Dipole-monopole criticality and chargeless half mode in an integrable gauge-coupled pseudoexciton-phonon system on a regular one-dimensional lattice

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A one-dimensional nonlinear dynamical system of gauge-coupled intrasite excitations and lattice vibrations on an infinite one-dimensional regular lattice is studied. The system as a whole is shown to be integrable in the Lax sense and it admits the exact four-component analytical solutions. Two mutually $\mathcal{PT}$-conjugated symmetry broken solutions are explicitly isolated in the framework of the Darboux-Bäcklund dressing technique. Each of the obtained four-component solutions demonstrates the pronounced interplay between the interacting subsystems in the form of an essentially nonlinear superposition of two principally distinct types of traveling waves characterized by two physically distinct spatial scales and by two distinct running velocities. Depending on the relationships between the spatial scaling parameters the system can manifest itself in three qualitatively distinct dynamical regimes referred to as the monopole regime, dipole regime, and threshold regime. The threshold value of the localization parameter separating the monopole and dipole dynamical regimes is strictly established in terms of basic physical parameters. The phenomenon of dipole-monopole crossover for the spatial distribution of pseudoexcitons is shown to initiate the partial splitting between the pseudoexcitonic and vibrational subsystems in the threshold dynamical regime specified by the threshold value of the localization parameter. This partial splitting is manifested by the complete elimination of one pseudoexcitonic component accompanied by the actual conversion of another pseudoexcitonic component into a pseudoexcitonic chargeless half mode. The integrable nonlinear pseudoexciton-phonon system under study is expected to be applicable for modeling the nonlinear dynamical properties of properly designed $\mathcal{PT}$-symmetric metamaterials.

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I. INTRODUCTION

The development of a nonlinear exciton-phonon dynamical system on a regular one-dimensional molecular chain by Davydov and Kyslukha about 50 years ago [1,2] is known to be in line with the pioneering works of Landau and Pekar [3], Bogolyubov [4], Fröhlich [5], Peierls [6], as well as Holstein [7] on the fundamental role of electron-phonon or exciton-phonon coupling in the formation of spatially confined nonlinear clusters (polarons or solitons). Moreover, such a type of interaction causes the Fröhlich-Peierls instability [5,6] originating superconducting states [5,8] or charge-density waves [9–11] in quasi-one-dimensional metals [10,11]. A similar sort of intersubsystem coupling is responsible also for the formation of charge-density wave packets in armchair silicene nanoribbons [12].

The Davydov-Kyslukha system [1,2] has been suggested to model the energy and charge transport in long macromolecules of both synthetic and natural (chiefly biological) origins. In its concise classical formulation it reads as a dynamical system characterized by the Hamiltonian function [13,14]

$$\mathrm{H} = -\mathrm{J}\sum_{m=-\infty}^{\infty}[\psi^*(m)\psi(m+1)+\psi^*(m)\psi(m-1)] + \sum_{m=-\infty}^{\infty}\left\{\pi^2(m)/2M+(\varkappa/2)[\beta(m)-\beta(m-1)]^2\right\} + \chi\sum_{m=-\infty}^{\infty}[\beta(m+1)-\beta(m-1)]\psi^*(m)\psi(m), \quad (1.1)$$

with the quantities $\psi^*(n)$ and $\psi(n)$ serving as the complex conjugate field amplitudes of an exciton (or electron) on the nth site of a lattice, while $\pi(n)$ and $\beta(n)$ stand for the momentum and coordinate variables associated with the displacement of the nth structural element (atom or molecule) from its equilibrium position. Each pair of quantities $\psi^*(n)$, $\psi(n)$ and $\pi(n)$, $\beta(n)$ serves as a pair of canonical field variables governed by a respective pair of Hamiltonian equations

$$+\mathrm{i}\hbar\, d\psi(n)/dt = \partial\mathrm{H}/\partial\psi^*(n), \quad (1.2)$$

$$-\mathrm{i}\hbar\, d\psi^*(n)/dt = \partial\mathrm{H}/\partial\psi(n), \quad (1.3)$$

