Editors' Suggestion

# Mesoscopic critical fluctuations

Saikat Banerjee [*] and Nikolai A. Sinitsyn [†]

*Theoretical Division, T-4, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

(Received 24 May 2023; accepted 11 August 2023; published 21 September 2023)

We investigate the magnetic fluctuations in a mesoscopic critical region formed at the interface due to smooth time-independent spatial variations of a control parameter around its critical value. In the proximity of the spatial critical point, the order parameter fluctuations exhibit a mesoscopic nature, characterized by their significant size compared to the lattice constant, while gradually decaying away from the critical region. To explain this phenomenon, we present a minimal model that effectively captures this behavior and demonstrates its connection to the integrable Painlevé-II equation governing the local order parameter. By leveraging the well-established mathematical properties of this equation, we gain valuable insights into the nonlinear susceptibilities exhibited within this region.

DOI: 10.1103/PhysRevE.108.034212

## I. INTRODUCTION

This article draws inspiration from experimental investigations into local magnetization fluctuations within a thin magnetic film exhibiting a *spatially* smooth variation in thickness [1–6]. The initial introduction of this spatial gradient was demonstrated in Ref. [1] to establish the presence of a critical region. In this context, the thin section of the film exhibited interactions favoring a phase characterized by in-plane magnetization. In contrast, the thicker segment preferred an antiferromagnetic out-of-plane order, as illustrated in Fig. 1.

Consistent with the expectations derived from the theory of critical phenomena [7–9], the film region corresponding to the critical thickness displayed pronounced dynamic magnetization fluctuations, facilitating the identification of the spatial boundary separating the two phases. Due to the smoothness of the film thickness, the position of this boundary could be changed continuously by varying externally controlled parameters, such as temperature and the external magnetic field. This unique attribute was harnessed in Ref. [6], where the noise power of frequency-integrated magnetization—directly proportional to the linear magnetic susceptibility—was measured within a designated observation zone. In a varying temperature, the critical point has passed through the observation spot and thus produced a peak of the integrated noise power as a function of the temperature.

In contrast to the susceptibility near a phase transition in a uniform system, this peak exhibited a finite amplitude. Moreover, Ref. [6] unveiled that the fluctuations at critical parameter values displayed pronounced non-Gaussian behavior. These fluctuations generated a distinct and discernible bispectrum pattern of the fluctuations, which rapidly diminished as the control parameter deviated from its critical point. The intensified fluctuations in proximity to a critical point have been observed through spin noise spectroscopy in other magnetic systems, such as spin ice [10].

Here, we argue that the standard theory of critical phenomena cannot be straightforwardly applied in order to provide a quantitative explanation for the fluctuations within the spatial critical region. The reason is that when some parameter undergoes spatial variation, the scenario deviates from that of a uniform bulk sample. As the thickness in Ref. [1] changes linearly along some axis $x$, the critical region within a $D$-dimensional sample effectively reduces to a $(D-1)$-dimensional surface. Consequently, it becomes inappropriate to anticipate phenomena such as the emergence of a divergent correlation length along the $x$ axis, for instance.

Conversely, certain aspects of the critical phenomena theory are expected to retain their validity near the spatial critical point. Consider, for instance, a scenario where the film thickness undergoes significant variations over distances substantially exceeding the atomic lattice constant. In this context, at the interface between the two phases, the magnetization fluctuations are anticipated to exhibit sizes considerably larger than the lattice constant. Consequently, it is reasonable to infer that the universality and scaling hypotheses, intrinsic to the critical phenomena theory, should persist in some form to describe the experimental observations.

Therefore, when a critical interface is set by introducing a weak spatial parameter gradient, the critical fluctuations aligned with this gradient manifest an inherently mesoscopic nature. These fluctuations neither resemble the microscopic nature (decaying at the scale of the lattice constant) nor the macroscopic type (exceeding the entire sample size). This prompts the following inquiry: How can we effectively describe the physics in proximity to a critical point when a control parameter changes slowly and continuously throughout the critical point?

