Powering quantum Otto engines only with q-deformation of the working substance

Fatih Ozaydin,[1,2,*] Özgür E. Müstecaplıoğlu,[3,4] and Tuğrul Hakioğlu[5,6,7]

[1]*Institute for International Strategy, Tokyo International University, 4-42-31 Higashi-Ikebukuro, Toshima-ku, Tokyo 170-0013, Japan*
[2]*Department of Information Technologies, Isik University, Sile, Istanbul 34980, Turkey*
[3]*Department of Physics, Koç University, Sarıyer, İstanbul 34450, Türkiye*
[4]*TÜBİTAK Research Institute for Fundamental Sciences, 41470 Gebze, Türkiye*
[5]*Energy Institute, İstanbul Technical University, Sarıyer, İstanbul 34467, Türkiye*
[6]*Department of Physics Engineering, İstanbul Technical University, Sarıyer, İstanbul 34467, Türkiye*
[7]*Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA*

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We consider a quantum Otto cycle with a q-deformed quantum oscillator working substance and classical thermal baths. We investigate the influence of the quantum statistical deformation parameter q on the work and efficiency of the cycle. In usual quantum Otto cycle, a Hamiltonian parameter is varied during the quantum adiabatic stages while the quantum statistical character of the working substance remains fixed. We point out that even if the Hamiltonian parameters are not changing, work can be harvested by quantum statistical changes of the working substance. Work extraction from thermal resources using quantum statistical mutations of the working substance makes a quantum Otto cycle without any classical analog.

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I. INTRODUCTION

Quantum heat engines (QHEs) are devices that can harvest work using a quantum working substance between hot and cold reservoirs [1,2]. After the foundations of the quantum engines have been established, many researchers have devoted intense theoretical and experimental efforts to find new breakthroughs [3–17]. Enhancement of work and efficiency of such quantum machines, together with exploring their fundamental bounds, are among the significant goals of the emerging field of quantum thermodynamics. For that aim, nonlinear, many-body, fermionic or bosonic working systems have been studied to reveal their differences and relative advantages [3]. Here, we contribute to these research endeavors by addressing two questions. First, how does the engine performance depend on quantum statistics in general if we mutate the particle statistics? Second, can we consider quantum statistics as another control parameter such that if all the system parameters remain the same, we can harvest work from a heat bath by only changing the quantum statistics of the working substance?

In the 1970s, the concept of deformed algebras was first initiated as a generalization of Weyl-Heisenberg algebra [18,19]. The theory of the q-oscillators was stated previously by Mcfarlane and Biedenharn [20,21]. Since then, q-deformation has been considered in various research areas including statistical physics and quantum information [22], nuclear and atomic physics [23], thermodynamics [24], and open quantum systems and optomechanical systems [25].

It is pointed out that there is a correspondence between q-deformed Heisenberg algebra and effective nonlinear interaction of the cavity mode [26] and an isomorphism between the q-deformed harmonic oscillator and an anharmonic oscillator model was discovered [27]. Physical realization of the deformation parameter q has been searched for heavily, among which are the quantum Yang-Baxter equation [28], deformed Jaynes-Cummings model [29], quantum phase problem [30–32], relativistic q-oscillator [33,34], Morse oscillator [35], and Kepler problem [36]. Deformed algebras have been explored by subjecting the nondeformed ones to nonlinear invertible transformations [37–39]. The q-deformation parameter was considered for deriving generalized uncertainty and information relations [40], Tsallis entropy, and other relative entropy measures [41–43].

In atomic and nuclear physics, q-deformation was considered from theoretical and experimental perspectives [23,44–49]. It was considered for obtaining generalizations of quantum spin chains with exact valence-bond ground states [50] and self-localized solitons of q-deformed quantum systems have been recently explored [51]. Deformed algebra is also used in open quantum systems to show the relationship between the efficiency of QHEs and the degree of the non-Markovianity cycle processes [24]. One of the recent works showed that two linearly coupled q-deformed cavities could be tuned to provide enhancement of nonclassical phenomena [25].

In one of the earliest works on q-deformed quantum information, entanglement and noise reduction techniques were studied between q-deformed harmonic oscillators [52]. Nonclassical properties of noncommutative states [53,54], coherent and cat states in Fock representation [55,56],

*fatih@tiu.ac.jp

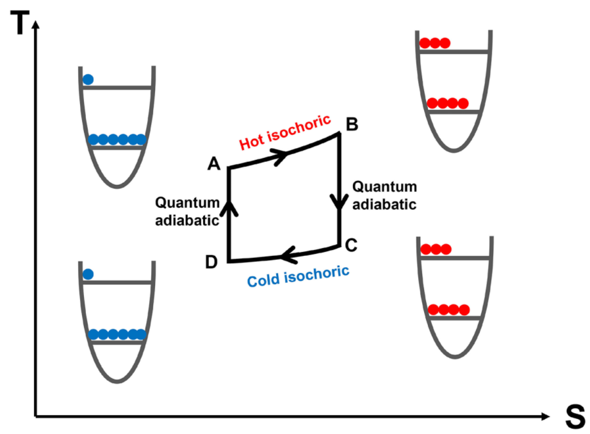

FIG. 1. Entropy-temperature (S-T) diagram of the quantum Otto cycle. In the isochoric processes, the system is in contact with hot and cold baths, while during the quantum adiabatic stages, quantum statistical character of the oscillator changes. Populations, therefore the entropies, remain the same during the quantum adiabatic stages, differentiating the quantum Otto cycle from its classical counterpart.

and entanglement in nonlinear quantum systems were analyzed in q-deformed settings [57]. Quantum states and logic gates were defined for two- and three-level q-deformed systems [22,58,59] and q-deformed relative entropies were studied in quantum metrology [60].

