of $1, 2, \ldots, n$ and $\tilde{l}_i$'s are additional nonnegative integers. For $j = 0$ we have the trivial composition $\tilde{l}_1 = 2n$.

As q' is arbitrary, for simplicity of calculation we set $q' = q$ in the sequel. The second trigonometric sum in (6) is expanded to be

$$\frac{1}{q}\sum_{k'=1}^{q}\tilde{s}_{k,k'}^{\tilde{l}_1}\tilde{s}_{k+1,k'}^{\tilde{l}_2}\cdots\tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}}=\frac{1}{q}\sum_{\tilde{l}'_i+\tilde{l}''_i=\tilde{l}_i}\mathrm{Q}^{(\tilde{l}'_2-\tilde{l}''_2)+2(\tilde{l}'_3-\tilde{l}''_3)+\cdots+j(\tilde{l}'_{j+1}-\tilde{l}''_{j+1})+k[(\tilde{l}'_1+\cdots+\tilde{l}'_{j+1})-(\tilde{l}''_1+\cdots+\tilde{l}''_{j+1})]}$$
$$\times\sum_{k'=1}^{q}\mathrm{Q}^{k'[(\tilde{l}'_1+\cdots+\tilde{l}'_{j+1})-(\tilde{l}''_1+\cdots+\tilde{l}''_{j+1})]}\binom{\tilde{l}_1}{\tilde{l}'_1}\binom{\tilde{l}_2}{\tilde{l}'_2}\cdots\binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}},$$

with $\tilde{l}'_i,\tilde{l}''_i\geqslant 0,\ i=1,\dots,j+1$. Since $\sum_{k'=1}^{q}\mathrm{Q}^{k'[(\tilde{l}'_1+\cdots+\tilde{l}'_{j+1})-(\tilde{l}''_1+\cdots+\tilde{l}''_{j+1})]}$ is nonvanishing only when $\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}=\tilde{l}''_1+\cdots+\tilde{l}''_{j+1}$ we obtain

$$\frac{1}{q}\sum_{k'=1}^{q}\tilde{s}_{k,k'}^{\tilde{l}_1}\tilde{s}_{k+1,k'}^{\tilde{l}_2}\cdots\tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}}=\sum_{\tilde{l}'_i+\tilde{l}''_i=\tilde{l}_i}\mathrm{Q}^{(\tilde{l}'_2-\tilde{l}''_2)+2(\tilde{l}'_3-\tilde{l}''_3)+\cdots+j(\tilde{l}'_{j+1}-\tilde{l}''_{j+1})}\binom{\tilde{l}_1}{\tilde{l}'_1}\binom{\tilde{l}_2}{\tilde{l}'_2}\cdots\binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}}.$$

Finally, by recognizing that the binomial product $\binom{\tilde{l}_1}{\tilde{l}'_1}\binom{\tilde{l}_2}{\tilde{l}'_2}\cdots\binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}}$ can be absorbed into $c_{1,2}$, as well as changing the notation $\tilde{l}'_i\to\tilde{l}_i,\tilde{l}''_i\to\tilde{l}'_i$, we arrive at

$$\frac{1}{q^2}\mathrm{tr}\,H'^{2n}=2n\sum_{\substack{\tilde{l}_1,\dots,\tilde{l}_{j+1};\tilde{l}'_1,\dots,\tilde{l}'_{j+1};l_1,\dots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}}}c_{1,1,2}(\tilde{l}_1,\dots,\tilde{l}_{j+1};\tilde{l}'_1,\dots,\tilde{l}'_{j+1};l_1,\dots,l_j)\mathrm{Q}^{(\tilde{l}_2-\tilde{l}'_2)+2(\tilde{l}_3-\tilde{l}'_3)+\cdots+j(\tilde{l}_{j+1}-\tilde{l}'_{j+1})}\frac{1}{q}\sum_{k=1}^{q-j}s_k^{l_1}s_{k+1}^{l_2}\dots s_{k+j-1}^{l_j},\tag{7}$$

with the combinatorial coefficients

$$c_{1,1,2}(\tilde{l}_1,\dots,\tilde{l}_{j+1};\tilde{l}'_1,\dots,\tilde{l}'_{j+1};l_1,\dots,l_j)=\frac{(\tilde{l}_1+\tilde{l}'_1+l_1-1)!}{\tilde{l}_1!\tilde{l}'_1!l_1!}\prod_{k=2}^{j+1}\binom{l_{k-1}+\tilde{l}_k+\tilde{l}'_k+l_k-1}{l_{k-1}-1,\tilde{l}_k,\tilde{l}'_k,l_k}.$$