Optical magnetization noise spectroscopy has emerged as an ideal experimental tool to investigate such fluctuations. This is attributed to the fact that an optical probe beam effectively encompasses a mesoscopic area of the sample, allowing

[*]saikatb@lanl.gov

[†]nsinitsyn@lanl.gov

 

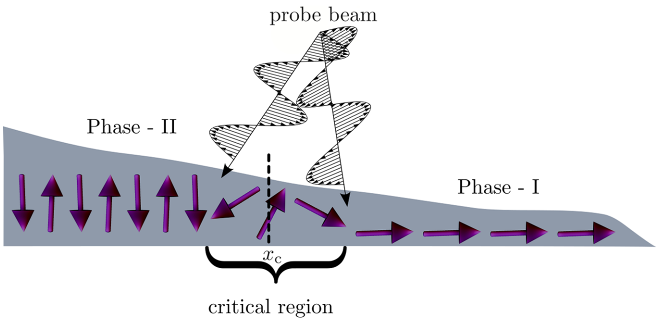


FIG. 1. A phase transition induced by a gradient of the width of a magnetic film. The critical point at $x = x_c$ separates two different magnetic phases. A mesoscopic region near this point is formed, which shows enhanced magnetic fluctuations, which are probed by an optical beam with a comparable width.

it to selectively capture the fluctuations inherent to the critical region while avoiding significant interference from the bulk to the sample. This technique naturally leads to the following inquiry: What can we learn about the parameters of the system by observing the enhanced magnetization fluctuations near a spatial critical point? Here, we hereby propose a minimal model based on the Ginzburg-Landau effective free-energy approach that contains the basic features of the problem.

## II. GINZBURG-LANDAU FREE ENERGY

Usually, phase transitions in quasi-one-dimensional (1D) systems can be understood with a free-energy functional [11]

$$\mathcal{L}_{1\mathrm{D}} = \int_{-L}^{L} dx \left[ \frac{D}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{r(x)}{2}\phi(x)^2 + \frac{\mathsf{g}}{4}\phi(x)^4 \right]. \tag{1}$$

Here, $\phi(x)$ represents a coarse-grained average order parameter. In the case of the ferromagnet-paramagnet phase transition, $\phi(x)$ is identified with the local magnetization $m_i = \langle \hat{S}_i^z \rangle$ in the original magnetic system (where $\mathbf{S}_i$'s denotes the spin degrees of freedom), $D$ is the diffusion coefficient, and $\mathsf{g} > 0$ is the coupling responsible for nonlinearity associated with the mutual interaction between the individual spins. In the thermodynamic equilibrium, the ground state of the system is dictated by the minimum of $\mathcal{L}_{1\mathrm{D}}$. To make the phase transition well defined, we assume a three-dimensional system along the transverse direction within a mesoscopic region of linear size much smaller than the quasi-1D length of the system.

We will assume that the phenomenological parameter $r$ is $x$ dependent. Without a diffusion term, the minimum of $\mathcal{L}_{1\mathrm{D}}$ is achieved at

$$\phi = 0, \quad \text{for} \quad r \geqslant 0, \tag{2a}$$

$$\phi = \pm\sqrt{\frac{r}{\mathsf{g}}}, \quad \text{for} \quad r < 0. \tag{2b}$$

However, disregarding the diffusion, the calculation of the local susceptibility would give diverging predictions due to the discontinuity of the solution (2a) and (2b) at the spatial point with $r = 0$. Indeed, we will show that the properties of dynamic fluctuations near this point depend on the diffusion coefficient $D$ essentially.