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and

$$d\pi(n)/dt = -\partial \mathrm{H}/\partial \beta(n), \tag{1.4}$$

$$d\beta(n)/dt = +\partial \mathrm{H}/\partial \pi(n). \tag{1.5}$$

The parameters J and χ, appearing in the Hamiltonian function (1.1), characterize the intersite resonant coupling in an exciton subsystem and the coupling between the involved subsystems (i.e., between the subsystem of displacements and the exciton subsystem), respectively. The parameters M and $\varkappa$ stand, respectively, for the mass of the structural element and for the elasticity coefficient related to the subsystem of displacements. The spatial variable n is supposed to be a discrete one and it spans all integers from minus to plus infinity, while the time variable t is assumed to be a continuous one.

The exact analytical solitonlike solutions to the Davydov-Kyslukha model (1.1)–(1.5) were found only in the continuous spatial approximation since the model as such is not integrable both in the Lax and Liouville sense. One of the way to circumvent this stumbling block is to construct an integrable nonlinear exciton-phonon system which could reproduce at least some features of a physically motivated one. Considerable progress in the development of integrable pseudoexciton-phonon nonlinear dynamical systems on quasi-one-dimensional lattices has been reported in several of our previous papers [15–18].

The main objective of the present paper is to analyze the most important implications concerning the basic properties of an integrable gauge-coupled pseudoexciton-phonon nonlinear dynamical system on a regular one-dimensional lattice, which will be interesting for the physical scientific community. In particular, we pay attention to the $\mathcal{PT}$ symmetry of an announced system widely requested in modeling the physical properties of metamaterials. We thoroughly describe the phenomenon of a dipole-monopole crossover in the spatial distribution of pseudoexcitons, and point out that the pseudoexcitonic subsystem degrades into a subsystem characterized by a chargeless half mode under a critical value of the localization parameter.

II. INTEGRABLE GAUGE-COUPLED PSEUDOEXCITON-PHONON NONLINEAR DYNAMICAL SYSTEM

The gauge-coupled pseudoexciton-phonon nonlinear dynamical system of our interest,

$$\begin{aligned} dg_+(n)/d\tau &= J\, g_+(n+1) - J\, g_+(n) \\ &\quad + [p(n) - g_+(n)g_-(n)]g_+(n), \end{aligned} \tag{2.1}$$

$$\begin{aligned} dg_-(n)/d\tau &= J\, g_-(n) - J\, g_-(n-1) \\ &\quad - [p(n) - g_+(n)g_-(n)]g_-(n), \end{aligned} \tag{2.2}$$

$$\begin{aligned} dp(n)/d\tau &= \Omega^2 \exp[+q(n+1) - q(n)] \\ &\quad - \Omega^2 \exp[+q(n) - q(n-1)], \end{aligned} \tag{2.3}$$

$$dq(n)/d\tau = p(n) - g_+(n)g_-(n), \tag{2.4}$$

is designed to be integrable in the Lax sense inasmuch as it admits the semidiscrete zero-curvature representation

$$\frac{d}{d\tau} L(n|\lambda) = A(n+1|\lambda)L(n|\lambda) - L(n|\lambda)A(n|\lambda), \tag{2.5}$$

with the spectral $L(n|\lambda)$ and evolution $A(n|\lambda)$ operators substantiated by the following 3×3 square matrices:

$$L(n|\lambda) = \begin{pmatrix} \lambda + g_+(n)g_-(n) - p(n) & g_+(n)\sqrt{J} & \Omega \exp[+q(n)] \\ g_-(n)\sqrt{J} & J & 0 \\ -\Omega \exp[-q(n)] & 0 & 0 \end{pmatrix}, \tag{2.6}$$

$$A(n|\lambda) = \begin{pmatrix} \lambda & g_+(n)\sqrt{J} & \Omega \exp[+q(n)] \\ g_-(n-1)\sqrt{J} & J & 0 \\ -\Omega \exp[-q(n-1)] & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Here, the two sets $g_+(n) \equiv g_+(n|\tau)$, $g_-(n) \equiv g_-(n|\tau)$ and $p(n) \equiv p(n|\tau)$, $q(n) \equiv q(n|\tau)$ of field functions are related to the subsystem of Dirac pseudoexcitons and to the subsystem of Toda vibrations, respectively. The greek letter τ stands for the continuous-time variable. The spatial position of a lattice site is marked by the integer n running from minus infinity to plus infinity. The real-valued constant parameters J and Ω are responsible for the intersite resonant coupling in a subsystem of pseudoexcitons and for the intersite elasticity in a vibrational subsystem, respectively. The auxiliary spectral parameter λ is assumed to be a time- and coordinate-independent one.

