

Geometric clusters in the overlap of the Ising model

Michail Akritidis,^{1,*} Nikolaos G. Fytas^{2,3,†} and Martin Weigel^{2,‡}

¹*Centre for Fluid and Complex Systems, Coventry University, Coventry, CV1 5FB, United Kingdom*

²*Institut für Physik, Technische Universität Chemnitz, 09107 Chemnitz, Germany*

³*Department of Mathematical Sciences, University of Essex, Colchester CO4 3SQ, United Kingdom*



(Received 13 June 2023; accepted 22 September 2023; published 27 October 2023)

We study the percolation properties of geometrical clusters defined in the overlap space of two statistically independent replicas of a square-lattice Ising model that are simulated at the same temperature. In particular, we consider two distinct types of clusters in the overlap, which we dub soft- and hard-constraint clusters, and which are subsets of the regions of constant spin overlap. By means of Monte Carlo simulations and a finite-size scaling analysis we estimate the transition temperature as well as the set of critical exponents characterizing the percolation transitions undergone by these two cluster types. The results suggest that both soft- and hard-constraint clusters percolate at the critical temperature of the Ising model and their critical behavior is governed by the correlation-length exponent $\nu = 1$ found by Onsager. At the same time, they exhibit nonstandard and distinct sets of exponents for the average cluster size and percolation strength.

DOI: [10.1103/PhysRevE.108.044145](https://doi.org/10.1103/PhysRevE.108.044145)

I. INTRODUCTION

In simple ferromagnets the definition of an order parameter is straightforward, and in the vast majority of cases one simply considers the magnetization [1]. For more intricate problems such as some frustrated systems with quenched disorder and certain quantum-spin models (easily measurable) order parameters are harder to come by [2–5]. In spin glasses the free-energy landscape has many minima that are occupied at low temperatures, but which are not related to each other by simple symmetry transformations (such as, e.g., spin-flip symmetry) [6]. For such systems it was proposed to consider *self-consistent* definitions of ordering by constructing order parameters that capture the tendency of such systems to occupy the same set of metastable configurations. As Parisi showed [7], such *overlap* definitions allow one to describe the transition to a short-range-ordered spin-glass phase [8]. Different order parameters in general might also lead to different scaling and associated critical exponents, however. One of the simplest examples of this type is the ordering in the overlap space of the square-lattice ferromagnetic Ising model which is studied here. The overlap is a more general and conceptually robust order parameter also for this ferromagnetic system, and so it is worthwhile to study its behavior. In addition, such setups might have some more general relevance, for example, for the description of layered Ising models with (asymptotically) vanishing coupling as present, for example, in multiplex networks [9], where ordering might occur independently in the different levels of the graphs. More generally, a system composed of two or several layers of magnetic material with

Ising-like anisotropy that are very weakly coupled would also be represented by such a model.

The study of ordering transitions of spin systems from a *geometrical* perspective has greatly enhanced our understanding of phase transitions. Such approaches naturally fall into the realm of percolation theory [10] in which spin systems can be described by appropriately defined clusters, capable of encoding the critical behavior of the system. Fortuin and Kasteleyn (FK) [11] showed that the q -state Potts model is equivalent to a site-bond correlated percolation problem, where clusters are defined as neighboring parallel spins, and the bonds between them are deleted with a certain temperature-dependent probability. Such clusters percolate at the transition temperature and, even more importantly, they encode the critical behavior of the system, as suitably defined cluster exponents are found to be identical to the thermal ones (such clusters were independently also analyzed by Coniglio and Klein [12]). Apart from the conceptual importance of these results, they also allowed for the construction of powerful Monte Carlo algorithms by Swendsen and Wang [13] and Wolff [14], where entire FK clusters are flipped in contrast to local update schemes, such as the Metropolis algorithm [15]. It is well established that the main advantage of this approach is the reduction of autocorrelation times in the vicinity of the critical point as compared to local update schemes.

While FK clusters encode the critical behavior of the system, this is generally not the case for the geometrical or spin clusters [16]. Instead, such clusters undergo a percolation transition that is normally distinct from the thermal one, with the related critical exponents also being different from the thermal ones (see Refs. [17,18] for a review). In two dimensions, however, the geometrical clusters percolate at the thermal transition point [19] and it has been shown that they encode the tricritical behavior of the site-diluted $q = 1$ Potts model [20]. In fact, analogous interrelations have

*michail.akritidis@coventry.ac.uk

†nikolaos.fytas@essex.ac.uk

‡martin.weigel@physik.tu-chemnitz.de

been reported for the more general q -state Potts model and its diluted version for $0 \leq q \leq 4$ in both analytical and numerical terms; see Refs. [20–24] and references therein. It is hence a natural question to investigate how such geometrical clusters defined in the overlap of the Ising model behave, and whether they percolate at the critical temperature of the ordering transition.

An additional motivation relates to the rather less clear connection between clusters and thermal phase transitions in spin-glass systems [25]. For such models the FK representation does not properly describe the phase transition, and the constructed clusters percolate at a temperature way above the spin-glass transition [26,27]. Several types of clusters in the overlap of two copies, including geometric clusters, have been considered as potential candidates for the construction of cluster updates [26,28,29]. It is found there that the spin-glass transition is connected to the onset of a density difference of the two largest clusters of a suitable type, while percolation of such clusters occurs already above the spin-glass transition point [26,30]. As we shall see below, the clusters considered for the Ising model in the present work are related to some of the cluster types discussed in the context of the spin-glass transition, cf. Ref. [30].

The rest of this paper is organized as follows. In Sec. II we introduce the replicated Ising model and the associated concept of soft- and hard-constraint clusters. We further outline the cluster-update Monte Carlo scheme used in the following to study the problem numerically. In Sec. III we elaborate on the relevant observables whose percolation properties are investigated in detail. In Sec. IV we report on a finite-size scaling analysis of the simulation data leading to estimates of the percolation temperature T_p and the critical exponents ν , β/ν , and γ/ν characterizing the transition for the two cluster types. We also comment on the influence of corrections to scaling in the estimation of the exponent ratios β/ν and γ/ν when different definitions are used for the involved observables. Finally, in Sec. V we provide a summary of our work and an outlook.

II. MODEL AND SIMULATION DETAILS

We study the nearest-neighbor, zero-field Ising model with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j, \quad (1)$$

where $J > 0$ indicates ferromagnetic interactions, $s_i = \pm 1$ denotes the spin on lattice site i , and $\langle \dots \rangle$ refers to a summation over nearest neighbors only.

We now consider two identical copies of the system, each being described by the Hamiltonian (1), resulting in two spin configurations, $s_i^{(1)}$ and $s_i^{(2)}$. Both systems are at the same temperature and do not interact with each other, thus being statistically independent. We then consider overlap variables q_i at each site, i.e.,

$$q_i = s_i^{(1)} s_i^{(2)}. \quad (2)$$

The behavior of this overlap parameter has been discussed in depth for the case of spin-glass systems (see, e.g., Ref. [8]).

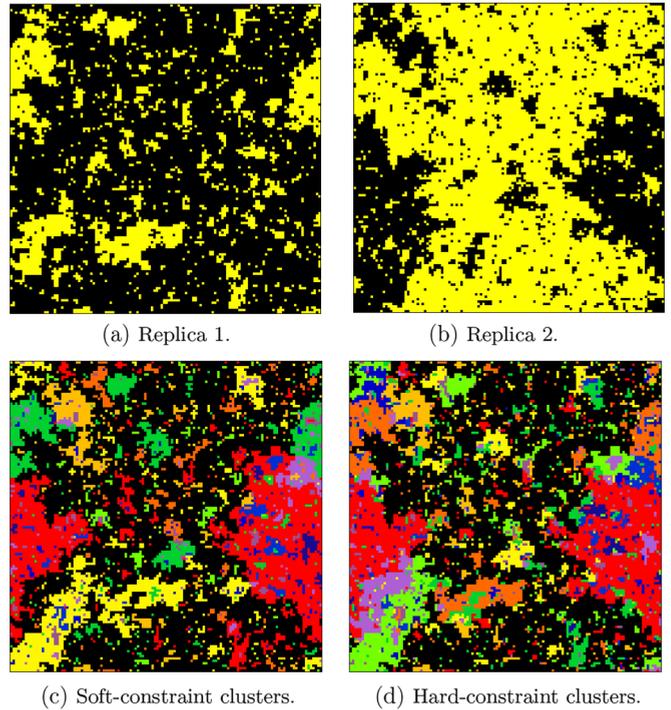


FIG. 1. Spin configurations as well as soft- and hard-constraint geometrical clusters of the two-dimensional Ising model at the Ising critical temperature T_c . (a) Spin configuration of the first replica. (b) Spin configuration of the second replica. (c) The resulting soft-constraint clusters. (d) The resulting hard-constraint clusters. In (c) and (d) all clusters, apart from the largest percolating one, are assigned colors at random. For the largest percolating cluster of both the soft- and hard-constraint definitions, the same color (black) is assigned.

Since $q_i = \pm 1$, the overlap configuration has the same configuration space $\{\pm 1\}^N$ as the spin lattices themselves (here, N denotes the total number of lattice sites). As a consequence, all derived observables usually considered for $\{s_i\}$ can also be defined for $\{q_i\}$. While for spin glasses this approach leads to the spin-glass susceptibility and related quantities such as the spin-glass correlation length, and exploitation of the available gauge symmetry of the couplings results in a possible approach towards understanding the spin-glass phase [31], the behavior of the overlap has hardly been studied for the case of ferromagnets.

In overlap space we may then define geometrical clusters, i.e., sets of neighboring lattice sites with the same values of the overlap, that can be formed in two particular ways:

(1) *Soft-constraint clusters* are created by joining spins on neighboring sites (i, j) with $q_i = q_j$.

(2) *Hard-constraint clusters* are formed by joining neighboring spins with $s_i^{(1)} = s_j^{(1)}$ and $s_i^{(2)} = s_j^{(2)}$.

