Transient solution of stochastic oscillators with both odd and even nonlinearity under modulated Gaussian white noise

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This study presents an improved solution procedure for determining the transient probability density function (PDF) solutions of the nonlinear oscillators with both odd and even nonlinearity under modulated random stimulation, which is an extension of the exponential-polynomial-closure approach. An evolutionary exponential-polynomial function with time-varying undetermined variables is considered as the transient probabilistic solution. By selecting a set of independent evolutionary base functions spanning a R^n space as weight functions, a set of ordinary differential equations can be formulated by integrating the weighted residual error. The undetermined variables can be determined numerically by solving those ordinary differential equations. Three numerical examples illustrate that the improved solution procedure can acquire the transient probabilistic responses of the stochastic dynamic systems effectively and efficiently even in the PDF solution tails when compared with Monte Carlo simulation. Moreover, the results indicate that the PDF solutions of the oscillators are asymmetrical at their nonzero means due to the influence of even nonlinearity. The nonstationary behaviors of the system responses are also investigated along with the behavior of modulated Gaussian white noise.

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I. INTRODUCTION

In the past several decades, the random stochastic oscillators subjected to white noise have been investigated by many researchers due to their broad applications in many fields of science and engineering [1–4]. However, most studies focus on stationary solutions, but the structure may break long before the structural responses arrive at stationary state in extreme conditions, such as seismic ground motion [5]. The main feature is that it only occurs for a short duration, which could result in tremendous losses during this duration. Therefore, modeling the stimulations by nonstationary processes in those cases is more appropriate and realistic. The nonstationary process is generally modeled by the product of a time-varying envelope function and a stationary process [6–8]. Several envelope functions such as triangular and exponential forms were introduced [9], which have been employed in the past to simulate the real circumstances when the duration of occurrence needs to be considered. On the other hand, both odd and even nonlinear terms exist in the oscillators frequently due to the influence of both static and dynamic loads [10]. When the stochastic dynamic systems are subjected to modulated Gaussian white noise, the transient system responses are a Markovian process [11,12], which is governed by a Fokker-Planck-Kolmogorov (FPK) equation. However, acquiring the analytical solution of the FPK equation about the nonlinear oscillator requires many restrictive conditions even for a stationary solution [13–17], which can rarely be satisfied for real problems. Therefore the transient probability density function (PDF) solutions need to be solved approximately for the nonlinear oscillators in many situations. The Monte Carlo simulation (MCS) is a straightforward numerical technique to acquire the PDF solutions of stochastic differential equations. However, large computational effort is needed to acquire an accurate solution when the probabilistic tails are required to be accurate in reliability analysis. The equivalent linearization (EL) is another frequently utilized approach for investigating the mean values and second moments of the oscillator [18,19], which is equal to the Gaussian closure approach when the oscillator is under purely external Gaussian white noise [20]. However, the result acquired by EL is a far from real solution when encountering strong nonlinearity in the oscillator. The path integral method is one of the earliest attempts [21–23], which was improved later by assuming the transition increment is Gaussian distribution [24,25]. The cell mapping method was proposed to improve the solution by fining cells [26]. The stochastic averaging method is widely utilized by analyzing the probability density of the system, which has slight damping and weak excitation [27–29]. In order to overcome the problem of negative values in the tails of PDF solutions given by the A-type Gram-Charlier expansion [30,31], an expansion of C-type form was adopted which can overcome the difficulty of negative PDF values for the single-degree-of freedom oscillator or two-dimensional FPK equation [32]. A more general method called the exponential-polynomial-closure (EPC) method was proposed simultaneously to address the PDF solutions of 2 ∼ 4-dimensional FPK equations [33]. It has been proved to be an efficient technique to acquire the PDF solutions of different stochastic systems effectively. Later, the EPC method was utilized to acquire the transition solutions of Duffing oscillator under Gaussian white noise [34]. However, no verification of the effectiveness of the presented solution was

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given by comparison to MCS. Some other methods are also frequently used, such as the maximum entropy method [35], multi-Gaussian closure methods [36,37], and neural network technique [38], which has been an interesting topic recently.

Until now, obtaining the transient PDF solutions of the nonlinear systems with both odd and even nonlinear terms under modulated random stimulation is still a challenging problem, though many real problems can be described by this type of oscillator. In this article, an evolutionary exponential-polynomial function with time-varying undetermined variables is considered as the transient probabilistic solution. Three nonlinear stochastic oscillators are analyzed to demonstrate the accuracy and efficiency of the improved solution procedure. Solving the problems by the improved solution procedure is also a preparation for extending the SSS-EPC technique to obtain the transient PDF solutions of the high-dimensional nonlinear systems in future work [39,40].

II. PROBLEM MODELING

Consider a stochastic oscillator with both odd and even nonlinearity and excited by modulated random noise,

$$\ddot{Z} + g_0(Z, \dot{Z}, t) = f(t)\eta(t), \tag{1}$$

in which $\ddot{Z}$, $\dot{Z}$, and Z are the oscillator responses, namely, the acceleration, velocity, and displacement, respectively; $g_0(Z, \dot{Z}, t)$ is a deterministic polynomial combination of the response $\mathbf{Z} = \{Z, \dot{Z}\}^T$; $f(t)$ is a time-varying function; $\eta(t)$ is the external random noise being Gaussian white noise; and the autocorrelation function is given as

$$E[\eta(t)\eta(t+\tau)] = K_0\delta(\tau), \tag{2}$$

where $E[\bullet]$ means the probabilistic average of $[\bullet]$, $\delta(\tau)$ is the Dirac delta function, and K_0 is the noise intensity. The time-varying intensity is given as [9]

$$K(t) = |f(t)|^2 K_0, \tag{3}$$

where $f(t)$ is the modulation function, which is given in the form as [6–9,41]

$$f(t) = \begin{cases} A_0(e^{-\alpha_1 t} - e^{-\alpha_2 t}), & t \in [0, +\infty) \\ 0, & t \in (-\infty, 0) \end{cases}, \tag{4}$$

in which A_0 is the scaling constant, and α_1 and α_2 ($\alpha_1, \alpha_2 > 0$) are the parameters that determine the envelope shape of $f(t)$. The maximum amplitude is given as

$$\max_t |f(t)| = 1, \tag{5}$$

in which max means the maximum value. It is noted that by selecting appropriate values of A_0, α_1, and α_2, the modulation function could be made to produce various evolutionary random noises. The modulated random noise with the parameters $A_0 = 3.2399$, $\alpha_1 = 0.1258$, and $\alpha_2 = 0.2988$ is shown in Fig. 1.

Equation (1) can be written in Stratonovich's type as

$$\begin{aligned} \dot{Z}_1 &= Z_2 \\ \dot{Z}_2 &= -g_0(Z_1, Z_2, t) + f(t)\eta(t), \end{aligned} \tag{6}$$

in which $Z_1 = Z$, $Z_2 = \dot{Z}$.

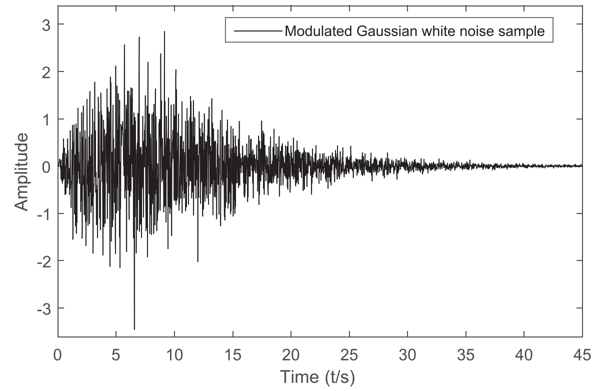

FIG. 1. Modulated random sample.