Due to its integrability the suggested system (2.1)–(2.4) possesses an infinite hierarchy of conservation laws. The most physically important conserved quantities are the total energy associated with the Hamiltonian function,

$$\begin{aligned} H &= \frac{1}{2}\sum_{m=-\infty}^{\infty} [p(m) - g_+(m)g_-(m)]^2 \\ &\quad + \Omega^2 \sum_{m=-\infty}^{\infty} \{\exp[+q(m) - q(m-1)] - 1\} \\ &\quad - J \sum_{m=-\infty}^{\infty} [g_+(m)g_-(m-1) - g_+(m)g_-(m)], \end{aligned} \tag{2.8}$$

as well as the total charge of the pseudoexcitonic subsystem,

$$C = \sum_{m=-\infty}^{\infty} g_+(m)g_-(m), \tag{2.9}$$

and the total momentum of the vibrational subsystem,

$$P = \sum_{m=-\infty}^{\infty} p(m). \tag{2.10}$$

III. BASIC FEATURES OF GAUGE-COUPLED PSEUDOEXCITON-PHONON NONLINEAR DYNAMICAL SYSTEM

It can be readily verified that the set of semidiscrete equations under consideration (2.1)–(2.4) corroborates its canonical dynamical formulation

$$dg_+(n)/d\tau = -\partial H/\partial g_-(n), \tag{3.1}$$

$$dg_-(n)/d\tau = +\partial H/\partial g_+(n), \tag{3.2}$$

$$dp(n)/d\tau = -\partial H/\partial q(n), \tag{3.3}$$

$$dq(n)/d\tau = +\partial H/\partial p(n), \tag{3.4}$$

with the Hamiltonian function H specified in the previous section (2.8). Considering the expression (2.8) for the Hamiltonian function H we clearly disclose the origin of the interaction between the involved subsystems as a sort of gaugelike coupling known from the field theory [19,20]. Evidently, this type of coupling is principally distinct from that exhibited by the Davydov-Kyslukha model (1.1)–(1.5). On the other hand, the parts performed by the involved subsystems in the gauge coupling typifying our Hamiltonian function (2.8) turn out to be opposite to those performed by the involved subsystems in the gauge coupling typifying the famous polaron Hamiltonian function considered by Lee, Low, and Pines [21,22].

Another unusual feature of our dynamical system (2.1)–(2.4) is the pseudoexcitonic form of intersite resonant coupling between the intrasite excitations, which proved to be distinct from the standard form of intersite resonant coupling typical of molecular excitons [14,23]. To elucidate this statement one simply should compare the structures of resonant coupling terms $-J\sum_{m=-\infty}^{\infty}[g_+(m)g_-(m-1) - g_+(m)g_-(m)]$ and $-\mathrm{J}\sum_{m=-\infty}^{\infty}[\psi^*(m)\psi(m+1) + \psi^*(m)\psi(m-1)]$ in expressions (2.8) and (1.1) for the Hamiltonian functions H and H, respectively.

The system under study (2.1)–(2.4) is proved to exhibit symmetry under space and time reversal ($\mathcal{PT}$ symmetry), implying that the transformed field functions $\mathrm{g}_+(n) \equiv \mathrm{g}_+(n|\tau)$, $\mathrm{g}_-(n) \equiv \mathrm{g}_-(n|\tau)$ and $\mathrm{p}(n) \equiv \mathrm{p}(n|\tau)$, $\mathrm{q}(n) \equiv \mathrm{q}(n|\tau)$, defined as

$$\mathrm{g}_+(n|\tau) = g_-(-n|-\tau)\exp(+\theta), \tag{3.5}$$

$$\mathrm{g}_-(n|\tau) = g_+(-n|-\tau)\exp(-\theta), \tag{3.6}$$

and

$$\mathrm{p}(n|\tau) = +p(-n|-\tau), \tag{3.7}$$

$$\mathrm{q}(n|\tau) = -q(-n|-\tau), \tag{3.8}$$

are governed by the same set of equations as that (2.1)–(2.4) for the original field functions $g_+(n), g_-(n)$ and $p(n), q(n)$. Here, θ is an arbitrary constant parameter.

