Six-photon resonant scattering of collinear laser pulses in plasma

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(Received 24 May 2023; accepted 25 September 2023; published 23 October 2023)

To achieve the highest possible laser intensities with the least laser energy, shorter-wavelengths lasers are advantaged if they can be focused to spots of a few laser wavelengths and durations of several laser periods. However, the top laser pulse energies available nowadays are megajoules at near-optical wavelengths and millijoules at shorter wavelengths. Thus, to produce the highest laser intensities, what is required is an efficient spectral transfer of the huge near-optical energies to shorter wavelengths. It is proposed here that the desired spectral transfer could occur via resonant photon interactions associated with nonlinearity of mildly relativistic motions of plasma electrons in intense laser fields, specifically via the six-photon resonant scattering of collinear laser pulses in plasma. The six-photon interaction can, in fact, be the dominant resonant photon interaction to achieve collinear frequency up-conversion.

DOI: 10.1103/PhysRevE.108.045208

I. INTRODUCTION

The top laser intensities achievable in laboratories were steadily growing through the last three decades following the invention of chirped pulse amplification (CRA) technique [1,2] and development of optical parametric CPA (OPCPA) [3,4]. The currently greatest intensity 10^{23} W/cm^2 was reached in the state-of-art experiment [5] where the 4 PW laser pulse of 800 nm wavelength was focused to a near-diffraction-limited μm spot and contracted to a near-Fourier-transform-limited 20 fs duration. The pulse energy $\sim$100 J was much smaller than the currently available energies, but the quality was superior. If it were possible to achieve a comparable quality for 10 kJ optical laser pulses, the focused intensity would reach 10^{25} W/cm^2 level. The 10 kJ benchmark is set by the National Ignition Facility (NIF) [6,7], which features 192 such lasers with total energy $\sim$2 MJ. However, producing a 10 kJ pulse in NIF takes a nearly half-meter-size gratings operating at the highest tolerable energy fluence on the surface, which makes getting the superior pulse quality challenging. Reaching 10^{25} W/cm^2 intensity, probably close to the limit achievable within the common technologies now, is aimed in a few exawatt-class laser facilities currently built around the world [8]. However, to achieve the highest possible laser intensities with the least laser energy, shorter-wavelength laser pulses are advantaged if they can be focused to spots of a few laser wavelengths and durations of several laser periods.

Intense shorter-wavelength pulses could be produced in plasma as a high-harmonic radiation coming from nonlinear ultrarelativistic electron oscillations in ultraintense optical laser pulses [9]. However, it is challenging to make the energy transfer into high harmonics sufficiently efficient to achieve the greatest intensities by focusing to smaller spots. It was proposed to increase the efficiency of relativistic third-harmonic generation by modulating plasma density along the pulse propagation direction in a way spatially synchronizing the driving pulse and the third harmonic [10]. Apart from the strong modulation, a more practical alternative was considered there to use a low-frequency small-amplitude density wave in an homogeneous plasma, such as, for example, a long-wavelength ion acoustic wave. However, relying on such waves, it would be challenging to control the stimulated Brillouin and Raman scatterings seeded by noise, along with other parasitic processes.

Higher efficiencies for spectral energy transfer are more readily achievable in resonant photon interactions driven by the nonlinearity of mildly relativistic motions, $v/c = a \ll 1$, of plasma electrons in intense laser fields. The lowest-order nonlinear process of possible interest here would be the paraxial four-photon scattering, capable of nearly doubling laser frequencies. However, it also appears to be challenging. On the one hand, the paraxial angle must be small, $\theta \ll 1$, to avoid excessive transverse slippage. On the other hand, the rate of four-photon scattering appears to contain an extra small factor θ^2 due to the mutual cancellation of the leading nonlinear terms [11], while the evolving nonlinear detuning of the resonance does not contain such a cancellation in general, thus tending to quickly ruin the resonance at $\theta \ll 1$.

These challenges, however, could be handled in the self-channeling regimes [12], at powers mildly exceeding the relativistic self-focusing critical power $P_{cr} \approx 17\omega^2/\omega_e^2$ GW [13–15], where ω is the laser frequency and ω_e is the plasma electron frequency. In these regimes, having $\theta^2 \sim a^2 \sim P/P_{cr} - 1 \ll 1$, the leading terms of relativistic nonlinearity and transverse dispersion cancel each other. The noncancelled terms are of the order of the quintic relativistic nonlinearity, producing the same order transverse bound frequency of the photons in the ground trapped state. The quintic nonlinearity can produce the resonant six-photon scattering, which is allowed by the classical dispersion law even for exactly collinear laser pulses, in contrast to the resonant four-photon scattering. The four-photon coupling does not exceed the quintic nonlinearity at $\theta^2 \sim a^2 \ll 1$. The higher-order interactions tend to be slower at $a^2 \ll 1$. Thus, our goal here is to calculate the rate of resonant collinear six-photon scattering to see if this previously unknown rate could provide noticeable amplifications within tabletop distances. The positive answer to this question would strongly motivate further research in more complex geometries, such as the paraxial channel.

The paper is organized as follows. In Sec. II, we show how the six-photon process can be both resonant and collinear under the classical dispersion law, even as the four-photon process cannot, and demonstrate that the six-photon resonance can multiply the input pulse frequency by as much as $2+\sqrt{3}\approx 3.73$ times. In Sec. III, we write the basic equations for the scattering rate. In Sec. IV, we calculate the general six-photon coupling coefficient for collinear laser pulses in homogeneous plasma. In Secs. V, VI, and VII we analyze the six-photon couplings and rates in three particular classes of the scattering regimes. In Sec. VIII, we summarize our results and discuss some complementary and alternative techniques.

II. COLLINEAR SIX-PHOTON RESONANCE

Consider resonant collinear six-photon interaction in which four photons 1, 2, 3, and 4 are scattered into a higher-frequency photon 5 and a lower-frequency photon 6. Momentum and energy conservation of photons are expressed by the following relations, also known as synchronism conditions for wave numbers and frequencies:

$$k_1+k_2+k_3+k_4=k_5+k_6\equiv 2K, \tag{1}$$

$$\omega_1+\omega_2+\omega_3+\omega_4=\omega_5+\omega_6\equiv 2\Omega. \tag{2}$$

The classical dispersion law for electromagnetic waves in uniform plasma of electron concentration n_0 is

$$\omega_j^2-k_j^2c^2=\omega_e^2\equiv\frac{4\pi n_0e^2}{m}\equiv k_e^2c^2,\quad j=1,2,3,4,5,6, \tag{3}$$

where c is the speed of light in vacuum, m is the mass, and $-e$ is the charge of electron. The laser frequencies are larger than the plasma frequency.

Equations (1)–(3) define unambiguously the resonant output frequencies ω_5 and ω_6 as functions of K and Ω,

$$\omega_5=\Omega+Kc\sqrt{1-\omega_e^2/(\Omega^2-K^2c^2)},$$
$$\omega_6=\Omega-Kc\sqrt{1-\omega_e^2/(\Omega^2-K^2c^2)}. \tag{4}$$

Incidentally, this explicitly shows that, for resonant four-photon interactions under the classical dispersion law (3), frequencies of collinear input and output pulses are the same. In contrast to it, resonant collinear six-photon interactions do allow significant frequency shifts even under the classical dispersion law.