Subjected to modulated random noise, the transient joint PDF $p(\dot{z}, z, t)$ of the stochastic oscillator is governed by the following FPK equation [42]:

$$\frac{\partial p}{\partial t} + z_2\frac{\partial p}{\partial z_1} - \frac{\partial}{\partial z_2}[g_0(z_1, z_2, t)p] - \frac{1}{2}K(t)\frac{\partial^2 p}{\partial z_2^2} = 0. \tag{7}$$

$\hat{p}(\dot{z}, z, t)$ should fulfill the following conditions:

$$\begin{aligned} &\hat{p}(z, \dot{z}, t) \geqslant 0, \quad z, \dot{z} \in R^2 \\ &\lim_{z,\dot{z}\to\pm\infty} \hat{p}(z, \dot{z}, t) = 0, \\ &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{p}(z, \dot{z}, t)d\dot{z}dz = 1. \end{aligned} \tag{8}$$

III. THE IMPROVED EPC SOLUTION PROCEDURE

Since the analytical transient solution to Eq. (7) is still not available at the moment, the approximate transient PDF solution to Eq. (7) is assumed to be a time-dependent exponential-polynomial function, which is composed of the system response $\mathbf{z}$ and corresponding undetermined variables,

$$\hat{p}(\mathbf{z}, t; \mathbf{a}) = c_0 e^{Q_n(\mathbf{z}, t; \mathbf{a})}, \tag{9}$$

where c_0 is a normalization coefficient, and $\mathbf{a} = \{a_0(t), a_1(t), \ldots, a_m(t)\}$, which includes $m+1$ time-varying unknown variables; $Q_n(\mathbf{z}, t; \mathbf{a})$ is an evolutionary nth-degree polynomial function of $\mathbf{z} = \{z_1, z_2\}^T$ as

$$Q_n(\mathbf{z}, t; \mathbf{a}) = \sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1} a_r(t) z_1^{r_1-r_2} z_2^{r_2}. \tag{10}$$

The following residual error is yielded after substituting the approximate solution $\hat{p}(\mathbf{z}, t; \mathbf{a})$ into Eq. (7):

$$\begin{aligned} \Upsilon(\mathbf{z}, t; \mathbf{a}) &= \frac{\partial \hat{p}}{\partial t} + z_2\frac{\partial \hat{p}}{\partial z_1} - \frac{\partial g_0}{\partial z_2}\hat{p} - g_0\frac{\partial \hat{p}}{\partial z_2} - \frac{1}{2}K(t)\frac{\partial^2 \hat{p}}{\partial z_2^2} \\ &= \left\{\frac{\partial Q_n}{\partial t} + z_2\frac{\partial Q_n}{\partial z_1} - \frac{\partial g_0}{\partial z_2} - g_0\frac{\partial Q_n}{\partial z_2}\right. \\ &\quad \left. - \frac{1}{2}K(t)\left[\frac{\partial^2 Q_n}{\partial z_2^2} + \left(\frac{\partial Q_n}{\partial z_2}\right)^2\right]\right\}\hat{p} \\ &= \delta(\mathbf{z}, t; \mathbf{a})\hat{p} \end{aligned} \tag{11}$$

where

$$\delta(\mathbf{z},t;\mathbf{a}) = \frac{\partial Q_n}{\partial t} + z_2\frac{\partial Q_n}{\partial z_1} - \frac{\partial g_0}{\partial z_2} - g_0\frac{\partial Q_n}{\partial z_2} - \frac{1}{2}K(t)\left[\frac{\partial^2 Q_n}{\partial z_2^2} + \left(\frac{\partial Q_n}{\partial z_2}\right)^2\right]. \quad (12)$$

Generally if $\hat{p} \neq 0$, the only choice to fit Eq. (7) is $\delta(\mathbf{z},t;\mathbf{a}) = 0$. However, $\delta(\mathbf{z},t;\mathbf{a}) \neq 0$ in general because $\hat{p}$ is an approximation. Thus a group of evolutionary independent functions $H_i(\mathbf{z},t)$ spanning the R^2 space are introduced here. The projection of $\delta(\mathbf{z},t;\mathbf{a})$ on the evolutionary R^2 space is made to vanish so that Eq. (7) is satisfied in the weak sense as follows:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(\mathbf{z},t;\mathbf{a})H_i(\mathbf{z},t)d\mathbf{z} = 0, \quad i = 0, 1, 2, \ldots, m, \quad (13)$$

where the independent functions $H_i(\mathbf{z},t)$ are chosen as

$$H_i(\mathbf{z},t) = z_1^{i_1-i_2}z_2^{i_2}h(\mathbf{z},t), \quad (14)$$

in which $i_1 = 0, 1, \ldots, n$ and $i_2 = 0, 1, \ldots, i_1$. $h(\mathbf{z},t)$ is a particular option of the PDF solution from EL, which can be expressed as

$$h(\mathbf{z},t) = \frac{1}{2\pi\sigma_1(t)\sigma_2(t)\sqrt{1-\rho^2(t)}} \times \exp\left\{-\frac{1}{2(1-\rho^2(t))} \times \left[\frac{(z_1-\mu_1(t))^2}{\sigma_1^2(t)} + \frac{(z_2-\mu_2(t))^2}{\sigma_2^2(t)} - \frac{2\rho(t)(z_1-\mu_1(t))(z_2-\mu_2(t))}{\sigma_1(t)\sigma_2(t)}\right]\right\}, \quad (15)$$

where $\rho(t)$, $\mu_i(t)$, and $\sigma_i(t)$ are the time variants of the correlation coefficient, mean value, and standard deviation, respectively, at the instant t.

Considering that the means of the responses are nonzero due to the even nonlinearity in the system, the terms in the integration of Eq. (13) are written as

$$\int_{R^2} z_1^{m_1}z_2^{m_2}h(\mathbf{z},t)d\mathbf{z} = \int_{R^2}(y_1+\mu_1)^{m_1}(y_2+\mu_2)^{m_2}h(\mathbf{z},t)d\mathbf{z} = \sum_{e_1=0}^{m_1}\sum_{e_2=0}^{m_2}\binom{m_1}{e_1}\binom{m_2}{e_2}\mu_1^{m_1-e_1}\mu_2^{m_2-e_2}\int_{R^2} y_1^{e_1}y_2^{e_2}h(\mathbf{z},t)d\mathbf{z}, \quad (16)$$

where $y_i = z_i - \mu_i$ and

$$\binom{m_1}{e_1} = \frac{m_1!}{e_1!(m_1-e_1)!}. \quad (17)$$

Thus Eq. (13) can be simply integrated by using Isserlis's theorem as [43]

$$\int_{R^2} y_1^{e_1}y_2^{e_2}h(\mathbf{z},t)d\mathbf{z} = \begin{cases}\sum\prod E[(z_i-\mu_i)(z_j-\mu_j);t], & \text{if } e_1+e_2 \text{ is even}\\ 0, & \text{otherwise.}\end{cases} \quad (18)$$

A set of first-order stochastic differential equations about the unknown variables $\mathbf{a}(t)$ can be acquired from Eq. (13), and those evolutionary variables can be determined by the Runge-Kutta method numerically. In view that the system is subjected to modulated random noise, the time-varying PDF solutions of the oscillator responses can result. Meanwhile, the even nonlinearity of the nonlinear oscillator can induce the transient nonzero means of both displacement and velocity.

IV. NUMERICAL EXAMPLES

To identify the suitability of the improved solution procedure for acquiring the transient solutions of stochastic oscillators with both odd and even nonlinearity subjected to modulated Gaussian white noise, three examples are given in the following numerical analysis. For all three oscillators, the mean values of the displacement and velocity are nonzero due to the even nonlinearity, and there are no stationary solutions due to the modulation envelope approaching zero as time elapses. The polynomial order $n = 4$ is then utilized in the improved solution procedure to obtain the transient PDFs. The numerical results from the improved solution procedure are compared with EL and MCS to validate the advantages of the improved solution procedure for transition PDFs of the oscillators. It is worth noting that the case of the order $n = 2$ of the approximate polynomial function adopted in the improved solution procedure is a special situation which is equivalent to EL. MCS is utilized to acquire the numerical PDFs with a sample size of 5×10^7, and the numerical procedure is detailed in Appendix A.

A. Example 1

Considering the following nonlinear oscillator with odd and even nonlinearity in displacement under modulated Gaussian white noise,

$$\ddot{Z} + 2\xi\omega_0\dot{Z} + \omega_0^2 Z + \varepsilon_1 Z^2 + \varepsilon_2 Z^3 = f(t)\eta(t), \quad (19)$$

where $\xi = 0.25$, $\omega_0 = 1$, $\varepsilon_1 = 0.3$, and $\varepsilon_2 = 0.5$; $\eta(t)$ is Gaussian white noise with autocorrelation function $E[\eta(t)\eta(t+\tau)] = \delta(\tau)$; and the deterministic time-varying function $f(t)$ is given with $A_0 = 3.2399$, $\alpha_1 = 0.1258$, and $\alpha_2 = 0.2988$. The PDFs of z_1 and z_2 at $t = 0s$ are assumed as Gaussian with $\mu_{z_1} = 0$, $\mu_{z_2} = 0$, $\sigma_{z_1}^2 = 0.05$, and $\sigma_{z_2}^2 = 0.05$. The time step Δt for both the improved solution procedure and MCS is taken to be $0.02s$.

The improved solution procedure is then utilized with the moments from EL at time instant t to acquire the transition PDFs. The mean value of the oscillator displacement is nonzero due to the impact of even nonlinear term $\varepsilon_1 Z^2$. The time-varying PDFs of the displacement at $t = 3s$-40 s are shown in Figs. 2–7. In particular, the PDF corresponding to the maximum variance is demonstrated in Fig. 3. It is noted that the transition PDFs of the oscillator displacement obtained by the improved solution procedure are not symmetric due to the existence of even nonlinearity. It is observed that the transition PDFs become Gaussian distribution when the noise becomes weak enough as time elapses, which means the results obtained by the improved solution procedure, EL, and MCS overlap. Therefore, EL is enough in this situation.