Presently the $\mathcal{PT}$-symmetric models become increasingly applicable in physical sciences [24–28] inasmuch as they permit to obtain physically meaningful results without invoking the rather restrictive condition of Hermiticity [24,25,27]. This trend is inspired by current progress in the fabrication and treatment of so-called metamaterials [29–38].

In order to estimate the potential place of our nonlinear dynamical system (2.1)–(2.4) in modeling the excited states of appropriate metamaterials it is reasonable to perform its low-amplitude analysis relying upon the following linearized equations,

$$dG_+(n)/d\tau = J\,G_+(n+1) - J\,G_+(n), \tag{3.9}$$

$$dG_-(n)/d\tau = J\,G_-(n) - J\,G_-(n-1), \tag{3.10}$$

$$d^2Q(n)/d\tau^2 = \Omega^2[Q(n+1) - 2Q(n) + Q(n-1)]. \tag{3.11}$$

The final result of the low-amplitude analysis is as follows,

$$G_+(n) = G_+\exp[-\gamma(k)\tau]\cos[kn - \omega(k)\tau + \varphi_+], \tag{3.12}$$

$$G_-(n) = G_-\exp[+\gamma(k)\tau]\cos[kn - \omega(k)\tau + \varphi_-], \tag{3.13}$$

$$Q(n) = Q\cos[kn - \omega_0(k)\tau + \varphi], \tag{3.14}$$

where

$$\gamma(k) = 2J\sin^2(k/2), \tag{3.15}$$

$$\omega(k) = -J\sin(k), \tag{3.16}$$

$$\omega_0(k) = 2|\Omega\sin(k/2)|, \tag{3.17}$$

and the quasimomentum k is assumed to be a real-valued one. Here, all the newly involved parameters G_+, G_-, Q, φ_+, φ_-, φ are the real-valued constants.

Irrespective of the sign of the hopping parameter J, the one pseudoexcitonic component is exponentially growing in time while the other pseudoexcitonic component is exponentially decaying in time. Nevertheless, their product $G_+(n)G_-(n)$ is seen to be finite and appears as a sort of charge-density wave. Due to the robust balance between the gain factor $\exp[+|\gamma(k)|\tau]$ and the loss factor $\exp[-|\gamma(k)|\tau]$ the respective charge current $-JG_+(n+1/2)G_-(n-1/2)$ also appears as a physically meaningful quantity. Such a gain-loss balance is known to be the most desirable property of real $\mathcal{PT}$-symmetric systems [27,30,39].

On the other hand, the low-amplitude pseudoexcitonic submodes $\omega(k) - i\gamma(k)$ and $\omega(k) + i\gamma(k)$ are characterized by the common cyclic frequency $\omega(k)$ which demonstrates the dependence $-J\sin(k)$ on the quasimomentum k typical of one-dimensional Dirac metamaterials.

In view of its above described basic features the suggested semidiscrete nonlinear integrable system of gauge-coupled Dirac pseudoexcitons and Toda-like vibrations (2.1)–(2.4) can apparently be applicable to modeling the physical properties of Dirac metamaterials characterized by properly designed quasi-one-dimensional lattice superstructures.

IV. MUTUALLY $\mathcal{PT}$-CONJUGATED SYMMETRY BROKEN ANALYTICAL SOLUTIONS

Due to its complete integrability the nonlinear dynamical system of our interest (2.1)–(2.4) admits the exact solutions to be found in the framework of one or another well-developed integration technique. In this respect, the most straightforward approach to isolate a certain nontrivial analytical solution explicitly is proved to be the Darboux-Bäcklund dressing method [17,18].

