Universality of explosive percolation under product and sum rule

Ziting Luo,[1] Wei Chen,[2,3,4,*] and Jan Nagler[5,†]

[1]*LMIB and School of Mathematical Sciences, Beihang University, Beijing 100191, China*
[2]*Institute of Artificial Intelligence, Beihang University, Beijing 100191, China*
[3]*Zhongguancun Laboratory, Beijing 100094, People's Republic of China*
[4]*Beijing Advanced Innovation Center for Big Data and Brain Computing, Beihang University, Beijing 100191, China*
[5]*Deep Dynamics, Centre for Human and Machine Intelligence, Frankfurt School of Finance and Management, Frankfurt am Main 60322, Germany*

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We study explosive percolation processes on random graphs for the so-called product rule (PR) and sum rule (SR), in which M candidate edges are randomly selected from all possible ones at each time step, and the edge with the smallest product or sum of the sizes of the two components that would be joined by the edge is added to the graph, while all other $M-1$ candidate edges are being discarded. These two rules are prototypical "explosive" percolation rules, which exhibit an extremely abrupt yet continuous phase transition in the thermodynamic limit. Recently, it has been demonstrated that PR and SR belong to the same universality class for two competing edges, i.e., $M=2$. Here we investigate whether the claimed PR-SR universality is valid for higher-order models with M larger than 2. Based on traditional finite-size scaling theory and largest-gap scaling, we obtain the percolation threshold and the critical exponents of the order parameter, susceptibility, and the derivative of entropy for PR and SR for M from 2 to 9. Our results strongly suggest PR-SR universality, for any fixed M.

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I. INTRODUCTION

Percolation, the emergence of large-scale connectivity, is widely used in many natural, technological, and social systems [1–7], including the spreading of forest fires [8–10], fluid flow through porous media [11–13], infiltrations in composite materials processing [14], epidemic spreading [15–17], and information diffusion [18,19]. One of the most classic percolation models is random percolation on Erdős-Rényi (ER) random graphs [20]. This model is initiated with N isolated nodes. From the modeling perspective, two randomly selected nodes at each step are being connected. In graph theory parlance, once the density of edges in the graph exceeds a critical threshold $t_c = 1/2$, a macroscopic giant component emerges in the graph in a continuous, second-order transition in the thermodynamic limit [21].

The Achlioptas process imposes a competition rule to the ER model where several candidate random edges are selected at each step but only one of them is added to the graph while all other candidate edges are being discarded, based on certain competition rules [22]. The first proposed and well-studied competition rules are product rule (PR) and sum rule (SR) which connect the edge with the smallest product or sum of the sizes of components they join. Due to this size-suppressive bias, the onset of the percolation is delayed and the giant component emerges in an abrupt, "explosive" phase transition. Explosive phenomena of percolation transition have come with a great deal of interest. The majority of proposed explosive percolation models show a continuous transition in the thermodynamic limit, while a few are genuinely discontinuous, showing gaps in the order parameter [23–29]. In addition, there are many investigations still underway to understand the critical and supercritical behaviors of explosive phase transition such as multiple transitions, multiple giant components, and discrete scale invariance [30–32]. In particular, genuinely discontinuous and continuous explosive percolation has enjoyed a number of applications, while a classification of critical percolation phenomena can be found in reviews [29,33,34].

Different percolation models may have the same critical behavior of phase transition and therefore belong to the same universality class [35]. Classification of percolation models to universality classes is an important topic in studying critical phenomena [36–38]. For explosive percolation, Grassberger *et al.* first reported that four Achlioptas-type processes with explosive percolation transitions are in different universality classes, yet with unusual finite-size behavior [39]. Bastas *et al.* found that site and bond explosive percolation on the lattice are not in the same universality class [40]. D'Souza and Nagler studied some explosive percolation models and showed that they are in different universality classes [33]. Recently, Sabbir and Hassan reported that explosive percolation on ER networks for PR and SR (for $M=2$) belong to the same universality class [41], meaning that these two different microscopic processes have the same critical exponents in the thermodynamic limit. This work was the first result showing

*chwei@buaa.edu.cn
†jan.nagler@gmail.com

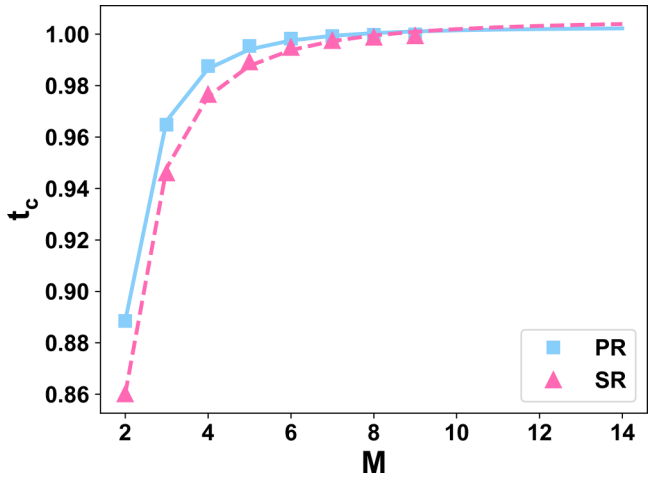

FIG. 1. Percolation threshold t_c as a function of M for PR and SR. Solid and dashed lines are fits according to Eqs. (4) and (5).

that different explosive-type percolation models can belong to the same universality class. However, they only refer to the case of randomly selecting two (but not more) candidate edges at each step. Here we further investigate whether PR-SR universality holds for the general case of randomly selecting M candidate edges at each step where M may be larger than 2. We use finite-size scaling analysis and largest-gap statistics [35] to measure the critical exponents about order parameter, susceptibility, and the derivative of entropy for explosive percolation models under PR and SR. We find that PR-SR universality holds for M between 2 and 9. The remainder of the paper is structured as follows. In Sec. II we introduce the Achlioptas process under PR and SR, and the Monte Carlo algorithm. In Sec. III we first measure the percolation threshold t_c for PR and SR. Then we study the critical exponents for order parameter, susceptibility, and derivative of entropy with finite-size scaling analysis. Finally, our results will be discussed and concluded in Sec. IV.