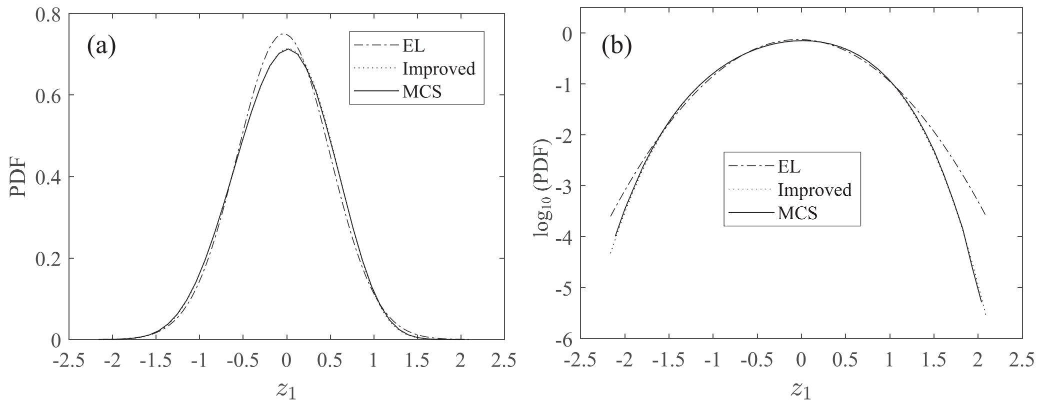

FIG. 2. PDF and $\log_{10}$(PDF) of z_1 at $t = 3$ s.

It is also noted that the probabilistic solutions of the system velocity by utilizing EL, MCS, and the improved solution procedure coincide, which means that the transition PDFs are always Gaussian. Therefore the probabilistic solutions of system velocity are not illustrated here.

The evolutionary mean and variance of z_1 are shown in Fig. 8. It is noticed that the mean value acquired by the improved solution procedure is nonzero since the nonlinear oscillator with the even nonlinear term is in displacement. The results obtained by the improved solution procedure agree well with MCS, even near the peak region of the evolutionary response process, which indicates that the improved solution procedure is suitable for analyzing the transition PDFs of stochastic oscillators with even nonlinearity in displacement. However, the results acquired by EL deviate significantly from those acquired by MCS and the improved solution procedure.

On the other hand, it is noted that the time required for acquiring the evolutionary PDFs at $t = 40$ s by utilizing the improved solution procedure is 79 s. However, the time required for obtaining the responses at $t = 40$ s by utilizing MCS is 1351 minutes on the same computer. The improved solution procedure increased the running efficiency by 1026 times.

B. Example 2

The following oscillator with odd and even nonlinear terms in both displacement and velocity under modulated random noise is given:

$$\ddot{Z}+2\xi\omega_0\dot{Z}+\omega_0^2 Z+\varepsilon_1\dot{Z}^2+\varepsilon_2\dot{Z}^3+\varepsilon_3 Z^2+\varepsilon_4 Z^3 = f(t)\eta(t), \tag{20}$$

where $\xi = 0.25$, $\omega_0 = 1$, $\varepsilon_1 = \varepsilon_3 = 0.3$, and $\varepsilon_2 = \varepsilon_4 = 0.5$; $\eta(t)$ is Gaussian white noise with autocorrelation function $E[\eta(t)\eta(t+\tau)] = \delta(\tau)$; and the other settings are given as the same as in example 1.

The improved solution procedure is then utilized with the moments from EL at time instant t to acquire the transition PDFs. The mean values of both system displacement and velocity are nonzero due to the influence of even nonlinear terms $\varepsilon_1\dot{Z}^2$ and $\varepsilon_3 Z^2$. The transition PDFs of the system responses are shown in Figs. 9–14. In particular, the transition PDFs corresponding to both the maximum variances are demonstrated in Fig. 10. Notably, the PDF solutions of both system displacement and velocity acquired by the improved solution procedure are not symmetric due to the existence of even nonlinearity. It is also observed that the transition PDFs of both the oscillator responses become Gaussian distribution when the noise becomes weak enough as time elapses, which means the results obtained by the improved solution procedure, EL, and MCS overlap. Therefore, utilizing EL is enough for this situation.

The evolutionary mean values and variances of both system displacement and velocity are shown in Fig. 15. It is noticed that both the means of the system responses acquired by the

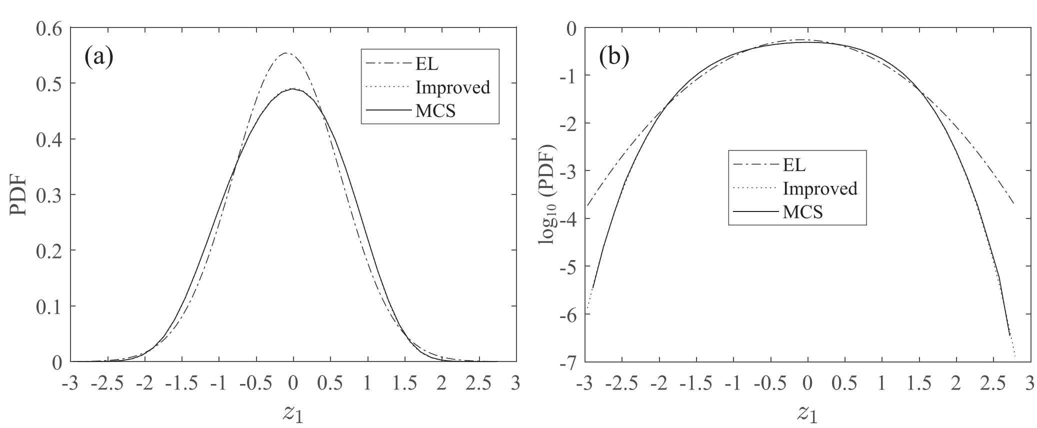

FIG. 3. PDF and $\log_{10}$(PDF) of z_1 at $t = 6.8$ s.

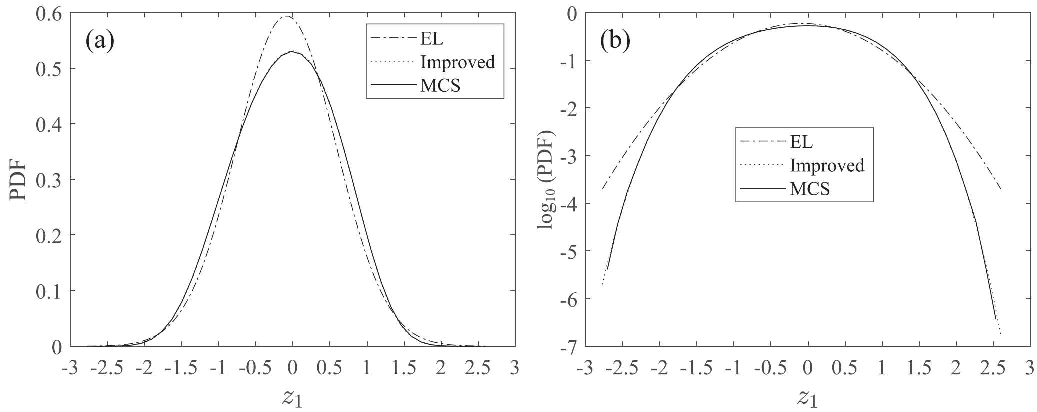

FIG. 4. PDF and $\log_{10}$(PDF) of z_1 at $t = 10$ s.

improved solution procedure are nonzero due to the even nonlinear terms in the oscillator. The results of both system displacement and velocity obtained by the improved solution procedure are still in good agreement with MCS. Due to the strong nonlinearity in this oscillator, the probabilistic tails of both system responses are not simulated enough by MCS, as shown in Figs. 9(b)–12(b) and Figs. 9(d)–12(d), which means the sample size 5×10^7 is not enough to prolong the probabilistic tails to the extendable ranges. Increasing the sample size can further result in a huge computational burden for MCS.

Notably, utilizing the improved solution procedure needs 80 s to acquire the evolutionary PDFs up to the time instant $t = 40$ s. However, utilizing MCS needs 1375 minutes to acquire the evolutionary PDFs up to the time instant $t = 40$ s in the same running environment. The improved solution procedure increased the efficiency by 1031 times.