From the above it is obvious that the hard-constraint clusters trivially satisfy $q_i = q_j$ and they are a subset of the soft-constraint clusters. The construction of these clusters is illustrated through the snapshots shown in Fig. 1, where configurations of the two replicas are shown (top row) along with the soft- and hard-constraint clusters (bottom row) for a system of linear size $L = 128$ at the critical temperature of the

square-lattice ferromagnetic Ising model. It is apparent that the typical clusters in the overlap are smaller than those in the two replicas, and that the hard constraint leads to smaller clusters than the soft constraint. We note that in the spin-glass setup, the prescription for soft-constrained clusters leads to what is there known as Houdayer clusters [28], while the hard constraint corresponds to a geometric-cluster version of the Chayes-Machta-Redner construction [30,32].

In the present work, we study these clusters for the case of the Ising model on the square lattice. In particular, we simulate the Ising model by considering two replicas on the square lattice with periodic boundary conditions for a range of temperatures including the exact Ising critical temperature, i.e., $T_c = 2/(1 + \sqrt{2}) \approx 2.269185$. Configurations were generated via the Swendsen-Wang algorithm [13] applied to systems of linear sizes in the range $8 \leq L \leq 2048$. For all system sizes and on each replica the total number of simulation steps was $1.1 \times \tau_{\text{int}, E} \times 10^5$ sweeps, of which $\tau_{\text{int}, E} \times 10^4$ sweeps were discarded during equilibration. Here, $\tau_{\text{int}, E}$ denotes the integrated autocorrelation time of the energy [33]. After every $\tau_{\text{int}, E}$ sweeps a measurement was taken, resulting in 10^5 measurements per run. The estimates of $\tau_{\text{int}, E}$, rounded up to the next largest integer, varied from 5 sweeps for $L = 8$ to 15 sweeps for $L = 2048$. Note that to identify wrapping clusters we employed the method of Machta *et al.* [34]. Finally, for all curve fitting performed throughout this paper we restricted ourselves to data with $L \geq L_{\text{min}}$, adopting the standard χ^2 test for goodness of the fit. Specifically, we considered a fit as being acceptable only if $Q > 0.01$, where Q is the quality-of-fit parameter [35].

III. OBSERVABLES

To investigate the percolation transition of geometrical overlap clusters, the main relevant quantities are the percolation strength P_∞ , the average cluster size S , and the wrapping probability R [10]. The last of these is defined as the probability that, given a spin configuration, at least one cluster wraps around the periodic boundaries of a finite lattice and is connected back to itself. In the thermodynamic limit, one expects that $R = 1$ for temperatures below the percolation transition at T_p , and $R = 0$ at temperatures above T_p . The wrapping of a cluster can occur in various ways, and here we consider the following cases, in analogy to Ref. [36].

(1) $R_{x \text{ or } y}$ is the probability that a cluster wraps around the lattice in horizontal or vertical (or both) direction(s).

(2) $R_{x \text{ and } y}$ is the probability that a cluster wraps in horizontal and in vertical direction.

(3) R_x is the probability that a cluster wraps in horizontal direction. Obviously, on the square lattice the probability that a cluster wraps in vertical direction $R_y \equiv R_x$.

(4) $R_{x \text{ and } \bar{y}}$ is the probability that a cluster wraps around one but not the other direction. Here we choose the probability that a cluster wraps around the horizontal and not the vertical direction. The symmetry of the square lattice indicates that $R_{x \text{ and } \bar{y}} \equiv R_{y \text{ and } \bar{x}}$.

The wrapping probability R is a dimensionless quantity, and so one expects finite-size scaling of the form [10,37]

$$R = \tilde{R}[(T - T_p)L^{1/\nu}]. \quad (3)$$

Hence, the R curves for systems of different sizes are expected to cross, up to finite-size corrections, at the same point, marking the transition temperature T_p . In addition, since the scaling function \tilde{R} is expected to be universal, so is the value of $R = \tilde{R}(0)$ at the crossing point [10]. This behavior is nicely verified in Figs. 2 and 3, where the various wrapping probabilities are plotted as a function of temperature T for the larger system sizes studied and for both soft- and hard-constraint clusters. Except for $R_{x \text{ and } \bar{y}}$, all wrapping probabilities increase with decreasing temperature, indicative of the onset of the percolating phase. The crossing of data sets for different system sizes is found to occur very close to the critical temperature T_c of the Ising model shown in Figs. 2 and 3 as a dashed vertical line. This later observation also holds for $R_{x \text{ and } \bar{y}}$ with the essential difference that this observable exhibits a maximum, the position of which is expected to shift to its asymptotic value as $L \rightarrow \infty$ [36]. In numerical studies of ordinary, uncorrelated site or bond percolation, however, it was shown that $R_{x \text{ and } \bar{y}}$ exhibits both crossing points and maxima only in three dimensions [38], whereas in two dimensions the crossing region is absent [36,38]. Thus, the existence of both maxima and crossing points in $R_{x \text{ and } \bar{y}}$ in two dimensions marks an interesting feature of the percolation signature of geometrical clusters in the overlap of the Ising model.

Of central importance in percolation theory is the *cluster number* n_s , denoting the expected number of clusters of s sites per lattice site [10]. Thus the average cluster size can be expressed as

$$S = \frac{\sum_{s'} s^2 n_s}{\sum_{s'} s n_s}, \quad (4)$$

where $s n_s$ corresponds to the probability of a randomly picked site to belong to a cluster of size s . The notation s' indicates that the sums are restricted to certain subsets of clusters. Denoting the set of clusters in a configuration as C and letting P be a subset of C containing the percolating clusters, we can introduce the following definitions for S .

(1) All clusters are included: C .

(2) Exclude the largest cluster in each measurement: $C \setminus \max C$.

(3) Exclude all percolating clusters: $C \setminus P$.

(4) Exclude all clusters percolating in horizontal and in vertical direction: $C \setminus P_{x \text{ and } y}$.

(5) Exclude all clusters percolating in one specific direction, e.g., horizontal: $C \setminus P_x$.

(6) Exclude all clusters percolating in one but not the other direction, e.g., horizontal and not vertical: $C \setminus P_{x \text{ and } \bar{y}}$.

In most numerical studies of percolation the employed definition of the average cluster size excludes the largest cluster in each measurement, corresponding to our case (2) [10]. With this convention, S has a maximum around the percolation point since in the nonpercolating phase the size of many contributing clusters increases, while in the percolating regime most spins belong to the largest cluster which is not counted towards the sum. This estimate is shown in Fig. 4 where S is plotted as a function of T for the larger system sizes and for both soft- and hard-constraint clusters. In the vicinity of the percolation point, the average cluster size is expected to

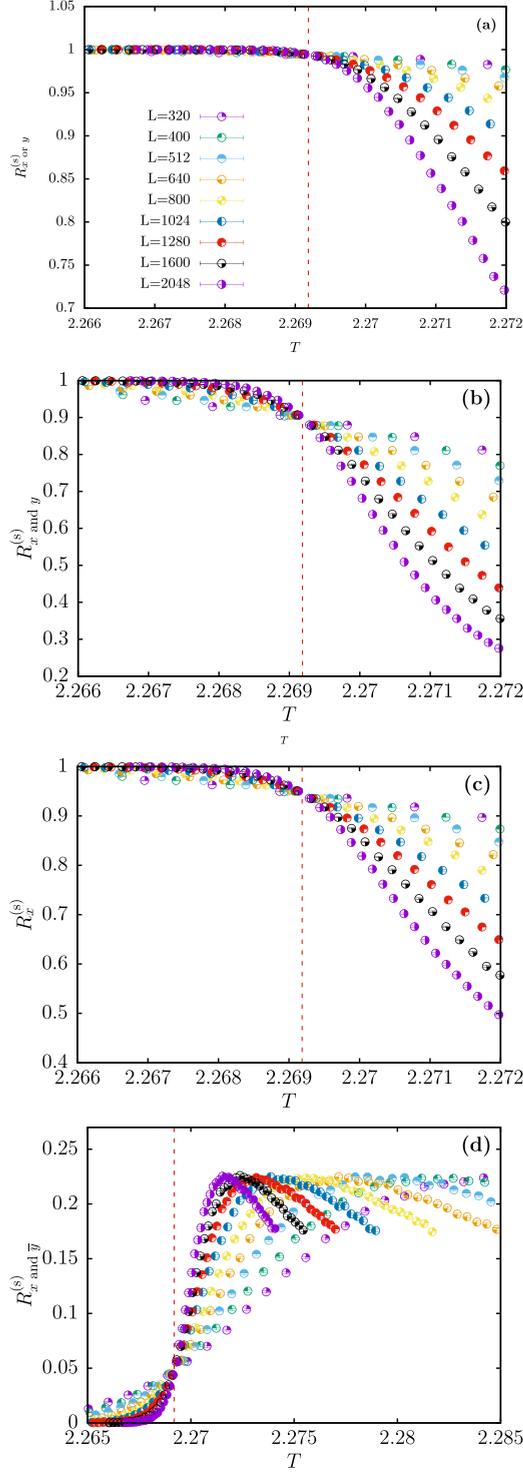


FIG. 2. Wrapping probabilities $R^{(s)}$ of the soft-constraint clusters as a function of temperature T for (a) $R_{x \text{ or } y}^{(s)}$, (b) $R_{x \text{ and } y}^{(s)}$, (c) $R_x^{(s)}$, and (d) $R_{x \text{ and } \bar{y}}^{(s)}$. Only results for the larger system sizes are shown ($L \geq 320$). Here and in the following, the dashed vertical lines mark the critical temperature T_c of the Ising model.

follow a scaling form according to [10]

$$S(L, T) = L^{\gamma/\nu} \tilde{S}[(T - T_p)L^{1/\nu}], \quad (5)$$

which can be used for determining the critical exponent ratio γ/ν conventionally associated to the scaling of the magnetic