II. NETWORK GROWTH MODELS

We study the Achlioptas process under PR and SR. We start with N isolated nodes and iteratively add one edge at each time step to the system. Therefore, after n edges have been added to the system, the edge density in the system is $t = n/N$. At each time step, M candidate edges, $e_{i_1 j_1}, e_{i_2 j_2}, e_{i_3 j_3}, \ldots, e_{i_M j_M}$, are randomly selected among all

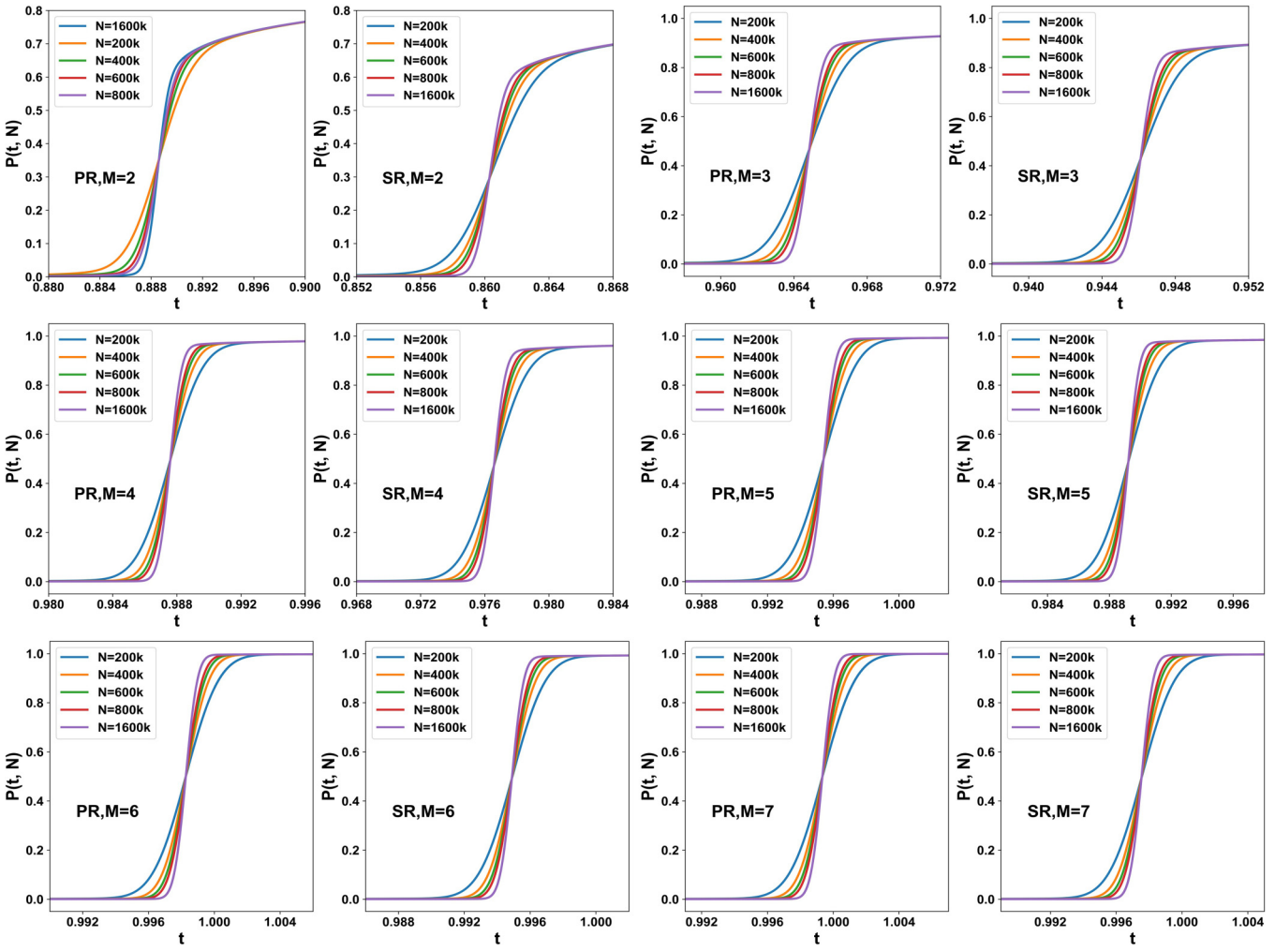

FIG. 2. Order parameter P versus t for PR and SR, for M competing edges, and system sizes N.

