Extremal statistics for a resetting Brownian motion before its first-passage time

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We study the extreme value statistics of a one-dimensional resetting Brownian motion (RBM) till its first passage through the origin starting from the position x_0 (>0). By deriving the exit probability of RBM in an interval $[0, M]$ from the origin, we obtain the distribution $P_r(M|x_0)$ of the maximum displacement M and thus gives the expected value $\langle M\rangle$ of M as functions of the resetting rate r and x_0. We find that $\langle M\rangle$ decreases monotonically as r increases, and tends to $2x_0$ as $r \to \infty$. In the opposite limit, $\langle M\rangle$ diverges logarithmically as $r \to 0$. Moreover, we derive the propagator of RBM in the Laplace domain in the presence of both absorbing ends, and then leads to the joint distribution $P_r(M, t_m|x_0)$ of M and the time t_m at which this maximum is achieved in the Laplace domain by using a path decomposition technique, from which the expected value $\langle t_m\rangle$ of t_m is obtained explicitly. Interestingly, $\langle t_m\rangle$ shows a nonmonotonic dependence on r, and attains its minimum at an optimal $r^* \approx 2.71691 D/x_0^2$, where D is the diffusion coefficient. Finally, we perform extensive simulations to validate our theoretical results.

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I. INTRODUCTION

Understanding extreme events that occur very infrequently is important as they may bring catastrophic consequences. From natural calamities like earthquake, tsunamis, and floods to economic collapses and outbreak of pandemic are all examples of extreme events which can lead to devastating consequences [1–6]. Extreme-value statistics (EVS) has been a branch of statistics which deals with the extreme deviations of a random process from its mean behavior. Gnedenko's classical law of extremes is a general result regarding the asymptotic distribution of extreme value for independent and identically distributed random variables [7]. The study of EVS has been extremely important in the field of disordered systems [8,9], fluctuating interfaces [10,11], interacting spin systems [12], stochastic transport models [13,14], random matrices [15–17], epidemic outbreak [18], binary search trees [19], and related computer search algorithms [20,21]. In recent years, there is an increasing interest in studying the extreme value for many weakly and strongly correlated stochastic processes [22–28]. We refer the readers to Refs. [29,30] for two recent reviews on the extreme-value statistics.

One of the central goals on this subject is to compute the statistics of extremes, i.e., the maximum M of a given trajectory $x(t)$ during an observation time window $[0, t]$, and the time t_m at which the maximum M is reached. A paradigmatic example is the one-dimensional Brownian motion for a fixed duration t starting from the origin. The joint distribution of M and t_m is given by [31]

$$P_0(M, t_m|t) = \frac{M}{2\pi D\sqrt{t_m^3(t-t_m)}} e^{-M^2/4Dt_m}, \qquad (1)$$

where D the diffusion coefficient. Integrating $P_0(M, t_m|t)$ over t_m from 0 to t, one can get the marginal distribution of M,

$$P_0(M|t) = \frac{1}{\sqrt{\pi Dt}} e^{-M^2/4Dt}, \quad M > 0, \qquad (2)$$

which is the one-sided Gaussian distribution. Integrating $P_0(M, t_m|t)$ over M from 0 to ∞, one can obtain the marginal distribution of t_m,

$$P_0(t_m|t) = \frac{1}{\pi\sqrt{t_m(t-t_m)}}, \quad 0 \leqslant t_m \leqslant t, \qquad (3)$$

which is often referred to as the "arcsine law" due to Lévy [32–34]. The name stems from the fact that the cumulative distribution of t_m reads $F(z) = \int_0^z P(t_m)dt_m = (2/\pi)\arcsin\sqrt{z/t}$. A counterintuitive aspect of the U-shaped distribution Eq. (3) is that its average value $\langle t_m\rangle = t/2$ corresponds to the minimum of the distribution, i.e., the less probable outcome, whereas values close to the extrema $t_m = 0$ and $t_m = t$ are much more likely. Recent studies led to many extensions of the law, such as in constrained Brownian motions [35], random acceleration process [36,37], fractional Brownian motion [38,39], run-and-tumble motion [40], resetting Brownian motion [41], and for general stochastic processes [31,42–49]. Extension to study the distribution of the time difference between the minimum and the maximum for stochastic processes has also been made in Refs. [50,51]. Quite remarkably, the statistics of t_m has found applications in convex hull problems [52] and also in detecting whether a stationary process is equilibrium or not [53,54].

While the statistics of M and t_m in a fixed duration time has been extensively studied, the study of these quantities for a stochastic process until a stopping time, e.g., the first-passage time when the process arrives at some threshold value brings some recent attention. This problem is relevant to some context. For instance, in queue theory the maximum queue