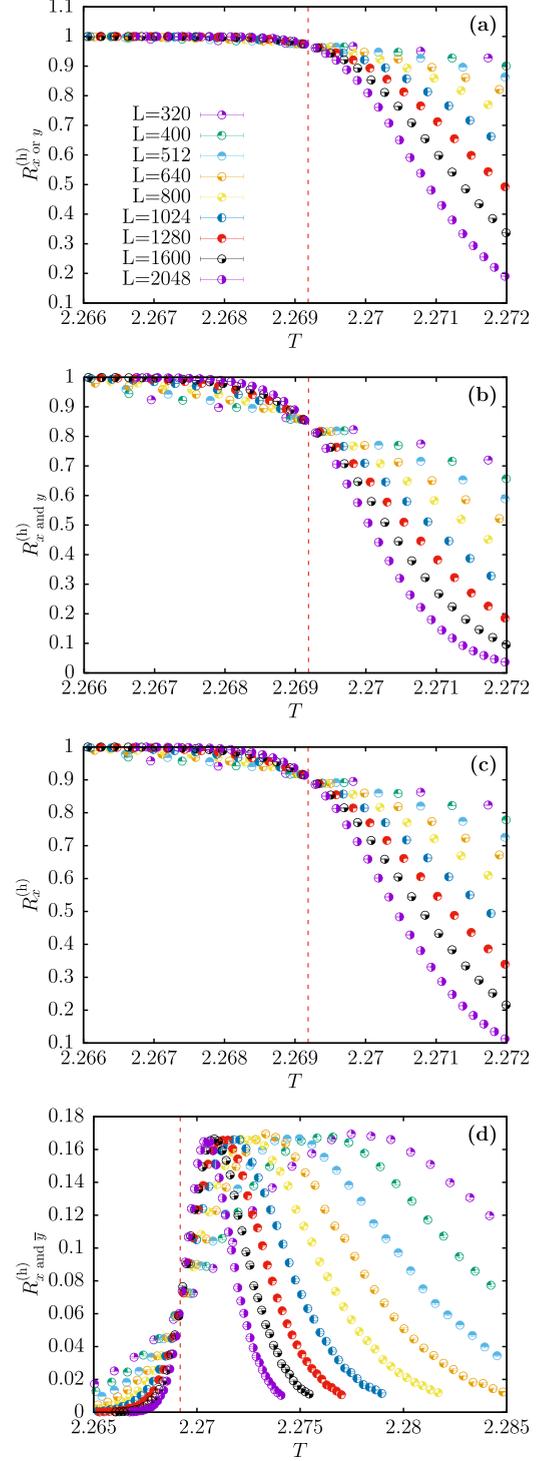


FIG. 3. Wrapping probabilities $R^{(h)}$ of the hard-constraint clusters as a function of temperature T , analogous to the data for soft-constraint clusters shown in Fig. 2.

susceptibility. The behaviors of the remaining definitions of S are shown for the soft-constraint and hard-constraint clusters in Figs. 5(a) and 5(b), respectively. Most definitions show a maximum of S , the exceptions being the full cluster set C as well as $C \setminus P_{x \text{ and } \bar{y}}$.

The percolation strength P_∞ corresponds to the fraction of sites belonging to the infinite cluster in the thermodynamic

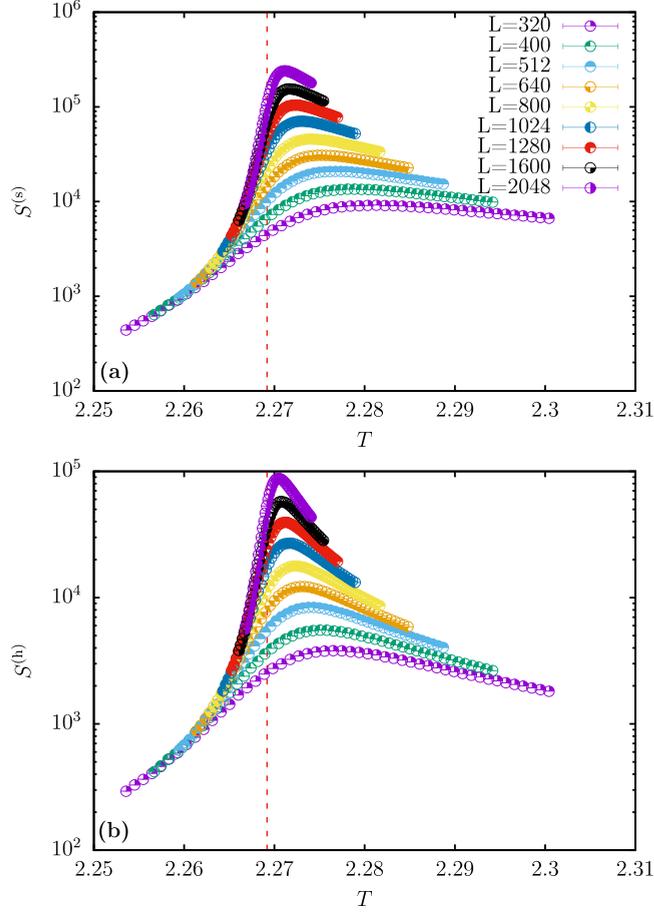


FIG. 4. Average cluster size S according to the definition $C \setminus \max C$ as a function of temperature on a semi-logarithmic scale. Results for both (a) soft-constraint and (b) hard-constraint clusters are presented for the larger system sizes studied ($L \geq 320$).

limit. For finite-size systems it is usually estimated from the fraction of sites belonging to the largest cluster. In Fig. 6, P_∞ is plotted against T for the full range of system sizes studied and for both soft- and hard-constraint clusters. Note that (i) as the temperature decreases P_∞ increases, indicating the appearance of a percolating cluster, and (ii) for $T = 0$ we have $P_\infty = 1$ as all spins belong to the percolating cluster. We remind that when studying the FK clusters in magnetic systems, the percolation strength corresponds to the magnetization of the system [11]. Finite-size scaling theory suggests a scaling form

$$P_\infty(L, T) = L^{-\beta/\nu} \tilde{P}_\infty[(T - T_p)L^{1/\nu}], \quad (6)$$

where β/ν denotes the corresponding critical exponent ratio.

Analogous to the treatment of R and S , it is natural to also consider modified percolation strengths P_∞ by studying the fractions of sites occupied by the following subsets.

- (1) Largest cluster: $\max C$.
- (2) Largest percolating cluster: $\max P$.
- (3) Largest cluster that percolates in horizontal and in vertical direction: $\max P_{x \text{ and } y}$.
- (4) Largest cluster that percolates in one specific direction, e.g., horizontal: $\max P_x$.

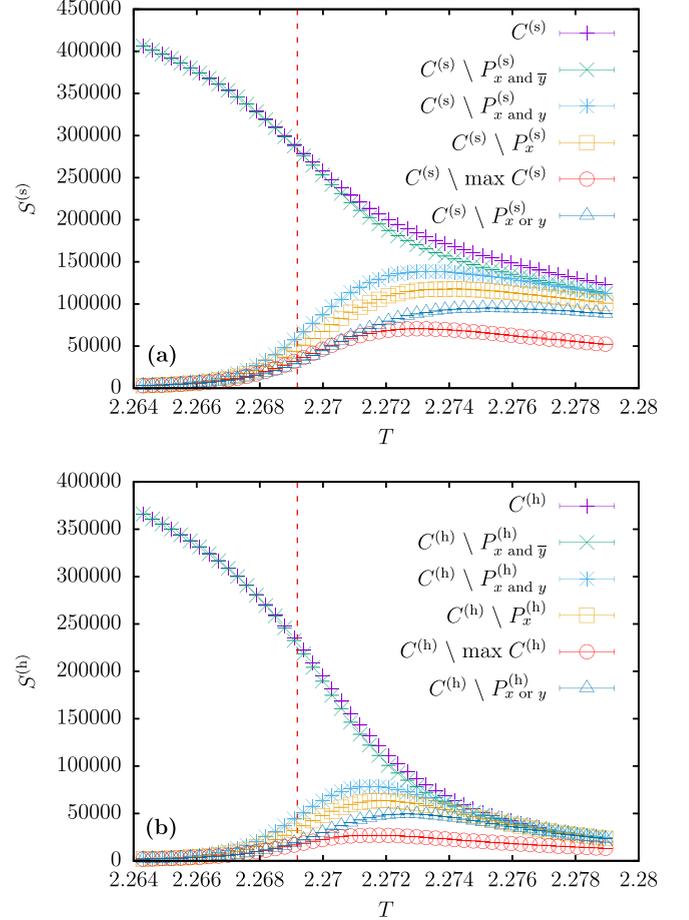


FIG. 5. Average cluster size S versus T of (a) the soft-constraint and (b) the hard-constraint clusters for $L = 1024$ using the different cluster sets defined in the text.

(5) Largest cluster that percolates in one but not the other direction, e.g., horizontal and not vertical: $\max P_x \text{ and } \bar{y}$.

The temperature dependence of the percolation strength according to these definitions is shown for an $L = 1024$ system and the soft-constraint and hard-constraint clusters in Figs. 7(a) and 7(b), respectively. With the exception of the unlikely $P_x \text{ and } \bar{y}$ clusters, the different definitions lead to the same behavior for temperatures below the percolation point, but visible differences above.

In the following, we will use the scaling forms (3), (5), and (6) with the universal scaling functions \tilde{R} , \tilde{P}_∞ , and \tilde{S} to determine the correlation-length critical exponent ν as well as the exponent ratios γ/ν and β/ν of the average cluster size and the percolation strength, respectively, as well as the percolation temperature T_p .