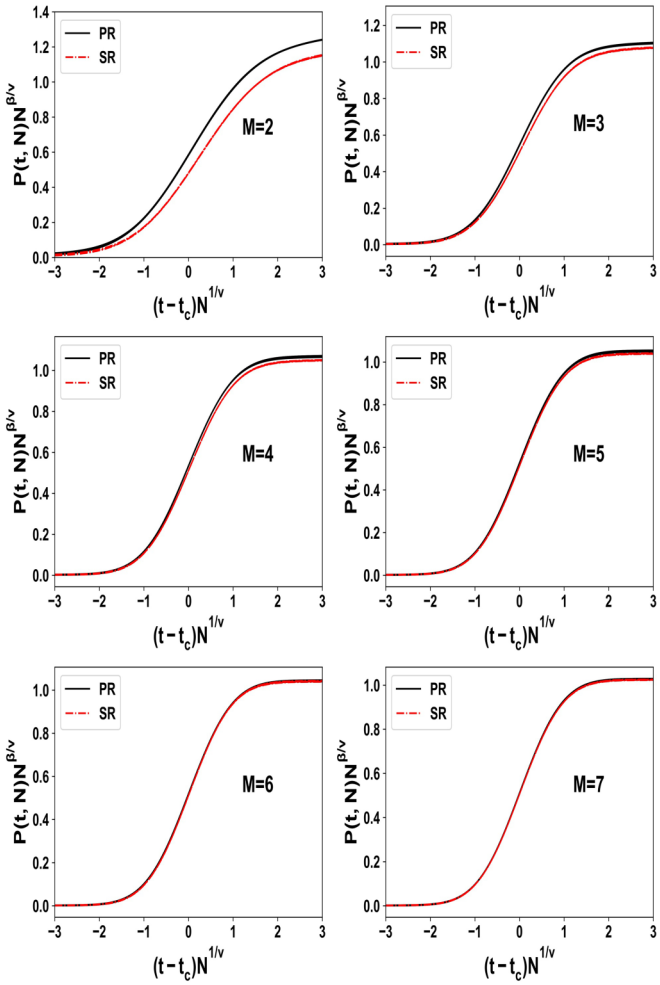

FIG. 3. Plots of $PN^{\beta/\nu}$ versus $(t-t_c)N^{1/\nu}$ for PR and SR with $M = 2, 3, 4, 5, 6, 7$ (for $M = 8, 9$, see Supplemental Material Fig. 1 [45]).

possible edges. For PR, we consider the size of the two components joined by each candidate edge, and the edge minimizing the product of the sizes of the two components that would be joined by the edge is occupied. For SR, we use the sum of the sizes of the two components that would be joined by the edge instead of the product. For both SR and PR, the growth of the largest component is suppressed due to the applied minimization criterion. The emergence of the giant component is delayed with an explosive transition, which for any finite system is substantially more abrupt than the percolation transition in the Erdős-Rényi model. The finite-system gap of the order parameter and its scaling for various models has been discussed in [35].

The fast Monte Carlo algorithm for percolation proposed by Newman and Ziff [42,43], which involves the union-find procedure to merge components, can allow us to keep track of the component size distribution. In all simulations, we use the algorithm for determining an observable $Q(p)$. Then, once accurately n edges have been occupied, we evaluate the "microcanonical" Q_n and find the "canonical" $Q(p)$ by performing a convolution with a binomial distribution

$$B(N, n, p) = \binom{N}{n} p^n (1-p)^{N-n},$$

$$Q(p) = \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} Q_n. \tag{1}$$

In the Monte Carlo simulations we run $2N$ steps for all N and 10 000 independent simulations have been performed for each network size.

III. NUMERICAL ANALYSIS

A. The percolation threshold

Several methods have been proposed to identify the percolation threshold t_c. The first method is based on the finite-size scaling theory. If the percolation transition is continuous, an observable X near the percolation threshold is said to satisfy the following scaling form:

$$X \sim N^{\omega/\nu} \Phi[(t-t_c)N^{1/\nu}], \tag{2}$$

where ω and ν are different critical exponents and Φ is the universal scaling function. If $t = t_c$, we have $X \sim N^{\omega/\nu}$. Therefore the percolation threshold can be determined by estimating the value of t at which the variable X and system size N satisfy the underlying scaling law [44]. The second method is based on the scaling window, $\Delta T = t_2 - t_1$, which is proposed by Achlioptas *et al.* [22]. ΔT is a function of system size N, where t_1 is the last step at which the largest connected component $S_{\max} < N^{1/2}$ and t_2 is the first step at which $S_{\max} > 0.5N$. When N tends to infinity, t_1 and t_2 converge to the critical threshold. The third method is based on largest-gap scaling [26,35]. We define the order parameter $P = S_{\max}/N$, where $S_{\max}$ is the largest connected component size and N is the number of nodes in the network. The edge density at which the largest gap of the order parameter occurs in a network of size N, denoted as $t_c(N)$, obeys the scaling relation

$$t_c(N) = t_c + bN^{-1/\nu}. \tag{3}$$

We use the third method to measure the percolation threshold for the Achlioptas process for PR and SR with the number of candidate edges M from 2 to 9. The value of t_c can be obtained with linear fitting, which is shown in Table I. It is worth noting that for the Achlioptas process of PR and SR with $M = 2$, the percolation thresholds and the critical exponents obtained in our paper are slightly different from those reported in Ref. [41]. Our method, however, based on largest gap averaging, is expected to provide better estimates and data collapses [35] than methods based on simple ensemble averaging, which is numerically confirmed by our results, showing clean data collapses together with a high consistence of scaling parameters, as we will demonstrate.