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length and the time at which this length is achieved before the queue length gets to zero [55]. In stock market, an agent can hold the stock till its price reaches a certain threshold value. A best time to sell the stock is when the price of the stock reaches its maximum before dropping to the threshold [56]. Another example arises in the biological context regarding the maximal excursions of the tracer proteins before binding at a site [57,58]. For a one-dimensional Brownian motion starting from the position $x_0(> 0)$, the statistics of M and t_m before its first passage through the origin has been studied, and the marginal distributions of M and t_m are given by [55]

$$P_0(M|x_0) = \frac{x_0}{M^2}, \quad M \geqslant x_0, \tag{4}$$

and

$$P_0(t_m|x_0) = \frac{1}{2\pi t_m}\left[\pi - \int_0^{\pi} dy\vartheta_4\big(y/2, e^{-Dt_m y^2/x_0^2}\big)\right], \tag{5}$$

where $\vartheta_4(y,z)$ is the fourth of Jacobi's theta functions. It was shown that $P_0(t_m|x_0)$ exhibits power-law forms at both large and small tails with $P_0(t_m|x_0) \sim t_m^{-1/2}$ as $t_m \to 0$ and $P_0(t_m|x_0) \sim t_m^{-3/2}$ as $t_m \to \infty$. A recent study has extended to compute the joint distribution of M and t_m and their marginal distributions for the run-and-tumble particle in one dimension [59].

Stochastic resetting is a renewal process in which the dynamics is interrupted stochastically followed by its starting anew. The subject has recently gained considerable attention due to wide applications in search problems [60,61], the optimization of randomized computer algorithms [62], and in the field of biophysics [63,64] (see Refs. [65,66] for two recent reviews). A paradigmatic example in statistical physics is resetting Brownian motions where a diffusing particle is reset to its starting point at random times but with a constant rate. A finite resetting rate leads to a nonequilibrium stationary state with non-Gaussian fluctuations for the particle position. The mean time to reach a given target for the first time can become finite and be minimized with respect to the resetting rate [67]. Some extensions have been made in the field, such as spatially [68] or temporally [69–73] dependent resetting rate, higher dimensions [74], complex geometries [75–78], noninstantaneous resetting [79–85], in the presence of external potential [82,86,87], other types of Brownian motion, like run-and-tumble particles [88–90], active particles [91,92], constrained Brownian particle [93], and so on [94]. These nontrivial findings have triggered an enormous recent activities in the field, including statistical physics [95–106], stochastic thermodynamics [107–109], chemical and biological processes [63,64,110–112], record statistics [113–116], optimal control theory [117], and single-particle experiments [118,119].

In the present work, we aim to study the statistics of M and t_m for a resetting Brownian particle in one dimension till it passes through the origin for the first time, starting from a positive position x_0. We first compute analytically the marginal distribution $P_r(M|x_0)$ of M by the splitting or exit probability from the origin when the resetting Brownian motion is confined in an interval $[0, M]$. For any nonzero resetting rate r, $P_r(M|x_0)$ decays exponentially in the large-M limit, such that the expectation $\langle M\rangle$ of M converges. The exact expression

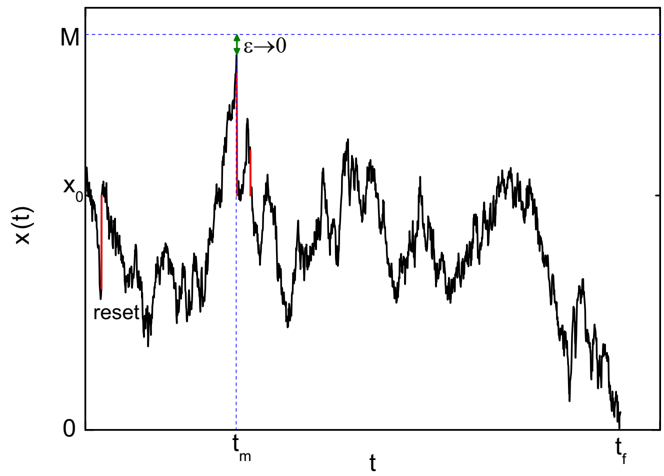

FIG. 1. A realization of a one-dimensional resetting Brownian motion in the presence of an absorbing wall at the origin. The stochastic process $x(t)$ starts from x_0 and reaches its maximum M at time t_m before the first-passage time t_f through the origin.

of $\langle M\rangle$ is also derived as functions of r and x_0. $\langle M\rangle$ diverges with $-\frac{x_0}{2}\ln r$ as $r \to 0$, and decreases monotonically with r and converges to $2x_0$ as $r \to \infty$. Using the path decomposition technique, we express the joint distribution $P_r(M, t_m|x_0)$ of M and t_m in the Laplace space, from which we can obtain the exact expression of the expectation $\langle t_m\rangle$ of t_m. Interestingly, $\langle t_m\rangle$ shows a unique minimum at an optimal resetting rate $r^* \approx 2.71691D/x_0^2$, reminiscent of another optimal resetting rate $\approx 2.53964D/x_0^2$ at which the mean first-passage time is a minimum [65,67].

II. MODEL

Let us consider a one-dimensional resetting Brownian motion (RBM), starting at $x_0(> 0)$, with resetting to the position $x_r > 0$ at rate r. The position $x(t)$ of the particle at time t is updated by the following stochastic rule [65,67]:

$$x(t+dt) = \begin{cases} x(t) + \sqrt{2D}\xi(t)dt, & \text{with prob.} \quad 1 - rdt, \\ x_r, & \text{with prob.} \quad rdt, \end{cases} \tag{6}$$

where D is the diffusion coefficient, and $\xi(t)$ is a Gaussian white noise with zero mean $\langle\xi(t)\rangle = 0$ and δ correlator $\langle\xi(t)\xi(t')\rangle = \delta(t - t')$.