IV. MAIN RESULTS

A. Correlation-length exponent

In Monte Carlo studies of phase transitions the temperature derivative of the Binder cumulant [39] provides a reliable estimate for the critical exponent ν [40]. In random percolation, a similar behavior is expected for the derivative of the wrapping probability with respect to the bond occupa-

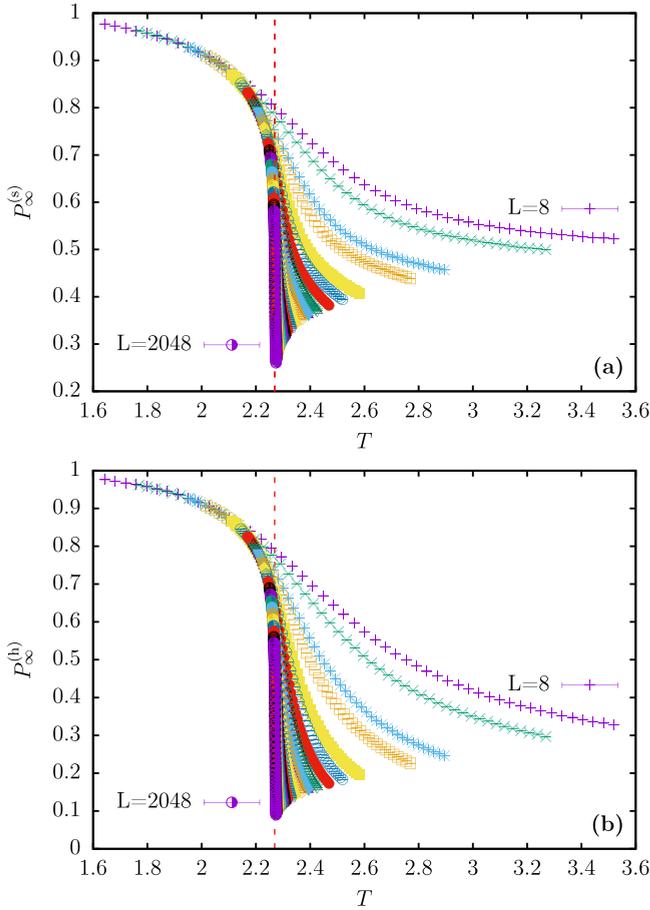


FIG. 6. Percolation strength P_∞ as a function of temperature for both (a) soft-constraint and (b) hard-constraint clusters. For clarity, only the legends of the smallest and largest system sizes are highlighted.

tion probability [38]. For the present problem of clusters in a thermal problem it is natural, in contrast, to consider the temperature derivative of the wrapping probabilities. As the wrapping probabilities are monotonic functions of the temperature (except for $R_{x \text{ and } \bar{y}}$) one expects that the maximum of the absolute value of its first derivative should scale as

$$\left| \frac{dR}{dT} \right|_{\max} \sim L^{1/\nu}. \quad (7)$$

As shown in Sec. III, the exception to this rule is $R_{x \text{ and } \bar{y}}$ which is a nonmonotonic function of T , showing both a maximum and a crossing region. Nevertheless, if we restrict ourselves to the vicinity of the crossing regime, this observable is also expected to follow the scaling behavior of Eq. (7). Importantly, Eq. (7) allows one to obtain estimates of ν without prior knowledge of the percolation temperature T_p . To determine the maximum of $|dR/dT|$, both the first and second derivatives are computed using the symmetric-finite-difference definition, and the root is located using the bisection method [35]. The required estimates at nearby temperatures are extracted from the simulation data by means of single-histogram reweighting [41], using a step size $\Delta T = 10^{-7}$.

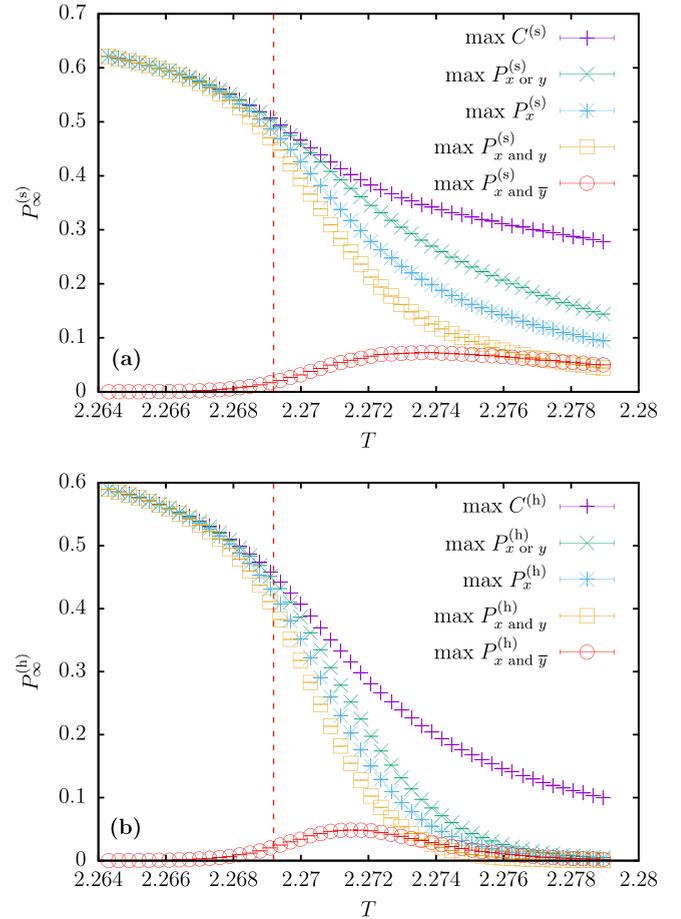


FIG. 7. Percolation strength P_∞ versus T of (a) the soft-constraint and (b) the hard-constraint clusters for $L = 1024$ and the cluster types discussed in the main text.

In Fig. 8, $|dR/dT|_{\max}$ is shown as a function of L for the different wrapping probabilities of the soft- and hard-constraint clusters. Fits of the form (7) were performed for system sizes on intervals $L_{\min} \leq L \leq L_{\max}$ by systematically increasing the lower cutoff L_{\min} , while keeping the upper cutoff fixed at $L_{\max} = 2048$. The resulting effective values of ν are shown in Fig. 9. The final estimates we quote for both the soft- and hard-constraint clusters using the $R_{x \text{ and } y}^{(s)}$ and $R_{x \text{ and } \bar{y}}^{(h)}$ definitions, respectively, are

$$\nu^{(s)} = 1.005(5) \quad (L_{\min} = 256), \quad (8a)$$

$$\nu^{(h)} = 1.00(3) \quad (L_{\min} = 800). \quad (8b)$$

These results suggest that the critical exponent of the correlation length is the same for both cluster types and consistent with that of the Ising model, i.e., $\nu = 1$.

B. Percolation temperature

For the estimation of the percolation temperature T_p we considered the intersection of the wrapping probabilities of pairs of system sizes (L, L') as a function of T , following the original prescription by Binder for the magnetization cumulant [39,40]. The points where these wrapping probabilities

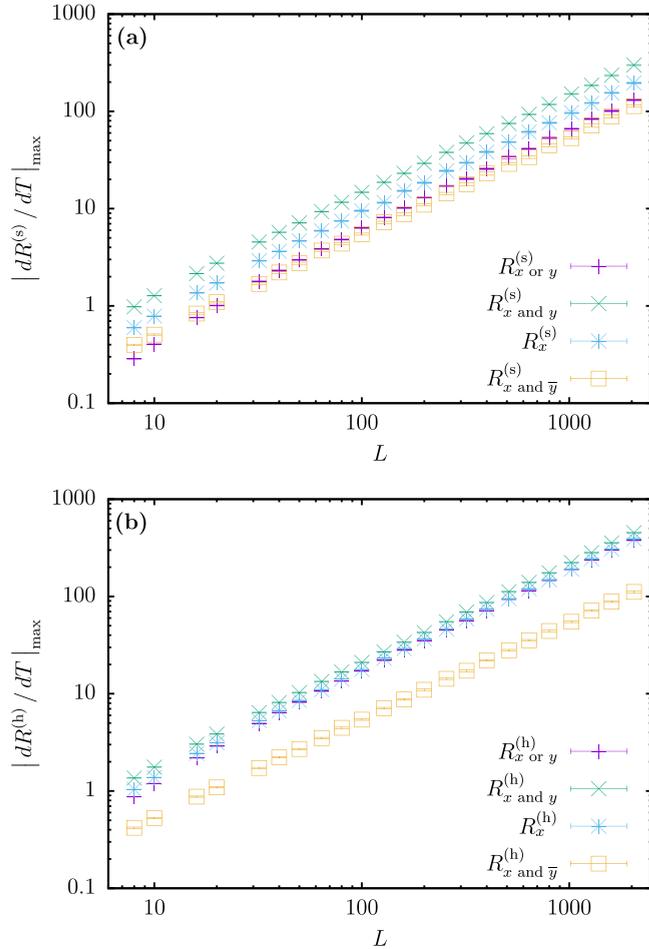


FIG. 8. Maximum slope of the wrapping probability $|dR/dT|_{\max}$ as a function of system size L on a double logarithmic scale for all variants of the wrapping probabilities and for (a) soft-constraint and (b) hard-constraint clusters.

cross scale as [39]

$$T_{\text{cross}}(L, b) = T_p + aL^{-(1/\nu+\omega)} \left(\frac{b^{-\omega} - 1}{b^{1/\nu} - 1} \right), \quad (9)$$

where a is a nonuniversal scaling parameter, ω is the corrections-to-scaling exponent, and $b = L'/L$ is the quotients ratio, fixed hereafter to $b = 2$. Crossings were determined using the bisection method [35] alongside the single-histogram reweighting technique [41].