C. Example 3

The last oscillator with complicated nonlinearity under modulated random noise is given,

$$\begin{aligned}\ddot{Z} + 2\xi\omega_0\dot{Z} + \omega_0^2 Z + \varepsilon_1\dot{Z}^2 + \varepsilon_2\dot{Z}^3 + \varepsilon_3 Z^2 \\ + \varepsilon_4 Z^3 + \gamma_1 Z^2\dot{Z} + \gamma_2\dot{Z}^2 Z \\ = f(t)\eta(t),\end{aligned} \tag{21}$$

where $\xi = 0.25$, $\omega_0 = 1$, $\varepsilon_1 = \varepsilon_3 = 0.3$, $\varepsilon_2 = \varepsilon_4 = 0.5$, and $\gamma_1 = \gamma_2 = 0.3$; $\eta(t)$ is Gaussian white noise with autocorrelation function $E[\eta(t)\eta(t+\tau)] = \delta(\tau)$; the other settings are given as the same as in example 1.

The improved solution procedure is then utilized with the moments from EL at time instant t to acquire the transition PDFs. The transition PDFs of the oscillator responses for time $t = 3$ s-40 s are shown in Figs. 16–21. Particularly, the transition PDFs related to both the maximum variances are demonstrated in Fig. 17. Notably, the transition PDFs of the oscillator responses acquired by the improved solution procedure are also not symmetric due to the existence of even nonlinear terms $\varepsilon_1\dot{Z}^2$ and $\varepsilon_3 Z^2$. It is observed that the transition PDFs of both the oscillator responses become Gaussian distribution when the noise becomes weak enough as time elapses, which means the results obtained by the improved solution procedure, EL, and MCS overlap. Therefore, utilizing EL is enough for this situation.

The evolutionary mean values and variances are shown in Fig. 22. It is noticed that both the mean values are also nonzero due to the even nonlinearity. The results of both system displacement and velocity obtained by the improved solution procedure are still in good agreement with MCS. Due to the strong nonlinearity in this oscillator, the probabilistic tails of both system responses are not simulated enough by MCS, as demonstrated in Figs. 16(b)–19(b) and Figs. 16(d)–19(d), which means the sample size 5×10^7 is also not enough

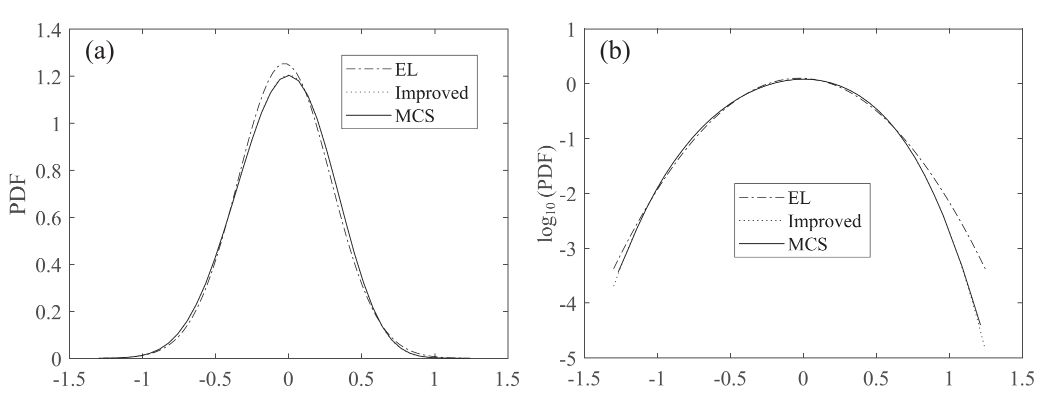

FIG. 5. PDF and $\log_{10}$(PDF) of z_1 at $t = 20$ s.

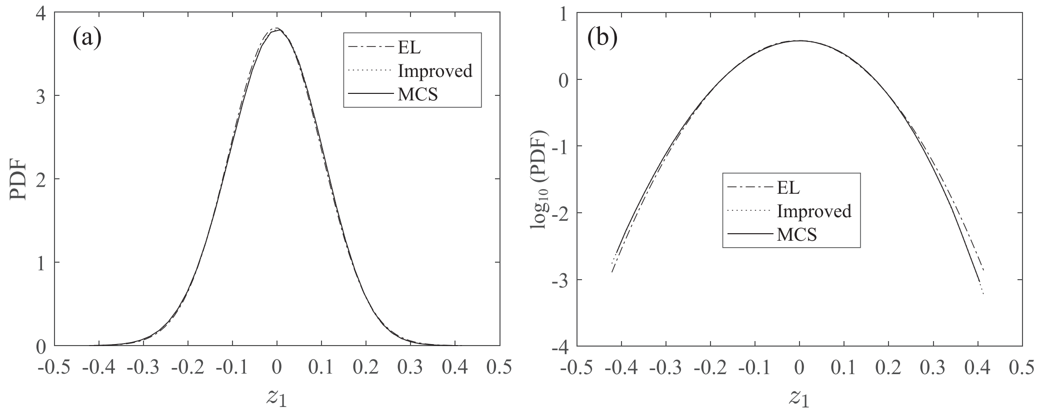

FIG. 6. PDF and $\log_{10}$(PDF) of z_1 at $t = 30$ s.

to prolong the tails to the extendable ranges. Increasing the sample size can further result in a huge computational burden for MCS.

In addition, it takes 80 s to acquire the evolutionary PDFs at $t = 40$ s with the improved solution procedure and 1383 minutes with MCS, respectively. The improved solution procedure increased the efficiency by about 1037 times.

V. CONCLUSIONS

In this paper, the transient PDF solutions of the stochastic oscillators with various odd and even nonlinear terms and subjected to modulated random noise are studied. The transition PDFs of three oscillators acquired by the improved solution procedure are compared with EL and MCS. It reveals the workability of the improved solution procedure for obtaining transient PDFs of the nonlinear dynamic systems under modulated random noise despite the odd and even nonlinearity. The results show that the probabilistic solutions and moments of the oscillator responses fluctuate as the modulation function fluctuates in a similar manner. In addition, the probabilistic solutions obtained by the improved solution procedure are in good agreement with MCS, even in the tails. Accurately predicting the extendable probabilistic tails of the oscillators with strong nonlinearity needs a large sample size, which can result in a heavy computational burden for MCS in return. In this situation, the computational efficiency by utilizing the improved solution procedure is increased significantly by over 1000 times without losing the PDF precision. In addition, the results indicate that the probabilistic solutions of the oscillators are asymmetric about their nonzero means due to the influence of even nonlinear terms.

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APPENDIX: THE PROCEDURE OF MONTE CARLO SIMULATION

The numerical process of Monte Carlo simulation for the oscillator under modulated random noise is performed in the following procedures:

(1) Create one sequence of pseudo-random numbers $\{Y\}_{1\times n}$-$N[0, 1]$ which is normally distributed.

(2) Assume the time step for Monte Carlo simulation is Δt. The stationary random stimulation $\{\eta(t)\}_{1\times n}$ is obtained by

$$\{\eta(t)\}_{1\times n} = \sqrt{\frac{K_0}{\Delta t}}\{Y\}_{1\times n}.$$

(3) Create a sequence of the modulation function $\{f(t)\}_{1\times n}$ with the same time step being Δt from its initial value being zero.

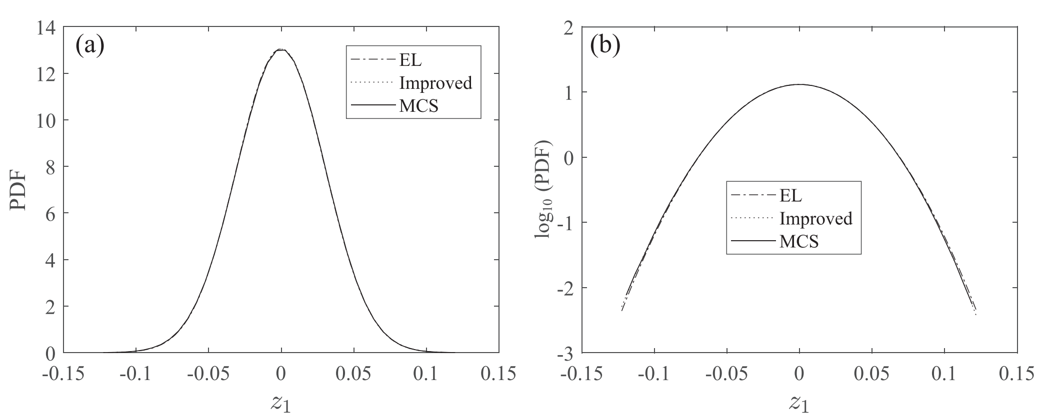

FIG. 7. PDF and $\log_{10}$(PDF) of z_1 at $t = 40$ s.