In the course of searching for a physically meaningful solution to the system under study (2.1)–(2.4), we have revealed that the nontrivial result for the crop solution is obtainable by dressing the symmetry broken trivial seed solution within the Darboux-Bäcklund integration technique. Having applied the Darboux-Bäcklund method we have managed to obtain both irregular and regular solutions to our system (2.1)–(2.4) in closed analytical forms. Bearing in mind its potential physical applicability we prefer to present here the regular four-component analytical solution,

$$g_+(n) = g_+ - g_+\exp(-\nu)\frac{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)+1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)-1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)-1/2)]}, \tag{4.1}$$

$$g_-(n) = \frac{2\sigma g_-|\Omega|[\cosh(\nu)-\cosh(\mu)]\exp[\nu(n-y(\tau)+1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)+1/2)]}, \tag{4.2}$$

$$\begin{aligned} p(n) = \sigma\,\Omega&\frac{|\Omega|\cosh[\mu(n-x(\tau)+3/2)] + |g_+g_-|[2\cosh(\mu)-\exp(-\nu)]\exp[\nu(n-y(\tau)+1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)+1/2)]} \\ &-\sigma\,\Omega\frac{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|[2\cosh(\mu)-\exp(-\nu)]\exp[\nu(n-y(\tau)-1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)-1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)-1/2)]}, \end{aligned} \tag{4.3}$$

$$q(n) = q + \ln\left\{\frac{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)+1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)-1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)-1/2)]}\right\}, \tag{4.4}$$

with the running position coordinates $x(\tau)$ and $y(\tau)$ of two nonlinearly superposed waves specified by the formulas

$$\mu x(\tau) = -\sigma\,\tau\Omega\sinh(\mu) + \mu x(0), \tag{4.5}$$

$$\nu y(\tau) = +\sigma\tau\Omega[\cosh(\mu) - \exp(+\nu)] + \nu y(0). \tag{4.6}$$

Here, the free constant parameters g_+, g_-, $x(0)$, $y(0)$, and μ are assumed to be the real-valued ones, while the parameter ν is defined via the physical parameters J and Ω by the formula

$$\exp(+\nu) = \frac{|J|}{|\Omega|}. \tag{4.7}$$

In addition, the sign parameter

$$\sigma = \frac{J\Omega}{|J\Omega|} \tag{4.8}$$

has been introduced and the regularizing constraint relation

$$g_+g_-|\Omega| = -|g_+g_-|\Omega \tag{4.9}$$

has been adopted.

We see that each of the four components $g_+(n)$, $g_-(n)$, $p(n)$, $q(n)$ of the inspected solution (4.1)–(4.4) comprises the nonlinear superposition of two qualitatively distinct waves characterized by two physically distinct spatial scales $1/|\mu|$ and $1/|\nu|$ as well as by two distinct velocities $dx(\tau)/d\tau = -(\sigma\Omega/\mu)\sinh(\mu)$ and $dy(\tau)/d\tau = +(\sigma\Omega/\nu)[\cosh(\mu) - \exp(+\nu)]$. The first spatial scale $1/|\mu|$ can be regulated by the initial conditions attributed to the problem of nonlinear wave packet generation as that, for example, has been established by the theory of soliton generation in the noninertial spatially continuous version of the Davydov-Kyslukha model [40]. On the contrary, the second spatial scale $1/|\nu|$ is strictly prescribed by the system's physical parameters J and Ω according to the previously written expression (4.7).

Having analyzed the suggested four-component solution (4.1)–(4.4) we clearly observe that $g_+(n|\tau) \neq g_-(-n|-\tau)\exp(+\theta)$, $g_-(n|\tau) \neq g_+(-n|-\tau)\exp(-\theta)$, $p(n|\tau) \neq +p(-n|-\tau)$, $q(n|\tau) \neq -q(-n|-\tau)$. Thus, the $\mathcal{PT}$ symmetry within the inspected solution (4.1)–(4.4) is seen to be broken.