In order to capture this physics, we include the diffusion term into consideration but use another simplification. Namely, near the critical point with $r = 0$, we assume that the control parameter is only linearly varied along $x$,

$$r(x) = \theta x, \tag{3}$$

where we set the critical point at $x = 0$. The free-energy functional in Eq. (1) then modifies as

$$\mathcal{L}_{1\mathrm{D}} = \int_{-L}^{L} dx \left[ \frac{D}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\theta x}{2}\phi(x)^2 + \frac{\mathsf{g}}{4}\phi(x)^4 - h\phi(x) \right], \tag{4}$$

where we added an external "magnetic field" $h$.

Far away from the critical point $x = 0$, we have two phases: the phase-I for $x > 0$ with $\phi \to 0$ as $x \to +\infty$, and the phase-II for $x < 0$ with $\phi(x) \to \pm\sqrt{\theta|x|/\mathsf{g}}$ as $x \to -\infty$. In what follows, we will assume the symmetry to be broken in phase-II spontaneously so that

$$\phi(x) \to \sqrt{\theta|x|/\mathsf{g}}, \quad x \to -\infty, \tag{5a}$$

$$\phi(x) \to 0, \quad x \to +\infty. \tag{5b}$$

To find the equilibrium $\phi(x)$, we should minimize the new functional in Eq. (4). This gives us an equation

$$-D\frac{d^2\phi}{dx^2} + \theta x\phi(x) + \mathsf{g}\phi(x)^3 = h. \tag{6}$$

Note that by dividing this equation by $D$, we find that only the combinations $\theta/D$, $g/D$, and $h/D$ matter. Then, by rescaling space, $x \to \lambda x$, we get the same equation but with $\theta \to \theta\lambda^3$ and $\mathsf{g} \to \mathsf{g}\lambda^2$. Choosing

$$\lambda = (D/\theta)^{1/3}, \tag{7}$$

we get rid of the coefficient near $-x\phi(x)$. The coefficient $\mathsf{g}$ then becomes $\mathsf{g} \to \mathsf{g}/(D\theta^2)^{1/3}$. Finally, by rescaling $\phi \to \mu\phi$, we can set the coefficient near $\phi^3$ to be equal to 2. This is achieved for

$$\mu = \sqrt{\frac{2D}{\mathsf{g}\lambda^2}} = \frac{\sqrt{2}D^{1/6}\theta^{1/3}}{\mathsf{g}^{1/2}}. \tag{8}$$

In terms of these scaled variables and parameters, the differential Eq. (6) is modified with a rescaled average local magnetization, $u(x, \alpha)$, satisfying

$$\frac{d^2u(x,\alpha)}{dx^2} = 2u(x,\alpha)^3 + xu(x,\alpha) - \alpha, \tag{9}$$

where

$$\alpha = h\frac{\lambda^2}{D\mu} = \frac{h\sqrt{\mathsf{g}}}{\theta\sqrt{2D}}. \tag{10}$$

Then, the solution of the original Eq. (6) is given by

$$\phi(x|h) = \mu u[x/\lambda, \alpha(h)]. \tag{11}$$

Note that this reduction predicts the characteristic scales for the magnetic field

$$h_0 = \frac{\theta\sqrt{2D}}{\sqrt{g}}, \quad \alpha = h/h_0, \tag{12}$$

and also the length $\lambda$ in Eq. (7), as well as the magnetization $\mu$ in Eq. (8). All these scales depend on the diffusion coefficient $D$ and the slope $\theta$. Hence, they must be related to the observable physics of the magnetic fluctuations within the length $\lambda$ near the critical point.

## III. SOLUTION OF THE PAINLEVÉ EQUATION

Equation (9) is known as the Painlevé-II ($P_{II}$) nonlinear differential equation. The fact that this equation can be obtained by minimizing a Ginzburg-Landau functional is well known [12–14]. However, here our interest is more specific.