For example, in the regime $\omega_1=\omega_2=\omega_3=\omega_4$,

$$K=2k_1,\quad \Omega=2\omega_1,$$
$$\omega_5=2\omega_1+\sqrt{3}\,k_1c,\quad \omega_6=2\omega_1-\sqrt{3}\,k_1c. \tag{5}$$

Frequency multiplication factors ω_5/ω_1 and ω_6/ω_1 as functions of the input laser-to-plasma frequency ratio ω_1/ω_e are shown in Fig. 1 by solid lines. At $\omega_1/\omega_e\gg 1$, these factors

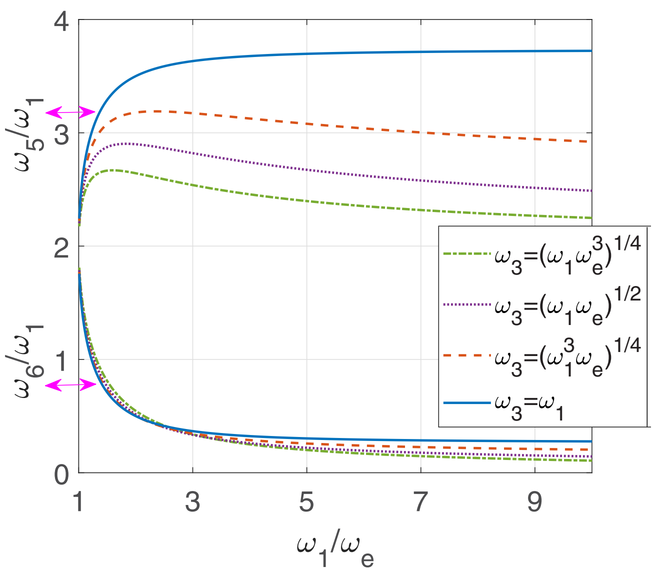

FIG. 1. Frequency multiplication factors ω_5/ω_1 and ω_6/ω_1 as functions of ω_1/ω_e in six-photon scattering of two photons of frequency ω_1 and two photons of frequency ω_3 into a photon of frequency $\omega_5>2\omega_1$ and a photon of frequency $\omega_6<2\omega_1$.

tend to the limits

$$\omega_5/\omega_1\to 2+\sqrt{3}\approx 3.73,$$
$$\omega_6/\omega_1\to 2-\sqrt{3}\approx 0.27. \tag{6}$$

As will be seen from what follows, the six-photon coupling in this regime quickly decreases with the increase of laser-to-plasma frequency ratio $\omega_1/\omega_e\gg 1$ (mostly due to the large frequencies and, hence, small amplitudes of the beatings between input and output waves). The decrease could be mitigated in regimes where some of the input-output beatings stay at relatively small frequencies even at $\omega_1\gg\omega_e$. Consider, for example, regimes where $\omega_1=\omega_2=\omega_3$, while ω_4 is selected such that the frequency of beatings between pulses 4 and 6,

$$\omega_b\equiv\omega_4-\omega_6, \tag{7}$$

is relatively small. Equations (1)–(3) and (7), define unambiguously the resonant output frequencies ω_5 and ω_6 as functions of ω_1 and ω_b,

$$\omega_5=3\omega_1+\omega_b,\quad k_b\equiv k_5-3k_1\ \Rightarrow \tag{8}$$

$$ck_b=c(k_4-k_6)=\sqrt{\omega_4^2-\omega_e^2}-\sqrt{\omega_6^2-\omega_e^2}\ \Rightarrow$$
$$\omega_6=ck_b\sqrt{\frac{\omega_e^2}{c^2k_b^2-\omega_b^2}+\frac{1}{4}}-\frac{\omega_b}{2}. \tag{9}$$

At $\omega_1\gg\omega_e$,

$$ck_b=\sqrt{\omega_5^2-\omega_e^2}-3\sqrt{\omega_1^2-\omega_e^2}\approx\omega_b+4\omega_e^2/3\omega_1\ \Rightarrow$$
$$\omega_6\approx\sqrt{3\omega_1\omega_b/8}-\omega_b/2.$$

This regime frequency multiplication factors ω_5/ω_1 and ω_6/ω_1 as functions of the input laser-to-plasma frequency ratio ω_1/ω_e are plotted in the Fig. 2, for a few values of parameter ω_b/ω_e.

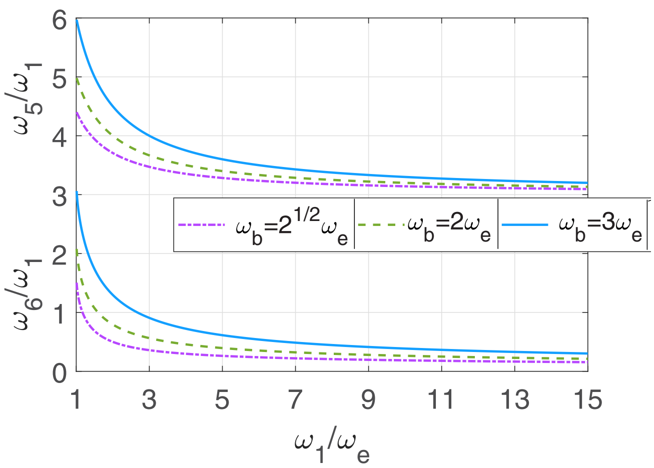

FIG. 2. Frequency multiplication factors ω_5/ω_1 and ω_6/ω_1 as functions of ω_1/ω_e in six-photon scattering of three photons of frequency ω_1 and a photon of frequency ω_4 into a photon of frequency ω_5 and a photon of frequency $\omega_6 = \omega_4 - \omega_b$, for a few values of parameter $\omega_b/\omega_e = \sqrt{2}$, 2, and 3.

As yet another example, consider regimes where $\omega_1 = \omega_2$ and $\omega_3 = \omega_4 < \omega_1$. In such regimes (some of which are shown in Fig. 1)

$$\begin{aligned} K &= k_1 + k_3, \quad \Omega = \omega_1 + \omega_3, \\ \omega_5 &= \omega_1 + \omega_3 + (k_1 + k_3)c \\ &\quad \times \sqrt{1 - \omega_e^2/\left[2\left(\omega_1\omega_3 - k_1k_3c^2 + \omega_e^2\right)\right]}, \\ \omega_6 &= \omega_1 + \omega_3 - (k_1 + k_3)c \\ &\quad \times \sqrt{1 - \omega_e^2/\left[2\left(\omega_1\omega_3 - k_1k_3c^2 + \omega_e^2\right)\right]}. \end{aligned} \tag{10}$$

At $\omega_3 = \omega_1$, this reduces to (5). At $\omega_1 \gg \omega_3$, this reduces to

$$\omega_5 \approx 2\omega_1 + 3(\omega_3 + k_3c)/4, \quad \omega_6 \approx 2\omega_3 - 3(\omega_3 + k_3c)/4.$$

For $\omega_3 \gg \omega_e$, it gives $\omega_6 \approx \omega_3/2$. For $\omega_3 = 2\omega_e \Rightarrow k_3c = \omega_e\sqrt{3}$, it gives $\omega_6 \approx \omega_e\,(5/2 - 3\sqrt{3}/4) \approx 1.2\omega_e$.