Figure 10 showcases the scaling of the crossing points of the various wrapping probabilities for both the soft- and hard-constrained clusters, which appear to be consistent with the critical temperature of the Ising model, up to surprisingly small finite-size effects. To obtain more accurate estimates of T_p we performed fits using Eq. (9). Due to the smallness of the corrections, however, the accuracy of our data did not allow us to resolve their detailed form, resulting in fits of poor quality and consequently in unreliable estimates of the involved parameters. As an alternative, we fixed ν to the expected value $\nu = 1$, so that Eq. (9) (ignoring scaling corrections) simplifies to

$$T_{\text{cross}} = T_p + a/L. \quad (10)$$

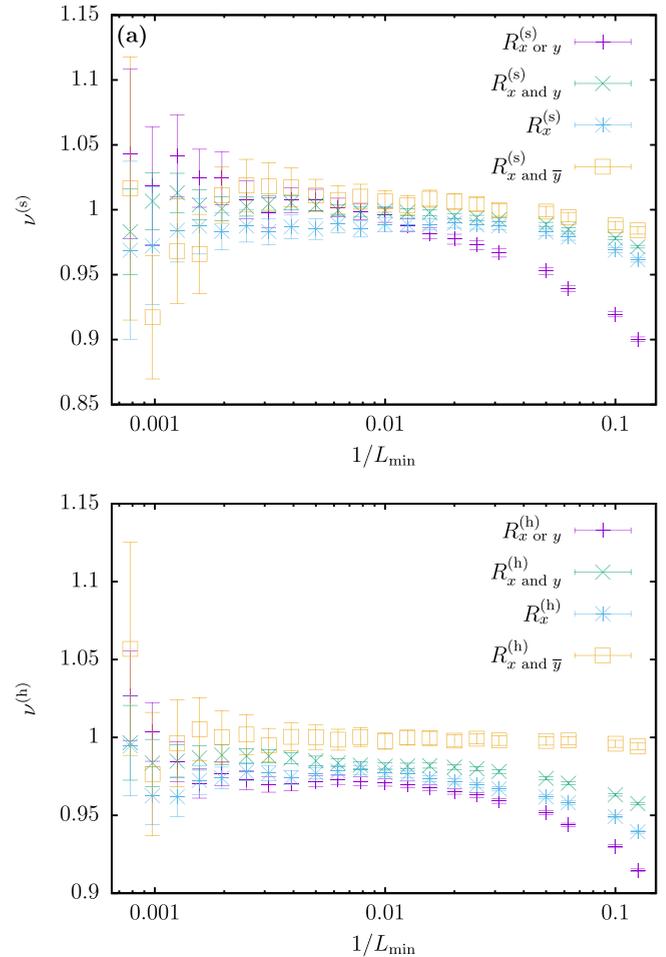


FIG. 9. Effective values of the exponent ν versus the inverse lower cutoff size $1/L_{\min}$ on a semi-logarithmic scale. The estimates were obtained from fits of the form (7) to the data of Fig. 8.

Estimates of the percolation temperature resulting from the linear fits of Eq. (10) are shown in Fig. 11 for both soft- and hard-constraint clusters; they are found to be consistent with the critical temperature of the Ising ferromagnet, i.e., $T_c = 2.269185\dots$. We note that the results for $R_{x \text{ or } y}$ have slightly elevated statistical errors as compared to the other definitions, a feature that can be traced back to the fact that $R_{x \text{ or } y}$ is very close to 1 at T_c such that the curves cross at a smaller angle, cf. Figs. 2(a) and 3(a). In addition, the parameter a of Eq. 10 for all fits is consistent with zero within error bars, indicating that the data can also be described by a constant $T_{\text{cross}}(L) = T_p = T_c$, independent of L . For random percolation it has been observed that most estimates of pseudocritical points on the square lattice converge even more quickly to the thermodynamic limit than expected from the $L^{-1/\nu}$ scaling [42], namely, proportional to $L^{-(1/\nu+1)}$. Performing fits with a corresponding variant of Eq. (10) with the $1/L$ term replaced by $1/L^2$ yield perfectly compatible results, indicating that our data are in fact not able to resolve these very small deviations from the infinite-size behavior. At this point we may safely conclude that soft- and hard-constraint clusters undergo a percolation transition at the critical temperature T_c of the Ising model.

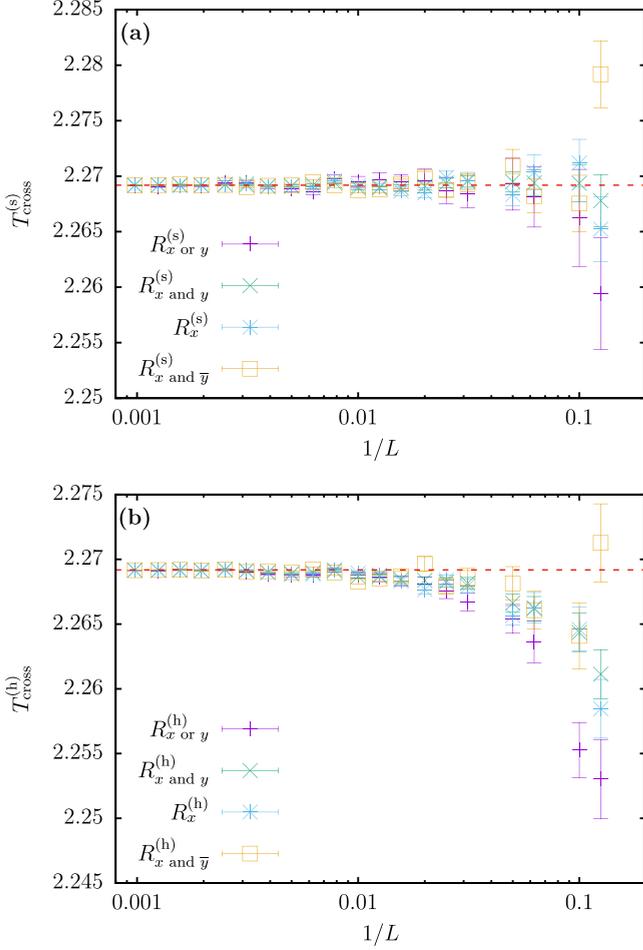


FIG. 10. Estimates of crossing temperatures for (a) soft-constraint and (b) hard-constraint clusters on a semi-logarithmic scale. The dashed horizontal line marks the transition temperature of the Ising model.

C. Scaling at criticality

We proceed with the computation of critical exponents related to the percolation strength, P_∞ and the average cluster size S for the two types of clusters at criticality. Since we found convincing evidence of the identity of the percolation and critical temperatures $T_p = T_c = 2.269185\dots$ in the previous section, we consider the scaling of P_∞ and S at the fixed temperature $T = T_c$; the result is shown for the soft- and hard-constraint clusters in Figs. 12 and 13, respectively. For both observables, all data appear to follow straight parallel lines, suggesting minor corrections to scaling and universal exponents, independent of the definition used.

At the percolation point, according to Eqs. (5) and (6) the scaling functions become constant, thus allowing for the estimation of the involved exponents from finite-size scaling. In the following, we present the results of systematically fitting those functional forms to the data for P_∞ and S for all sets of definitions, following the established protocol of varying L_{\min} as described above, allowing us to monitor the influence of corrections to scaling.

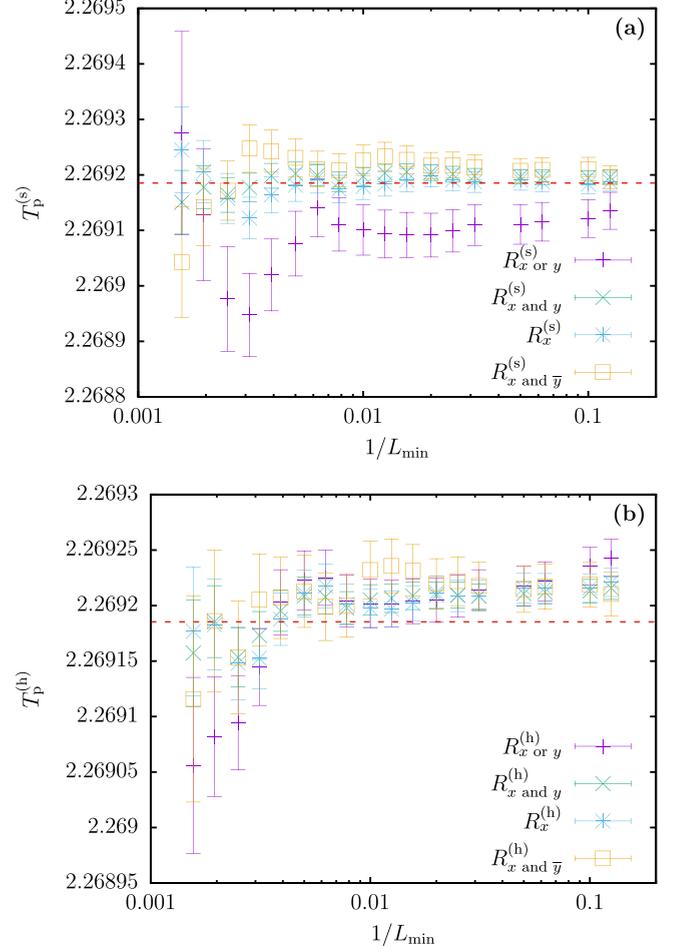


FIG. 11. Estimates of the percolation temperature T_p versus $1/L_{\min}$ on a semi-logarithmic scale. Results obtained from fits of the form (10) to the data of Fig. 10. The dashed horizontal line marks here as well as the transition temperature of the Ising model.

Figure 14 shows our estimates of the effective exponent ratios $(\beta/\nu)^{(s)}$ and $(\beta/\nu)^{(h)}$ while varying the cutoff L_{\min} . In addition to the max P_x and \bar{y} data, it is evident that the exponents converge relatively quickly to the values $(\beta/\nu)^{(s)} \approx 0.095$ and $(\beta/\nu)^{(h)} \approx 0.12$ for the soft- and hard-constraint clusters, respectively. The fact that the numerical data for the max P_x and \bar{y} definition provide unreliable estimates of the involved exponent can be readily understood: clusters percolating in one but not the other direction are sparse, leading to poor statistics in the estimation of the percolation strength and the associated exponent. This observation has also been reported in Ref. [43] for the Ising model. We note, however, that it is also conceivable that the rather special constraint “ x and \bar{y} ” leads to an asymptotically different fractal dimension for such clusters as compared to the unconstrained case.