By convention $l_k=0$ for $k>j$. We define the sequence of integers $\tilde{l}_1,\dots,\tilde{l}_{j+1};\tilde{l}'_1,\dots,\tilde{l}'_{j+1};l_1,\dots,l_j$ as a (1,1,2)-composition of $2n$ if they satisfy the conditions

$$2n=(\tilde{l}_1+\cdots+\tilde{l}_{j+1})+(\tilde{l}'_1+\cdots+\tilde{l}'_{j+1})+2(l_1+\cdots+l_j),\ \tilde{l}_i,\tilde{l}'_i\geqslant 0,\ l_i>0,$$

i.e., l_i's are the usual compositions of $1,2,\dots,n$ and $\tilde{l}_i,\tilde{l}'_i$'s are nonnegative integers. We also include, with constraint $\tilde{l}_1=\tilde{l}'_1$, the trivial composition $(n;n;0)$. A combinatorial interpretation of the (1,1,2)-composition and $c_{1,1,2}$ will be discussed in Sec. II C.

B. (1,1,2)-exclusion statistics

Now we take a step further by defining $t_k=\mathrm{Q}^k$. Given that for $\tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}$

$$\frac{1}{q}\sum_{k=1}^{q-j}t_k^{\tilde{l}_1}\bar{t}_k^{\tilde{l}'_1}s_k^{l_1}t_{k+1}^{\tilde{l}_2}\bar{t}_{k+1}^{\tilde{l}'_2}s_{k+1}^{l_2}\dots=\mathrm{Q}^{(\tilde{l}_2-\tilde{l}'_2)+2(\tilde{l}_3-\tilde{l}'_3)+\cdots+j(\tilde{l}_{j+1}-\tilde{l}'_{j+1})}\frac{1}{q}\sum_{k=1}^{q-j}s_k^{l_1}s_{k+1}^{l_2}\dots s_{k+j-1}^{l_j},$$

we rewrite (7) in its standard form that consists solely of compositions, a combinatorial coefficient, and a trigonometric sum, as follows:

$$\frac{1}{q^2}\mathrm{tr}\,H'^{2n}=2n\sum_{\substack{\tilde{l}_1,\dots,\tilde{l}_{j+1};\tilde{l}'_1,\dots,\tilde{l}'_{j+1};l_1,\dots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}}}c_{1,1,2}(\tilde{l}_1,\dots,\tilde{l}_{j+1};\tilde{l}'_1,\dots,\tilde{l}'_{j+1};l_1,\dots,l_j)\frac{1}{q}\sum_{k=1}^{q-j}t_k^{\tilde{l}_1}\bar{t}_k^{\tilde{l}'_1}s_k^{l_1}t_{k+1}^{\tilde{l}_2}\bar{t}_{k+1}^{\tilde{l}'_2}s_{k+1}^{l_2}\dots,\tag{8}$$

which indicates a mixture of $g=1$, $g=1$, and $g=2$ exclusion. We call it (1,1,2)-exclusion statistics. Therefore,

$$\mathbf{Tr}\,H^{2n}=\frac{1}{q^2}\mathrm{tr}\,H^{2n}=\frac{1}{q^2}\mathrm{tr}\,H'^{2n}=-\frac{2n}{q}b'_{2n}.\tag{9}$$

That is, $\mathbf{Tr}\,H^{2n}$ is equivalent, up to a trivial factor, to the cluster coefficient b'_{2n} associated with the $2n$-body partition function for particles in a one-body spectrum ϵ_k ($k=1,\dots,q$) obeying a mixture of three statistics: fermions with Boltzmann factor $\mathrm{e}^{-\beta\epsilon_k}=t_k$, fermions of another type with Boltzmann factor $\mathrm{e}^{-\beta\epsilon_k}=\bar{t}_k$, and two-fermion bound states occupying one-body energy levels k and $k+1$ with Boltzmann factor $\mathrm{e}^{-\beta\epsilon_{k,k+1}}=-s_k$ behaving effectively as $g=2$ exclusion particles. b'_{2n} is *constrained* by the requirement that