Our work presents a unique perspective on the quantumness of heat engines, which reflects the genuine quantum statistical character of the working system in harvesting work from classical thermal resources. The usual method to characterize the quantum nature of a heat engine is to look for quantum-enhanced performance over its classical analog. Our case is another yet more direct reflection of the quantumness of a heat engine as the cycle mechanism, which is based upon harvesting work by changing the quantum statistical character of its working substance, has no classical analog.

This paper is organized as follows. Section II introduces the q-deformed quantum oscillator as the working system of the quantum Otto cycle. Section III presents the necessary tools to construct a q-deformed heat engine by discussing the fundamental thermodynamic quantities such as entropy and internal energy. Then, equipped with the theoretical tools presented in previous sections, we present our results in Sec. IV, where we explicitly show how work harvesting from thermal resources can be achieved by varying the particle statistics. We compare our results with previous investigations and discuss the effectiveness of our approach in Sec. V. We conclude in Sec. VI.

II. WORKING SYSTEM: q-DEFORMED OSCILLATOR

We consider a q-deformed harmonic oscillator as the working substance of the Otto cycle as illustrated in Fig. 1. The usual commutation relation of the Weyl-Heisenberg algebra of the quantum harmonic oscillator is deformed in the case of a q-deformed quantum oscillator according to [24,61]

$$[\hat{a}, \hat{a}^\dagger]_q = \hat{a}\hat{a}^\dagger - q^{-1}\hat{a}^\dagger\hat{a} = q^{\hat{N}}, \tag{1}$$

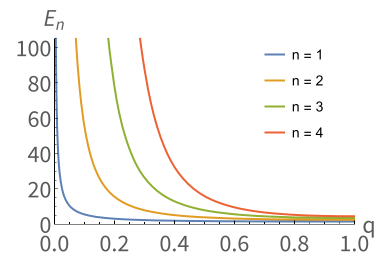

FIG. 2. Energy eigenvalues as a function of the deformation parameter q for energy eigenindex $n = 1, 2, 3, 4$.

where $\hat{N}$ is a number operator with eigenstates $|n\rangle$ such that $\hat{N}|n\rangle = n|n\rangle$, a and $a^\dagger$ are lowering and raising operators in the spectrum of $\hat{N}$ such that $a^\dagger a|n\rangle = [n]|n\rangle$, and q is the deformation parameter.

In terms of $\hat{a}$ and $\hat{a}^\dagger$, the Hamiltonian of the q-deformed quantum oscillator with natural frequency ω is written as

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}a^\dagger + \hat{a}^\dagger\hat{a}). \tag{2}$$

Using the q-number notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{3}$$

the eigenenergies of the Hamiltonian (2) can be expressed in the form

$$E_n = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{4}$$

These eigenenergies are associated with the deformed Fock number eigenstates

$$|n\rangle = \frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle, \tag{5}$$

where the q-factorial is defined to be $[n]! = [n][n-1]\cdots[1]$.

With a real valued r, the deformation parameter can be considered to be a pure phase factor as $q = \exp(ir)$ or to be a real number as $q = \exp(r)$ [62].

Here we focus on the nonlinear characteristics of the quantum oscillator associated with q-deformation and the deformation of the bosonics system; hence we consider only real values with $0 < q < 1$ [61,62]. The usual quantum harmonic oscillator relations can be recovered by substituting $q = 1$ corresponding to the nondeformed case,

$$E_n = \frac{\hbar\omega}{2}\frac{\sinh\left[r\left(n+\frac{1}{2}\right)\right]}{\sinh\left(\frac{r}{2}\right)}, \tag{6}$$

where $r = \ln(q)$ and $n = 0, 1, \ldots < \infty$ is the energy eigenindex. E_n is plotted in Fig. 2 as a function of q for $n = 1, 2, 3, 4$.

Deformation parameter q gives strong nonlinear character to the quantum oscillator, reflected in the anharmonicity of the energy spectrum. Without the deformation ($q = 1$), the quantum Otto cycle can only be implemented by changing ω in the quantum adiabatic stages. When ω is increased, the uniform energy gaps ($\hbar\omega$) between the eigenenergies increase, which can be compared to the case of compressing the volume of the oscillator. In contrast, if ω is decreased, the energy gaps shrink as if the volume of the oscillator is increasing. Accordingly, an effective pistonlike behavior can be translated to the quantum oscillator by ω variation. For the case of a deformed quantum oscillator ($q \neq 1$), the oscillator has nonuniform energy gaps and their splitting can be further controlled by q. By keeping ω constant, we can induce a pistonlike behavior to a deformed oscillator by changing its quantum statistics *per se*. Such a q-deformed oscillator uses q as the quantum adiabatic control (expansion and compression) parameter and it can exploit the anharmonic energy spectrum (for more expansion and compression effects) to enhance the work harvesting and efficiency. Furthermore, alternatively we can still keep the q parameter constant and vary ω as usual, but determine a critical quantum statistics (critical q) for which the work harvesting and (or) efficiency would be maximum.

III. QUANTUM OTTO CYCLE

The canonical partition function of the working system with Hamiltonian $\hat{H}$ in Eq. (2) at temperature T is given by

$$Z = \mathrm{Tr}[\exp(-\hat{H}/T)] = \sum_n e^{-E_n/T}, \tag{7}$$

where we take the Boltzmann constant k_B as unity. Such a nondeformed structure of the partition function relies on the assumption of the nondeformed Gibbsian form of the thermal equilibrium state $\rho \sim \exp(-\beta\hat{H})$ with $\beta = 1/T$ and the expectation value $\langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$; it is associated with the assumption that the Boltzmann-Gibbs form of the entropy function $S = \sum p_i ln p_i$ is preserved [63], where p_is are the associated probabilities.