As there is no systematic trend visible in our data that could possibly reveal the existence of corrections to scaling for $(\beta/\nu)^{(s)}$ and $(\beta/\nu)^{(h)}$, we do not attempt to perform fits including correction terms. Instead, as a trade-off between unavoidable corrections to scaling and reasonable values of χ^2 , we choose sufficiently large L_{\min} for our final estimates of

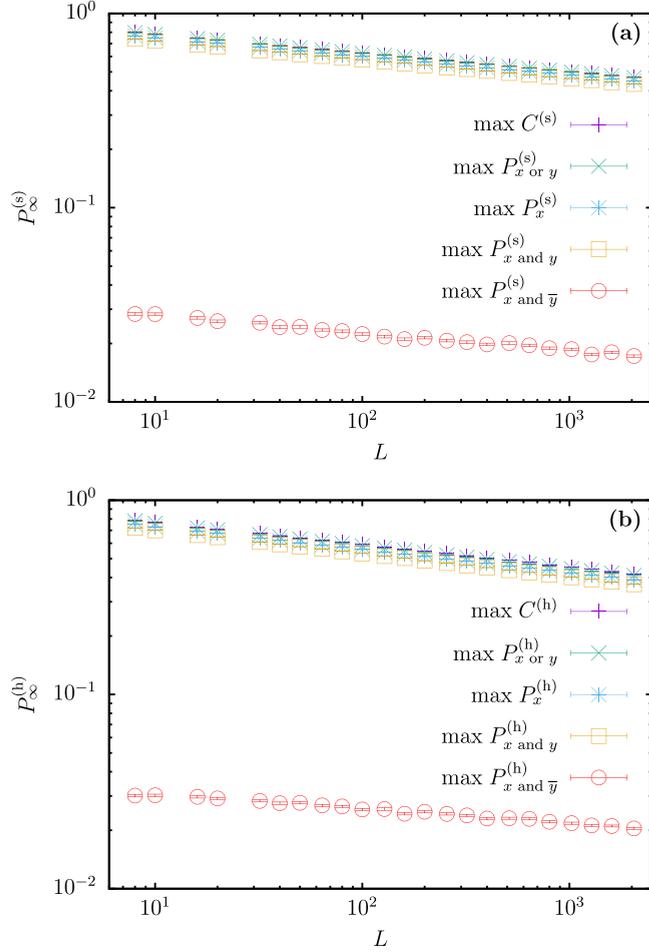


FIG. 12. Percolation strength P_∞ versus L at the critical temperature of the Ising model on a double logarithmic scale for (a) soft-constraint and (b) hard-constraint clusters.

$(\beta/\nu)^{(s)}$ and $(\beta/\nu)^{(h)}$,

$$\left(\frac{\beta}{\nu}\right)^{(s)} = 0.0950(7) \quad (L_{\min} = 320), \quad (11a)$$

$$\left(\frac{\beta}{\nu}\right)^{(h)} = 0.1184(11) \quad (L_{\min} = 512). \quad (11b)$$

Estimates for the effective exponent ratios $(\gamma/\nu)^{(s)}$ and $(\gamma/\nu)^{(h)}$ are shown in Fig. 15. For the C and $C \setminus P_{x \text{ and } \bar{y}}$ definitions, the exponent ratio converges relatively quickly to the values $(\gamma/\nu)^{(s)} \approx 1.81$ and $(\gamma/\nu)^{(h)} \approx 1.77$ for the soft- and hard-constraint clusters respectively, indicating that corrections to scaling are not substantial. On the other hand, for the rest of the definitions, corrections to scaling become important and the convergence to an asymptotic value is rather slow. The fact that C and $C \setminus P_{x \text{ and } \bar{y}}$ give similar results is to be expected, as the later definition excludes clusters that rarely appear, thus not altering significantly the sums in Eq. (4). Note that the reduction of scaling corrections for the average cluster size through the choice of the C definition was also reported in Ref. [24] for the Ising model, while a systematic study of the behavior for the Ising model from all of the above definitions was presented in Ref. [43].

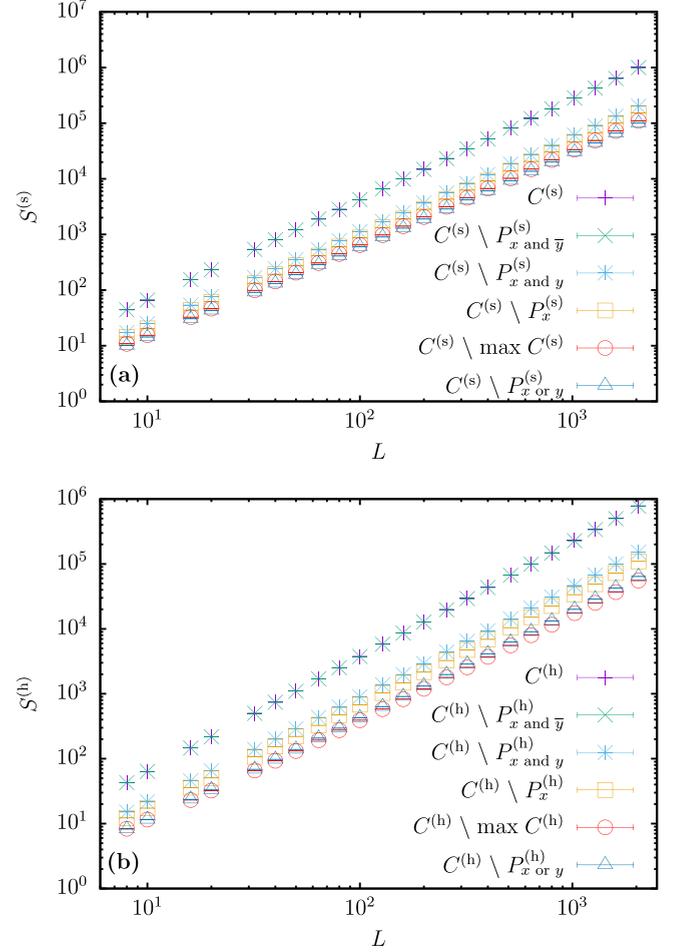


FIG. 13. Average cluster size S versus L at the critical temperature of the Ising model on a double logarithmic scale for (a) soft-constraint and (b) hard-constraint clusters.

More reliable estimates for the exponent ratio γ/ν can be retrieved by taking into account corrections to scaling for both soft- and hard-constraint clusters. To arrive at somewhat stable estimates for the correction-to-scaling exponent, we performed joint fits to the data for the different definitions of S including one correction term using the *Ansatz*

$$S = aL^{\gamma/\nu}(1 + bL^{-\omega}), \quad (12)$$

where a and b are the noncommon fitting parameters, and γ/ν as well as ω the shared parameters. Since data from different definitions are not statistically independent, as they result from the same Monte Carlo series, a naive implementation of the above fitting procedure will result in erroneous error estimation of the fit parameters. Thus, to provide reliable estimates for the errors of the involved parameters, we employ the jackknife method [44]. For all values of L_{\min} , estimates of γ/ν agree within error bars as is shown in Fig. 16. Additionally, in Fig. 17 the exponent ω is plotted as a function of L_{\min} . The fact that the error bars in the estimates of γ/ν and ω are increasing with L_{\min} is of course a consequence of the decreasing number of degrees of freedom. However, the errors in ω increase rapidly, and for $L_{\min} \geq 320$ they are comparable with the absolute values of ω

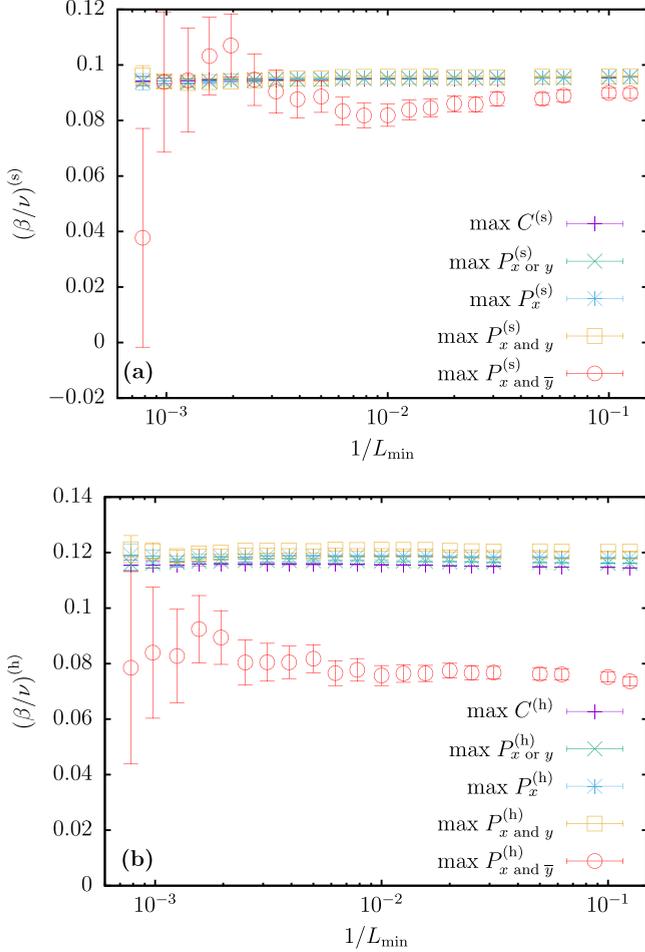


FIG. 14. Effective exponent ratios β/ν extracted from fits of the functional forms (5) and (6) as a function of the inverse lower size cutoff $1/L_{\min}$ for (a) soft-constraint and (b) hard-constraint clusters.

for both cluster types. Our final estimates of the involved exponent ratio γ/ν for the soft- and hard-constraint clusters are

$$\left(\frac{\gamma}{\nu}\right)^{(s)} = 1.814(5) \quad (L_{\min} = 200), \quad (13a)$$

$$\left(\frac{\gamma}{\nu}\right)^{(h)} = 1.765(4) \quad (L_{\min} = 200). \quad (13b)$$

We note that the estimated value of $\omega \approx 0.2$ for both soft- and hard-constraint clusters represents a rather slow decay of corrections, consistent with the slow convergence of γ/ν for most definitions of S that is clearly visible in Fig. 15. We are not aware of any theoretical estimates relating to the value of ω .