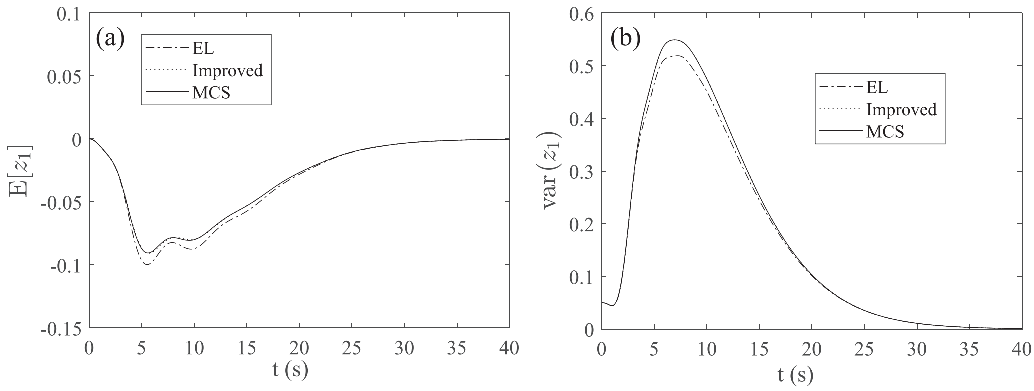

FIG. 8. Evolutionary (a) mean value and (b) variance of z_1 acquired by various approaches in example 1.

(4) The modulated Gaussian white noise at time instant t_i can be generated by multiplying the modulation function and the stationary random stimulation as $f(t_i)\eta(t_i)$.

(5) The fourth-order Runge-Kutta technique is adopted to obtain the numerical solution of the stochastic oscillator.

(6) Statistical analysis is conducted to obtain the mean, variance, and PDF of the oscillator responses.

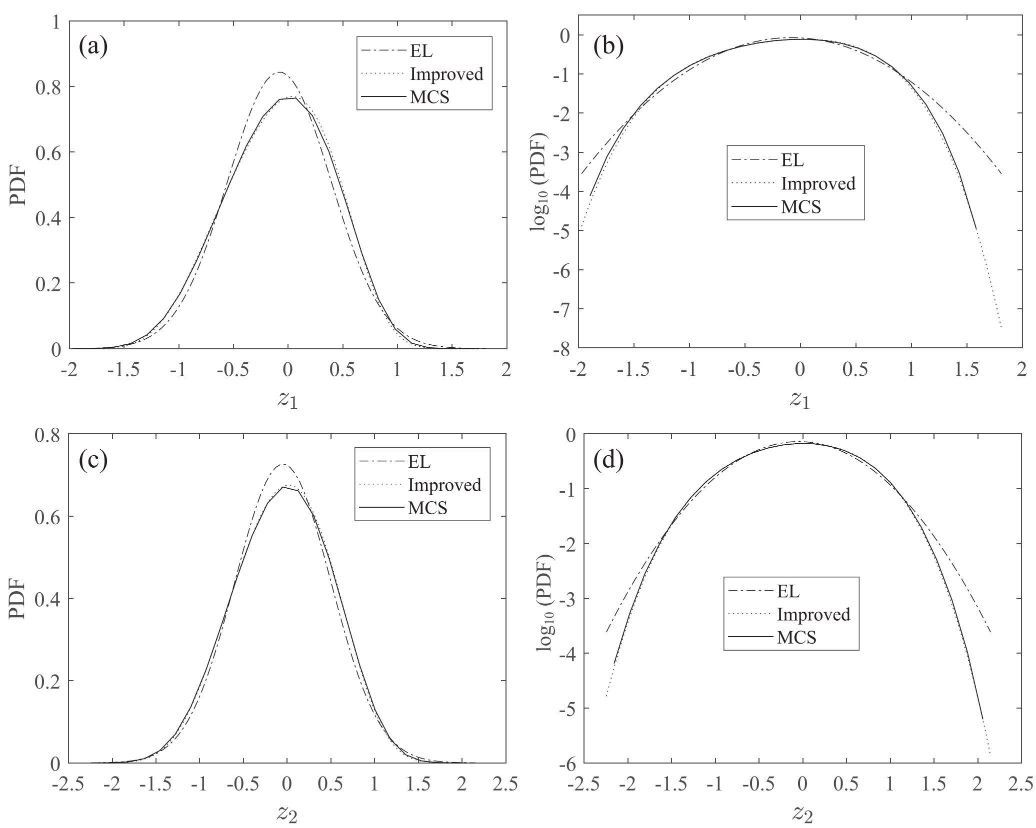

FIG. 9. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 3$ s.

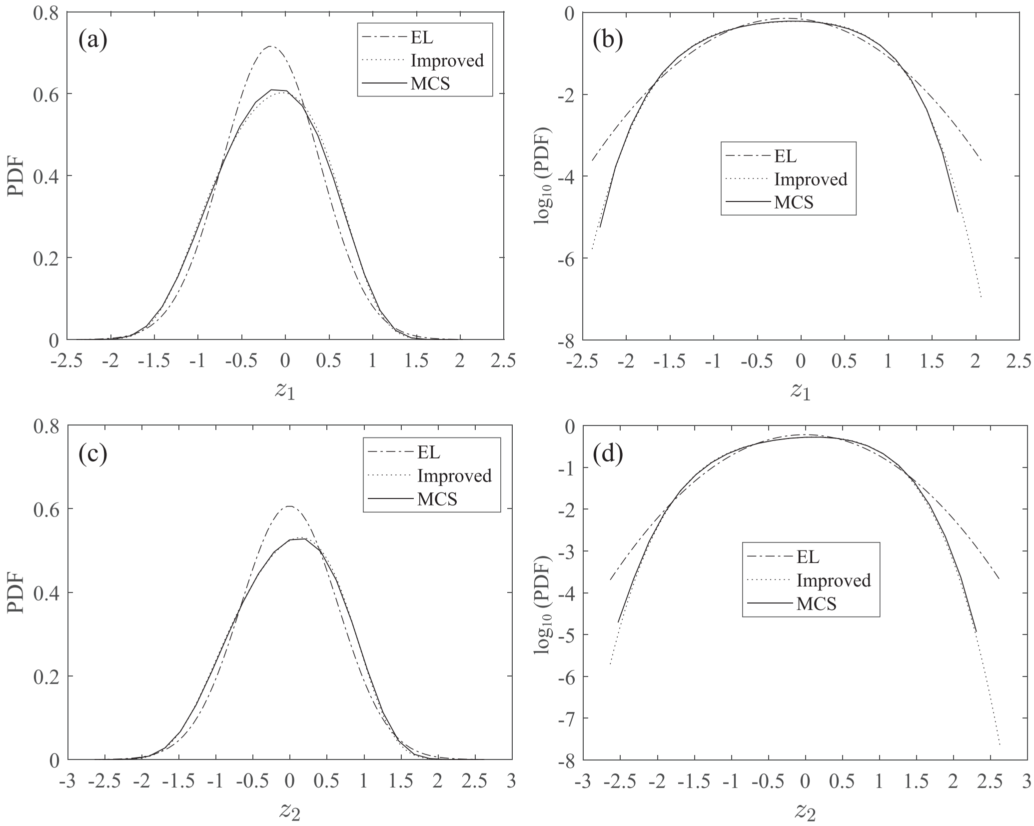

FIG. 10. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 5.5$ s.

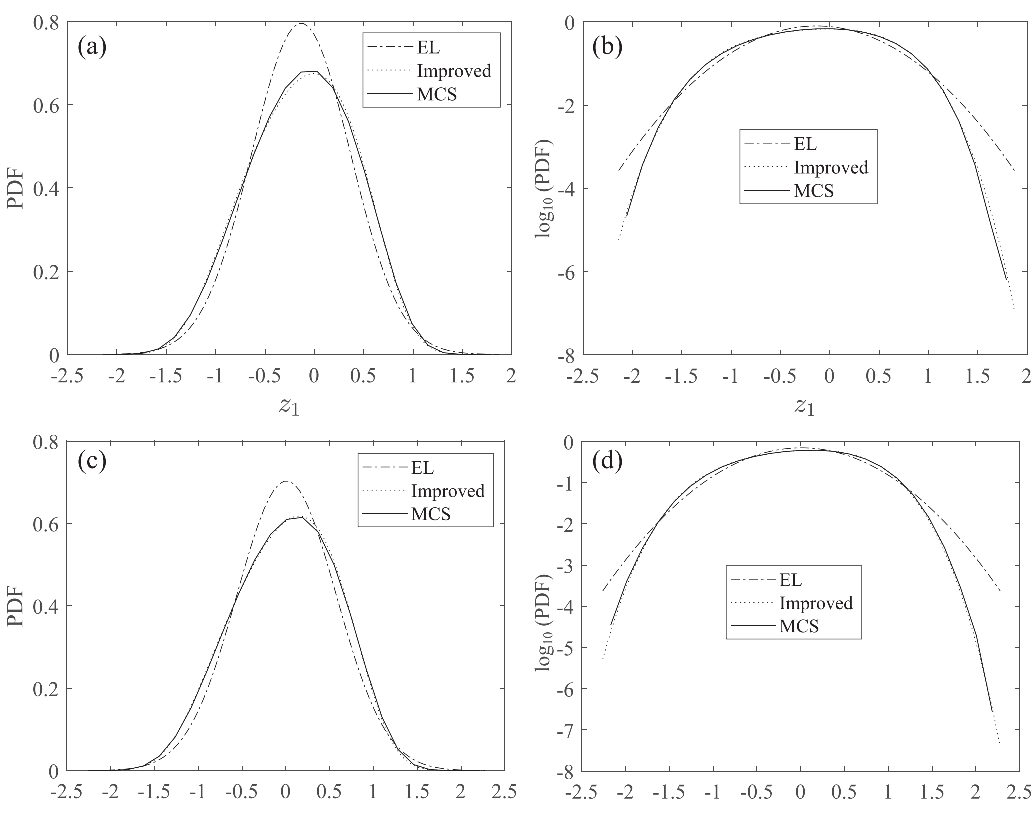

FIG. 11. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 10$ s.