The quantum version of the classical Otto cycle [64] has been experimentally realized with quantum working substances [17,65]. As illustrated in Fig. 1, it consists of two quantum adiabatic and two isochoric heating and cooling stages. We can describe the cycle in energy-population space for a quantum number n. At point A, the system is given with energy levels $E_n(A)$ and their populations $P_n(A)$. Under isochoric heating, the system is transformed ($A \rightarrow B$) to a thermal state at point B such that $E_n(B) = E_n(A)$ and the population is changed to $P_n(B)$. Subsequently, the bath is removed and the system is quantum adiabatically transformed ($B \rightarrow C$) obeying the conditions $P_n(C) = P_n(B)$, while the energy eigenvalues are adiabatically changed to $E_n(C)$. In the third stage, the system is isochorically cooled ($C \rightarrow D$) to point D such that $E_n(D) = E_n(C)$, while the population is modified to $P_n(D)$. Finally, the system is reset to its starting point A, ($D \rightarrow A$), by closing the cycle with another quantum adiabatic transformation under the condition $P_n(A) = P_n(D)$.

In classical Otto cycles, the control parameter in the isentropic steps is typically the physical volume of the system. Here, we consider two types of quantum Otto cycles. (i) We keep the frequency of the quantum oscillator ω constant and vary the deformation parameter q during the quantum adiabatic steps. (ii) We keep q constant and change ω in the quantum adiabatic stages. During these steps, the system is uncoupled from the thermal baths and the system is quantum adiabatically transformed such that occupation probabilities P_n of the eigenenergies E_n do not change.

For a classical heat engine, it is sufficient to make the transformation faster than the rate of heat exchange, instead of physically uncoupling the system from the environment, to ensure the adiabatic condition. In the quantum case, the transformation needs to be slower than the characteristic time scale for the transitions between the energy levels to ensure P_n remains the same. The quantum case is also an isentropic process, while the classical case does not necessarily satisfy the constant population condition. During the other two steps, isochoric heating and cooling, thermal baths are coupled to the oscillator system while the parameters of the Hamiltonian and the quantum statistical parameter q are kept constant.

Using the energy eigenvalues E_n in Eq. (4) and associated populations

$$P_n = \exp(-E_n/T)/Z, \tag{8}$$

we evaluate the work output (W) of the quantum Otto cycle according to the formula [3]

$$W = \sum_n [E_n(A) - E_n(C)][P_n(B) - P_n(A)]. \tag{9}$$

$P_n(A)$ is the population of nth energy level at the beginning of the isochoric heating. We can use the quantum adiabatic condition $P_n(A) = P_n(D)$ to evaluate it by using the thermal distribution as the system is in thermal equilibrium at point D. $P_n(B)$ is calculated from a thermal distribution as well, as it is the level population at the end of isochoric heating. The factor $E_n(A) - E_n(C)$ is the net variation of the energy eigenvalue during the quantum adiabatic transformations.

Thermal efficiency of the engine is defined by

$$\eta = \frac{W}{Q_{\mathrm{in}}}. \tag{10}$$

Here, the injected heat into the system is given by

$$Q_{\mathrm{in}} = \sum_n E_n(A)[P_n(B) - P_n(A)], \tag{11}$$

where again we can use $P_n(A) = P_n(D)$ to evaluate it using thermal population distribution at point D.

IV. RESULTS

A. Work harvesting by statistical mutation

Let us start by changing the deformation parameter in the quantum adiabatic stages of the cycle. For a q-deformed quantum oscillator, a decrease in q yields an exponential increase in the energy gaps at higher energy levels. Thus we take a smaller q value (q_A) for hot isochore (T_h) and a higher q value (q_C) for cold isochore (T_c). During the isentropic stages, q varies between the values used in hot and cold isochores. Hamiltonian parameters remain fixed at $\omega = 1$ and only the q-deformation parameter of the working substance is varied in the cycle.

Population differences $P_n(B) - P_n(A)$, the factor $E_n(A) - E_n(C)$, and their multiplication in Eq. (9) for the nth energy level and q_A at $T_h = 0.5$ are presented in Figs. 3–5, respectively, for $q_C = 1$ at $T_c = 0.1$. Here, $q_C = 1$ corresponds to a nondeformed case showing that positive work can also be extracted by changing the statistics of the substance only for the hot isochore. Efficiency is meaningful and well defined only when this positive work condition is satisfied. Analytical expression for the positive work condition could be possible

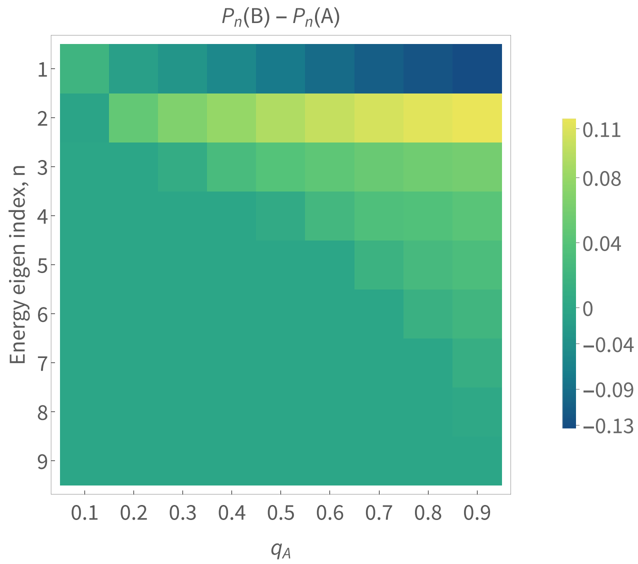

FIG. 3. Population differences $P_n(B) - P_n(A)$ for n and q_A at $T_h = 0.5$ with $q_C = 1$ at $T_c = 0.1$.

if the energy gaps would change uniformly with q [3], which is not the case for our deformed oscillator. Accordingly, we resort to the numerical calculations of the work for a range of temperature and q to determine the positive work domains.