We assume that there is an absorbing boundary located at the origin $x = 0$, such that the stochastic process is terminated once the particle hits the boundary. In Fig. 1, we show a realization of the RBM $x(t)$, starting at x_0 (>0) and terminating whenever $x(t)$ reaches the origin. The displacement $x(t)$ reaches its maximum M at time t_m. We are interested in the joint distribution $P_r(M, t_m|x_0)$ of the maximum displacement M of the particle and the time t_m at which the maximum is reached before the first-passage time to the absorbing boundary at the origin, and their marginal distributions of M and t_m, $P_r(M|x_0)$ and $P_r(t_m|x_0)$.

III. EXIT PROBABILITY

Consider the RBM starting from $x_0 \in [0, M]$, and both ends of the interval are absorbing boundaries. Let us denote by $\mathcal{E}_r(x_0; x_r)$ the splitting or exit probability that the particle exits the interval for the first time through the origin, i.e., the probability that the maximum before the first-passage time is less than or equal to M. The exit probability has been obtained in a recent work [120]. For completeness, we derive the expression but use a different method. Considering an infinitesimal time increment dt from beginning, the position of the particle will move either to $x_0 + \Delta$ from x_0 with probability $1 - rdt$, or to x_r with probability rdt, where Δ is a random variable and is distributed by a Gaussian function with zero mean and variance $\langle \Delta^2 \rangle = 2Ddt$. $\mathcal{E}_r(x_0; x_r)$ satisfies the following equation:

$$\mathcal{E}_r(x_0; x_r) = (1 - rdt)\langle \mathcal{E}_r(x_0 + \Delta; x_r)\rangle + rdt\mathcal{E}_r(x_r; x_r), \quad (7)$$

where $\langle \cdots \rangle$ denotes the average over Δ. Expanding $\mathcal{E}_r(x_0 + \Delta; x_r)$ to the second order in Δ, and then performing the average in Eq. (7), we obtain [121]

$$D\frac{d^2\mathcal{E}_r(x_0; x_r)}{dx_0^2} - r\mathcal{E}_r(x_0; x_r) + r\mathcal{E}_r(x_r; x_r) = 0. \quad (8)$$

Equation (8) can be solved subject to the boundary conditions

$$\mathcal{E}_r(0; x_r) = 1, \quad \mathcal{E}_r(M; x_r) = 0, \quad (9)$$

which yields

$$\mathcal{E}_r(x_0; x_r) = \frac{\sinh[\alpha_0(M - x_r)] + \sinh[\alpha_0(x_r - x_0)]}{\sinh[\alpha_0(M - x_r)] + \sinh(\alpha_0 x_r)}, \quad (10)$$

where $\alpha_0 = \sqrt{r/D}$. For the special case when the resetting position coincides with the starting position, $x_r = x_0$, Eq. (10) simplifies to

$$\mathcal{E}_r(x_0; x_0) = \frac{\sinh[\alpha_0(M - x_0)]}{\sinh[\alpha_0(M - x_0)] + \sinh(\alpha_0 x_0)}. \quad (11)$$

IV. MARGINAL DISTRIBUTION OF THE MAXIMUM DISPLACEMENT BEFORE THE FIRST PASSAGE TO THE ORIGIN

Differentiating Eq. (10) with respect to M gives the probability density function of M [55],

$$P_r(M|x_0; x_r) = \frac{\partial \mathcal{E}_r(x_0; x_r)}{\partial M} = \frac{\alpha_0 \cosh[\alpha_0(M - x_r)](\sinh(\alpha_0 x_r) - \sinh[\alpha_0(x_r - x_0)])}{(\sinh[\alpha_0(M - x_r)] + \sinh(\alpha_0 x_r))^2}. \quad (12)$$

In the limit $r \to 0$, Eq. (12) recovers to the result when the resetting is absent, see Eq. (4). For the special case $x_r = x_0$, Eq. (12) simplifies to

$$P_r(M|x_0; x_0) = \frac{\alpha_0 \sinh[\alpha_0 x_0] \cosh[\alpha_0(M - x_0)]}{(\sinh[\alpha_0(M - x_0)] + \sinh[\alpha_0 x_0])^2}. \quad (13)$$

In Fig. 2, we plot the distribution $P_r(M|x_0; x_0)$ for four different values of r but for the fixed $x_0 = 1$ and $D = 1/2$. For small resetting rates, $P_r(M|x_0; x_0)$ decays with M very slowly like a power law, similar to the case without resetting [see Eq. (4)]. For relatively larger resetting rates, the resetting to the starting position can decrease the long-range meandering of diffusing particle, so that the fat-tailed distribution is cut off in the large-M limit. Interestingly, as r increases, there exists a critical $r = r_c$ at which $P_r(M|x_0; x_0)$ changes from a monotonically decreasing function into a nonmonotonic function. Such a nonmonotonicity of $P_r(M|x_0; x_0)$ for $r > r_c$ is characterized by a maximum at $M = M_+$ and a minimum at $M = M_-$, see an illustration for the case $r = 5$ shown in Fig. 2. $M_\pm$ can be obtained by the condition $\partial P_r(M|x_0; x_0)/\partial M|_{M=M_\pm} = 0$, given by