In Fig. 1 we study the percolation threshold t_c as a function of M for both PR and SR. Through least-square fitting, the fitting functions read

$$\text{SR: } t_c = -0.9047(e^{-0.9395M}) + 0.9987, \tag{4}$$

$$\text{PR: } t_c = -1.0936(e^{-1.1442M}) + 0.9995. \tag{5}$$

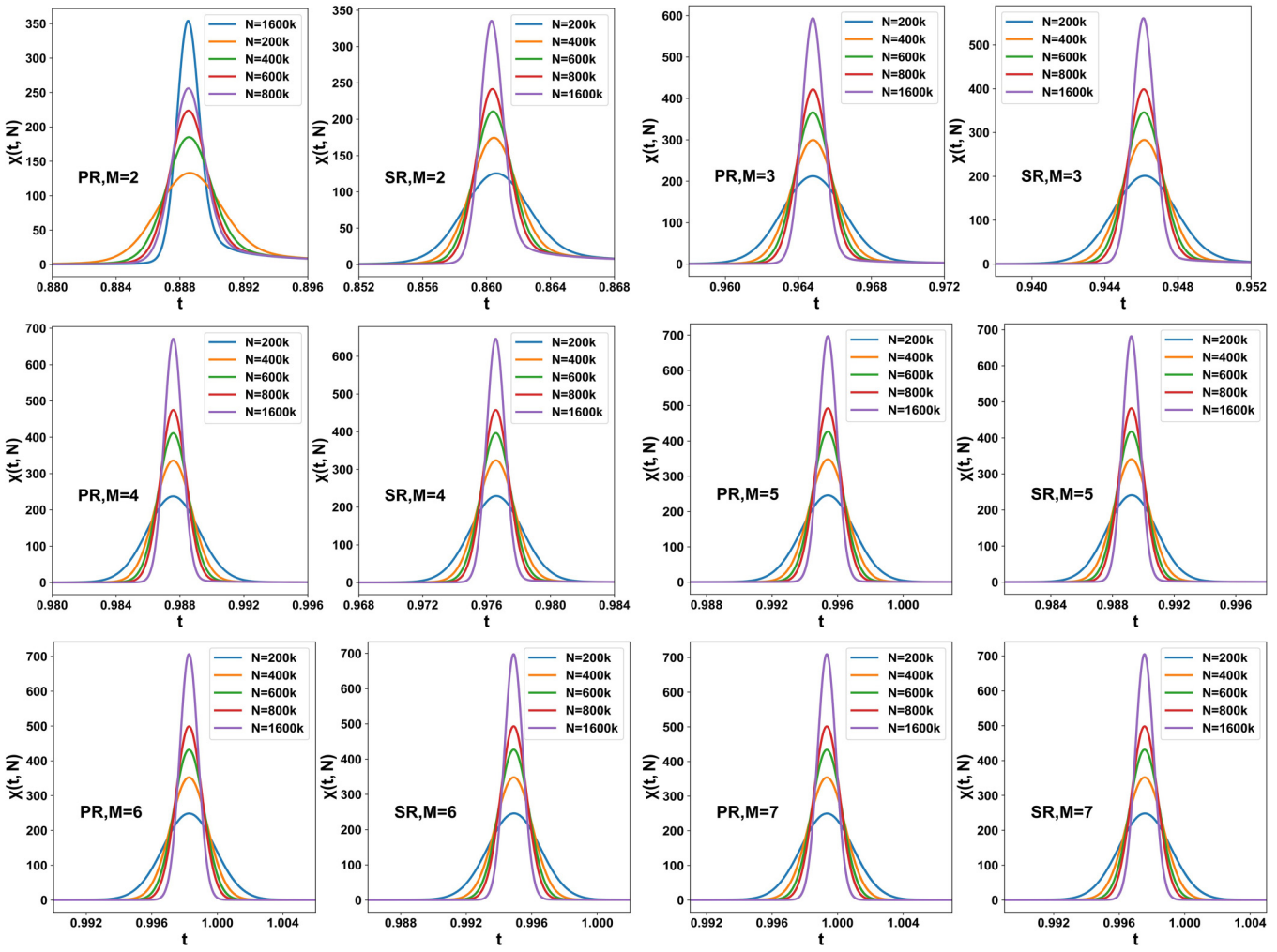

FIG. 4. Susceptibility χ versus t for PR and SR, for different M and N.

In agreement with existing literature [29], for a fixed M, the percolation threshold of SR is smaller than that of PR, which indicates that PR is more effective in delaying the onset of the giant component as shown in Fig. 1. Moreover, the onset of the giant component is delayed for larger values of M, due to a higher number of edges that compete for addition.

B. The order parameter

Here, we study the order parameter as a function of the edge density near the percolation threshold for PR and SR, for M varying from 2 to 9 and different system sizes. Unlike random percolation, Achlioptas rules change the growth process of networks and delay the formation of the giant component. For finite systems, for increasing system sizes, the order parameter P exhibits sharper changes, and the curves of the order parameters versus edge density under different system sizes cross at the same point near the phase transition point t_c in Fig. 2 and the Supplemental Material [45].

If the percolation transition is continuous, according to the finite-size scaling theory [46,47], the order parameter satisfies the following scaling form:

$$P \sim N^{-\beta/\nu}\Phi^{(1)}[(t-t_c)N^{1/\nu}], \tag{6}$$

where t_c is the critical threshold of percolation, β and ν are the critical exponents for the phase transition, and $\Phi^{(1)}$ is the universal scaling function. At $t = t_c$, $\log(P)$ versus $\log(N)$ is a straight line, and through linear regression, we can find β/ν. For $M = 3$, $\beta/\nu = 0.0144$ for PR, while $\beta/\nu = 0.0153$ for SR.

The calculated values for other values of M are shown in Table II. The ratio β/ν remains within error bars for PR and SR. As M increases, β/ν decreases. From

TABLE I. Percolation thresholds for PR and SR for M competing edges.

	M	2	3	4	5	6	7	8	9
t_c	PR	0.888470(60)	0.964798(7)	0.987575(2)	0.995408(1)	0.998263(1)	0.999335(1)	0.999741(1)	0.999895(1)
	SR	0.860159(10)	0.946069(2)	0.976553(2)	0.989212(2)	0.994879(3)	0.997533(3)	0.998797(1)	0.999406(3)

TABLE II. Critical exponents for PR and SR for different M.