Summing the work values over n in Fig. 5, the extractable work (W) with respect to q_A is presented in Fig. 6 at $T_h = 0.5$ for three choices of q_C at $T_c = 0.1$. Figure 7 presents the efficiency as a function of q_A with the same choices of q_C and bath temperatures. Quantum Otto engine's efficiency approaches the Carnot limit only by q deformation as q_A gets smaller within the positive work domain, which is numerically found and shown in the inset of Fig. 6. Note that it is straightforward to check numerically for any choice of q and bath temperatures that, in the positive work condition, the Carnot limit is satisfied. For example, setting the hot isochore $T_h = 1$ with the same choices as in Fig. 6, the positive work condition is satisfied until $q_A \approx 0.05$, at which the quantum Otto efficiency approaches the Carnot limit $\eta_{\text{Carnot}} = 0.9$.

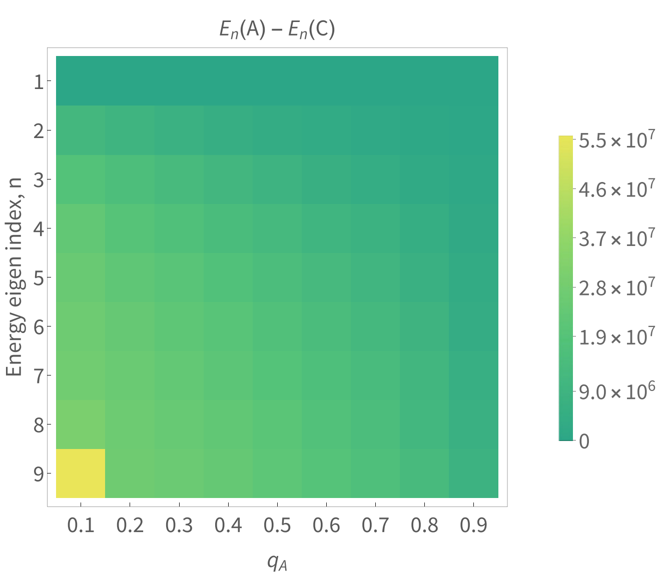

FIG. 4. Net variation of energy eigenvalue $E_n(A) - E_n(C)$ during the adiabatic transformations for n and q_A at $T_h = 0.5$ with $q_C = 1$ at $T_c = 0.1$.

FIG. 5. Extractable work for n and q_A at $T_h = 0.5$ with $q_C = 1$ at $T_c = 0.1$.

B. Critical quantum statistics for optimal work harvesting

For a given temperature difference between hot and cold reservoirs as a classical resource, we can point out a critical quantum statistics of a deformed quantum oscillator to maximize the harvested work or efficiency. Reference [66] has shown that a bosonic system has higher engine performance than a fermionic system and stated that the performance difference occurs due to the difference in internal energies arising from the Pauli exclusion principle. In the same spirit,

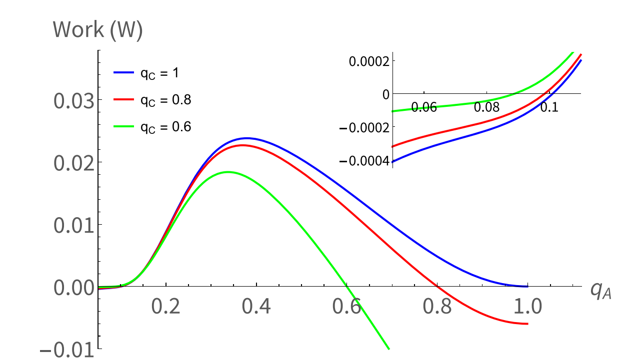

FIG. 6. Work (W) by only changing the quantum statistics of the working substance (q_A) for the hot isochore $(T_h = 0.5)$ for three choices of q_C for the cold isochore $(T_c = 0.1)$ with fixed oscillator frequency $\omega = 1$. Global maxima for W are at $q_A = 0.379$, $q_A = 0.369$, and $q_A = 0.338$, respectively, for the first three choices in descending order of q_C. The inset shows the domain that the positive work condition is not satisfied, i.e., $q_A < 0.101$, $q_A < 0.0984$, and $q_A < 0.0887$, respectively, for $q_C = 1$, $q_C = 0.8$, and $q_C = 0.6$.

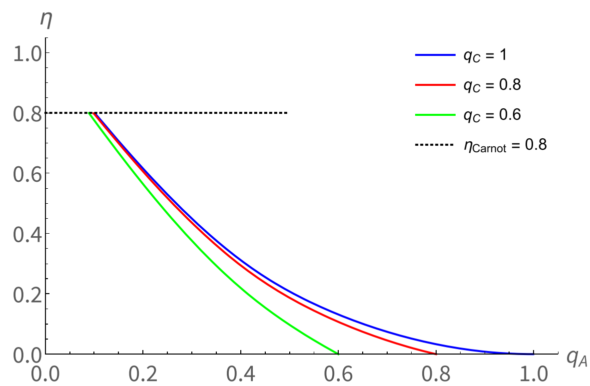

FIG. 7. Efficiency of quantum Otto cycle as a function of q_A (at $T_c = 0.1$) with three choices of q_C (at $T_h = 0.5$). The functions are plotted not for the entire range of q_A but only in the positive work domain as presented in the inset of Fig. 6. As q_A approaches its minimum within the positive work domain, the quantum Otto engine's efficiency η approaches the Carnot limit $\eta_{\rm Carnot} = 1 - T_c/T_h = 0.8$.

we find out that one can optimize the engine performance for a given thermal resource in terms of the quantum statistical character of the working substance. As shown in Fig. 6, $q_A \approx 0.4$ is the optimal value for the working substance for the hot isochore. We now keep $q_A = 0.4$ for various q_C and induce the cycle for different ω. As shown in Fig. 8, for $\omega < 0.2$, there is an optimal q_C that achieves maximum extractable work.