$$M_\pm = \frac{1}{\alpha_0} \ln\left(\frac{1}{2}B_\pm + \frac{1}{2}\sqrt{B_\pm^2 + 4}\right) + x_0, \quad (14)$$

with

$$B_\pm = \sinh(\alpha_0 x_0) \pm \sqrt{\sinh^2(\alpha_0 x_0) - 8}. \quad (15)$$

In the limit of $r \to \infty$, M_+ and M_- tend to $2x_0$ and x_0, respectively. As r decreases, M_+ and M_- get close to each other, until they colloid and annihilate at $r = r_c$, as shown in the inset of Fig. 2. The critical value r_c is determined by $B_+ = B_-$, which yields

$$r_c = \frac{D}{x_0}[\ln(3 + 2\sqrt{2})]^2 \approx 3.10728\frac{D}{x_0}. \quad (16)$$

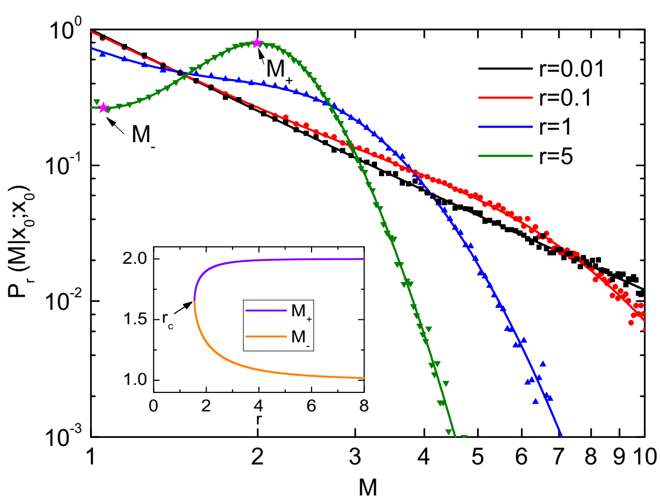

FIG. 2. The marginal distribution $P_r(M|x_0; x_r)$ of the maximum displacement M for four different resetting rate r, where $x_0 = x_r = 1$ and $D = 1/2$. The lines [see Eq. (12)] and symbols correspond to the theoretical and simulation results, respectively. For $r = 5$, $P_r(M|x_0; x_r)$ is a nonmonotonic function of M, and possesses a maximum at $M = M_+$ and a minimum $M = M_-$ (shown by asterisks), where $M_\pm$ is determined according to Eqs. (14) and (15). Inset: $M_\pm$ as a function of r. M_1 and M_2 colloid at a critical value of $r = r_c$, where r_c is given in Eq. (16).

The expectation of M can be computed as

$$\langle M(x_0;x_r)\rangle = \int_{x_0}^{\infty} dM M P_r(M|x_0;x_r) = \alpha_0^{-1} F_1(\gamma_0;\gamma_r), \tag{17}$$

where

$$\gamma_0 = x_0\alpha_0 = x_0\sqrt{r/D}, \quad \gamma_r = x_r\alpha_0 = x_r\sqrt{r/D}, \tag{18}$$

and

$$F_1(\gamma_0;\gamma_r) = \gamma_0 + \mathrm{sech}(\gamma_r)[\sinh\gamma_r - \sinh(\gamma_r-\gamma_0)] \times \left[\gamma_r + \ln\left(\cosh\gamma_r \coth\left(\frac{\gamma_0}{2}\right) - \sinh(\gamma_r)\right)\right]. \tag{19}$$

For the special case $x_r = x_0$, Eq. (19) reduces to

$$F_1(\gamma_0;\gamma_0) = \gamma_0 + \tanh(\gamma_0)\left[\gamma_0 + \ln\left(\coth\frac{\gamma_0}{2}\right)\right]. \tag{20}$$

For $r = 0$, $\langle M(x_0;x_r)\rangle$ is divergent. This is because that the standard Brownian motion can diffuse for increasingly long times in the positive direction without touching the absorbing boundary at the origin. For any nonzero r, $P_r(M|x_0;x_r)$ decays exponentially with M in the large-M limit, $P_r(M|x_0;x_r) \sim e^{-\alpha_0 M}$, and thus $\langle M(x_0;x_r)\rangle$ becomes convergent. In the limit of $r \to 0$,

$$\langle M(x_0;x_r)\rangle \sim -\frac{x_0}{2}\ln r, \quad r \to 0. \tag{21}$$

Therefore, the expected maximum $\langle M(x_0;x_r)\rangle$ diverges logarithmically as $r \to 0$. In the opposite limit $r \to \infty$,

$$\langle M(x_0;x_r)\rangle = 2x_r, \quad r \to \infty. \tag{22}$$

This result is a bit surprising as one may expect intuitively that the expected maximum is x_r rather than $2x_r$ as $r \to \infty$. The result in Eq. (22) can be understood as follows. In the limit $r \to \infty$, the exit probability in Eq. (10) tends to a step-like function, $\mathcal{E}_r(x_0;x_r) = 1 - \theta(x_r - \frac{M}{2})$, where $\theta(x)$ is the Heaviside step function defined as $\theta(x) = 1$ if $x \geqslant 0$ and zero otherwise. Taking the derivative of $\mathcal{E}_r(x_0;x_r)$ with respect to M to yield the Dirac δ distribution of M, i.e., $P_r(M|x_0;x_r) = \delta(M - 2x_r)$ as $r \to \infty$. Finally, Eq. (22) can be easily obtained by inserting the δ distribution into the definition of $\langle M(x_0;x_r)\rangle$ given in the first line of Eq. (17).