	M	2	3	4	5	6	7	8	9
β/ν	PR	0.045(3)	0.0144(7)	0.0071(9)	0.0045(3)	0.0034(1)	0.0021(4)	0.0014(3)	0.0011(9)
	SR	0.044(2)	0.0153(12)	0.0073(8)	0.0045(10)	0.0037(8)	0.0021(5)	0.0014(6)	0.0012(7)
γ/ν	PR	0.472(1)	0.4944(8)	0.5001(9)	0.5017(6)	0.5022(26)	0.5024(3)	0.5024(2)	0.5024(2)
	SR	0.474(1)	0.4932(9)	0.4985(8)	0.5008(5)	0.5017(3)	0.5021(3)	0.5023(2)	0.5024(2)
α/ν	PR	0.532(1)	0.5629(11)	0.5740(13)	0.5776(11)	0.5788(12)	0.5791(11)	0.5791(9)	0.5792(9)
	SR	0.534(1)	0.5646(9)	0.5726(7)	0.5763(10)	0.5779(11)	0.5785(11)	0.5789(9)	0.5791(9)
$1/\nu$	PR	0.517(4)	0.5087(15)	0.5072(18)	0.5062(9)	0.5056(27)	0.5048(7)	0.5038(5)	0.5035(11)
	SR	0.518(3)	0.5085(21)	0.5063(16)	0.5053(15)	0.5045(11)	0.5047(8)	0.5037(8)	0.5036(9)

$t_c(N) = t_c + bN^{-1/\nu}$ and the behavior of the giant component, we obtain the critical exponent β.

We plot $PN^{\beta/\nu}$ versus $(t - t_c)N^{1/\nu}$ to test that $P(t, N)$ obeys finite-size scaling, using the same critical exponents γ and ν for both the PR and the SR case. All the curves collapse into a single curve, as shown in Fig. 3, which is consistent with the universal scaling function $\Phi^{(1)}$ that does not depend on N.

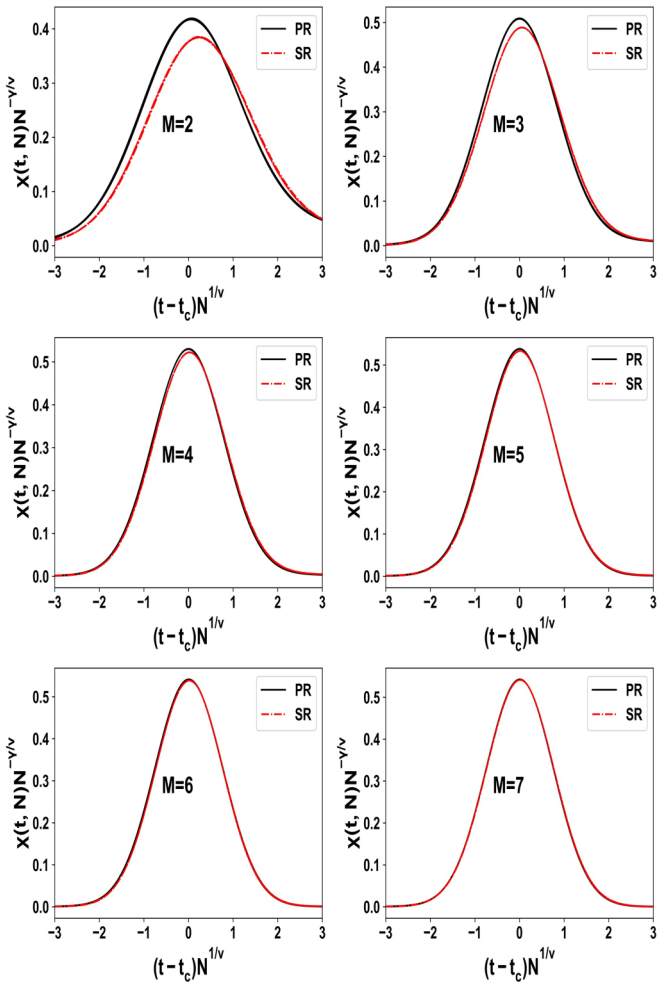

FIG. 5. Plots of $\chi N^{-\gamma/\nu}$ versus $(t - t_c)N^{1/\nu}$ for PR and SR, $M = 2, 3, 4, 5, 6, 7$.

C. The susceptibility of percolation

Another important method is to study the susceptibility of percolation, which has been proposed [48,49] to discern continuous and discontinuous percolation transitions. The susceptibility characterizes the range of fluctuations of the order parameter P, defined as [26,35,41]

$$\chi(t) = \frac{\Delta P}{\Delta t}. \tag{7}$$

Using $\Delta P = \Delta S_{\max}/N$ and $\Delta t = 1/N$, we have $\chi = \Delta S_{\max}$, which becomes the derivative of the order parameter P in the thermodynamic limit $N \to \infty$, for which $\Delta t \to 0$.

In Fig. 4 we study the susceptibility χ as a function of the edge density t around the critical threshold t_c for PR and SR, with M ranging from 2 to 9. When $t = t_c$, the size of the second-largest component in the network reaches the maximum, and it merges with the largest component, which leads to the maximum of χ at critical threshold t_c [26]. For all investigated M, around the critical threshold the susceptibility increases with N.