Namely, spin noise spectroscopy generally measures the noise power spectra, whose integrals over frequency return linear and nonlinear susceptibilities of a mesoscopic probed region. The solution of Painlevé-II is generally considered as independent special function because many of its properties are known analytically. In particular, very recently, it was shown that the integrals of the Painlevé-II solutions could be written analytically in terms of standard special functions [15]. A real solution of Eq. (9) that would be consistent with the conditions Eqs. (5a) and (5b) is unique and known as the Hastings-McLeod solution [16]. Its asymptotic behavior as $x \to \pm\infty$ and $\alpha \in (-1/2, 1/2)$ is also well known [17]:

$$u(x;\alpha)_{x\to+\infty} = B(\alpha;x) + \cos(\pi\alpha)[\mathrm{Ai}(x) + \mathcal{O}(x^{-3/4})], \tag{13a}$$

$$u(x;\alpha)_{x\to-\infty} = \sqrt{-x/2}\left(1 + \frac{\alpha}{\sqrt{2}|x|^{3/2}} - \frac{1+6\alpha^2}{8|x|^3} + \mathcal{O}(1/|x|^{9/2})\right) + [\text{exponentially small terms}], \tag{13b}$$

$$B(\alpha;x) = \frac{\alpha}{x}\sum_{n=0}^{\infty}\frac{a_n}{x^{3n}}, \quad a_0 = 1, \quad a_{n+1} = (3n+1)(3n+2)a_n - 2\alpha^2\sum_{k,l,m=0}^{n} a_k a_l a_m. \tag{13c}$$

At this stage, we note that the asymptotic behavior depends on two types of terms, i.e., the terms that decay as a power law and the exponentially suppressed terms.

Examination of the leading-order power-law decay terms shows that it is not related to the intrinsic to $x = 0$ physics. Namely, the leading asymptotic as $x \to -\infty$ in Eq. (13b) corresponds to the $\phi \to \sqrt{|x|}$ behavior, which is expected from Eq. (5a). For $x \to +\infty$, the term $B(\alpha;x)$ becomes zero at $\alpha = 0$, i.e., it describes a diamagnetic response in the nonmagnetic phase. Its appearance is expected because, for large $x$ and small $\phi(x)$, we can disregard the diffusion and nonlinear contributions so that the energy density becomes $\mathcal{E}(x) \sim \theta x\phi(x)^2 - h\phi(x)$, which is minimized at $\phi \sim h/(\theta x)$, and which reproduces the leading asymptotic in Eq. (13c) if we use it in the solution in Eq. (11).

The other terms are decaying exponentially. They play an important role for $\alpha = 0$, i.e., in the absence of the external magnetic field [see Eq. (10)]. The limit $x \to +\infty$ is then dominated by the Airy function in Eq. (5a), which leads to the asymptotic

$$u(x;\alpha=0)_{x\to+\infty} \sim \frac{x^{-1/4}}{2\sqrt{3\pi}}e^{-\frac{2}{3}|x|^{3/2}}. \tag{14}$$

Returning to the nonrescaled variables, we find

$$\phi(x;h=0)_{x\to+\infty} \sim \frac{\mu(x/l)^{-1/4}}{2\sqrt{3\pi}}e^{-\frac{2}{3}(|x|/\lambda)^{3/2}}. \tag{15}$$

Thus, we find that the magnetization does not drop to zero instantly in phase-I but rather persists up to the distance $l = (D/\theta)^{1/3}$ from the critical point with a characteristic amplitude $\mu$, and only then decays faster than exponentially with growing $x$. This means that the behavior near the critical point is dominated by the proximity effect, in which extension and amplitude are influenced by both the diffusion $D$ and the ramp of the transition $\theta$. Hence, already at the level of the asymptotic analysis, we find that the behavior of the magnetization contains the terms that are strongly localized near the critical point and therefore can be attributed only to the physics in its vicinity.