In Fig. 3, we show $\langle M(x_0;x_0)\rangle$ as a function of r for three different values of x_0. Clearly, $\langle M(x_0;x_0)\rangle$ decreases monotonically with r and approaches to $2x_0$ in the limit of $r \to \infty$. Simulation results are also shown in Fig. 3 and support the theoretical predictions.

V. SURVIVAL PROBABILITY

In this section, we compute the survival probability $Q_r(x_0,t;x_r)$ of the RBM confined in an interval $[0,M]$ with the absorbing boundaries at both ends, defined as the probability that the particle has hit neither of the boundaries until time t starting from $x_0 \in (0,M)$. In the next two sections, the survival probability will be used to deduce the propagator of RBM and the joint distribution of the maximum displacement

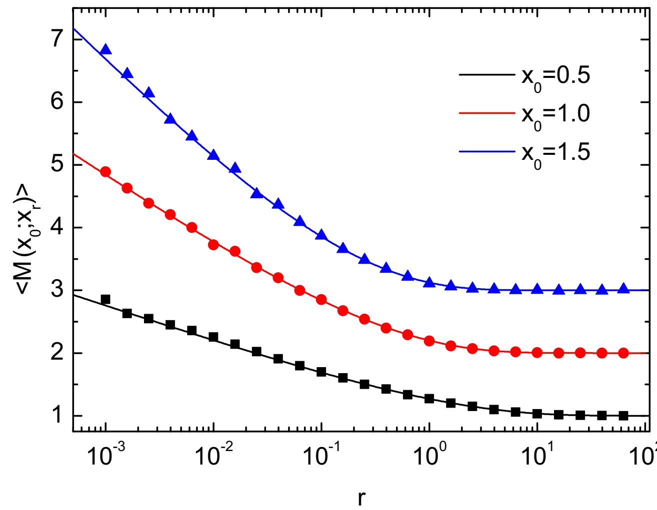

FIG. 3. The expected value $\langle M(x_0;x_r)\rangle$ of the maximum displacement M as a function the resetting rate r for three different values of x_0, where $x_r = x_0$ and $D = 1/2$. The lines [see Eq. (17)] and symbols correspond to the theoretical and simulation results, respectively.

M and the time t_m at which the maximum is reached. The backward equation for $Q_r(x_0,t;x_r)$ reads [65,67,68]

$$\frac{\partial Q_r(x_0,t;x_r)}{\partial t} = D\frac{\partial^2 Q_r(x_0,t;x_r)}{\partial x_0^2} - rQ_r(x_0,t;x_r) + rQ_r(x_r,t;x_r), \tag{23}$$

where the boundary conditions are $Q_r(0,t;x_r) = Q_r(M,t;x_r) = 0$ and the initial condition is $Q_r(x_0,0;x_r) = 1$.

Performing the Laplace transform $\tilde{Q}_r(x_0,s;x_r) = \int_0^\infty e^{-st} Q_r(x_0,t;x_r)dt$, Eq. (23) becomes

$$D\frac{d^2\tilde{Q}_r(x_0,s;x_r)}{dx_0^2} - (r+s)\tilde{Q}_r(x_0,s;x_r) + r\tilde{Q}_r(x_r,s;x_r) = -1, \tag{24}$$

subject to the boundary conditions $\tilde{Q}_r(0,s;x_r) = \tilde{Q}_r(M,s;x_r) = 0$. The solution of Eq. (24) reads

$$\tilde{Q}_r(x_0,s;x_r) = \frac{\sinh(\alpha M) - \sinh(\alpha x_0) - \sinh[\alpha(M-x_0)]}{s\sinh(\alpha M) + r\sinh(\alpha x_r) + r\sinh[\alpha(M-x_r)]}, \tag{25}$$

where $\alpha = \sqrt{(r+s)/D}$. For $x_r = x_0$, Eq. (25) simplifies to

$$\tilde{Q}_r(x_0,s;x_0) = \frac{\sinh(\alpha M) - \sinh(\alpha x_0) - \sinh[\alpha(M-x_0)]}{s\sinh(\alpha M) + r\sinh(\alpha x_0) + r\sinh[\alpha(M-x_0)]}. \tag{26}$$

VI. PROPAGATOR

In this section, we will derive the propagator $G_r(x,t|x_0;x_r)$ of the RBM in an interval $[0,M]$ with the absorbing boundaries at both ends. It is proved to be useful in the next section to compute the joint distribution of M and t_m. One can write a time-dependent equation for the propagator $G_r(x,t|x_0;x_r)$ using a last renewal formalism [65,67,68]

$$G_r(x,t|x_0;x_r) = e^{-rt}G_0(x,t|x_0) + r\int_0^t d\tau e^{-r\tau}G_0(x,\tau|x_r)Q_r(x_0,t-\tau;x_r). \tag{27}$$

Here, $G_0(x,t|x_0)$ is the propagator in the absence of resetting. Also, recall that $Q_r(x_0,t;x_r)=\int_0^M dx G_r(x,t|x_0;x_r)$ is the survival probability until time t. The first term on the right-hand side of Eq. (27) corresponds to the case when no resetting happens in the interval $[0,t]$ with the probability e^{rt}. The second term refers to the case when at least one resetting happens in the interval $[0,t]$, where the last resetting occurs at $t-\tau$ with the probability $re^{-r\tau}d\tau$. The factor $Q_r(x_0,t-\tau;x_r)$ is the survival probability during this interval $[0,t-\tau]$.