Lastly, for the second type of quantum Otto engine, we keep q constant (i.e., $q_A = q_C$) and change ω in the quantum adiabatic steps, i.e., we choose different ω_A and ω_C. As can be seen in Fig. 9, greater q implies greater extractable work and maximum in the nondeformed case ($q_A = q_C = 1$). This result shows that the advantage of q-deformation with no classical analog is due not to a simply deformed working substance, but rather to utilizing the working substance with different deformation levels at the adiabatic stages even with the same ω.

V. DISCUSSION

We have presented a general formalism showing the advantages of q-deformed oscillators in harvesting work efficiently

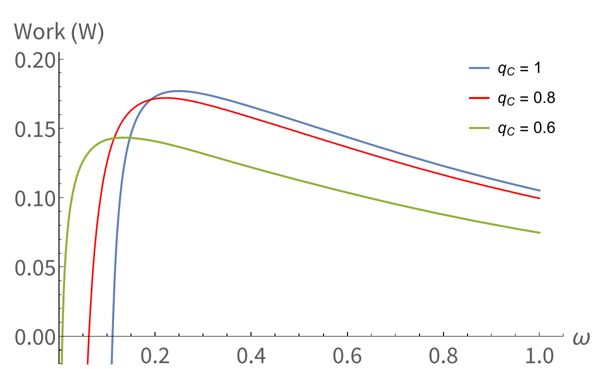

FIG. 8. Extractable work as a function of ω with $q_A = 0.4$ at $T_h = 1$ and q_C at $T_c = 0.1$. For small ω, an optimal choice of q_C yields maximum extractable work.

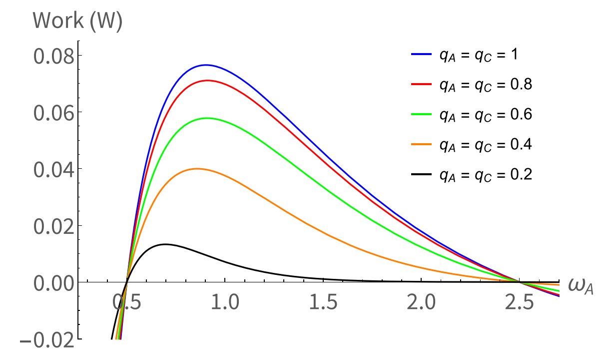

FIG. 9. Extractable work as a function of ω_A with $\omega_C = 0.5$ with $T_c = 0.1$ and $T_h = 0.5$ with the substance deformed at the same level in adiabatic steps ($q_A = q_C$).

from thermal resources in a quantum Otto cycle. Our results are independent of different physical embodiments of the q-deformed working substance. On the other hand, here we can suggest that the q-deformation method can be used to study specific nonlinear systems [67–70] and hence we can use such nonlinear systems to simulate our quantum Otto cycle via tailored nonlinear interactions. Possible examples could be s-wave scattering in atomic BEC [71,72] and Kerr nonlinearity in cavity-QED [73,74].

We remark that we do not include explicitly the work reservoir into the engine model as it is a common treatment in quantum heat engines. It was shown in [75] that, in a thermodynamic cycle using particles with nonlinear interactions as the working substance, adjusting the Feshbach resonances [76] can tune the nonlinear interaction strength to produce work by modifying the volume of the gas. Similarly in our model, it should be understood that the work of the engine will be done against the magnetic field used to modify Feshbach resonances to change the q-parameter associated with the nonlinearity of the working substance.

It is found in accordance with previous works [4,77–80] that the extractable work and efficiency are not monotonic with respect to the interaction parameter, but rather exhibit optimum values as presented in Fig. 6. The effect of q-deformation due to nonlinearity on the nonmonotonicity of the extractable work and efficiency becomes more visible with greater temperature difference between the cold and the hot bath as presented in Figs. 10 and 11, respectively.

Algebras corresponding to both q-deformed fermions and q-deformed bosons, as well as their transformations and unification, are widely studied [23,62,81–83]. Considering potential realizations in atomic BEC systems, also in accordance with the findings of Ref. [66] that the bosonic working system achieves a higher performance than fermionic systems in the nondeformed setting ($q = 1$), in this work we focused on a q-deformed bosonic system as the working substance. Nevertheless, it would be interesting as future research to study different performance characteristics of complex deformations, and q-deformed fermionic systems as well, especially following the unified representation by Lavagno and Swamy [83].

In another work, the effect of q-deformation was studied to show a relationship between the efficiency of QHE and the

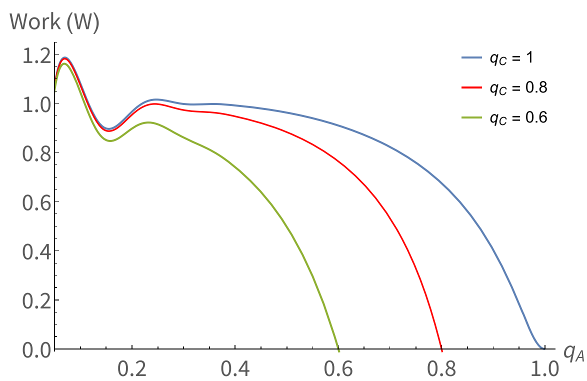

FIG. 10. Work with respect to q_A with a greater temperature difference between hot isochore ($T_h = 5$) and cold isochore ($T_c = 0.1$).