D. Fractal dimension and hyperscaling

At the percolation point the incipient spanning cluster is a fractal object and its mass M (i.e., the number of spins that belong to it) scales with the system size as $M \sim L^D$. Here, D denotes the fractal dimension of the incipient spanning cluster. Since from the discussion in Sec. III the percolation strength P_∞ corresponds to the fraction of spins in the percolating cluster, it follows that $D = d - \beta/\nu$. From the estimates of β/ν for the soft- and hard-constraint clusters provided in

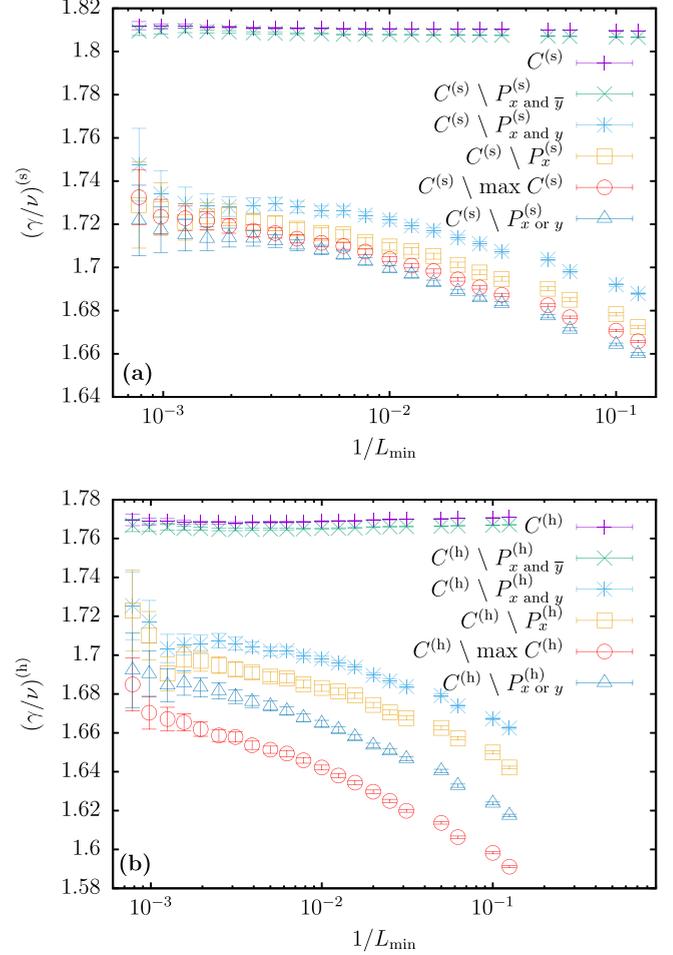


FIG. 15. Effective exponent ratios γ/ν versus $1/L_{\min}$ for (a) soft-constraint and (b) hard-constraint clusters.

Eqs. (11a) and (11b) we hence obtain

$$D^{(s)} = 1.9050(7), \quad D^{(h)} = 1.8816(11), \quad (14)$$

suggesting that the fractal dimension is different for the two cluster types. Also, the fact that $D^{(s)} > D^{(h)}$ highlights that the soft-constraint clusters are denser than the hard-constraint ones, a reasonable expectation as the hard-constraint clusters are a subset of the soft-constraint ones. Additionally, let us point out that the fractal dimensions for both cluster types are smaller than $D = 187/96 \approx 1.9479$, the fractal dimension of geometrical clusters in the Ising model [20].

On the other hand, if hyperscaling is valid, the fractal dimension can also be estimated from $D = \beta/\nu + \gamma/\nu$ [10]. Combining our estimates from Eqs. (11a) and (11b) with Eqs. (13a) and (13b), we arrive at

$$D^{(s)} = 1.909(5), \quad D^{(h)} = 1.883(4), \quad (15)$$

which is consistent with the estimates of Eq. (14), thus illustrating that hyperscaling is valid for both cluster types.

V. DISCUSSION

We studied the percolation properties of clusters defined in the overlap space of two statistically independent systems

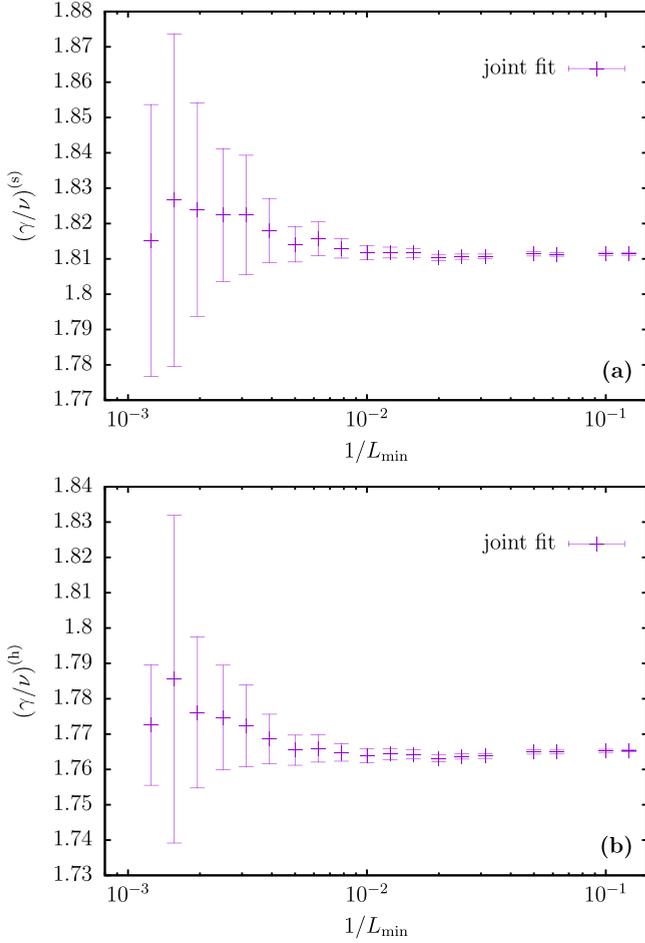


FIG. 16. Exponent ratio γ/ν resulting from the joint fit of all definitions as a function of $1/L_{\min}$ for (a) soft-constraint and (b) hard-constraint clusters.

of the square-lattice ferromagnetic Ising model. To this end, two distinct cluster types were introduced which we dubbed soft- and hard-constraint clusters. After a short exposition of the behavior of the main observables, i.e., the wrapping probabilities, average cluster sizes, and percolation strengths, the critical behavior of the system was investigated. Our results indicate that both cluster types are described by the same correlation length exponent which is found to be in agreement with the value $\nu = 1$ of the Ising ferromagnet. Additionally, both cluster types percolate at a temperature that is indistinguishable from the transition point of the two-dimensional Ising model for the lattice sizes up to $L = 2048$ that we considered in our study. In marked contrast with the exponent ν , our analysis for the exponent ratios β/ν and γ/ν manifests the following: (i) the exponent values are clearly different from those that characterize the geometrical clusters of the Ising model, and (ii) there is a small but seemingly systematic difference in the exponents estimated for the soft-constraint and the hard-constraint clusters, cf. the overview of exponent estimates provided in Table I.

Under the assumption that $T_p = T_c$ that is so well corroborated by our numerical results, the scaling exponents for the overlap clusters can be deduced from the following argument:

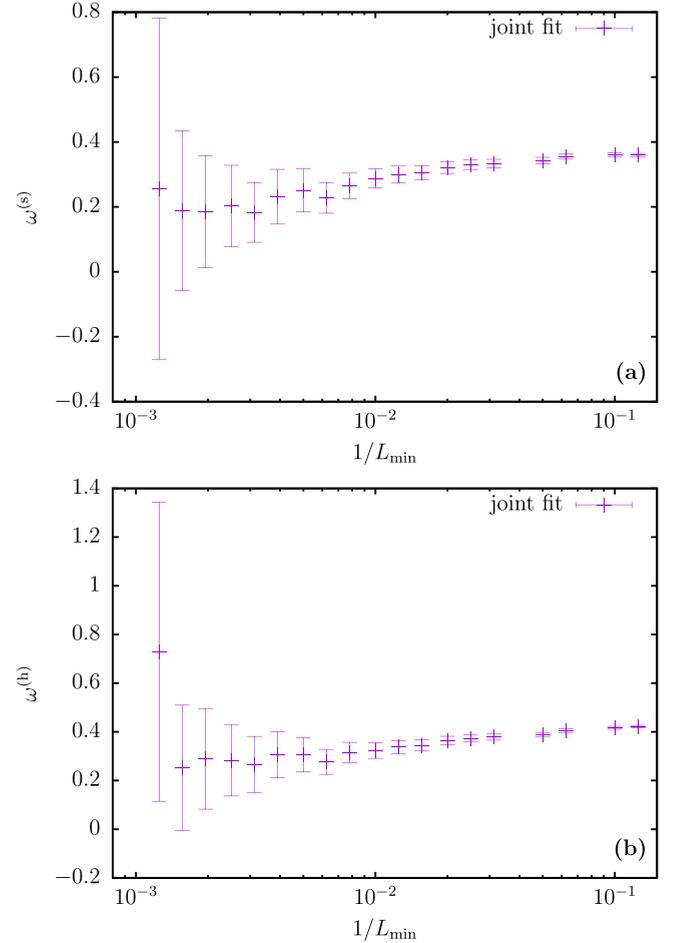


FIG. 17. Corrections-to-scaling exponent ω resulting from the joint fit of all definitions as a function of $1/L_{\min}$ for (a) soft-constraint and (b) hard-constraint clusters.

according to the expected scaling of P_∞ , at T_c the probability of a randomly picked site to be in the percolating cluster of replica one scales as $\sim L^{-(\beta/\nu)_{IG}}$, where $(\beta/\nu)_{IG} = 5/96$ corresponds to the exponent of the geometrical clusters in the Ising model. As the same holds for replica two and the two replicas are uncorrelated, the probability of the site being in the percolating cluster of *both* replicas decays as $\sim L^{-2(\beta/\nu)_{IG}}$, such that $\beta/\nu = 2(\beta/\nu)_{IG} = 10/96 \approx 0.1042$. As a consequence, one expects $\gamma/\nu = 2 - 4(\beta/\nu)_{IG} = 43/24 \approx 1.792$

TABLE I. Critical exponents of the soft- and hard-constraint clusters in comparison to the exact values of the geometrical clusters of the square-lattice Ising model model.