Fortunately, the above observation opens the door to initiate one more symmetry broken solution being $\mathcal{PT}$-conjugated to the one already considered (4.1)–(4.4). Precisely, relying upon the formulas of $\mathcal{PT}$ conjugation (3.5)–(3.8), we obtain

$$\mathrm{g}_+(n) = \frac{2\sigma g_-|\Omega|[\cosh(\nu)-\cosh(\mu)]\exp[+\theta-\nu(n-\mathrm{y}(\tau)-1/2)]}{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)-1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)-1/2)]}, \tag{4.10}$$

$$\mathrm{g}_-(n) = g_+\exp(-\theta) - g_+\exp(-\theta-\nu)\frac{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)-1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)-1/2)]}{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)+1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)+1/2)]}, \tag{4.11}$$

$$\mathrm{p}(n) = \sigma\,\Omega\frac{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)-3/2)] + |g_+g_-|[2\cosh(\mu)-\exp(-\nu)]\exp[-\nu(n-\mathrm{y}(\tau)-1/2)]}{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)-1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)-1/2)]}$$
$$-\sigma\,\Omega\frac{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)-1/2)] + |g_+g_-|[2\cosh(\mu)-\exp(-\nu)]\exp[-\nu(n-\mathrm{y}(\tau)+1/2)]}{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)+1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)+1/2)]}, \tag{4.12}$$

$$\mathrm{q}(n) = -q - \ln\left\{\frac{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)-1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)-1/2)]}{|\Omega|\cosh[\mu(n-\mathrm{x}(\tau)+1/2)] + |g_+g_-|\exp[-\nu(n-\mathrm{y}(\tau)+1/2)]}\right\}, \tag{4.13}$$

with the running position coordinates $\mathrm{x}(\tau)$ and $\mathrm{y}(\tau)$ specified by the formulas

$$\mu\mathrm{x}(\tau) = -\sigma\,\tau\Omega\sinh(\mu) - \mu x(0), \tag{4.14}$$

$$\nu\mathrm{y}(\tau) = +\sigma\tau\Omega[\cosh(\mu) - \exp(+\nu)] - \nu y(0). \tag{4.15}$$

The obtained four-component analytical solution (4.10)–(4.13) should be treated as the symmetry broken counterpart solution $\mathcal{PT}$-conjugated to the initiating one (4.1)–(4.4).

V. MONOPOLE, DIPOLE, AND THRESHOLD REGIMES OF SYSTEM'S DYNAMICS

In general, the product of two pseudoexcitonic amplitudes $g_+(n)g_-(n)$ is not obliged to be a positively defined function of its arguments n and τ. This property can be convincingly elucidated on an example of the first system's solution (4.1)–(4.4). For this purpose let us consider two alternative but algebraically equivalent expressions,

$$\begin{aligned} g_+(n)g_-(n) = {} & 4\,\sigma\,g_+g_-\,\Omega^2[\cosh(\nu)-\cosh(\mu)]\cosh(\nu/2)\cosh(\mu/2) \\ & \times\frac{\exp[\nu(n-y(\tau))]\cosh[\mu(n-x(\tau))]}{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)+1/2)]} \\ & \times\frac{\tanh(\nu/2)-\tanh(\mu/2)\tanh[\mu(n-x(\tau))]}{|\Omega|\cosh[\mu(n-x(\tau)-1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)-1/2)]}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} g_+(n)g_-(n) = {} & 2\,\sigma\,g_+g_-|\Omega|[\cosh(\nu)-\cosh(\mu)] \\ & \times\left\{\frac{\exp[\nu(n-y(\tau)+1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)+1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)+1/2)]}\right. \\ & \left. -\frac{\exp[\nu(n-y(\tau)-1/2)]}{|\Omega|\cosh[\mu(n-x(\tau)-1/2)] + |g_+g_-|\exp[\nu(n-y(\tau)-1/2)]}\right\}, \end{aligned} \tag{5.2}$$

for the product $g_+(n)g_-(n)$ of pseudoexcitonic field components $g_+(n)$ and $g_-(n)$.

Having analyzed these formulas (5.1) and (5.2) we are able to reveal two principally distinct regimes of pseudoexcitonic dynamics separated by the threshold condition $|\mu| = |\nu|$.