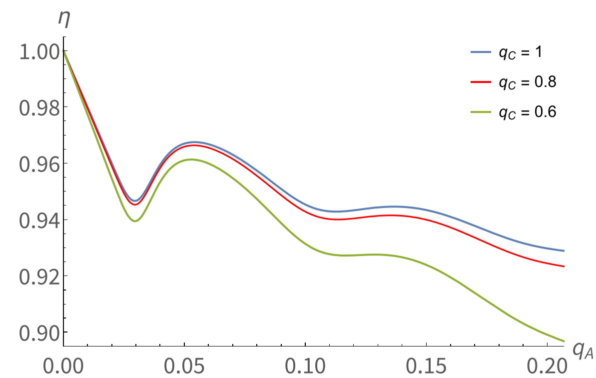

FIG. 11. Efficiency of quantum Otto cycle as a function of q_A at $T_c = 0.1$ and three choices of q_C at $T_h = 100$ with $\omega = 1$.

non-Markovianity in the engine cycle [24]. It states that the q-parameter, which causes nonequilibrium dynamics, helps to build a relationship between theoretical and experimental results. Here, we did not examine finite time engine cycles; the q-parameter, in our case, plays a more active and direct role as the engine cycle's control parameter.

VI. CONCLUSION

Classical heat engines harvest work from a thermal resource by converting a heat flow between a hot and cold bath to an ordered work. To realize this task, parameters of their classical working system, for example, the volume of working gas, are varied in an engine cycle. Quantum heat engines use a quantum working material and, in addition to external degrees of freedom, Hamiltonian parameters and internal degrees of freedom can also be utilized for work extraction. Profound quantum effects, particularly improving engine performance over its classical counterpart, such as via quantum correlations [14], are possible with quantum heat engines.

Here, we show that the quantum statistical character of the working substance can be used as another control parameter of a quantum heat engine. Specifically, we consider a quantum oscillator with a fixed frequency ω but deformed quantum statistics characterized by the q-deformation parameter and verify that such a q-deformed oscillator can harvest work from a thermal resource by variation of the q-parameter. Alternatively, we can optimize the harvested work or efficiency for a given thermal resource by choosing a critical quantum statistics of the deformed oscillator.

While we consider the Otto cycle as a paradigmatic model, we expect that our fundamental conclusion holds for other engine cycles as well. Variation of particle statistics can be experimentally challenging relative to the traditional way of variation of Hamiltonian parameters. However, q-deformation can be envisioned and mapped to nonlinear terms in Hamiltonians and effective engineered deformed oscillator models ranging from semiconductor cavity QED [74] to atomic Bose-Einstein condensates [72] could be explored for physical embodiment of the statistical mutation route of work extraction.

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[1] H. E. D. Scovil and E. O. Schulz-DuBois, Three-level masers as heat engines, Phys. Rev. Lett. **2**, 262 (1959).

[2] J. E. Geusic, E. O. Schulz-DuBios, and H. E. D. Scovil, Quantum equivalent of the Carnot cycle, Phys. Rev. **156**, 343 (1967).

[3] H. T. Quan, Y. X. Liu, C. P. Sun, and F. Nori, Quantum thermodynamic cycles and quantum heat engines, Phys. Rev. E **76**, 031105 (2007).

[4] F. Altintas, A. U. C. Hardal, and O. E. Müstecaplıoğlu, Quantum correlated heat engine with spin squeezing, Phys. Rev. E **90**, 032102 (2014).

[5] F. Altintas, A. U. C. Hardal, and O. E. Müstecaplıoğlu, Rabi model as a quantum coherent heat engine: From quantum biology to superconducting circuits, Phys. Rev. A **91**, 023816 (2015).

[6] J. Roßnagel, O. Abah, F. Schmidt-Kaler, K. Singer, and E. Lutz, Nanoscale heat engine beyond the Carnot limit, Phys. Rev. Lett. **112**, 030602 (2014).

[7] M. T. Naseem and O. E. Müstecaplıoğlu, Quantum heat engine with a quadratically coupled optomechanical system, J. Opt. Soc. Am. B **36**, 3000 (2019).

[8] T. Hugel, N. B. Holland, A. Cattani, L. Moroder, M. Seitz, and H. E. Gaub, Single-molecule optomechanical cycle, Science **296**, 1103 (2002).

[9] R. Kosloff and A. Levy, Quantum heat engines and refrigerators: Continuous devices, Annu. Rev. Phys. Chem. **65**, 365 (2014).

[10] G. Barontini and M. Paternostro, Ultra-cold single-atom quantum heat engines, New J. Phys. **21**, 063019 (2019).

[11] O. Abah, J. Roßnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz, Single-ion heat engine at maximum power, Phys. Rev. Lett. **109**, 203006 (2012).

[12] I. A. Martínez, É. Roldán, L. Dinis, D. Petrov, J. M. R. Parrondo, and R. A. Rica, Brownian Carnot engine, Nat. Phys. **12**, 67 (2016).

[13] J. Roßnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer, A single-atom heat engine, Science **352**, 325 (2016).

[14] A. Tuncer, M. Izadyari, C. B. Dağ, F. Ozaydin, and O. E. Müstecaplıoğlu, Work and heat value of bound entanglement, Quantum Inf. Process. **18**, 373 (2019).

[15] C. B. Dag, W. Niedenzu, F. Ozaydin, O. E. Müstecaplıoğlu, and G. Kurizki, Temperature control in dissipative cavities by entangled dimers, J. Phys. Chem. C **123**, 4035 (2019).

[16] S. Çakmak, Benchmarking quantum Stirling and Otto cycles for an interacting spinsystem, J. Opt. Soc. Am. B **39**, 1209 (2022).

[17] J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental characterization of a spin quantum heat engine, Phys. Rev. Lett. **123**, 240601 (2019).

[18] M. Arik, D. Coon, and Y.-m. Lam, Operator algebra of dual resonance models, J. Math. Phys. **16**, 1776 (1975).