III. BASIC EQUATIONS

To calculate the scattering rate, we use the Maxwell equations in Coulomb gauge and Hamilton-Jacobi equation for electron motion in electromagnetic fields [16]:

$$\boldsymbol{H} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\partial_{ct}\boldsymbol{A} - \nabla\Phi, \tag{11}$$

$$\nabla \cdot \boldsymbol{A} = 0, \quad \partial_{ct}^2\boldsymbol{A} - \Delta\boldsymbol{A} = \frac{4\pi\boldsymbol{J}}{c} - \partial_{ct}\nabla\Phi, \tag{12}$$

$$\boldsymbol{J} = \frac{-enc\boldsymbol{P}}{\sqrt{m^2c^2 + P^2}}, \quad \Delta\Phi = 4\pi e(n - n_0), \tag{13}$$

$$\boldsymbol{P} = \frac{e\boldsymbol{A}}{c} + \nabla S, \quad \partial_t S = e\Phi - c\sqrt{m^2c^2 + P^2} + mc^2. \tag{14}$$

For the dimensionless electromagnetic potentials, and the electron momentum,

$$\boldsymbol{A} = mc^2\boldsymbol{a}/e, \quad \Phi = mc^2\phi/e, \quad \boldsymbol{P} = mc\boldsymbol{p}, \tag{15}$$

and the rescaled action, $S = mc^2 s$, the equations can be rewritten in the form

$$\nabla \cdot \boldsymbol{a} = 0, \quad \left(\partial_t^2 - c^2\Delta\right)\boldsymbol{a} = -\boldsymbol{p}\,\frac{\omega_e^2 + c^2\Delta\phi}{\sqrt{1 + p^2}} - c\partial_t\nabla\phi, \tag{16}$$

$$\boldsymbol{p} = \boldsymbol{a} + c\nabla s, \quad \partial_t s = \phi - \sqrt{1 + p^2} + 1. \tag{17}$$

For collinear plane waves, all quantities depend only on the time t and the longitudinal coordinate z. Then, according to the equation $\nabla \cdot \boldsymbol{a} = 0$, the z component of $\boldsymbol{a}$ is zero, so that

$$\boldsymbol{p}_\perp = \boldsymbol{a}, \quad p_z = c\partial_z s,$$

$$\left(\partial_t^2 - c^2\partial_z^2 + \omega_e^2\right)\boldsymbol{a} = \omega_e^2 u\boldsymbol{a}, \tag{18}$$

$$u \equiv 1 - \frac{1 + \omega_e^{-2}c^2\partial_z^2\phi}{\sqrt{1 + p^2}},$$

$$\omega_e^2 p_z + c\partial_t\partial_z\phi = \omega_e^2 u p_z, \tag{19}$$

$$\partial_t p_z = c\partial_z\left(\phi - \sqrt{1 + p^2}\right). \tag{20}$$

Equations (19)–(20) can be rearranged as

$$\left(\partial_t^2 + \omega_e^2\right)c\partial_z\phi = \omega_e^2\left(\partial_t u p_z + c\partial_z\sqrt{1 + p^2}\right), \tag{21}$$

$$\left(\partial_t^2 + \omega_e^2\right)p_z = \omega_e^2 u p_z - c\partial_z\partial_t\sqrt{1 + p^2}. \tag{22}$$

Well off the Raman resonances, these equations define ϕ and p_z as functionals of a^2, thus turning (18) into a nonlinear equation for $\boldsymbol{a}$. For $a^2 \ll 1$, the nonlinear functional u can be expanded in the powers of a^2,

$$u = u_2 + u_4 + \cdots, \tag{23}$$

where u_2 is linear in a^2, u_4 is quadratic in a^2, and so on.

IV. SIX-PHOTON COUPLING

Solutions of Eq. (18) can be searched in the form

$$\boldsymbol{a} = \sum_j\left(\boldsymbol{a}_j e^{\imath\alpha_j} + c.c\right) + \delta\boldsymbol{a}, \tag{24}$$

$$\partial_z\alpha_j = k_{jn}, \quad \partial_t\alpha_j = -\omega_{jn}, \tag{25}$$

where envelopes a_j slowly vary in space time, k_{jn} and ω_{jn} are wave numbers and frequencies slightly modified by the nonlinearity, producing also small nonresonant beatings $\delta\boldsymbol{a}$. The modification (so-called renormalization) of the dispersion law comes from the terms in $u\boldsymbol{a}$ proportional to $\boldsymbol{a}_j e^{\imath\alpha_j}$, and can be described by slightly adjusting the plasma frequency, $\omega_e \to \omega_{ej}$, for each $\boldsymbol{a}_j$. The resonant variation of $\boldsymbol{a}_j \equiv \boldsymbol{e}_j a_j$ comes from the terms in $u\boldsymbol{a}$ not proportional to $a_j e^{\imath\alpha_j}$, but matching space-time synchronism conditions with $e^{\imath\alpha_j}$. In particular, variations of a_5 and a_6, associated with the renormalized resonance (1)–(2), can be described by equations of the form

$$\begin{aligned} &\left(\partial_t^2 - c^2\partial_z^2 + \omega_{e5}^2\right)a_5\exp(\imath\alpha_5) \\ &\quad = \omega_e^2 V_5\left(\prod_{j=1}^{j=4} a_j\right)a_6^*\exp\left[\imath\left(\sum_{j=1}^{j=4}\alpha_j - \alpha_6\right)\right], \end{aligned} \tag{26}$$

$$\left(\partial_t^2 - c^2\partial_z^2 + \omega_{e6}^2\right)a_6 e^{\imath\alpha_6} = \omega_e^2 V_5 \left(\prod_{j=1}^{j=4} a_j\right) a_5^* \exp\left[\imath\left(\sum_{j=1}^{j=4}\alpha_j - \alpha_5\right)\right]. \quad (27)$$

In the exact resonance,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \alpha_5 + \alpha_6 + \text{const}, \quad (28)$$

these equations can be reduced to

$$\omega_{5n}^2 = k_{5n}^2 c^2 + \omega_{e5}^2, \quad \omega_{6n}^2 = k_{6n}^2 c^2 + \omega_{e6}^2, \quad (29)$$

$$-2\imath(\omega_5\partial_t + k_5 c\partial_z)a_5 \approx \omega_e^2 V_5 a_1 a_2 a_3 a_4 a_6^*, \quad (30)$$

$$-2\imath(\omega_6\partial_t + k_6 c\partial_z)a_6 \approx \omega_e^2 V_6 a_1 a_2 a_3 a_4 a_5^*. \quad (31)$$

The energy density in pulse j is proportional to $\omega_j^2|a_j|^2$, so that the number-of-quanta density is proportional to $\omega_j|a_j|^2$. Since numbers of quanta in pulses 5 and 6 are changing identically in the resonant scattering (1)–(2), the difference $\omega_5|a_5|^2 - \omega_6|a_6|^2$ is conserved, as long as the slippage between envelopes of pulses 5 and 6 is much smaller than the scale of envelopes variation. This means that $V_5 \approx V_6$. For such relatively long pulses, in the mildly relativistic regimes considered here, the expected spectra of the up-converted radiation are narrowly located around the resonant frequency. Small amplitudes of the resonant pulses 5 and 6 can grow exponentially with the rate

$$\Gamma \approx \frac{\omega_e^2|V_5 a_1 a_2 a_3 a_4|}{2\sqrt{\omega_5\omega_6}}, \quad (32)$$

as long as the resonance is kept and depletion of pulses 1, 2, 3, and 4 is negligible. An exponentiation occurs within the propagation distance

$$L = \frac{c}{\Gamma} \approx \frac{2c\sqrt{\omega_5\omega_6}}{\omega_e^2|V_5 a_1 a_2 a_3 a_4|} = \lambda_1 \frac{\omega_1\sqrt{\omega_5\omega_6}}{\pi\omega_e^2|V_5 a_1 a_2 a_3 a_4|}, \quad (33)$$

where $\lambda_1 = 2\pi c/\omega_1$ is the pulse 1 wavelength.