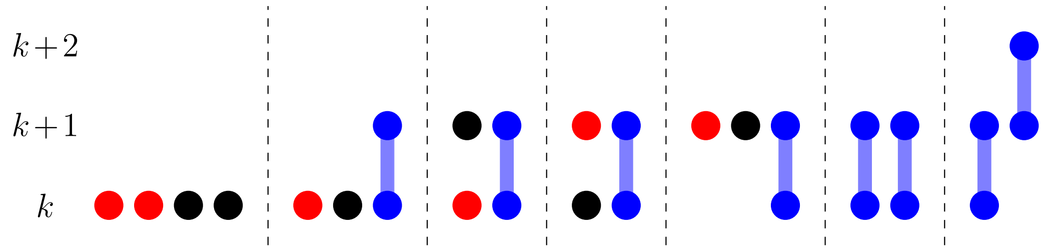

FIG. 3. Seven (1,1,2)-compositions of 4 with $\tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}$: (2; 2; 0), (1, 0; 1, 0; 1), (1, 0; 0, 1; 1), (0, 1; 1, 0; 1), (0, 1; 0, 1; 1), (0, 0; 0, 0; 2), (0, 0, 0; 0, 0, 0; 1, 1), illustrated by two types of fermions (red, black) and two-fermion bound states (blue).

the numbers of the two types of fermions are equal, implying $\mathbf{Tr}\, H^{2n+1}=0$ as expected. Note that setting $t_k=\bar{t}_k=0$ in (8) eliminates all terms with nonzero $\tilde{l}_i, \tilde{l}'_i$'s and (9) effectively reduces to (5).

C. Combinatorial interpretation

The (1,1,2)-compositions with the constraint $\tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}$ have a combinatorial interpretation, which can be derived from their relation to cluster coefficients of (1,1,2)-exclusion statistics. Specifically, (1,1,2)-compositions of $2n$ with constraints correspond to all distinct connected arrangements of $2n$ particles on a one-body spectrum, consisting of two types of fermions (with equal numbers) and two-fermion bound states. In other words, they represent all the possible ways to place two types of particles and bound states on the spectrum such that they cannot be separated into two or more mutually nonoverlapping groups. For example, as shown in Fig. 3, there are seven (1,1,2)-compositions of 4 with $\tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}$, which contribute to

$$-b'_4=\frac{1}{4q}\mathrm{tr}\, H'^4=\frac{3}{2}\sum_{k=1}^{q}t_k^2\bar{t}_k^2+2\sum_{k=1}^{q-1}t_k\bar{t}_k s_k$$

$$+\sum_{k=1}^{q-1}t_k s_k\bar{t}_{k+1}+\sum_{k=1}^{q-1}\bar{t}_k s_k t_{k+1}$$

$$+2\sum_{k=1}^{q-1}s_k t_{k+1}\bar{t}_{k+1}+\frac{1}{2}\sum_{k=1}^{q-1}s_k^2+\sum_{k=1}^{q-2}s_k s_{k+1}.$$

Following the argument in [14], $2n\, c_{1,1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j)$ admits an interpretation as the number of periodic generalized Motzkin paths of length $2n$ with $\tilde{l}_i$ horizontal steps (1,0), $\tilde{l}'_i$ horizontal steps (1,0) of another type, and l_i up steps (1,1) originating from the ith floor (see Fig. 4 for an example). By "periodic generalized Motzkin paths" we refer to generalized Motzkin bridges (and excursions) that start and end on the same floor.

III. CONCLUSION

Based on (4), (7), (9), and the fact that the trigonometric sum $\frac{1}{q}\sum_{k=1}^{q-j}s_k^{l_1}s_{k+1}^{l_2}\ldots s_{k+j-1}^{l_j}$ can be computed [2,3], we deduce the desired counting for closed random walks on a cubic lattice with given length $2n$ and algebraic area A