[19] M. Arik and D. D. Coon, Hilbert spaces of analytic functions and generalized coherent states, J. Math. Phys. **17**, 524 (1976).

[20] L. C. Biedenharn, The quantum group $SU_q(2)$ and a q-analogue of the boson operators, J. Phys. A: Math. Gen. **22**, L873 (1989).

[21] A. J. Macfarlane, On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, J. Phys. A: Math. Gen. **22**, 4581 (1989).

[22] A. A. Altintas, F. Ozaydin, C. Yesilyurt, S. Bugu, and M. Arik, Constructing quantum logic gates using q-deformed harmonic oscillator algebras, Quantum Inf. Process. **13**, 1035 (2014).

[23] D. Bonatsos and C. Daskaloyannis, Generalized deformed oscillators for vibrational spectra of diatomic molecules, Phys. Rev. A **46**, 75 (1992).

[24] H. Naseri-Karimvand, B. Lari, and H. Hassanabadi, Non-Markovianity and efficiency of a q-deformed quantum heat engine, Physica A **598**, 127408 (2022).

[25] A. Kundu and J. A. Miszczak, Transparency and enhancement in fast and slow light in q-deformed optomechanical system, Ann. Phys. (Leipzig) **534**, 2200026 (2022).

[26] V. Bužek, The Jaynes-Cummings model with a q analogue of a coherent state, J. Mod. Opt. **39**, 949 (1992).

[27] M. Artoni, J. Zang, and J. L. Birman, Anharmonic and nonclassical effects of the quantum-deformed harmonic oscillator, Phys. Rev. A **47**, 2555 (1993).

[28] Z. Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras* (World Scientific, Singapore, 1993), Vol. 1.

[29] M. Chaichian, D. Ellinas, and P. Kulish, Quantum algebra as the dynamical symmetry of the deformed Jaynes-Cummings model, Phys. Rev. Lett. **65**, 980 (1990).

[30] D. Ellinas, Quantum phase and a q-deformed quantum oscillator, Phys. Rev. A **45**, 3358 (1992).

[31] T Hakioğlu, Admissible cyclic representations and an algebraic approach to quantum phase, J. Phys. A: Math. Gen. **31**, 707 (1998).

[32] T. Hakioğlu, Finite-dimensional Schwinger basis, deformed symmetries, Wigner function, and an algebraic approach to quantum phase, J. Phys. A: Math. Gen. **31**, 6975 (1998).

[33] R. M. Mir-Kasimov, $SU_q(1,1)$ and the relativistic oscillator, J. Phys. A: Math. Gen. **24**, 4283 (1991).

[34] M. Arik and M. Mungan, q-oscillators and relativistic position operators, Phys. Lett. B **282**, 101 (1992).

[35] I. L. Cooper and R. K. Gupta, q-deformed Morse oscillator, Phys. Rev. A **52**, 941 (1995).

[36] O. F. Dayi and I. H. Duru, $sl_q(2)$ realizations for Kepler and oscillator potentials and q-canonical transformations, J. Phys. A: Math. Gen. **28**, 2395 (1995).

[37] T. Curtright, D. Fairlie, and C. Zachos, *Quantum Groups: Proceedings of the Argonne Workshop, Argonne National Laboratory, 16 April–11 May 1990* (World Scientific, Singapore, 1991).

[38] T. Curtright and C. Zachos, Deforming maps for quantum algebras, Phys. Lett. B **243**, 237 (1990).

[39] T. Hakioğlu and M. Arik, Quantum stereographic projection and the homographic oscillator, Phys. Rev. A **54**, 52 (1996).

[40] J. C. van der Lubbe, Y. Boxma, and D. E. Boekee, A generalized class of certainty and information measures, Inf. Sci. **32**, 187 (1984).

[41] J. Naudts, *Generalized Thermostatistics* (Springer, London, 2011).

[42] E. P. Borges and I. Roditi, A family of nonextensive entropies, Phys. Lett. A **246**, 399 (1998).

[43] P. T. Landsberg and V. Vedral, Distributions and channel capacities in generalized statistical mechanics, Phys. Lett. A **247**, 211 (1998).

[44] D. Bonatsos, D. Lenis, P. P. Raychev, and P. A. Terziev, Deformed harmonic oscillators for metal clusters: Analytic properties and supershells, Phys. Rev. A **65**, 033203 (2002).

[45] A. Georgieva, K. Sviratcheva, M. Ivanov, and J. Draayer, q-deformation of symplectic dynamical symmetries in algebraic models of nuclear structure, Phys. At. Nucl. **74**, 884 (2011).

[46] A. A. Altintaş, M. Arik, A. S. Arikan, and E. Dil, Inhomogeneous quantum invariance group of multi-dimensional multi-parameter deformed boson algebra, Chin. Phys. Lett. **29**, 010203 (2012).

[47] M. Hammad, S. Fawaz, M. El-Hammamy, H. Motaweh, and S. Doma, q-deformed vibrational limit of interacting boson model, J. Phys. Commun. **3**, 085019 (2019).

[48] M. Jafarizadeh, N. Amiri, N. Fouladi, M. Ghapanvari, and Z. Ranjbar, Study of phase transition of even and odd nuclei based on q-deformed SU(1,1) algebraic model, Nucl. Phys. A **972**, 86 (2018).

[49] N. Boutabba, S. Grira, and H. Eleuch, Analysis of a q-deformed hyperbolic short laser pulse in a multi-level atomic system, Sci. Rep. **12**, 9308 (2022).

[50] M. T. Batchelor and C. M. Yung, q-deformations of quantum spin chains with exact valence-bond ground states, Int. J. Mod. Phys. B **08**, 3645 (1994).

[51] C. Bayındır, A. A. Altintas, and F. Ozaydin, Self-localized solitons of a q-deformed quantum system, Commun. Nonlinear Sci. Numer. Simul. **92**, 105474 (2021).