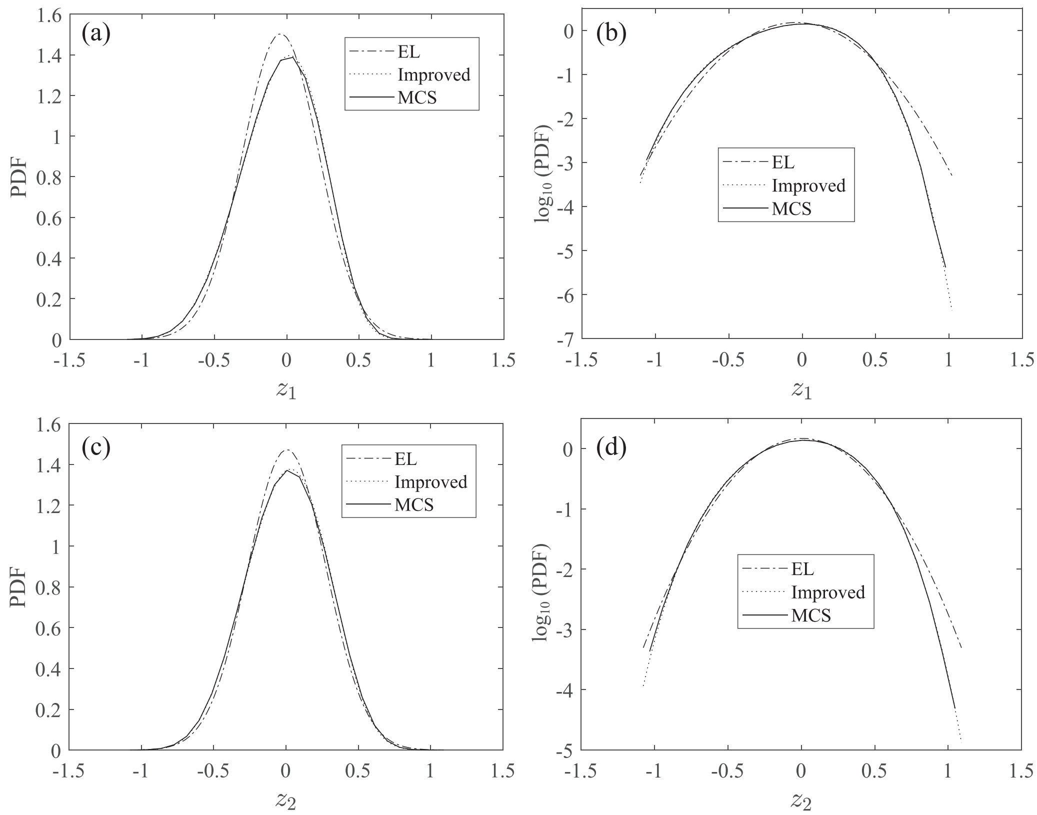

FIG. 12. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 20$ s.

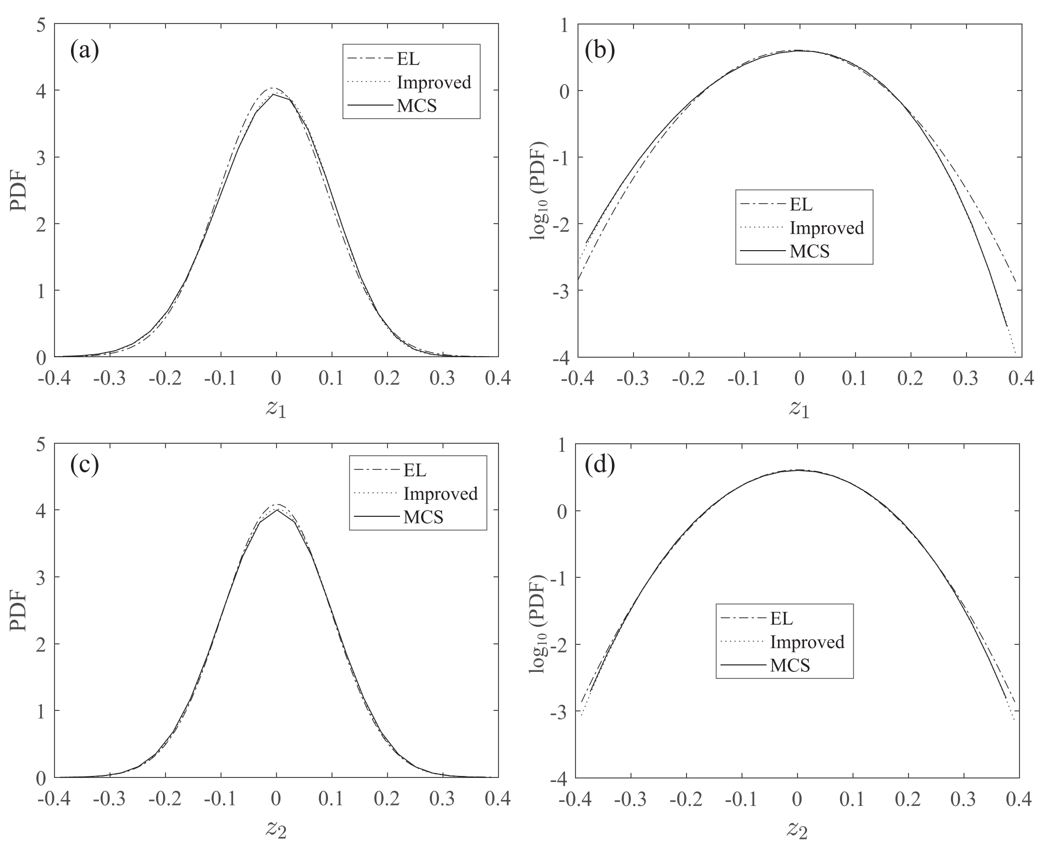

FIG. 13. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 30$ s.

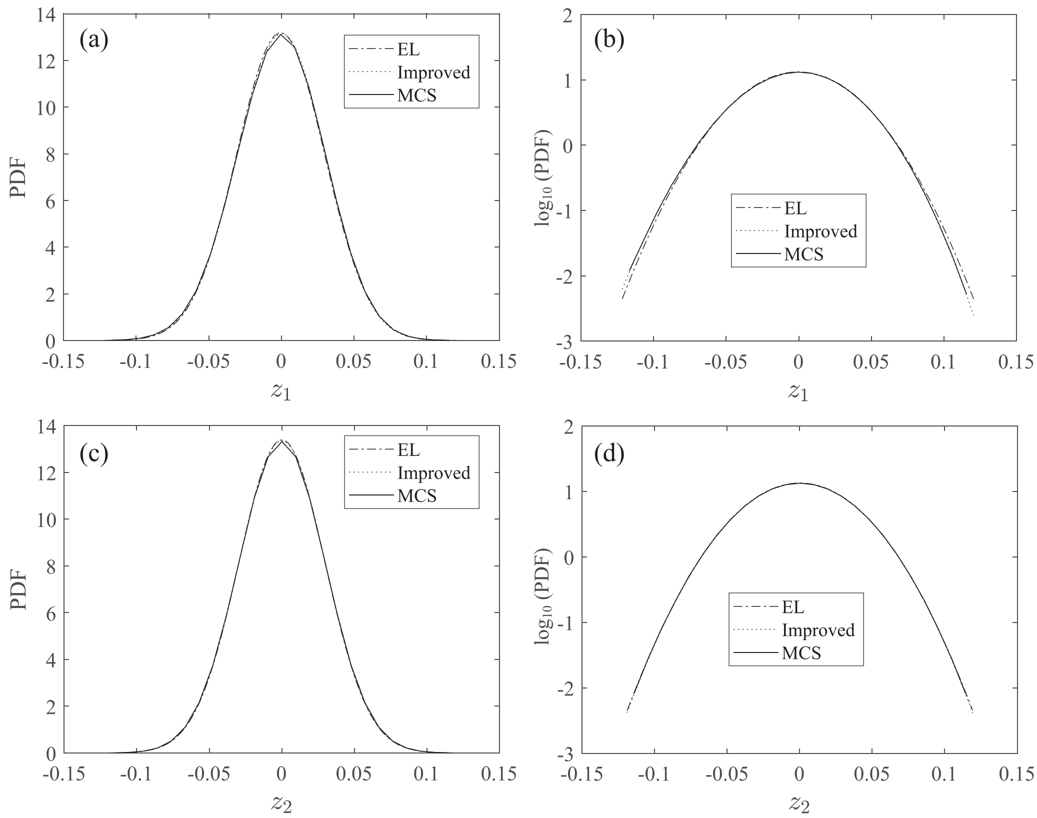

FIG. 14. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 40$ s.