$$C_{2n}(A)=2n\sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}}}\frac{(\tilde{l}_1+\tilde{l}'_1+l_1-1)!}{\tilde{l}_1!\tilde{l}'_1!l_1!}\prod_{k=2}^{j+1}\binom{l_{k-1}+\tilde{l}_k+\tilde{l}'_k+l_k-1}{l_{k-1}-1,\tilde{l}_k,\tilde{l}'_k,l_k}\sum_{k_3=-l_3}^{l_3}\sum_{k_4=-l_4}^{l_4}\cdots\sum_{k_j=-l_j}^{l_j}$$
$$\binom{2l_1}{l_1+A-\sum_{i=2}^{j+1}(i-1)(\tilde{l}_i-\tilde{l}'_i)+\sum_{i=3}^{j}(i-2)k_i}\binom{2l_2}{l_2-A+\sum_{i=2}^{j+1}(i-1)(\tilde{l}_i-\tilde{l}'_i)-\sum_{i=3}^{j}(i-1)k_i}\prod_{i=3}^{j}\binom{2l_i}{l_i+k_i}. \qquad (10)$$

Note that the enumeration can be computed recursively as well. See Appendix A for further details and several examples of $C_{2n}(A)$. In Appendix B, we present some combinatorial results for (1,1,2)-compositions and $c_{1,1,2}$, where the overall counting of closed $2n$-step cubic lattice walks is recovered to be $\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$.

In the continuum limit, in which the lattice spacing $a\to 0$, closed cubic lattice walks become 3D closed Brownian curves. The probability distribution of the enclosing an algebraic area A for a closed Brownian curve after a time t is given by

$$P'(A)=\frac{\pi}{2\sqrt{3}t}\frac{1}{\cosh^2[\pi A/(\sqrt{3}t)]}. \qquad (11)$$

Note that this distribution is simply the Fourier transform of the partition function of a charged particle moving in continuous 3D space coupled to a uniform magnetic field $\mathbf{B}=(1,1,1)$. By aligning the magnetic field with the z direction through a change of coordinates, we obtain the standard Landau levels plus free motion in the z direction. This

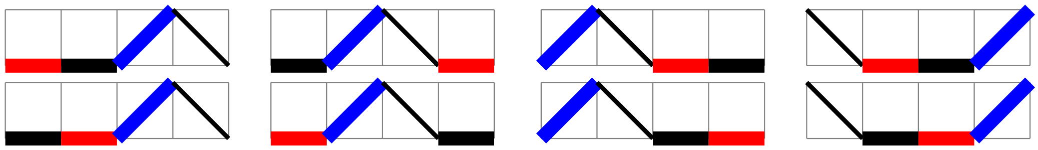

FIG. 4. All the $4c_{1,1,2}(1, 0; 1, 0; 1) = 8$ periodic generalized Motzkin paths of length $2n = 4$ with $\tilde{l}_1 = 1$ horizontal step (red), $\tilde{l}'_1 = 1$ horizontal step of another type (black), and $l_1 = 1$ up step (blue) originating from the first floor.

explains why (11) coincides with Lévy's law (1) for 2D closed Brownian curves, up to a rescaling of A due to the normalization of $\mathbf{B}$. With the scaling $\mathbf{n}a^2 = 3t$, we infer from (11) the asymptotics for (10) as the walk length $\mathbf{n} = 2n \to \infty$

$$C_{\mathbf{n}}(A) \sim \frac{\sqrt{3}\pi}{2\mathbf{n}\cosh^2(\sqrt{3}\pi A/\mathbf{n})}\binom{\mathbf{n}}{\mathbf{n}/2}\sum_{k=0}^{\mathbf{n}/2}\binom{\mathbf{n}/2}{k}^2\binom{2k}{k}, \tag{12}$$

where $A = 0, \pm 1, \pm 2, \dots$ is dimensionless. The asymptotics (12) has been checked numerically for $\mathbf{n}$ up to 50. However, deriving (12) directly from (10) is nontrivial and remains an open problem.

It is natural to extend the definition of the algebraic area for a cubic lattice walk to the sum of projection areas with *arbitrary* weights, which is equivalent to specifying an arbitrary magnetic field $\mathbf{B}$. For instance, when $\mathbf{B} = (0, 0, 1)$, the algebraic area is defined as the area of the walk projected onto the xy plane. The counting for closed $\mathbf{n}$-step cubic lattice walks enclosing an algebraic area A under this definition turns out to be

$$C'_{\mathbf{n}}(A) = \sum_{l=0}^{\mathbf{n}/2}\binom{\mathbf{n}}{2l, \mathbf{n}/2 - l, \mathbf{n}/2 - l}C_{2l,\mathrm{sq}}(A),$$

where $C_{2l,\mathrm{sq}}(A)$ is the number of closed $2l$-step square lattice walks enclosing an algebraic area A. Similarly, as $\mathbf{n} \to \infty$,

$$C'_{\mathbf{n}}(A) \sim \frac{3\pi}{2\mathbf{n}\cosh^2(3\pi A/\mathbf{n})}\binom{\mathbf{n}}{\mathbf{n}/2}\sum_{k=0}^{\mathbf{n}/2}\binom{\mathbf{n}/2}{k}^2\binom{2k}{k}.$$