Thus, for the underthreshold values $|\mu| < |\nu|$ of the localization parameter μ, the sign of charge density $\rho(n) = g_+(n)g_-(n)$ (5.1) is preserved on the whole infinite spatial interval so that the total charge C (2.9) must be of an essentially nonzero value. Therefore, in the underthreshold region $|\mu| < |\nu|$ of the free parameter μ, the charge density of pseudoexcitons manifests itself as a spatially extended monopole.

In contrast, for the overthreshold values $|\mu| > |\nu|$ of the localization parameter μ, the sign of charge density $\rho(n) = g_+(n)g_-(n)$ (5.1) changes its sign in a single spatial position,

$$\overline{n}(\tau) = x(\tau) + \frac{1}{\mu}\,\mathrm{artanh}\left\{\frac{\tanh(\nu/2)}{\tanh(\mu/2)}\right\}, \tag{5.3}$$

running along the chain with a constant velocity $d\overline{n}(\tau)/d\tau = -(\sigma\Omega/\mu)\sinh(\mu)$. On the other hand, in this overthreshold region $|\mu| > |\nu|$ each of the two terms in curly brackets of the second expression (5.2) for the charge density $\rho(n) = g_+(n)g_-(n)$ is finite and quickly tends to zero at both spatial infinities. Moreover, the functional forms of these two terms differ only by the primitive translation along the spatial coordinate. Inasmuch as the signs before these terms are distinct we promptly conclude that the total charge of pseudoexcitons C (2.9) calculated on the considered charge density (5.2) with $|\mu| > |\nu|$ must be equal to zero. Hence, in the overthreshold region $|\mu| > |\nu|$ of the localization parameter μ the charge density of pseudoexcitons manifests itself as a spatially extended dipole.

Consequently, the threshold point $|\mu| = |\nu|$ should be treated as a sort of critical point where the entire system's dynamics described by the four-component symmetry broken solution (4.1)–(4.4) undergoes a substantial qualitative rearrangement caused by the crucial crossover between the monopole and dipole scenarios of the charge-density spatial distribution.

In the very threshold point $|\mu| = |\nu|$ the $g_-(n)$ component (4.2) of the pseudoexcitonic subsystem is vanished and the pseudoexcitonic mode is shrunk to a single $g_+(n)$ component (4.1). The surviving component $g_+(n)$ can be referred to as the pseudoexcitonic chargeless half mode since it is unable to maintain the nonzero value of the charge density (5.1). Moreover, the running position coordinates $x(\tau)$ and $y(\tau)$ calculated in the critical point $|\mu| = |\nu|$ according to formulas (4.5) and (4.6) are characterized by the same velocity $dx(\tau)/d\tau = -(\sigma\Omega/\nu)\sinh(\nu) = dy(\tau)/d\tau$. As a consequence, the simple manipulations with expressions (4.3) and (4.4) for the components $p(n)$ and $q(n)$ of a vibrational subsystem renormalize them into a two-component solution typical of the standard Toda model [41–43]. In other words, the evolution of a vibrational subsystem up to the mere renormalizing spatial shift turns out to be independent of the pseudoexcitonic chargeless half mode, although the evolution of a pseudoexcitonic chargeless half mode is still essentially dictated by the vibrational subsystem. This conclusion is also confirmed by the inspection of a basic semidiscrete nonlinear system (2.1)–(2.4) with the component $g_-(n)$ being compulsorily eliminated. The dynamical regime specified by the threshold value $|\mu| = |\nu|$ of the localization parameter μ can be referred to as the threshold dynamical regime.

From a physical standpoint the effect of a dipole-monopole alternative encompassing the dipole, monopole, and threshold regimes of the system's dynamics is caused by the interplay between the two waves nonlinearly superposed in each of four components of the inspected solution (4.1)–(4.4). The variability of this interplay is supported by the two principally distinct spatial scales $1/|\mu|$ and $1/|\nu|$ characterizing the two physically distinct origins of the involved superposed waves.

VI. DISCUSSION

The crossover phenomenon between the monopole and dipole regimes of nonlinear dynamics admitting the existence of a quite unusual dipole regime in the spatial distribution of pseudoexcitons is found to be the basic physical property inherent to the suggested integrable gauge-coupled pseudoexciton-phonon nonlinear system on a regular one-dimensional lattice (2.1)–(2.4). Here, it is worth noticing that the dipole-monopole crossover phenomenon has also been discovered in the integrable standard-coupled pseudoexciton-phonon nonlinear system on a regular one-dimensional lattice [17,18].