## IV. LINEAR AND NONLINEAR SUSCEPTIBILITIES OF A MESOSCOPIC CRITICAL REGION

### A. Linear susceptibility far from critical point

Spin noise spectroscopy is the method that allows studies of the local spin susceptibilities, which are averaged over a mesoscopic region in space [18]. This technique can generally resolve the full frequency dependence of magnetization time correlators. However, the ground state configuration of the magnetic system, obtained by minimizing $\mathcal{L}_{1D}$ as in Eq. (6), contains information only about the static characteristics, such as the local linear and the nonlinear susceptibilities:

$$\chi_x^{(1)} = \left(\frac{\partial\phi(x)}{\partial h}\right)_{h=0}, \quad \chi_x^{(2)} = \frac{1}{2}\left(\frac{\partial^2\phi(x)}{\partial h^2}\right)_{h=0}, \quad \cdots, \tag{16}$$

and which are obtained experimentally by calculating the spectral volumes of the measured correlators.

Due to the finite size of the optical beam width ($L > 1$ μm), the measured magnetization is averaged over the length $L > \lambda$. If we are far from the point $x = 0$, the order parameter changes slowly with $x$. So, asymptotically we find the local susceptibility [from Eqs. (13a) and (13b)],

$$\chi_x^{(1)} \propto \frac{1}{\theta x}, \quad x \to +\infty, \tag{17a}$$

$$\chi_x^{(1)} \propto \frac{1}{2\theta x}, \quad x \to -\infty, \tag{17b}$$

which is a typical $\lambda$-like power-law behavior, near a critical point. However, this behavior is not related to a specific physics near the critical point. Moreover, it shows that the linear local susceptibility is likely not a good characteristic to explore these physics because the integrated over the macroscopic region susceptibility diverges with the growing width of the probe beam. Indeed, let the probe beam cut the susceptibility from the region $x \in (-c, s)$, where $c$ and $s$ are much larger than $\lambda$. Then, the integrated susceptibility

$$\chi^{(1)}_{\text{total}} = \int_{-c}^{s} \chi^{(1)}_x \, dx \tag{18}$$

behaves as

$$\chi^{(1)}_{\text{total}} \sim \chi_0 + \frac{1}{\theta}\left(\frac{\ln c}{2} + \ln s\right), \tag{19}$$

where $\chi_0$ is some constant that depends on the intermediate profile of $\chi^{(1)}_x$. This expression is dominated by the logarithmic tails with the probe-dependent cutoff. We will show that the constant part of such an integrated susceptibility is not very valuable either.

### B. Excess magnetization and linear susceptibility

Let us define the net magnetization in the observation region

$$M = \lim_{c,s\to\infty} \int_{-c}^{s} \phi(x)dx. \tag{20}$$

Using Eq. (11) we can rewrite

$$M = \mu\lambda \int_{-c'}^{s'} u[x, \alpha(h)]dx, \tag{21}$$

where $c' = c/\lambda$, $s' = s/\lambda$. The integrated susceptibilities are then defined as

$$\chi^{(n)}_{\text{total}} = \frac{1}{n!}\left(\frac{\partial^n M}{\partial h^n}\right)_{h=0} = \frac{1}{n!}\frac{\mu\lambda}{h_0^n}\left(\frac{\partial^n \mathcal{M}}{\partial \alpha^n}\right)_{\alpha=0}, \tag{22}$$

where

$$\mathcal{M} = \lim_{c',s'\to\infty} \int_{-c'}^{s'} u(x, \alpha)\, dx. \tag{23}$$

According to Ref. [15], this integral is known:

$$\begin{aligned}\mathcal{M} = {} & \frac{\sqrt{2}c'^{3/2}}{3} + \frac{\alpha}{2}\ln c' + \alpha \ln s' + \frac{1}{2}\ln(2\pi) \\ & - \ln\Gamma(\alpha + 1/2) - \frac{\alpha \ln 2}{2} + O(1/s'^{3/2}, 1/c'^{3/2}).\end{aligned} \tag{24}$$