The methodology used to define an algebraic area for a cubic lattice walk can be extended to other 3D lattices, such as deformed triangular and honeycomb lattices (see Fig. 5). However, the associated enumeration formulas and their connection with quantum exclusion statistics remain unresolved issues that require further study. Additionally, exploring the algebraic area enumeration for *open* random walks on various 3D lattices would also be of interest (see [15,16] for open walks on a square lattice).

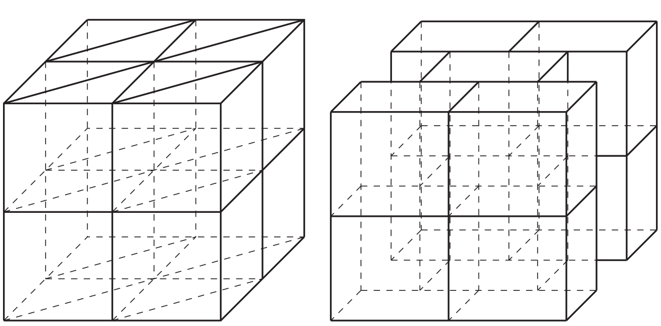

FIG. 5. Deformed triangular and honeycomb lattices in three dimensions.

In addition to the algebraic area enumeration, the explicit expression for $\operatorname{tr} H^{2n}$ can also be regarded as a term, up to a factor, in the expansion of the partition function $\operatorname{tr}(e^{-\beta H})$ for the 3D Hofstadter model, which provides a reference from a different perspective for investigating the spectrum and its intriguing fractal properties. Furthermore, our results have potential implications for the study of spin models with complex frustration graphs. In a frustration graph, each vertex corresponds to a Hamiltonian term. The edges connect all the terms that do not commute. Unconnected terms mutually commute. Analyzing the 3D Hofstadter Hamiltonian (3) offers insights into solving the spectrum of spin models whose frustration graphs contain even holes (see [17] for free-fermionic systems, each with an even-hole-free and claw-free frustration graph).

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APPENDIX A: RECURRENCE RELATION FOR ENUMERATION OF CUBIC LATTICE WALKS

Consider an $\mathbf{n}$-step cubic lattice walk that consists of m_1 steps in the direction (1, 0, 0), m_2 steps in the

TABLE I. $C_{2n}(A)$ up to $2n = 10$ for cubic lattice walks of length $2n$.

	$2n = 2$	4	6	8	10
$A = 0$	6	66	948	16 626	338 616
± 1		24	756	19 392	483 420
± 2			144	6744	230 340
± 3			12	1584	82 980
± 4				336	27 000
± 5				48	7740
± 6					1980
± 7					420
± 8					60
Total counting	6	90	1860	44 730	1 172 556

direction $(-1,0,0)$, l_1 steps in the direction (0,1,0), l_2 steps in the direction $(0,-1,0)$, r_1 steps in the direction (0,0,1), r_2 steps in the direction $(0,0,-1)$, where $m_1+m_2+l_1+l_2+r_1+r_2=\mathbf{n}$. If the walk is open, we can close it by adding a straight line that connects the endpoint to the starting point. Let $C_{m_1,m_2,l_1,l_2,r_1,r_2}(A)$ denote the number of such walks that enclose an algebraic area A. The generating function $Z_{m_1,m_2,l_1,l_2,r_1,r_2}(\mathrm{Q})=\sum_A C_{m_1,m_2,l_1,l_2,r_1,r_2}(A)\mathrm{Q}^A$ can be computed by the recursion