It is convenient to take the Laplace transform of Eq. (27) so that the convolution structure in the second term on the right-hand side of Eq. (27) can be exploited. By defining $\tilde{G}_r(x,s|x_0;x_r)=\int_0^\infty e^{-st}G(x,t|x_0;x_r)dt$, Eq. (27) becomes

$$\tilde{G}_r(x,s|x_0;x_r) = \tilde{G}_0(x,r+s|x_0) + r\tilde{G}_0(x,r+s|x_r)\tilde{Q}_r(x_0,s;x_r). \tag{28}$$

Here, the Laplace transform $\tilde{G}_0(x,s|x_0)$ of the propagator $G_0(x,t|x_0)$ without resetting is a classical result and known from the literature [121],

$$\tilde{G}_0(x,s|x_0) = \frac{\cosh\left[\sqrt{\frac{s}{D}}(M-|x-x_0|)\right]-\cosh\left[\sqrt{\frac{s}{D}}(M-x-x_0)\right]}{2\sqrt{sD}\sinh\left(\sqrt{\frac{s}{D}}M\right)}. \tag{29}$$

For simplicity, we consider the case when $x_r=x_0$. Thus, by substituting Eqs. (26) and (29) into Eq. (28) we obtain

$$\tilde{G}_r(x,s|x_0;x_0) = \frac{\alpha}{2}\frac{\cosh\left[\alpha(M-|x-x_0|)\right]-\cosh\left[\alpha(M-x-x_0)\right]}{s\sinh(\alpha M)+r\sinh(\alpha x_0)+r\sinh\left[\alpha(M-x)\right]}, \tag{30}$$

where $\alpha=\sqrt{(r+s)/D}$ again. Note that Eqs. (26) and (30) were also obtained in a previous work [120].

VII. JOINT DISTRIBUTION OF M AND t_m BEFORE ITS FIRST PASSAGE TO THE ORIGIN

In this section, we only consider the special case when the reseting position is the same as the starting one, i.e., $x_r=x_0$. Therefore, in the following we will suppress the x_r dependence on most of quantities for the sake of brevity. However, it is straightforward to generalize the subsequent results to the case when x_r is not necessarily the same as x_0.

Let us define $P_r(M,t_m|x_0)$ as the joint probability density function that the RBM reaches its maximum M at time t_m before passing through the origin for the first time t_f, providing that the Brownian starts from the position x_0 (>0). To compute the joint distribution $P_r(M,t_m|x_0)$, we can decompose the trajectory into two parts: a left-hand segment for which $0<t<t_m$, and a right-hand segment for which $t_m<t<t_f$, as shown in Fig. 1. Due to the Markovian property of the resetting Brownian trajectory, once the position of the particle is specified at t_m, the weights of the left and the right segments become completely independent and the total weight is just proportional to the product of the weights of the two separate segments. For the first segment, we have a process that propagates from x_0 at $t=0$ to M at $t=t_m$ without hitting the origin. The statistical weight of the first segment thus equals to the propagator $G_r(M,t_m|x_0)$. However, it turns out that $G_r(M,t_m|x_0)=0$ which implies that the contribution from this part is zero. To circumvent this problem, we compute $G_r(M-\epsilon,t_m|x_0)$ and later take the limit $\epsilon\to 0$ [11]. For the second segment, the process propagates from $M-\epsilon$ at t_m to 0 at t_f, where $t_f\geqslant t_m$ without crossing the level M and the level 0 in between. The statistical weight of the second segment is given by the exit probability $\mathcal{E}_r(M-\epsilon)$. Therefore, the joint probability density $P_r(M,t_m|x_0)$ can be written as the product of the statistical weights of two segments [55],

$$P_r(M,t_m|x_0) = \lim_{\epsilon\to 0}\mathcal{N}(x_0,\epsilon)G_r(M-\epsilon,t_m|x_0)\mathcal{E}_r(M-\epsilon), \tag{31}$$

where the normalization factor $\mathcal{N}(x_0,\epsilon)$ will be determined later.