We tried to trace similar properties among the model systems of a nonlinear Schrödinger type considered by some other authors [44–48]. However, a close inspection of the above listed papers [44–48] shows that their term "dipole soliton" is actually attributed separately to each of two envelope functions related to the two complex conjugated components in a particular solution considered by the respective article. Evidently, the physically meaningful product of such two components is obligated to be a positively defined quantity serving for the positively defined local density. Unfortunately, this positively defined density has nothing to do with the dipolelike density distribution associated with zero total charge. Therefore, the term "dipole soliton" used at least in the above quoted articles [44–48] appears to be a physically inadequate or even a misleading one. In contrast, our model system (2.1)–(2.4) was shown to admit the solutions with a true dipole spatial distribution of relevant local density.

As for the terms "dipole-mode spatial solitons" and "dipole soliton-vortices" used in other articles on nonlinear optics [49,50], we are unable to analyze properly their physical adequacy due to the lack of analytical expressions for the respective solutions. Bearing in mind that the intensity of any optical beam is an essentially positive quantity, we think the term "dipole soliton" widely circulated in the field of nonlinear optics [51,52] to represent the pair of out-of-phase solitons such as vortex solitons and azimutons sounds somewhat controversial.

VII. CONCLUSION

Inspired by the physical applicability of the Davydov-Kyslukha soliton model we have suggested a gauge-coupled exciton-phonon nonlinear dynamical system settled on a regular one-dimensional infinite lattice and characterized by two physical parameters. The system is shown to be integrable in the Lax sense and consequently it admits exact analytical solutions. In addition, it possesses an infinite hierarchy of conserved quantities. The three basic physically important conserved quantities as well as the canonical Hamiltonian formulation of the system's dynamics have been explicitly presented. The system as a whole clearly demonstrates the symmetry under space and time reversal referred to as the $\mathcal{PT}$ symmetry. This property allows to generate another symmetry broken solution $\mathcal{PT}$-conjugated to its presumably known symmetry broken counterpart by a simple reversal of spatial and temporal variables. To realize this recipe one of the symmetry broken four-component solutions has been isolated in the framework of the Darboux-Bäcklund dressing technique, whose rather extended calculation procedure has been omitted for the sake of brevity. Each of the symmetry broken solutions has been presented in concise analytical form with several strictly defined physical parameters. Each solution clearly demonstrates the pronounced mutual influence between the constituent subsystems via the essentially nonlinear superposition of two principally distinct types of traveling waves characterized by two spatial scales and two velocities of two distinct physical origins. We have thoroughly analyzed one of the obtained four-component symmetry broken solutions and observed the remarkable threshold phenomenon consisting in a crossover between the monopole and dipole regimes being the most pronounced in the dynamical behavior of the pseudoexcitonic subsystem. Here, we would like to stress that the threshold value $|\nu|$ of the localization parameter $|\mu|$ is determined exclusively by the system's physical parameters J and Ω via formula (4.7). The threshold dynamical regime corresponding to the threshold value of the localization parameter is characterized by a partial splitting of the involved pseudoexcitonic and vibrational subsystems giving rise to the formation of a chargeless pseudoexcitonic half mode.

In view of its unusual and somewhat unexpected properties the suggested semidiscrete nonlinear integrable system (2.1)–(2.4) can apparently be applicable for modeling the nonlinear properties of appropriate $\mathcal{PT}$-symmetric metamaterials, demonstrating the strict gain-loss balance in the temporal evolution of their excited states. As we have already mentioned, such a gain-loss balance is known to be the most remarkable characteristic of already existing $\mathcal{PT}$-symmetric metamaterial systems [27,30,39]. Thus, one of the items of our present research is to encourage the fabrication of different regular quasi-one-dimensional metamaterial lattice superstructures distinguished by the $\mathcal{PT}$ symmetry, gain-loss balance and specific nearest-neighbor interactions of their effective excitations [see expression (2.8) for the model Hamiltonian function H].

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