For a continuous phase transition, according to the finite-size scaling theory, the susceptibility χ satisfies

$$\chi(t, N) \sim N^{\gamma/\nu}\Phi^{(2)}[(t - t_c)N^{1/\nu}], \tag{8}$$

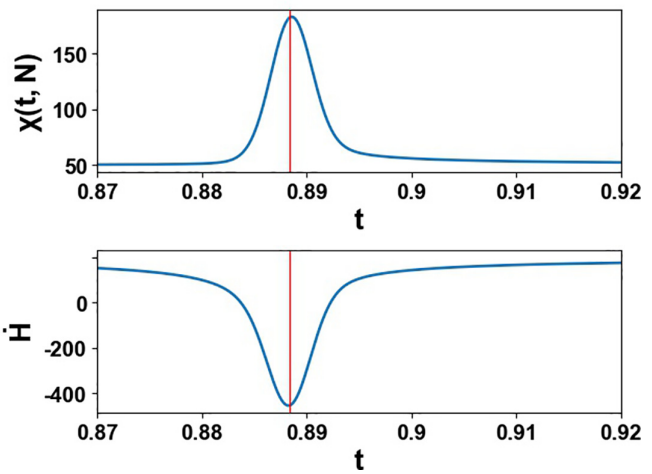

FIG. 6. Plots of susceptibility χ and the derivative of entropy $\dot{H}$ versus t for PR with $M = 2$ and for system size $N = 200\,000$. The red vertical line is $t = t_c$; in this case it is 0.888 470.

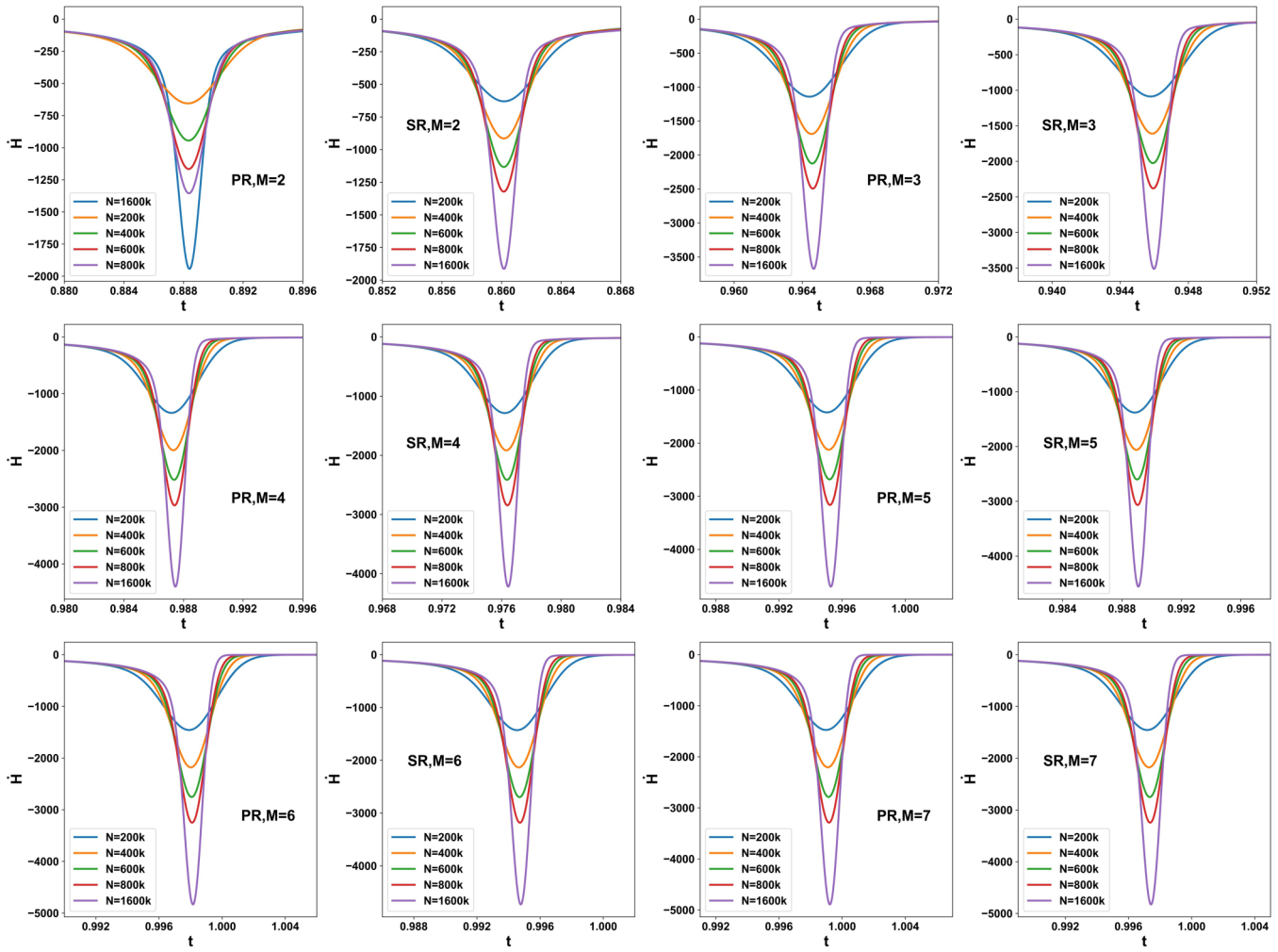

FIG. 7. Derivative of entropy, $\dot{H}$ versus t for PR and SR, for different M and N.

where t_c is the critical threshold of percolation, γ and ν are the critical exponents, and $\Phi^{(2)}$ is the universal scaling function. At $t = t_c$, we have $\chi(t, N) \sim N^{\gamma/\nu}$, and plotting $\log(\chi)$ vs $\log(N)$ gives a straight line with a slope of γ/ν. We obtain the same values of γ/ν for PR and SR, as shown in Table II.