Exponent	Overlap		Ising
	Soft	Hard	—
ν	1.005(5)	1.00(3)	1
β/ν	0.0950(7)	0.1184(11)	$5/96 \approx 0.052$
γ/ν	1.814(5)	1.765(4)	$91/48 \approx 1.895$
$D = \gamma/\nu + \beta/\nu$	1.909(5)	1.883(4)	$187/96 \approx 1.947$
$D = d - \beta/\nu$	1.9050(7)	1.8816(11)	$187/96$

and $D = 2 - 2(\beta/\nu)_{IG} = 91/48 \approx 1.8958$. As a glance at Table I shows, these values are quite compatible with our findings, in particular, for the hard-constraint clusters to which this type of argument applies. The small deviations observed might be a consequence of the slow decay of scaling corrections expressed in the small value $\omega \approx 0.2$ of the Wegner exponent. The clusters in the soft-constraint problem are, by construction, larger than those of the hard-constraint variant, but their scaling does not directly follow from the above argument, so it is possible that they show asymptotically different exponents.

At the same time, the density of the percolating cluster in the overlap is below that of the percolating spin cluster in a single Ising model, while the clusters of the first are also found to be more compact than those in the second. This is a natural consequence of the superimposition of the

two fractal structures such that the objects investigated here correspond to their intersection. It would be most intriguing to study how clusters in the mutual overlap of more than two copies of the system behave; this task is left for future work, however.

ACKNOWLEDGMENTS

The authors thank L. Münster for useful discussions. We acknowledge the provision of computing time on the parallel computer cluster Zeus of Coventry University. N.G.F. acknowledges support through the Visiting Scholar Program of Chemnitz University of Technology and is grateful to the colleagues at the Institute of Physics for their warm hospitality, which he enjoyed while part of this work was completed.

-
- [1] W. Janke, Monte Carlo simulations of spin systems, in *Computational Physics*, edited by K. H. Hoffmann and M. Schreiber (Springer, Berlin, 1996), pp. 10–43.
 - [2] F. J. Wegner, Duality in generalized Ising models and phase transitions without local order parameters, *J. Math. Phys.* **12**, 2259 (1971).
 - [3] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, *Ann. Phys. (NY)* **303**, 2 (2003).
 - [4] C. Castelnovo and C. Chamon, Topological order and topological entropy in classical systems, *Phys. Rev. B* **76**, 174416 (2007).
 - [5] D. A. Johnston, Goniherdic (and Fuki-Nuke) order, *J. Phys. A: Math. Theor.* **45**, 405001 (2012).
 - [6] K. Binder and A. P. Young, Spin glasses: Experimental facts, theoretical concepts, and open questions, *Rev. Mod. Phys.* **58**, 801 (1986).
 - [7] G. Parisi, Order parameter for spin-glasses, *Phys. Rev. Lett.* **50**, 1946 (1983).
 - [8] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
 - [9] S. Boccaletti, G. Bianconi, R. Criado, C. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, and M. Zanin, The structure and dynamics of multilayer networks, *Phys. Rep.* **544**, 1 (2014).
 - [10] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, Revised, 2nd ed. (Taylor & Francis, London, 1994).
 - [11] C. M. Fortuin and P. W. Kasteleyn, On the random-cluster model: I. Introduction and relation to other models, *Physica* **57**, 536 (1972).
 - [12] A. Coniglio and W. Klein, Clusters and Ising critical droplets: A renormalisation group approach, *J. Phys. A: Math. Gen.* **13**, 2775 (1980).
 - [13] R. H. Swendsen and J.-S. Wang, Nonuniversal critical dynamics in Monte Carlo simulations, *Phys. Rev. Lett.* **58**, 86 (1987).
 - [14] U. Wolff, Collective Monte Carlo updating for spin systems, *Phys. Rev. Lett.* **62**, 361 (1989).
 - [15] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, *J. Chem. Phys.* **21**, 1087 (1953).
 - [16] K. Binder, “Clusters” in the Ising model, metastable states and essential singularity, *Ann. Phys. (NY)* **98**, 390 (1976).
 - [17] A. Coniglio and A. Fierro, Correlated percolation, in *Encyclopedia of Complexity and Systems Science*, edited by R. A. Meyers (Springer, New York, 2009), pp. 1596–1615.
 - [18] A. A. Saberi, Recent advances in percolation theory and its applications, *Phys. Rep.* **578**, 1 (2015).
 - [19] A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo, Percolation points and critical point in the Ising model, *J. Phys. A: Math. Gen.* **10**, 205 (1977).
 - [20] A. L. Stella and C. Vanderzande, Scaling and fractal dimension of Ising clusters at the $d = 2$ critical point, *Phys. Rev. Lett.* **62**, 1067 (1989).
 - [21] C. Vanderzande and A. L. Stella, Bulk, surface and hull fractal dimension of critical Ising clusters in $d = 2$, *J. Phys. A: Math. Gen.* **22**, L445 (1989).
 - [22] C. Vanderzande, Fractal dimensions of Potts clusters, *J. Phys. A: Math. Gen.* **25**, L75 (1992).
 - [23] W. Janke and A. M. Schakel, Geometrical vs. Fortuin-Kasteleyn clusters in the two-dimensional q -state Potts model, *Nucl. Phys. B* **700**, 385 (2004).
 - [24] W. Janke and A. M. J. Schakel, Fractal structure of spin clusters and domain walls in the two-dimensional Ising model, *Phys. Rev. E* **71**, 036703 (2005).
 - [25] L. de Arcangelis, A. Coniglio, and F. Peruggi, Percolation transition in spin glasses, *Europhys. Lett.* **14**, 515 (1991).
 - [26] J. Machta, C. M. Newman, and D. L. Stein, The percolation signature of the spin glass transition, *J. Stat. Phys.* **130**, 113 (2007).
 - [27] H. Fajen, A. K. Hartmann, and A. P. Young, Percolation of Fortuin-Kasteleyn clusters for the random-bond Ising model, *Phys. Rev. E* **102**, 012131 (2020).
 - [28] J. Houdayer, A cluster Monte Carlo algorithm for 2-dimensional spin glasses, *Eur. Phys. J. B* **22**, 479 (2001).
 - [29] T. Jörg, Cluster Monte Carlo algorithms for diluted spin glasses, *Prog. Theor. Phys. Suppl.* **157**, 349 (2005).
 - [30] L. Münster and M. Weigel, Cluster percolation in the two-dimensional Ising spin glass, *Phys. Rev. E* **107**, 054103 (2023).

- [31] F. Haake, M. Lewenstein, and M. Wilkens, Relation of random and competing nonrandom couplings for spin-glasses, *Phys. Rev. Lett.* **55**, 2606 (1985).
- [32] L. Chayes, J. Machta, and O. Redner, Graphical representations for Ising systems in external fields, *J. Stat. Phys.* **93**, 17 (1998).
- [33] M. Weigel and W. Janke, Error estimation and reduction with cross correlations, *Phys. Rev. E* **81**, 066701 (2010).
- [34] J. Machta, Y. S. Choi, A. Lucke, T. Schweizer, and L. M. Chayes, Invaded cluster algorithm for Potts models, *Phys. Rev. E* **54**, 1332 (1996).
- [35] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd ed. (Cambridge University Press, Cambridge, England, 2007).
- [36] M. E. J. Newman and R. M. Ziff, Fast Monte Carlo algorithm for site or bond percolation, *Phys. Rev. E* **64**, 016706 (2001).
- [37] V. Privman, Finite-size scaling theory, in *Finite Size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990), pp. 1–98.
- [38] P. H. L. Martins and J. A. Plascak, Percolation on two- and three-dimensional lattices, *Phys. Rev. E* **67**, 046119 (2003).
- [39] K. Binder, Finite size scaling analysis of Ising model block distribution functions, *Z. Phys. B* **43**, 119 (1981).
- [40] A. M. Ferrenberg and D. P. Landau, Critical behavior of the three-dimensional Ising model: A high-resolution Monte Carlo study, *Phys. Rev. B* **44**, 5081 (1991).
- [41] A. M. Ferrenberg and R. H. Swendsen, New Monte Carlo technique for studying phase transitions, *Phys. Rev. Lett.* **61**, 2635 (1988).
- [42] R. M. Ziff and M. E. J. Newman, Convergence of threshold estimates for two-dimensional percolation, *Phys. Rev. E* **66**, 016129 (2002).
- [43] M. Akritidis, N. G. Fytas, and M. Weigel, Corrections to scaling in geometrical clusters of the 2D Ising model, *J. Phys.: Conf. Ser.* **2207**, 012004 (2022).
- [44] B. Efron, Bootstrap methods: Another look at the jackknife, *Ann. Statist.* **7**, 1 (1979).