To evaluate the six-photon coupling coefficient V_5, we need explicit expressions for functionals u_2 and u_4, which can be obtained from the definition of u and Eqs. (21)–(22),

$$u_2 = \left[1 - c^2\partial_z^2\left(\partial_t^2 + \omega_e^2\right)^{-1}\right]a^2/2, \quad (34)$$

$$\begin{aligned} u_4 = {} & \left[1 - c^2\partial_z^2\left(\partial_t^2 + \omega_e^2\right)^{-1}\right]\left\{\left[c\partial_z\partial_t\left(\partial_t^2 + \omega_e^2\right)^{-1}a^2\right]^2 - a^4\right\}/8 \\ & + a^2\left[c^2\partial_z^2\left(\partial_t^2 + \omega_e^2\right)^{-1} - 1\right]a^2/4 + c\partial_z\partial_t\left(\partial_t^2 + \omega_e^2\right)^{-1} \\ & \times\left\{\left[a^2 - c^2\partial_z^2\left(\partial_t^2 + \omega_e^2\right)^{-1}a^2\right]\left[c\partial_z\partial_t\left(\partial_t^2 + \omega_e^2\right)^{-1}a^2\right]\right\}/4. \end{aligned} \quad (35)$$

The cubic term $u_2\boldsymbol{a}$ in Eq. (18) does not enable collinear four-photon resonances producing significant frequency upshifts, except for special conditions considered in the paper [17] on the cubic model of Eq. (18). Well off such special conditions, the cubic term directly generates only nonresonant collinear beatings

$$\delta\boldsymbol{a} \approx \omega_e^2\left(\partial_t^2 - c^2\partial_z^2 + \omega_e^2\right)^{-1}u_2\boldsymbol{a}. \quad (36)$$

Substituting these beatings into the cubic term modifies the quintic term in Eq. (18):

$$(u\boldsymbol{a})_5 = u_4\boldsymbol{a} + u_2\delta\boldsymbol{a} + \boldsymbol{a}\delta u_2, \quad (37)$$

$$\delta u_2 = \left[1 - c^2\partial_z^2\left(\partial_t^2 + \omega_e^2\right)^{-1}\right](\boldsymbol{a}\cdot\delta\boldsymbol{a}). \quad (38)$$

Equations (34)–(35) with $\boldsymbol{a}$ in the form (24) and the procedure leading to (30) give an explicit formula for V_5.

V. 3.7+ FREQUENCY MULTIPLICATION

In particular, for all pulses having the same linear polarization $\boldsymbol{a}_j = \boldsymbol{e}_x a_j$, in the regime (5):

$$\begin{aligned} V_5 = {} & \frac{48\omega_e^2\omega_1^2\left(\omega_1^2 - \omega_e^2\right) - 15\omega_e^4\left(\omega_1^2 + \omega_e^2/8\right)}{\left(16\omega_1^2 - \omega_e^2\right)\left(4\omega_1^2 - \omega_e^2\right)^2} - \frac{3\omega_e^2}{4\left(4\omega_1^2 - \omega_e^2\right)} + \frac{2\omega_1\omega_6 - 2c^2k_1k_6 - \omega_e^2}{2\left[(\omega_1 - \omega_6)^2 - \omega_e^2\right]}\left\{1 - \frac{4ck_1\omega_1 c(3k_1 - k_6)(3\omega_1 - \omega_6)}{\left(4\omega_1^2 - \omega_e^2\right)\left[(3\omega_1 - \omega_6)^2 - \omega_e^2\right]}\right\} \\ & + \frac{3\omega_e^2}{2\left(4\omega_1^2 - \omega_e^2\right)}\left\{\frac{c(3k_1 - k_6)(3\omega_1 - \omega_6)c(k_1 - k_6)(\omega_1 - \omega_6)}{\left[(3\omega_1 - \omega_6)^2 - \omega_e^2\right]\left[(\omega_1 - \omega_6)^2 - \omega_e^2\right]} - 1\right\} + \left\{\frac{4c^2k_1(k_1 - k_6)\omega_1(\omega_1 - \omega_6)}{\left(4\omega_1^2 - \omega_e^2\right)\left[(\omega_1 - \omega_6)^2 - \omega_e^2\right]} - 1\right\}\frac{9\omega_e^2 - 6\omega_1\omega_6 + 6c^2k_1k_6}{2\left[(3\omega_1 - \omega_6)^2 - \omega_e^2\right]} \\ & + \frac{3\omega_e^2\left(2\omega_1\omega_6 - 2c^2k_1k_6 - \omega_e^2\right)}{16\left(4\omega_1^2 - \omega_e^2\right)\left[(\omega_1 - \omega_6)^2 - \omega_e^2\right]}\left(1 + \frac{2\omega_e^2}{\omega_1\omega_6 - c^2k_1k_6 - \omega_e^2}\right) + \frac{9\omega_e^6}{16\left(\omega_1\omega_6 - c^2k_1k_6 - \omega_e^2\right)\left(4\omega_1^2 - \omega_e^2\right)^2} \\ & - \frac{45\omega_e^4}{16\left(4\omega_1^2 - \omega_e^2\right)\left(16\omega_1^2 - \omega_e^2\right)} + \frac{9\omega_e^2\left(2\omega_1\omega_6 - 2c^2k_1k_6 - 3\omega_e^2\right)}{16\left(4\omega_1^2 - \omega_e^2\right)\left[(3\omega_1 - \omega_6)^2 - \omega_e^2\right]} \\ & + \frac{3\omega_e^2\left(2\omega_1\omega_6 - 2c^2k_1k_6 - 3\omega_e^2\right)}{4\left[(3\omega_1 - \omega_6)^2 - \omega_e^2\right]\left(\omega_1\omega_6 - c^2k_1k_6 - \omega_e^2\right)}\left[\frac{2\omega_1\omega_6 - 2c^2k_1k_6 - \omega_e^2}{(\omega_1 - \omega_6)^2 - \omega_e^2} - \frac{3\omega_e^2/2}{4\omega_1^2 - \omega_e^2}\right], \end{aligned} \quad (39)$$

where the first two lines come from the term u_4a, the third line comes from the term $u_2\delta a$, and the last two lines come from the term $a\delta u_2$ of (37). This V_5 of the regime (5), proportional to ω_e^4/ω_1^4 at $\omega_1 \gg \omega_e$, can be conveniently put in the form

$$V_5 = V\omega_e^4/\omega_1^4 \quad (40)$$

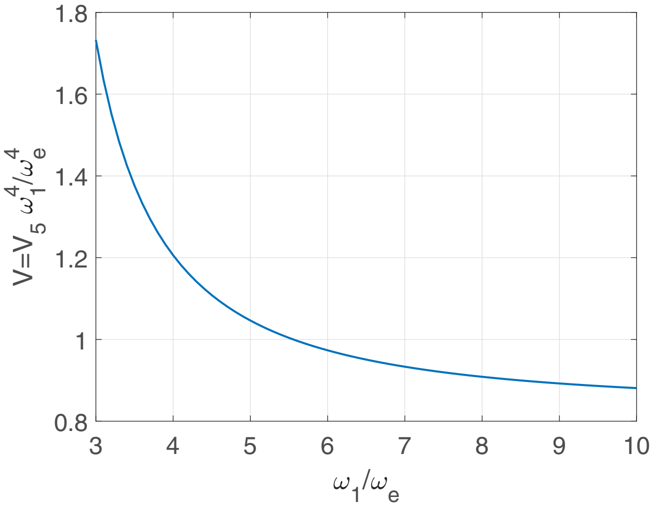

FIG. 3. The six-photon coupling coefficient $V = V_5\,\omega_1^4/\omega_e^4$ as a function of the laser-to-plasma frequency ratio ω_1/ω_e for the regime (5).

with function V shown in Fig. 3. To stay off the Raman resonances, we limit here the parameter range by $\omega_1/\omega_e > 3$, so that $\omega_1 - \omega_6 > (\sqrt{24} - 3)\,\omega_e \approx 1.9\,\omega_e$ and $\omega_6 > (6 - \sqrt{24})\,\omega_e \approx 1.1\,\omega_e$.