Setting $\alpha = 0$ in Eq. (24), and using that $\mu\lambda = \sqrt{2D/\mathsf{g}}$, we find the net magnetization

$$M = \sqrt{\frac{2D}{\mathsf{g}}}\left(\frac{\sqrt{2}c'^{3/2}}{3} + \frac{\ln 2}{2}\right). \tag{25}$$

The term $\sim c'^{3/2}$ is expected from the diffusionless limit. Indeed, using that $c' = c/\lambda$, and $\lambda^{3/2} \sim \sqrt{D}$, we find that this term does not depend on the diffusion coefficient. The contribution

$$\delta M = \sqrt{\frac{D}{2\mathsf{g}}}\ln 2 \tag{26}$$

is, however, a correction due to the proximity effects near the critical point. Interestingly, this "excess magnetization" correction does not depend on the slope $\theta$, which is our first prediction for the property of the critical region.

Potentially, $\delta M$ can be measured optically by scanning the dependence of the average Kerr rotation on the area of the probe beam and fitting the result to a power law with an offset. However, on the background of the dominating trivial $\sim c'^{3/2}$ contribution, this would be hard. Hence, let us look now at the linear susceptibility. Using that $\mu\lambda/h_0 = 1/\theta$, we find

$$\chi^{(1)}_{\text{total}} = \frac{1}{\theta}\left[\frac{1}{2}\ln c' + \ln s' + \left(\gamma_e + \frac{3\ln 2}{2}\right)\right], \tag{27}$$

where $\gamma_e \approx 0.577$ is the Euler-Mascheroni constant. This expression shows that the integrated linear susceptibility of the critical region is quite uninformative. In addition to the logarithmic cutoff-dependent terms, it has a subdominant contribution that depends only on the gradient of the control parameter, $\theta$.

Still, this susceptibility is a valuable pointer to the critical point, as it reaches the maximum when this point is placed inside the integration interval $(-c, s)$. However, as a small warning to experiments, we note that within our model, the factor $1/2$ in (27) leads to a mismatch of the critical point from the center of this interval. For example, if the width of the interval is $L$, then $s = L - c$, and the maximum of the linear susceptibility is found at $c = L/3$ rather than $L/2$.

### C. Nonlinear susceptibilities

The behavior that is intrinsic only to the critical region is found in all higher-order susceptibilities. Thus

$$\chi^{(2)}_{\text{total}} = -\frac{\pi^2\mu\lambda}{4h_0} = -\frac{\pi^2\sqrt{\mathsf{g}}}{4\theta^2\sqrt{2D}}, \tag{28a}$$

$$\chi^{(3)}_{\text{total}} = \frac{7\zeta(3)\mu\lambda}{3h_0^3} = \frac{7\zeta(3)\mathsf{g}}{6\theta^3 D}, \tag{28b}$$

where $\zeta(3) \approx 1.20$ is a special value of the Riemann zeta function. First, we note the absence of the cutoff-dependent contributions to $\chi^{(2)}_{\text{total}}$ and $\chi^{(3)}_{\text{total}}$. This means that they are dominated by the physics near the critical point. Indeed, far away from the critical point, the magnetic fluctuations are essentially Gaussian because they are dominated by numerous uncorrelated microscopic events. Near the critical point, the fluctuations are not only enhanced, but they also become highly non-Gaussian. Therefore, the nonlinear susceptibilities acquire finite contributions precisely near the critical region, in agreement with the experimental observation in Ref. [6].

In addition, we note the singular dependence of the nonlinear susceptibilities on the diffusion coefficient $D$. Due to the critical slowing down, the fluctuations at the critical point have a formally infinite lifetime, which would lead to diverging bispectra near the frequency $\omega_{1,2} = 0$ point. However, the diffusion introduces a new characteristic lifetime that smears this divergence and makes the integrated susceptibility finite. Such behavior is consistent with the appearance of $D$ in the denominators of (28a) and (28b).