[52] G. Dattoli and A. Torre, q-harmonic-oscillator entangled states, Nuovo Cimento B (1971-1996) **111**, 731 (1996).

[53] S. Dey, q-deformed noncommutative cat states and their nonclassical properties, Phys. Rev. D **91**, 044024 (2015).

[54] K. Berrada and H. Eleuch, Noncommutative deformed cat states under decoherence, Phys. Rev. D **100**, 016020 (2019).

[55] H. Fakhri and S. Mousavi-Gharalari, q-cat states revisited: two families in a Fock representation space of q-oscillator algebra with different nonclassical behaviors, Eur. Phys. J. Plus **136**, 1 (2021).

[56] H. Fakhri and S. Mousavi-Gharalari, Nonclassical properties of two families of q-coherent states in the Fock

representation space of q-oscillator algebra, Eur. Phys. J. Plus **135**, 1 (2020).

[57] K. Berrada, M. E. Baz, H. Eleuch, and Y. Hassouni, Bipartite entanglement of nonlinear quantum systems in the context of the q-Heisenberg Weyl algebra, Quantum Inf. Process. **11**, 351 (2012).

[58] A. Filippov, D. Gangopadhyay, and A. Isaev, Harmonic oscillator realization of the canonical q-transformation, J. Phys. A: Math. Gen. **24**, L63 (1991).

[59] A. A. Altintas, F. Ozaydin, and C. Bayındır, q-deformed three-level quantum logic, Quantum Inf. Process. **19**, 247 (2020).

[60] H. Hasegawa, Quantum Fisher information and q-deformed relative entropies: Additivity vs nonadditivity, Prog. Theor. Phys. Suppl. **162**, 183 (2006).

[61] A. Boumali and H. Hassanabadi, The statistical properties of the deformed dirac oscillator in one and two dimensions, Adv. High Energy Phys. **2017**, 9371391 (2017).

[62] D. Bonatsos and C. Daskaloyannis, Quantum groups and their applications in nuclear physics, Prog. Part. Nucl. Phys. **43**, 537 (1999).

[63] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. **52**, 479 (1988).

[64] R. Dittman and M. Zemansky, *Heat and Thermodynamics: An Undergraduate and Intermediate Textbook* (Independently Published, 2020).

[65] Q. Bouton, J. Nettersheim, S. Burgardt, D. Adam, E. Lutz, and A. Widera, A quantum heat engine driven by atomic collisions, Nat. Commun. **12**, 2063 (2021).

[66] N. M. Myers and S. Deffner, Bosons outperform fermions: The thermodynamic advantage of symmetry, Phys. Rev. E **101**, 012110 (2020).

[67] M. Momeni-Demneh, A. Mahdifar, and R. Roknizadeh, Nonlinear optical effects on the atom–field interaction based on the nonlinear coherent states approach, J. Opt. Soc. Am. B **39**, 1353 (2022).

[68] M. Momeni-Demneh, A. Mahdifar, and R. Roknizadeh, Control of the nonclassical properties of the two-mode kerr nonlinear optical system based on the nonlinear coherent states approach, Int. J. Opt. Photon. **16**, 27 (2022).

[69] V. I. Man'ko, G. Marmo, S. Solimeno, and F. Zaccaria, Physical nonlinear aspects of classical and quantum q-oscillators, Int. J. Mod. Phys. A **08**, 3577 (1993).

[70] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, f-oscillators and nonlinear coherent states, Phys. Scr. **55**, 528 (1997).

[71] Z. Haghshenasfard, M. Naderi, and M. Soltanolkotabi, Subluminal to superluminal propagation of an optical pulse in an f-deformed bose–einstein condensate, J. Phys. B: At., Mol., Opt. Phys. **41**, 165501 (2008).

[72] C. W. Gardiner, Particle-number-conserving Bogoliubov method which demonstrates the validity of the time-dependent Gross-Pitaevskii equation for a highly condensed Bose gas, Phys. Rev. A **56**, 1414 (1997).

[73] T. Birol and Ö. E. Müstecaplıoğlu, Phase diffusion of a q-deformed oscillator, Symmetry **1**, 240 (2009).

[74] Y.-X. Liu, C. P. Sun, S. X. Yu, and D. L. Zhou, Semiconductor-cavity QED in high-Q regimes with q-deformed bosons, Phys. Rev. A **63**, 023802 (2001).

[75] J. Li, T. Fogarty, S. Campbell, X. Chen, and T. Busch, An efficient nonlinear Feshbach engine, New J. Phys. **20**, 015005 (2018).

[76] H. Feshbach, Unified theory of nuclear reactions, Ann. Phys. (NY) **5**, 357 (1958).

[77] E. A. Ivanchenko, Quantum Otto cycle efficiency on coupled qudits, Phys. Rev. E **92**, 032124 (2015).

[78] G. Thomas and R. S. Johal, Coupled quantum Otto cycle, Phys. Rev. E **83**, 031135 (2011).

[79] S. Çakmak, D. Türkpençe, and F. Altintas, Special coupled quantum Otto and Carnot cycles, Eur. Phys. J. Plus **132**, 1 (2017).

[80] M. Boubakour, T. Fogarty, and T. Busch, Interaction-enhanced quantum heat engine, Phys. Rev. Res. **5**, 013088 (2023).

[81] P. Narayana swamy, Transformations of q-boson and q-fermion algebras, Mod. Phys. Lett. B **15**, 915 (2001).

[82] S. Sargolzaeipor, H. Hassanabadi, W. Chung, and A. Ikot, q-deformed oscillator algebra in fermionic and bosonic limits, Pramana **93**, 68 (2019).

[83] A. Lavagno and P. Narayana Swamy, Generalized thermodynamics of q-deformed bosons and fermions, Phys. Rev. E **65**, 036101 (2002).