At frequencies much greater than shown in Fig. 3, $\omega_1/\omega_e \ggg 1$, the factor V tends to the limit $V_\infty \approx 0.835$.

Using (5) and (40) in Eq. (33) gives

$$L = \lambda_1\,\frac{\omega_1^5\sqrt{\omega_1^2 + 3\omega_e^2}}{\pi\,\omega_e^6\big|Va_1^4\big|}. \tag{41}$$

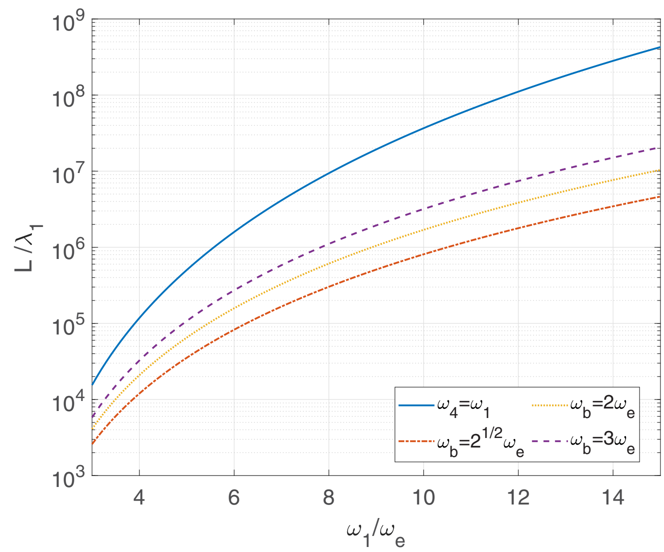

FIG. 4. The exponentiation length L to the laser wavelength λ_1 ratio, as a function of the laser-to-plasma frequency ratio ω_1/ω_e, for the regimes (5) and (8)–(9) at $|a_4|^2 = |a_1|^2 = 0.1$.

The ratio L/λ_1 is shown by the solid line in Fig. 4.

VI. 3+ HIGH-FREQUENCY MULTIPLICATION

In regimes (8)–(9), the above procedure gives the following formula for V_5:

$$
\begin{aligned}
V_5 = {} & \left(1 - \frac{c^2k_b^2}{\omega_b^2 - \omega_e^2}\right)\left\{\frac{2c^2k_1\omega_1(2k_1+k_b)(2\omega_1+\omega_b)}{(4\omega_1^2-\omega_e^2)\big[(2\omega_1+\omega_b)^2-\omega_e^2\big]} - \frac{1}{2}\right\} + \frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\left\{\frac{c^2k_b\omega_b(2k_1+k_b)(2\omega_1+\omega_b)}{\big[(2\omega_1+\omega_b)^2-\omega_e^2\big](\omega_b^2-\omega_e^2)} - 1\right.\\
& \left. + \frac{c^2(k_1+k_4)(\omega_1+\omega_4)(3k_1+k_4)(3\omega_1+\omega_4)}{\big[(3\omega_1+\omega_4)^2-\omega_e^2\big]\big[(\omega_1+\omega_4)^2-\omega_e^2\big]} - 1 + \frac{c^2(k_1-k_6)(\omega_1-\omega_6)(3k_1-k_6)(3\omega_1-\omega_6)}{\big[(3\omega_1-\omega_6)^2-\omega_e^2\big]\big[(\omega_1-\omega_6)^2-\omega_e^2\big]} - 1\right\}\\
& + \left[1 - \frac{c^2(2k_1+k_b)^2}{(2\omega_1+\omega_b)^2-\omega_e^2}\right]\left\{\frac{2c^2k_b\omega_bk_1\omega_1}{(\omega_b^2-\omega_e^2)(4\omega_1^2-\omega_e^2)} - \frac{1}{2} + \frac{c^2(k_1+k_4)(\omega_1+\omega_4)(k_1-k_6)(\omega_1-\omega_6)}{\big[(\omega_1+\omega_4)^2-\omega_e^2\big]\big[(\omega_1-\omega_6)^2-\omega_e^2\big]} - 1\right\}\\
& + \left[1 - \frac{c^2(3k_1+k_4)^2}{(3\omega_1+\omega_4)^2-\omega_e^2}\right]\left\{\frac{2c^2k_1\omega_1(k_1+k_4)(\omega_1+\omega_4)}{(4\omega_1^2-\omega_e^2)\big[(\omega_1+\omega_4)^2-\omega_e^2\big]} - \frac{1}{2}\right\} + \left[1 - \frac{c^2(3k_1-k_6)^2}{(3\omega_1-\omega_6)^2-\omega_e^2}\right]\left\{\frac{2c^2k_1\omega_1(k_1-k_6)(\omega_1-\omega_6)}{(4\omega_1^2-\omega_e^2)\big[(\omega_1-\omega_6)^2-\omega_e^2\big]} - \frac{1}{2}\right\}\\
& + \left[1 - \frac{c^2(k_1+k_4)^2}{(\omega_1+\omega_4)^2-\omega_e^2}\right]\left\{\frac{2c^2k_1\omega_1(3k_1+k_4)(3\omega_1+\omega_4)}{(4\omega_1^2-\omega_e^2)\big[(3\omega_1+\omega_4)^2-\omega_e^2\big]} - \frac{1}{2} + \frac{c^2(k_1-k_6)(\omega_1-\omega_6)(2k_1+k_b)(2\omega_1+\omega_b)}{\big[(\omega_1-\omega_6)^2-\omega_e^2\big]\big[(2\omega_1+\omega_b)^2-\omega_e^2\big]} - 1\right\}\\
& + \left[1 - \frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\right]\left\{\frac{2c^2k_1\omega_1(3k_1-k_6)(3\omega_1-\omega_6)}{(4\omega_1^2-\omega_e^2)\big[(3\omega_1-\omega_6)^2-\omega_e^2\big]} - \frac{1}{2} + \frac{c^2(k_1+k_4)(\omega_1+\omega_4)(2k_1+k_b)(2\omega_1+\omega_b)}{\big[(\omega_1+\omega_4)^2-\omega_e^2\big]\big[(2\omega_1+\omega_b)^2-\omega_e^2\big]} - 1\right\}\\
& + \left(1 - \frac{c^2k_b^2}{\omega_b^2-\omega_e^2}\right)\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\left(\frac{\omega_e^2}{c^2k_b^2-\omega_b^2+2c^2k_bk_1-2\omega_b\omega_1} - \frac{1}{8}\right)\\
& + \left[1 - \frac{c^2(k_1+k_4)^2}{(\omega_1+\omega_4)^2-\omega_e^2}\right]\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\left(\frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2} + \frac{\omega_e^2}{c^2k_b^2-\omega_b^2+2c^2k_bk_1-2\omega_b\omega_1}\right)\\
& + \left[1 - \frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\right]\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\left(\frac{\omega_e^2/4}{c^2k_1k_4-\omega_1\omega_4-\omega_e^2} + \frac{\omega_e^2}{c^2k_b^2-\omega_b^2+2c^2k_bk_1-2\omega_b\omega_1}\right)\\
& + \left[1 - \frac{c^2(k_1+k_4)^2}{(\omega_1+\omega_4)^2-\omega_e^2}\right]\left[1 - \frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\right]\left(\frac{\omega_e^2/4}{c^2k_1k_4-\omega_1\omega_4-\omega_e^2} + \frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg[1-\frac{c^2(2k_1+k_b)^2}{(2\omega_1+\omega_b)^2-\omega_e^2}\bigg]\bigg\{\bigg[1-\frac{c^2k_b^2}{\omega_b^2-\omega_e^2}+1-\frac{c^2(k_1+k_4)^2}{(\omega_1+\omega_4)^2-\omega_e^2}+1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\bigg]\frac{\omega_e^2}{c^2k_b^2-\omega_b^2+2c^2k_bk_1-2\omega_b\omega_1}\\
&+\frac{3\omega_e^2/8}{4\omega_1^2-\omega_e^2}\bigg(\frac{\omega_e^2}{c^2k_1k_4-\omega_1\omega_4-\omega_e^2}+\frac{\omega_e^2}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}\bigg)+\bigg[1-\frac{c^2(k_1+k_4)^2}{(\omega_1+\omega_4)^2-\omega_e^2}\bigg]\frac{\omega_e^2/4}{c^2k_1k_4-\omega_1\omega_4-\omega_e^2}+\bigg[1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\bigg]\\
&\times\frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}\bigg\}+\bigg[1-\frac{c^2(3k_1+k_4)^2}{(3\omega_1+\omega_4)^2-\omega_e^2}\bigg]\bigg\{\bigg[1-\frac{c^2(k_1+k_4)^2}{(\omega_1+\omega_4)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\bigg]\frac{\omega_e^2/4}{c^2k_1k_4-\omega_1\omega_4-\omega_e^2}-\frac{3\omega_e^2/16}{4\omega_1^2-\omega_e^2}\bigg\}\\
&+\bigg[1-\frac{c^2(3k_1-k_6)^2}{(3\omega_1-\omega_6)^2-\omega_e^2}\bigg]\bigg\{\bigg[1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}+\frac{3\omega_e^2/2}{\left(4\omega_1^2-\omega_e^2\right)}\bigg]\frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}-\frac{3\omega_e^2/16}{4\omega_1^2-\omega_e^2}\bigg\},
\end{aligned}
\tag{42}
$$