One can compute the Laplace transform of $G_r(M-\epsilon,t_m|x_0)$ in terms of Eq. (30) and $\mathcal{E}_r(M-\epsilon)$ by Eq. (10). Up to the leading order in ϵ, they are

$$\tilde{G}_r(M-\epsilon,s|x_0) = \frac{\alpha^2\sinh(\alpha x_0)}{s\sinh(\alpha M)+r\sinh(\alpha x_0)+r\sinh\left[\alpha(M-x_0)\right]}\epsilon, \tag{32}$$

and

$$\mathcal{E}_r(M-\epsilon) = \frac{\alpha_0\cosh\left[\alpha_0(M-x_0)\right]}{\sinh\left[\alpha_0(M-x_0)\right]+\sinh(\alpha_0 x_0)}\epsilon. \tag{33}$$

It is convenient to perform the Laplace transform for $P_r(M,t_m|x_0)$ over t_m,

$$\begin{aligned}\tilde{P}_r(M,s|x_0) &= \int_0^\infty dt_m e^{-st_m}P_r(M,t_m|x_0)\\ &= \mathcal{N}(x_0,\epsilon)\tilde{G}_r(M-\epsilon,s|x_0)\mathcal{E}_r(M-\epsilon).\end{aligned} \tag{34}$$

Letting $s \to 0$, the left hand side of Eq. (34) is just the marginal distribution $P_r(M|x_0)$, which yields

$$\begin{aligned} P_r(M|x_0) &= \int_0^\infty dt_m P_r(M, t_m|x_0) \\ &= \mathcal{N}(x_0, \epsilon)\tilde{G}_r(M-\epsilon, 0|x_0)\mathcal{E}_r(M-\epsilon). \end{aligned} \tag{35}$$

Substituting Eqs. (13), (32), and (33) into Eq. (35), we obtain

$$\mathcal{N}(x_0, \epsilon) = \frac{D}{\epsilon^2}, \tag{36}$$

which is independent of the starting position x_0. In fact, $\mathcal{N}$ can be also obtained by the normalization condition of $P_r(M|x_0)$, i.e., $\int_{x_0}^\infty dM P_r(M|x_0) = 1$. This can be easily accomplished by integrating the second line of Eq. (35) over M. The two strategies produce the consistent result.

Substituting Eqs. (32), (33), and (36) into Eq. (34), we obtain the joint distribution $P_r(M, t_m|x_0)$ in the Laplace space,

$$\tilde{P}_r(M, s|x_0) = \frac{(r+s)\sinh(\alpha x_0)}{s\sinh(\alpha M) + r\sinh(\alpha x_0) + r\sinh[\alpha(M-x_0)]}\frac{\alpha_0\cosh[\alpha_0(M-x_0)]}{\sinh[\alpha_0(M-x_0)] + \sinh(\alpha_0 x_0)}. \tag{37}$$

To obtain the joint distribution $P_r(M, t_m|x_0)$, one has to perform the inverse Laplace transformation for Eq. (37) with respect to s. Unfortunately, it turns out to be a challenging task. However, one may expect to obtain explicitly the statistics of t_m, such as the expectation value of t_m. To this end, by integrating Eq. (34) over M from x_0 to ∞, one obtain the Laplace transform of the marginal distribution $P_r(t_m|x_0)$,

$$\begin{aligned} \tilde{P}_r(s|x_0) &= \int_0^\infty dt_m e^{-st_m} P_r(t_m|x_0) \\ &= \int_{x_0}^\infty dM \tilde{P}_r(M, s|x_0). \end{aligned} \tag{38}$$

In particular, the expectation of the time t_m is given by

$$\langle t_m \rangle = -\lim_{s\to 0}\frac{\partial \tilde{P}_r(s|x_0)}{\partial s} = r^{-1}F_2(\gamma), \tag{39}$$

where $\gamma = x_0\sqrt{r/D}$ and

$$\begin{aligned} F_2(\gamma) = {} & \{\coth(\gamma)[1 + 2e^{2\gamma} + e^{4\gamma} - \gamma + 2e^{2\gamma} - \gamma e^{4\gamma} + (e^{2\gamma}-1)^2\ln((e^\gamma+1)(e^\gamma-1))] + 4e^{2\gamma}[\mathrm{Li}_2(e^{-\gamma}) - \mathrm{Li}_2(-e^\gamma)]\} \\ & \times\frac{e^\gamma\sinh(\gamma)}{2(1+e^{2\gamma})^3} - \frac{\gamma}{4}\coth(\gamma) + \frac{(e^\gamma-1)\mathrm{csch}(2\gamma)\sinh^2(\gamma)}{(1+e^\gamma)(1+e^{2\gamma})^2} \\ & \times[-2 - e^\gamma - 2e^{2\gamma} + e^{5\gamma} + 2e^\gamma(1+e^\gamma)^2(x + \ln(\coth(\gamma/2)))], \end{aligned} \tag{40}$$

where $\mathrm{Li}_2(z)$ is the polylogarithm function. In the limit of $r \to 0$, we have

$$\langle t_m \rangle = \frac{\pi^2 x_0}{16\sqrt{Dr}} \sim r^{-1/2}. \tag{41}$$

As expected, $\langle t_m \rangle$ diverges in the absence of resetting, and possesses the same asymptotic behavior as the mean first-passage time $\langle t_f \rangle$ with $r \to 0$ [65,67].