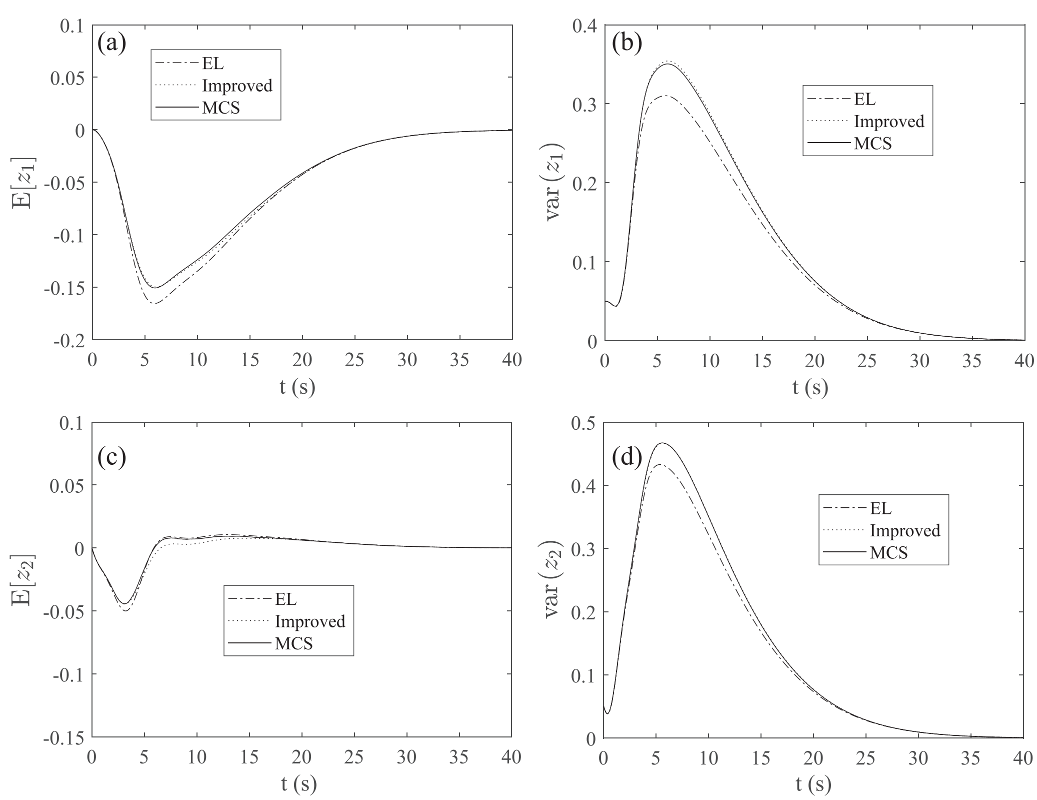

FIG. 15. Evolutionary (a), (c) mean values and (b), (d) variances of z_1 and z_2 acquired by various approaches in example 2.

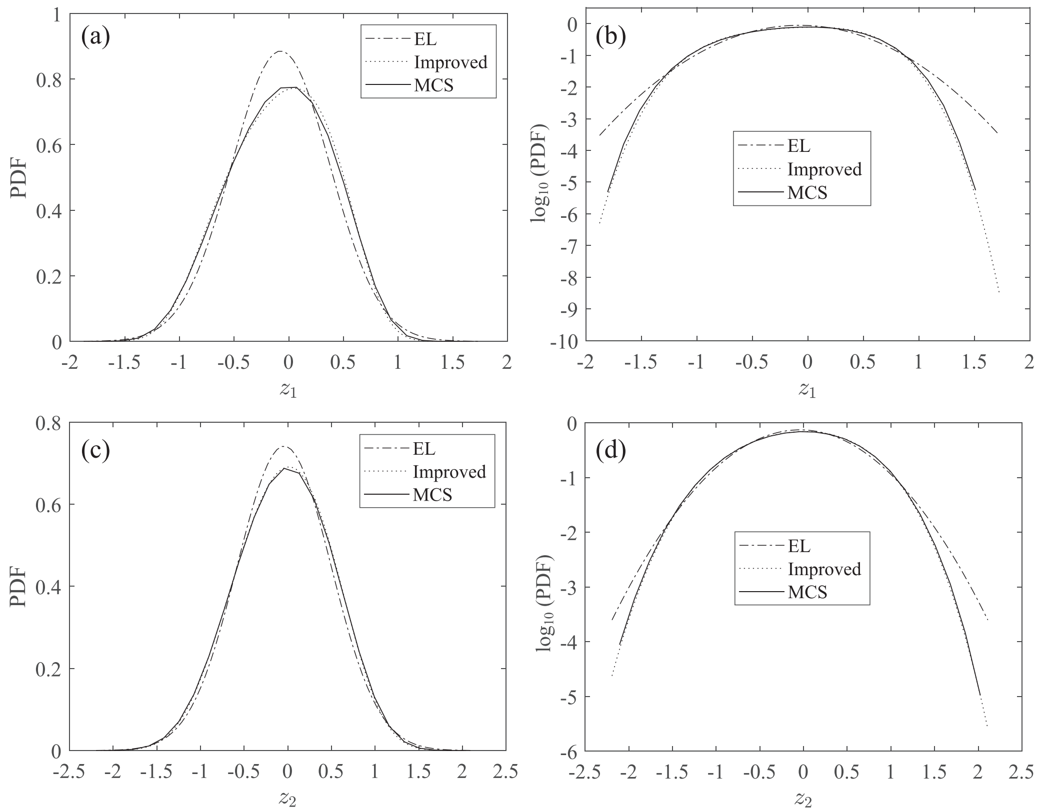

FIG. 16. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 3$ s.

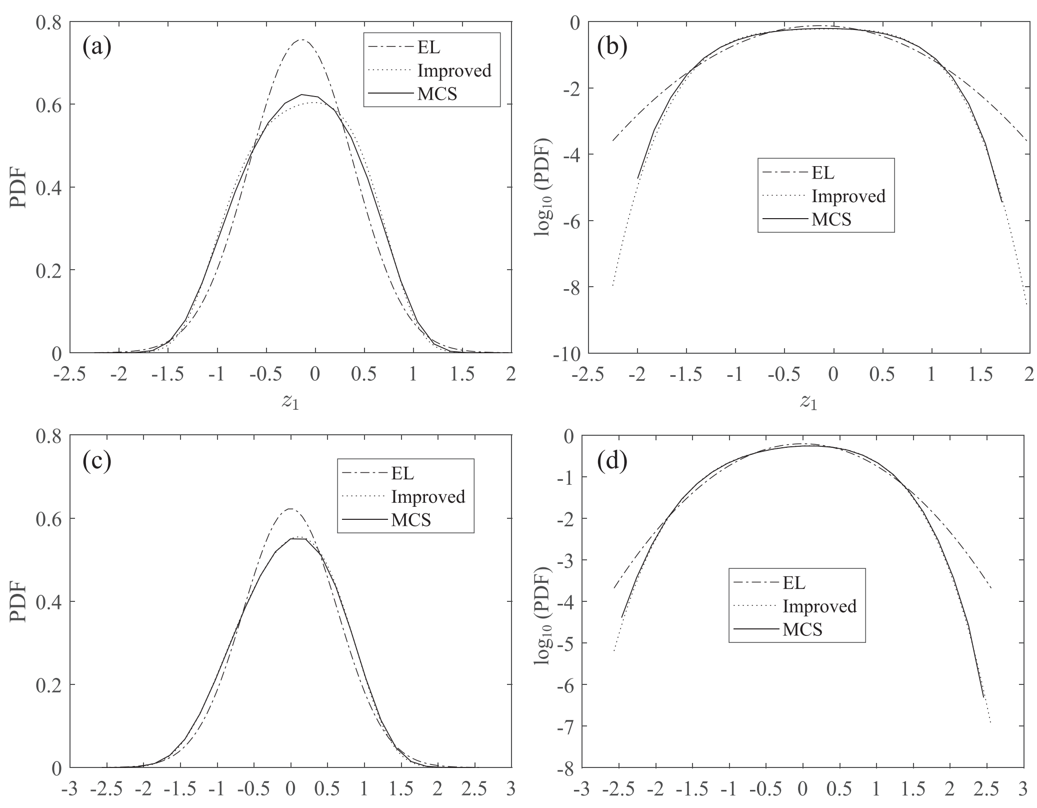

FIG. 17. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 5.5$ s.