As mentioned earlier, for $N \to \infty$, χ becomes the derivative of P. Then, we have $1/\nu = \gamma/\nu + \beta/\nu$, as summarized in Table II. For the same number of competing edges M, the values of $1/\nu$ are equal within error bars for PR and SR.

In Fig. 5, using the same critical exponents γ and ν for PR and SR, we plot the curves of $\chi N^{-\gamma/\nu}$ versus $(t - t_c)N^{1/\nu}$. The data well collapse into its scaling function $\Phi^{(2)}$. This suggests that $\chi(t, N) \sim (t - t_c)^{-\gamma}$ together with the divergence of susceptibility at the critical threshold.

D. The derivative of entropy

In addition to the order parameter and its susceptibility, entropy can also be used to study the critical behaviors of phase transitions [50]. In particular, it has been proposed for analyzing whether the phase transition is continuous or discontinuous [51–53]. There are various definitions for entropy. Here we use the Shannon entropy with respect to the cluster size distribution, defined as

$$H(t) = -K \sum_{i}^{k} \mu_i \log \mu_i, \tag{9}$$

where K is a constant and can be treated as being 1, for simplicity. For a given t, assuming that there are k clusters in the system, we label clusters $i = 1, 2, \ldots, k$ whose sizes are $s_1, s_2, \ldots, s_k$, while $\mu_i = s_i/N$ denotes the probability of selecting a node at random which belongs to cluster i. The Shannon entropy represents the degree of system disorder and stands in that way opposite to the percolation order parameter.

For further analyzing the critical behaviors,

in Fig. 6, we investigate the evolution of the derivative of entropy and the susceptibility. Our analysis suggests that the derivative of entropy is minimal where the susceptibility attains its maximum. Fundamentally, this behavior is expected since an Achlioptas rule exhibits a continuous percolation transition and the information entropy must attain a maximum at t_c in the thermodynamic limit. Yet, this expectation has been contrasted by the results by Viera *et al.* [51]. They point out that the entropy attains a maximum at the phase transition point for classical percolation but not necessarily for explosive percolation. They also conclude that the location

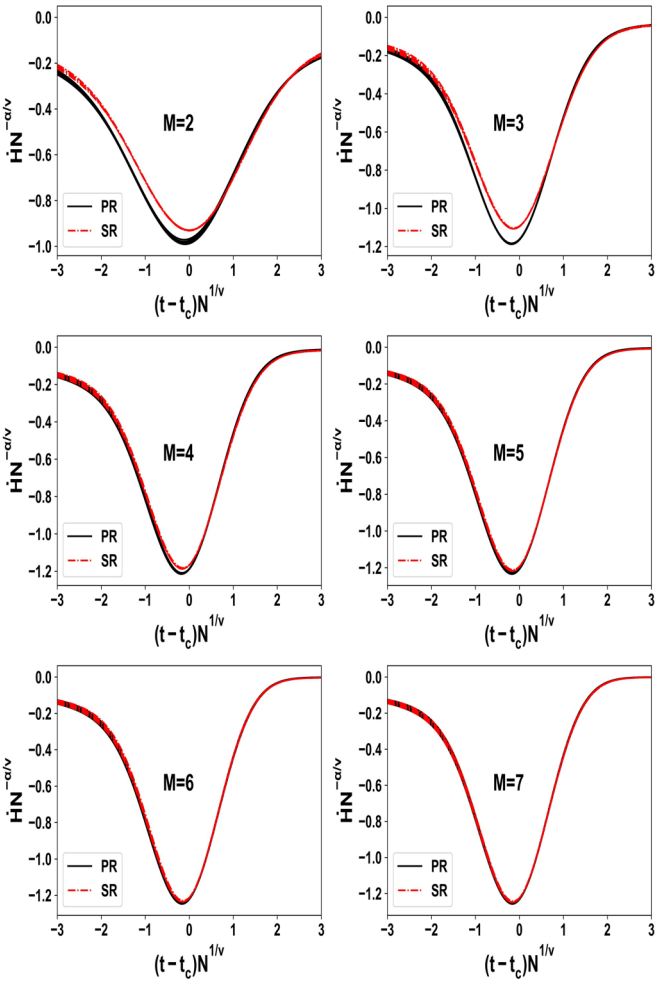

FIG. 8. Scaling function $\Phi^{(3)}$. We plot $\dot{H}N^{-\alpha/\nu}$ versus $(t-t_c)N^{1/\nu}$ for PR and SR, and $M = 2, 3, 4, 5, 6, 7$.