where the first six lines come from the term u_4a, the next four lines come from the term $u_2\delta a$, and the last four lines come from the term $a\delta u_2$ of (37). The exponentiation length (33) for this regime is shown in Fig. 4, along with the exponentiation length for the regime (5).

A far asymptotic expression for the coefficient V_5 at frequencies much greater than shown in Fig. 4, $\omega_1 \ggg \omega_b-\omega_e \sim \omega_e$, is

$$
V_{5f}\approx \omega_e^4\big/\big[8\big(\omega_b^2-\omega_e^2\big)\omega_1^2\big]. \tag{43}
$$

The ratio V_5/V_{5f} is shown in Fig. 5. The exponentiation length L, corresponding to the far asymptotic formula (43) for V_5 in the regimes (8)–(9), is proportional to $(\omega_1/\omega_e)^{11/4}$.

VII. 2+ HIGH-FREQUENCY MULTIPLICATION

In regimes (10), the above procedure gives the following formula for

$$
\begin{aligned}
V_5=&\bigg[1-\frac{c^2(k_3-k_6)^2}{(\omega_3-\omega_6)^2-\omega_e^2}\bigg]\bigg[\frac{c(k_1+k_3)(\omega_1+\omega_3)}{(\omega_1+\omega_3)^2-\omega_e^2}\frac{c(k_1+2k_3-k_6)(\omega_1+2\omega_3-\omega_6)}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}+\frac{2ck_1\omega_1}{4\omega_1^2-\omega_e^2}\frac{c(2k_1+k_3-k_6)(2\omega_1+\omega_3-\omega_6)}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}-\frac{3}{2}\bigg]\\
&+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\bigg[\frac{c(k_3-k_6)(\omega_3-\omega_6)}{(\omega_3-\omega_6)^2-\omega_e^2}\frac{c(2k_1+k_3-k_6)(2\omega_1+\omega_3-\omega_6)}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}+\frac{2ck_3\omega_3}{4\omega_3^2-\omega_e^2}\frac{4c(k_1+k_3)(\omega_1+\omega_3)}{4(\omega_1+\omega_3)^2-\omega_e^2}-\frac{3}{2}\bigg]\\
&+\bigg[1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\bigg]\bigg[\frac{c(k_1+k_3)(\omega_1+\omega_3)}{(\omega_1+\omega_3)^2-\omega_e^2}\frac{c(2k_1+k_3-k_6)(2\omega_1+\omega_3-\omega_6)}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}+\frac{2ck_3\omega_3}{4\omega_3^2-\omega_e^2}\frac{c(k_1+2k_3-k_6)(\omega_1+2\omega_3-\omega_6)}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}-\frac{3}{2}\bigg]\\
&+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}\bigg[\frac{c(k_1-k_6)(\omega_1-\omega_6)}{(\omega_1-\omega_6)^2-\omega_e^2}\frac{c(k_1+2k_3-k_6)(\omega_1+2\omega_3-\omega_6)}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}+\frac{2ck_1\omega_1}{4\omega_1^2-\omega_e^2}\frac{4c(k_1+k_3)(\omega_1+\omega_3)}{4(\omega_1+\omega_3)^2-\omega_e^2}-\frac{3}{2}\bigg]\\
&+\bigg[1-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}\bigg]\bigg\{\frac{c(k_3-k_6)(\omega_3-\omega_6)}{(\omega_3-\omega_6)^2-\omega_e^2}\frac{c(k_1+2k_3-k_6)(\omega_1+2\omega_3-\omega_6)}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}+\frac{c(k_1-k_6)(\omega_1-\omega_6)}{(\omega_1-\omega_6)^2-\omega_e^2}\frac{c(2k_1+k_3-k_6)(2\omega_1+\omega_3-\omega_6)}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}\\
&+\frac{4c^2(k_1+k_3)^2(\omega_1+\omega_3)^2}{\big[(\omega_1+\omega_3)^2-\omega_e^2\big]\big[4(\omega_1+\omega_3)^2-\omega_e^2\big]}-3\bigg\}+\bigg[1-\frac{4c^2(k_1+k_3)^2}{4(\omega_1+\omega_3)^2-\omega_e^2}\bigg]\bigg\{\frac{4c^2k_1k_3\omega_1\omega_3}{\big(4\omega_1^2-\omega_e^2\big)\big(4\omega_3^2-\omega_e^2\big)}+\frac{c^2(k_1+k_3)^2(\omega_1+\omega_3)^2}{2[(\omega_1+\omega_3)^2-\omega_e^2]^2}-\frac{3}{4}\bigg\}\\
&+\bigg[1-\frac{c^2(2k_1+k_3-k_6)^2}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}\bigg]\bigg\{\frac{c^2(k_1+k_3)(k_1-k_6)(\omega_1+\omega_3)(\omega_1-\omega_6)}{\big[(\omega_1+\omega_3)^2-\omega_e^2\big]\big[(\omega_1-\omega_6)^2-\omega_e^2\big]}+\frac{2c^2k_1(k_3-k_6)\omega_1(\omega_3-\omega_6)}{\big(4\omega_1^2-1\big)\big[(\omega_3-\omega_6)^2-\omega_e^2\big]}-\frac{3}{2}\bigg\}\\
&+\bigg[1-\frac{c^2(k_1+2k_3-k_6)^2}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}\bigg]\bigg\{\frac{c^2(k_1+k_3)(k_3-k_6)(\omega_1+\omega_3)(\omega_3-\omega_6)}{\big[(\omega_1+\omega_3)^2-\omega_e^2\big]\big[(\omega_3-\omega_6)^2-\omega_e^2\big]}+\frac{2c^2k_3(k_1-k_6)\omega_3(\omega_1-\omega_6)}{\big(4\omega_3^2-\omega_e^2\big)\big[(\omega_1-\omega_6)^2-\omega_e^2\big]}-\frac{3}{2}\bigg\}\\
&+\bigg[1-\frac{c^2(k_3-k_6)^2}{(\omega_3-\omega_6)^2-\omega_e^2}\bigg]\frac{\omega_e^2/4}{c^2k_1k_3-\omega_1\omega_3-\omega_e^2}\bigg[1-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\bigg]\\
&+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}\frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}\bigg[1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\bigg]\\