$$\begin{aligned}Z_{m_1,m_2,l_1,l_2,r_1,r_2}(\mathrm{Q}) =& \mathrm{Q}^{(l_2-l_1+r_1-r_2)/2}Z_{m_1-1,m_2,l_1,l_2,r_1,r_2}(\mathrm{Q})\\ &+\mathrm{Q}^{(l_1-l_2+r_2-r_1)/2}Z_{m_1,m_2-1,l_1,l_2,r_1,r_2}(\mathrm{Q})\\ &+\mathrm{Q}^{(m_1-m_2+r_2-r_1)/2}Z_{m_1,m_2,l_1-1,l_2,r_1,r_2}(\mathrm{Q})\\ &+\mathrm{Q}^{(m_2-m_1+r_1-r_2)/2}Z_{m_1,m_2,l_1,l_2-1,r_1,r_2}(\mathrm{Q})\\ &+\mathrm{Q}^{(m_2-m_1+l_1-l_2)/2}Z_{m_1,m_2,l_1,l_2,r_1-1,r_2}(\mathrm{Q})\\ &+\mathrm{Q}^{(m_1-m_2+l_2-l_1)/2}Z_{m_1,m_2,l_1,l_2,r_1,r_2-1}(\mathrm{Q}),\end{aligned}$$

with $Z_{0,0,0,0,0,0}(\mathrm{Q})=1$ and $Z_{m_1,m_2,l_1,l_2,r_1,r_2}(\mathrm{Q})=0$ whenever $\min(m_1,m_2,l_1,l_2,r_1,r_2)<0$.

For closed walks of length $\mathbf{n}=2n$, we have

$$\sum_A C_{2n}(A)\mathrm{Q}^A=\sum_{m=0}^{n}\sum_{l=0}^{n-m}Z_{m,m,l,l,n-m-l,n-m-l}(\mathrm{Q}). \tag{A1}$$

Table I lists some examples of $C_{2n}(A)$.

APPENDIX B: COMBINATIORIAL RESULTS FOR (1,1,2)-COMPOSITIONS AND $c_{1,1,2}$

By considering the combinatorial interpretation of cluster coefficient b'_{2n} as fermions of two types and two-fermion bound states, we can derive the counting of (1,1,2)-compositions of $2n$ with $\tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}$ to be

$$\begin{aligned}N_{1,1,2}(n)&=1+\sum_{k=0}^{n-1}\sum_{m=0}^{k}\binom{k}{m}\binom{n+m-k}{m+1}^2\\ &=1,2,7,27,108,443,\ldots,\end{aligned}$$

with the convention $N_{1,1,2}(0)=1$. Equivalently, the generating function of the $N_{1,1,2}(n)$'s is

$$\sum_{n=0}^{\infty}x^nN_{1,1,2}(n)=\frac{1-x}{\sqrt{x^4-2x^3+7x^2-6x+1}}.$$

We have

$$2n\sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}\\ l_1+\cdots+l_j=k}}c_{1,1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j)=\binom{2n}{n}\binom{n}{k}^2,$$

from which we infer

$$2n\sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}}}c_{1,1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j)=\sum_{k=0}^{n}\binom{2n}{n}\binom{n}{k}^2=\binom{2n}{n}^2.$$

In the limit $q\to\infty$, i.e., $\mathrm{Q}\to 1$ [2,3]

$$\frac{1}{q}\sum_{k=1}^{q-j}t_k^{l_1}\bar{t}_k^{\tilde{l}'_1}s_k^{l_1}t_{k+1}^{\tilde{l}_2}\bar{t}_{k+1}^{\tilde{l}'_2}s_{k+1}^{l_2}\ldots\to\binom{2(l_1+\cdots+l_j)}{l_1+\cdots+l_j},$$

we recover the overall counting to be

$$\begin{aligned}&2n\sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}}}c_{1,1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j)\binom{2(l_1+\cdots+l_j)}{l_1+\cdots+l_j}\\ &=\sum_{k=0}^{n}\sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j\\ (1,1,2)\text{-composition of }2n\\ \tilde{l}_1+\cdots+\tilde{l}_{j+1}=\tilde{l}'_1+\cdots+\tilde{l}'_{j+1}\\ l_1+\cdots+l_j=k}}2n\,c_{1,1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};\tilde{l}'_1,\ldots,\tilde{l}'_{j+1};l_1,\ldots,l_j)\binom{2k}{k}=\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k},\end{aligned}$$

which is indeed $[x^0y^0z^0](x+y+z+x^{-1}+y^{-1}+z^{-1})^{2n}=6,90,1860,44730,1172556,\ldots$ (see the OEIS sequence A002896).

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