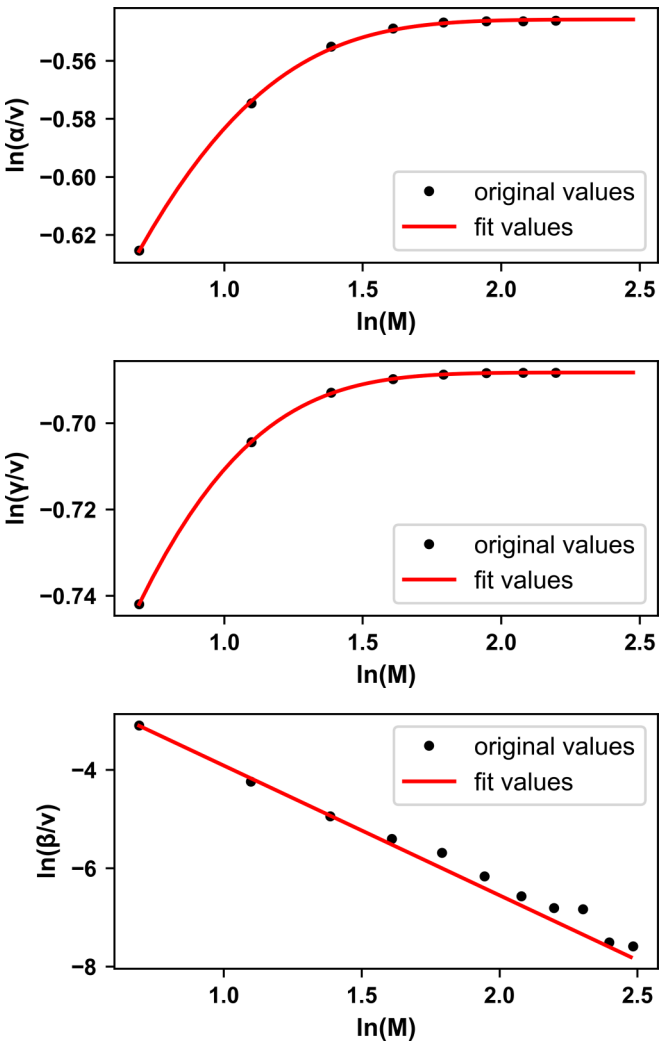

FIG. 9. Critical exponents versus M.

of the minimum of the derivative of entropy $\dot{H}$ can more robustly quantify the critical threshold for both explosive and traditional percolation models. Here, we note that the definition of entropy used in their paper is different from our definition. Specifically, Viera *et al.* defines the entropy as $H = -K\sum_i (p_i \log_2 p_i)$, based on the time-dependent probability distribution $\{p_i = n_t(i)/n(t)\}$, with $n(t) = \sum_i n_t(i)$ clusters at time t, $n_t(i)$ clusters of size i.

In Fig. 7 we find that $\dot{H}<0$ for the entire process, where the minimum is attained at t_c. For continuous percolation, according to the finite-size scaling theory, we can assume that

$$\dot{H} \sim N^{\alpha/\nu}\Phi^{(3)}[(t-t_c)N^{1/\nu}], \tag{10}$$

where α and ν are the standard critical exponents, and $\Phi^{(3)}$ is the universal scaling function. Therefore, whether $\dot{H}$ is asymmetric with respect to t_c is governed by the scaling function $\Phi^{(3)}$. Figure 8 suggests that $\Phi^{(3)}$ is asymmetric with respect to t_c. Hence, $\dot{H}$ adopts the asymmetry with respect to t_c, as shown in Fig. 7.

Similarly to the methodology for computing the critical exponents for the order parameter, α/ν is computed, as shown in Table II, which also displays no significant differences between PR and SR for the same M.

Using these calculated values, we study $\dot{H}N^{-\alpha/\nu}$ as a function of $(t-t_c)N^{1/\nu}$, for PR and SR, for which the curves become indistinguishable for larger M, as shown in Fig. 8. Taken together, we find that the universal function of $\dot{H}$ does not show any finite-size dependence, further supporting the continuity of the percolation transition of PR and SR. In Figs. 3, 5, 8, for $M < 5$, the universal functions of PR and SR show only slight differences from each other. For $M \geqslant 5$, we observe that the curves of PR and SR basically overlap, which indicates that the universal functions of PR and SR become practically identical for the already moderately large values of M.

E. Critical exponents versus number of competing edges

The values of critical exponents as a function of M are shown in Fig. 9, whereas the regression results are shown in Eqs. (11)–(13). We observe that the values of β/ν, γ/ν, and α/ν lie between 0 and 1. The values of β/ν decrease continuously with M while the values of γ/ν and α/ν increase continuously with M. Moreover, for $M \geqslant 6$, γ/ν settles around 0.5 while α/ν approach a plateau around 0.58. Strikingly, for the entire range of M, $1/\nu$ remains invariant at about

0.5. Evidently, Eq. (11) implies that as M tends to infinity, β/ν approaches 0. This indicates that the phase transition becomes discontinuous in the limit $M \to \infty$, which is consistent with the results in Refs. [23,26] for Achlioptas processes.

$$\frac{\beta}{\upsilon} = 0.2784 * (M^{-2.6366}), \tag{11}$$

$$\frac{\gamma}{\upsilon} = -0.2829 * (e^{-1.1887M}) + 0.5024, \tag{12}$$

$$\frac{\alpha}{\upsilon} = -0.3403 * (e^{-1.0172M}) + 0.5794. \tag{13}$$

IV. CONCLUSION

In this paper, we have studied two families of explosive percolation models, where M edges compete for addition: the product rule (PR) and the sum rule (SR). In agreement with existing literature, for PR and SR, we demonstrated that the critical percolation threshold t_c increases as M increases, that is, percolation is delayed. We have also numerically studied several observables, including order parameter, susceptibility, entropy, and the derivatives of entropy. We proposed a slightly different definition of the entropy, which led to clean data collapses and consistent results. We found that for all studied values of M, order parameter, susceptibility, and the derivative of entropy show power-law scaling at the critical threshold, as one would expect for a continuous or second-order phase transition.

By definition, for $M = 1$, both the PR rule and the SR rule are identical to classical random percolation, exhibiting the same universality class. Previous work suggested that PR and SR belong to the same universality class for $M = 2$. We find that the critical exponents are the same with error bars for PR and SR, for all studied $M \geqslant 2$. This strongly suggests that PR and SR, for the same M, belong to the same universality class, which is our main result.

Future work must establish how prominent universality is in explosive percolation, whose models have been known to crucially depend on their microscopic details. In that light, we believe that PR and SR were worth revisiting regarding universality.

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