&+\bigg[1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}\bigg]\frac{\omega_e^2/4}{c^2k_1k_3-\omega_1\omega_3-\omega_e^2}\bigg[1-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}\bigg]\\
&+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\frac{\omega_e^2/4}{\omega_3\omega_6-c^2k_3k_6-\omega_e^2}\bigg[1-\frac{c^2(k_3-k_6)^2}{(\omega_3-\omega_6)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}\bigg]\\
&+\bigg[1-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}\bigg]\frac{\omega_e^2}{c^2(k_1+k_3-k_6)^2-(\omega_1+\omega_3-\omega_6)^2+\omega_e^2}\bigg[3-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}-\frac{c^2(k_3-k_6)^2}{(\omega_3-\omega_6)^2-\omega_e^2}\bigg]\\
&+\bigg[1-\frac{4c^2(k_1+k_3)^2}{4(\omega_1+\omega_3)^2-\omega_e^2}\bigg]\frac{\omega_e^2/4}{c^2k_1k_3-\omega_1\omega_3-\omega_e^2}\bigg[2-\frac{2c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[2-\frac{c^2(2k_1+k_3-k_6)^2}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}-\frac{c^2(k_1+2k_3-k_6)^2}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}\right]\frac{\omega_e^2}{c^2(k_1+k_3-k_6)^2-(\omega_1+\omega_3-\omega_6)^2+\omega_e^2}\\
&\times\left[3-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}-\frac{c^2(k_3-k_6)^2}{(\omega_3-\omega_6)^2-\omega_e^2}\right]+\left\{\frac{\omega_e^2/4}{c^2k_1k_3-\omega_1\omega_3-\omega_e^2}\left[1-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\right]\right.\\
&\left.+\frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}\left[1-\frac{c^2(k_1-k_6)^2}{(\omega_1-\omega_6)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_1^2-\omega_e^2}\right]\right\}\left[1-\frac{c^2(2k_1+k_3-k_6)^2}{(2\omega_1+\omega_3-\omega_6)^2-\omega_e^2}\right]+\left[1-\frac{c^2(k_1+2k_3-k_6)^2}{(\omega_1+2\omega_3-\omega_6)^2-\omega_e^2}\right]\\
&\times\left\{\frac{\omega_e^2/4}{c^2k_1k_3-\omega_1\omega_3-\omega_e^2}\left[1-\frac{c^2(k_1+k_3)^2}{(\omega_1+\omega_3)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}\right]+\frac{\omega_e^2/4}{\omega_1\omega_6-c^2k_1k_6-\omega_e^2}\left[1-\frac{c^2(k_3-k_6)^2}{(\omega_3-\omega_6)^2-\omega_e^2}+\frac{3\omega_e^2/2}{4\omega_3^2-\omega_e^2}\right]\right\}, \qquad (44)
\end{aligned}
$$

where the first eight lines come from the term u_4a, the next five lines come from the term $u_2\delta a$, and the last five lines come from the term $a\delta u_2$ of (37). The exponentiation length L (33) for this regime is shown in Fig. 6.

The coupling coefficient V_5 behavior at frequencies ω_1 much greater than shown in Fig. 6 is shown in Fig. 7. Figure 7 contains a few plots of the coefficient V_5 divided over the far asymptotic expression for it V_a,

$$
\begin{aligned}
V_a = &\frac{5(\omega_6-ck_6)-13(\omega_3-ck_3)}{4\omega_1}\\
&\times\frac{4(\omega_3-\omega_6)^2-3\omega_e^2-[\omega_3-\omega_6+c(k_3-k_6)]^2}{(\omega_3-\omega_6)^2-\omega_e^2}\\
&+\frac{4(\omega_6-ck_6)-5(\omega_3-ck_3)}{2\omega_1}\frac{4\omega_3^2-4ck_3\omega_3+\omega_e^2/2}{4\omega_3^2-\omega_e^2}. \qquad (45)
\end{aligned}
$$

This formula assumes that $\omega_3-\omega_e$ is about ω_e, but off the Raman resonance $\omega_3-\omega_6\approx\omega_e$ located at $\omega_3\approx\omega_e(\sqrt{27/8}+1/2)\approx 2.337\omega_e$. The exponentiation length L, corresponding to the far asymptotic formula (45) for V_5 in the regimes (10), is proportional to $(\omega_1/\omega_e)^{3/2}$.

VIII. SUMMARY AND DISCUSSION

We identified several resonant six-photon scattering regimes of collinear laser pulses in plasma and calculated the respective scattering rates. In these regimes, the upshifted output pulse frequency ω_5 ranges from 2–3.73 times multiplied top input pulse frequency ω_1. The input does not need to contain a seed of the high frequency ω_5, since the resonant amplification can be initiated by the seed of lower input frequency $\omega_6<\omega_1$. The exponentiation length L tends to be larger for larger frequency upshifts and laser-to-plasma frequency ratios ω_1/ω_e.

In regimes exhibiting the largest upshift (5), the pulse amplitude exponentiation length increases at $\omega_1/\omega_e\gg 1$ like $L/\lambda_1\propto\omega_1^6/\omega_e^6$. The increase can be mitigated in regimes (8)–(9) to be like $L/\lambda_1\propto\omega_1^{11/4}/\omega_e^{11/4}$, and even more mitigated in regimes (10) to be like $L/\lambda_1\propto\omega_1^{3/2}/\omega_e^{3/2}$. This may be of interest for handling greater laser powers $P\approx 17\omega_1^2/\omega_e^2$ GW in a single self-made channel. However, for getting greater single-step frequency upshifts, the regimes (5) are more favorable. These single-pump-pulse regimes may be also easier for the experimental implementation.