In Fig. 4, we show $\langle t_m \rangle$ as a function of r for three different values of x_0, but for a fixed $D = 1/2$. Interestingly, $\langle t_m \rangle$ show a nonmonotonic dependence on r or equivalently on γ for the fixed x_0 and D. There exist an optimal r at which $\langle t_m \rangle$ attains its unique minimum. Also, we have performed simulations (shown by symbols in Fig. 4), which are in excellent agreement with our theory (shown by lines in Fig. 4). Taking the derivative of $\langle t_m \rangle$ with respect to r, and then letting the derivative equal to zero, we obtain the optimal r, determined by the equation $2F_2(\gamma^*) = \gamma^* F_2'(\gamma^*)$. The equation can be solved by numeric to yield $\gamma^* \approx 1.64831$, or equivalently, the optimal resetting rate

$$r^* \approx 2.71691 D/x_0^2. \tag{42}$$

We note that the mean first-passage time through the origin shows a nonmonotonic change with r as well, and has a minimum at an optimal r, which equals approximately to $2.53964 D/x_0^2$ [65,67], different from the value given in Eq. (42).

VIII. CONCLUSIONS

In conclusion, we have studied the extremal statistics of the maximal displacement M of the RBM starting from a positive position x_0 and the time t_m at which the maximum is reached before the first-passage time through the origin. In the first part of this paper, we compute the exit probability of the RBM in an interval $[0, M]$ from the origin [Eq. (10)], and thus obtain the marginal distribution $P_r(M|x_0)$ [Eq. (13)]. In particular, the expectation of M, $\langle M \rangle$, is obtained explicitly [Eq. (17)]. We find that $\langle M \rangle$ decreases monotonically with the resetting

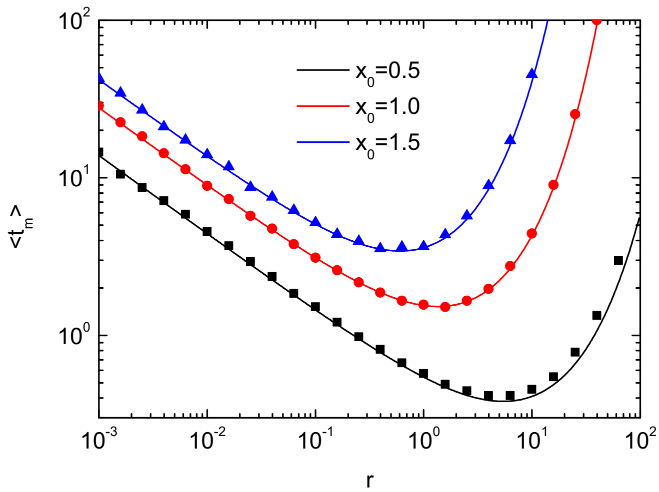

FIG. 4. The expected value $\langle t_m \rangle$ of the time t_m at which the RBM reaches its maximum before its first passage through the origin, having starting from $x_0 > 0$, as a function of the resetting rate r, where $D = 1/2$ is fixed. The lines [see Eq. (39)] and symbols correspond to the theoretical and simulation results, respectively.

rate r, and converges to its asymptotic value $2x_0$ as $r \to \infty$, but diverges with the negative logarithm of r as $r \to 0$. In the second part of this paper, we compute the survival probability and propagator of the RBM in an interval $[0, M]$ with absorbing boundaries at both ends. However, these quantities are obtained explicitly only in the Laplace space. Subsequently, we obtain in the Laplace space the joint distribution of M and t_m based on a path decomposition technique for Markov processes. Fortunately, the expectation $\langle t_m \rangle$ of t_m is obtained explicitly [Eq. (39)]. $\langle t_m \rangle$ diverges as $r \to 0$ with $\langle t_m \rangle \sim r^{-1/2}$, as the diffusing particle in the absence of resetting can meander indefinitely in the positive direction without reaching its maximum displacement. Also $\langle t_m \rangle$ diverges as $r \to \infty$, because that the diffusing particle has less time between resets to reach the origin as the resetting rate increases. In between these two divergences there is an optimal resetting rate [Eq. (42)] at which $\langle t_m \rangle$ attains its unique minimum. Such a phenomenon is reminiscent of mean first-passage time $\langle t_f \rangle$ of the RBM, wherein $\langle t_f \rangle$ also possesses a single minimum at some resetting rate. We should emphasize that the two optimal resetting rates are quantitatively different. The optimal resetting for $\langle t_m \rangle$ is slightly larger than that for $\langle t_f \rangle$. Therefore, our study provides an additional example regarding the nontrivial effects of stochastic resetting.

In the future, it would be interesting to investigate the extremal statistics of other types of Brownian motions under resetting before their first passage to an absorbing boundary, such as active Brownian motions [91,92] and run-and-tumble motions [59,88–90]. While the instantaneous resetting is easy to analyze mathematically in studying resetting models, it is only justified when the time scale of return is far less than the time scale of the dynamics itself. In reality the return time is not usually ignored, and thus noninstantaneous resetting is common [79–85]. Recently, a noninstantaneous resetting controlled by an intermittent potential has been proposed as a feasible protocol in experiments [83,84]. Concerned with extremal statistics of noninstantaneous resetting models, one should compute time-dependent quantities in the presence of absorbing ends, such as survival probability and propagator. This maybe posed some theoretical challenges in the future. Finally, we hope that our results are able to be applied to some other models under stochastic resetting, such as economic models [122].

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