## V. DISCUSSION

Our theoretical investigations substantiate the experimental observation concerning the increasing prominence of non-Gaussian fluctuations within the mesoscopic critical region. We have introduced a minimal model that effectively showcases this phenomenon and offers the ability to characterize it quantitatively.

We have demonstrated that the nonlinear susceptibilities contain valuable insights into the intrinsic parameters of the interacting spins within the critical region. Furthermore, we predict a remarkably universal trend in the behavior of the integrated linear susceptibility and the excess magnetization within the mesoscopic critical region. Our theory uncovers a power-law correlation between the measurable susceptibilities and the gradient of the control parameter $\theta$. Collectively, these observations suggest the potential universality of such fluctuations across diverse systems.

Drawing from our findings, we put forth the conjecture that the existence of long-range correlations likely gives rise to effective field theories that can describe these critical regions. These theories might encompass a few important universality classes, akin to the framework in conventional critical phenomena theory. Owing to the chiral symmetry breaking induced by a gradient of the parameter, these theories might also manifest exotic topologically protected excitations. Importantly, these insights could translate into practical applications for experimental investigations of such regions. For instance, these regions can be dynamically manipulated within samples by modulating externally controlled parameters such as temperature and external fields. This dynamic manipulation might be harnessed for the adiabatic transport of novel quasiparticles or for the strategic generation of substantial, localized magnetic fluctuations.

## ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy (DOE), Office of Science, and Office of Advanced Scientific Computing Research, through the Quantum Internet to Accelerate Scientific Discovery Program.

[1] A. L. Balk, M. D. Stiles, and J. Unguris, Phys. Rev. B **90**, 184404 (2014).
[2] M.-Y. Im, P. Fischer, D.-H. Kim, and S.-C. Shin, Appl. Phys. Lett. **95**, 182504 (2009).
[3] D. A. Christian, K. S. Novoselov, and A. K. Geim, Phys. Rev. B **74**, 064403 (2006).
[4] M. Liebmann, A. Schwarz, U. Kaiser, R. Wiesendanger, D.-W. Kim, and T.-W. Noh, Phys. Rev. B **71**, 104431 (2005).
[5] S.-C. Shin, K.-S. Ryu, D.-H. Kim, S.-B. Choe, and H. Akinaga, J. Magn. Magn. Mater. **310**, 2599 (2007).
[6] A. L. Balk, F. Li, I. Gilbert, J. Unguris, N. A. Sinitsyn, and S. A. Crooker, Phys. Rev. X **8**, 031078 (2018).
[7] V. I. Yukalov, Int. J. Mod. Phys. B **17**, 2333 (2003).
[8] V. I. Yukalov and E. P. Yukalova, Phys. Part. Nuclei **54**, 1 (2023).
[9] V. Yukalov, Phys. Rep. **208**, 395 (1991).
[10] M. Goryca, X. Zhang, J. Li, A. L. Balk, J. D. Watts, C. Leighton, C. Nisoli, P. Schiffer, and S. A. Crooker, Phys. Rev. X **11**, 011042 (2021).
[11] L. D. Landau, Zh. Eksp. Teor. Fiz. **7**, 19 (1937).
[12] R. Conte and M. Musette, Physica D **69**, 1 (1993).
[13] G. Profilo and G. Soliani, Nuovo Cimento B **106**, 307 (1991).
[14] J.-G. Liu, Y.-Z. Li, and B. Tian, Commun. Nonlinear Sci. Numer. Simul. **14**, 1214 (2009).
[15] D. Dai, S.-X. Xu, and L. Zhang, J. Diff. Equ. **269**, 2430 (2020).
[16] S. P. Hastings and J. B. McLeod, Arch. Ration. Mech. Anal. **73**, 31 (1980).
[17] D. Dai and W. Hu, Nonlinearity **30**, 2982 (2017).
[18] N. A. Sinitsyn and Y. V. Pershin, Rep. Prog. Phys. **79**, 106501 (2016).