Table I gives a few numerical examples of possible pulse parameters in regimes (5).

These examples use the small amplitude seed exponentiation length L shown by the solid line in Fig. 4. The respective intensity exponentiation length is $L/2$. The longitudinal slippage is associated primarily with the lower frequency ω_6 seed, which slips backward for the fraction $L_{slip}/L\approx\omega_e^2/(\omega_6^2+$

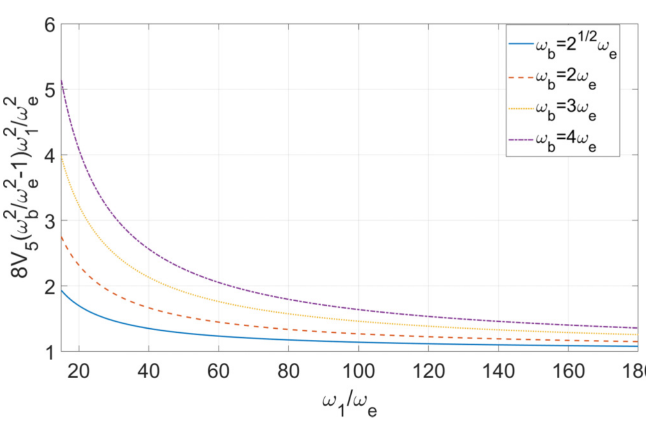

FIG. 5. Far asymptotic behavior of the six-photon coupling coefficient V_5 in the regimes (8)–(9).

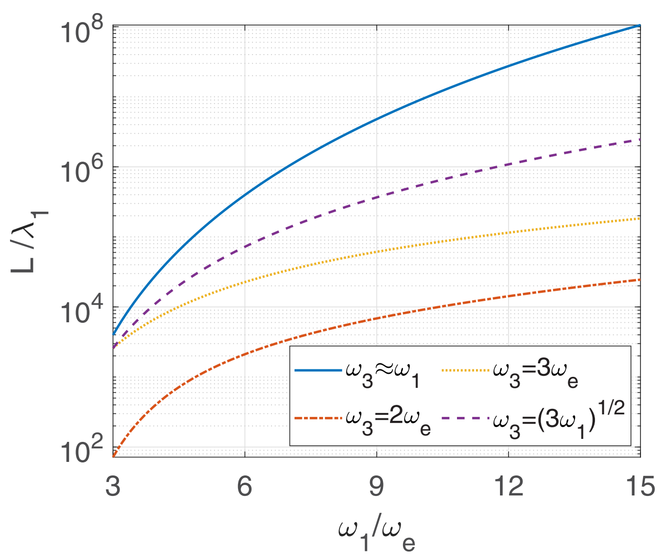

FIG. 6. The exponentiation length L ratio to the laser wavelength λ_1, as a function of the laser-to-plasma frequency ratio ω_1/ω_e, for the regimes (10) at $|a_3|^2=|a_1|^2=0.1$ and a few different ω_3. The solid line corresponds to ω_3 close to, but different from ω_1. This line is nearly the same as one obtained by formally replacing in the regime (5) $|a_1|^4$ for $6|a_1|^2|a_3|^2$, so that at $|a_3|^2=|a_1|^2$ the resulting L is six times smaller than in the regime (5).

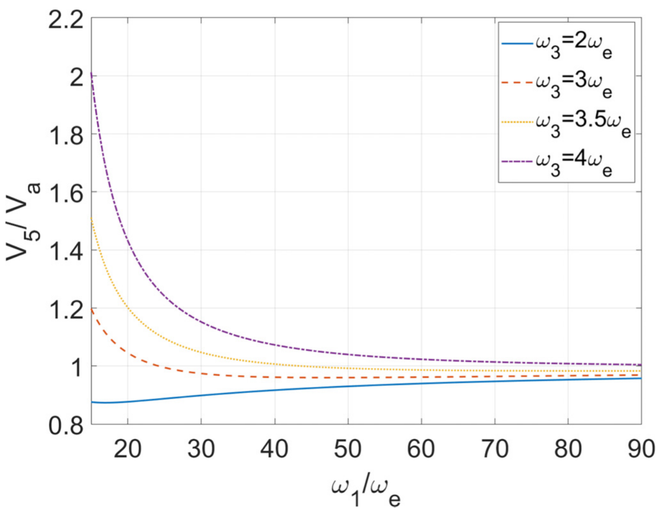

FIG. 7. Far asymptotic behavior of the six-photon coupling coefficient V_5 in the regimes (10).

$ck_6\omega_6$) of the propagation length L. The conventionally chosen pulse length is $L_{pulse} \approx L/2$. The last two lines show the pulse energy $\mathcal{E}$ and the intensity I, which would be achieved if the energy were compressed to the $1000\lambda_5^3$ volume. Some of these speculative intensities are much greater than speculative intensities envisioned in earlier schemes combining CPA and backward Raman amplification (BRA) of laser pulses in plasma [18–20] techniques [21]. Additionally, the device size of the present concept could be far smaller than that of the earlier schemes.

The proposed scheme builds on methods of manipulating intense laser pulses in plasma via the resonant nonlinear Raman and Brillouin scattering. A recent review of these methods may be found in Ref. [22]. The crucial new element added here is the possibility of resonant transfer of huge near-optical laser energies to shorter wavelengths via the six-photon scattering of mildly relativistic-intense collinear laser pulses in plasma. This six-photon scheme avoids the most serious challenges experienced by the four-photon scheme [11,17,23,24].

TABLE I. Examples of pulse parameters.

ω_1/ω_e	4	5	6	7	8	9
ω_5/ω_1	3.677	3.697	3.708	3.714	3.718	3.721
ω_1/ω_6	3.096	3.001	3.423	3.500	3.552	3.589
P [TW]	0.28	0.43	0.62	0.84	1.1	1.4
$10^{-5}L/\lambda_1$	1.2	5	16	41	94	193
L_{slip}/L	0.37	0.25	0.18	0.13	0.1	0.08
λ_1 [nm]	850	200	63	24	11	5
λ_5 [nm]	230	54	17	6.5	3	1.4
L [cm]	10	10	10	10	10	10
$\mathcal{E}$ [J]	45	70	100	140	180	230
I [10^{25}W/cm^2]	0.0002	0.03	1	30	450	5000

Apart from the ultrahigh intensity laser applications proposed here, the current calculations may be of broader interest. The six-photon resonance creates an unusual frequency output. A signature of the six-photon resonant scattering, in principle, might be detectable in such processes as collinear radiation traveling through the solar corona. The unusual frequency multiplication factors that are predicted here might also be used for diagnosing laboratory plasma, especially under conditions when only axial access to the plasma is possible, like in Z-pinch experiments [25,26].

A leading alternative approach to achieving ultrahigh intensities is by free electron lasers (FEL) [27,28]. Currently, the greatest focused intensities produced by the most powerful large-size FEL are a few orders smaller than the top focused intensities of optical lasers. However, it might be possible to increase the intensities and reduce the sizes of FEL via plasma wake-field acceleration of the driving electron beams [29]. Wake-field-making techniques might also enable a more direct energy transfer from the driving laser to shorter wavelengths via photon acceleration in the evolving density gradient of wake-field plasma [30].

ACKNOWLEDGMENT

This work was supported by grant NNSA DE-SC0021248 and grant NNSA 83228-10966 [Prime No. DOE (NNSA) DE-NA0003764].

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