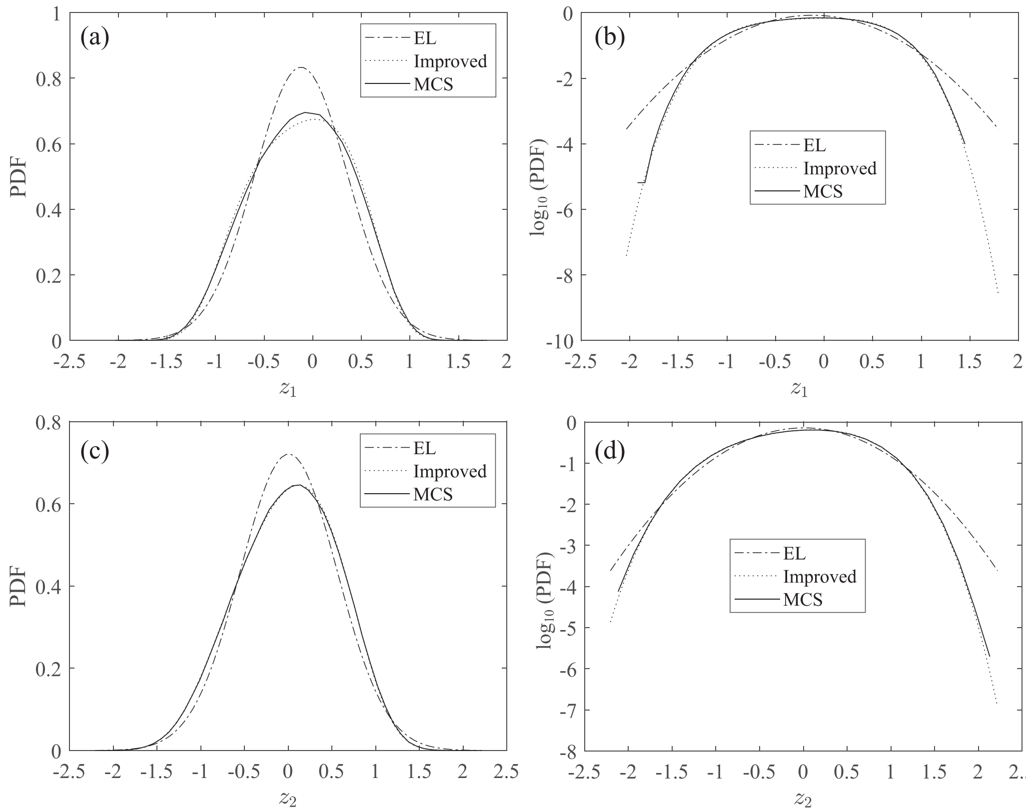

FIG. 18. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 10$ s.

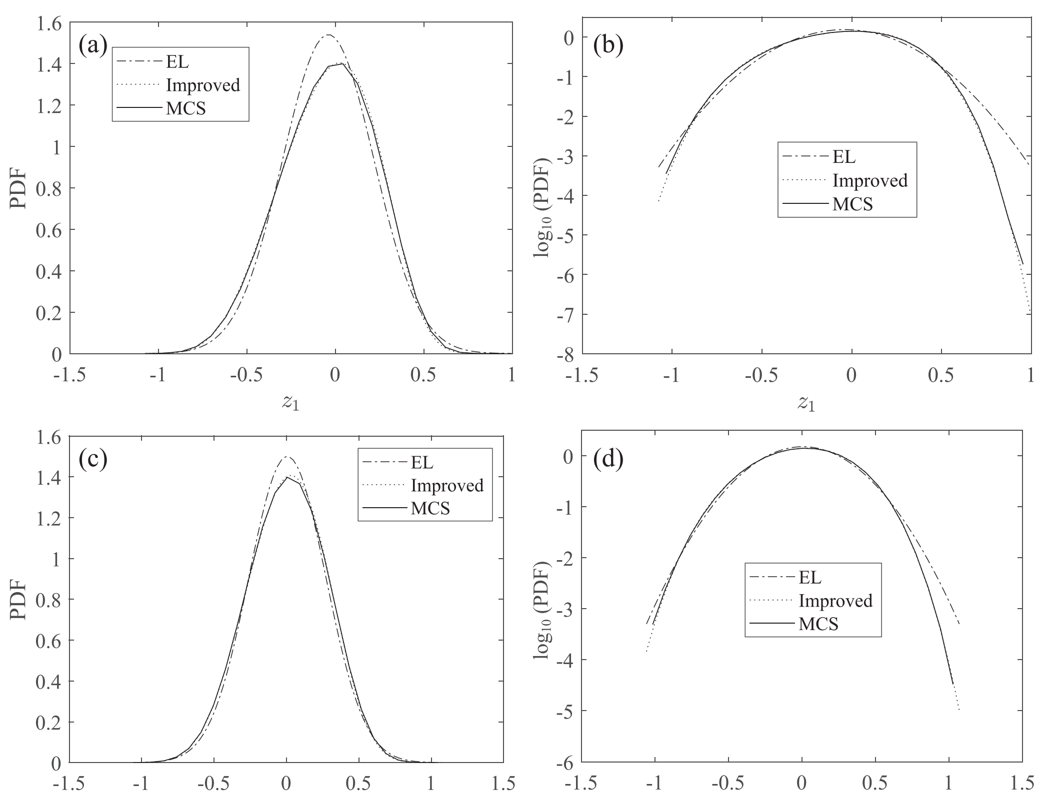

FIG. 19. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 20$ s.

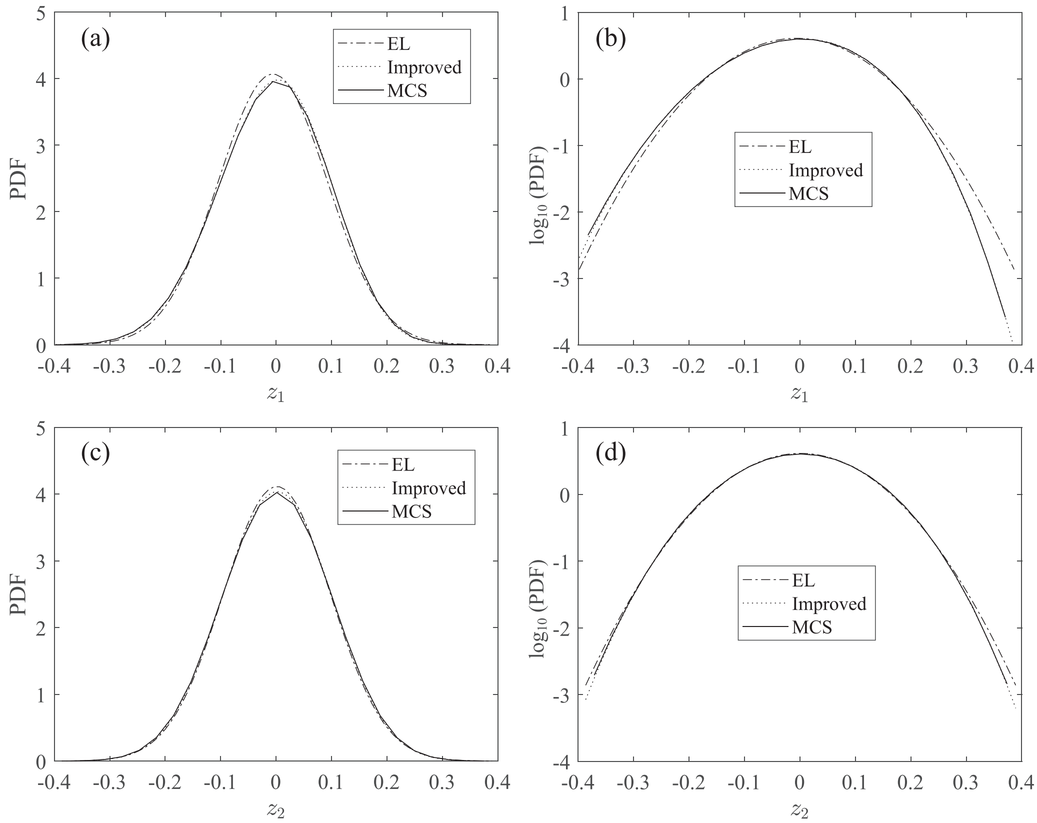

FIG. 20. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 30$ s.

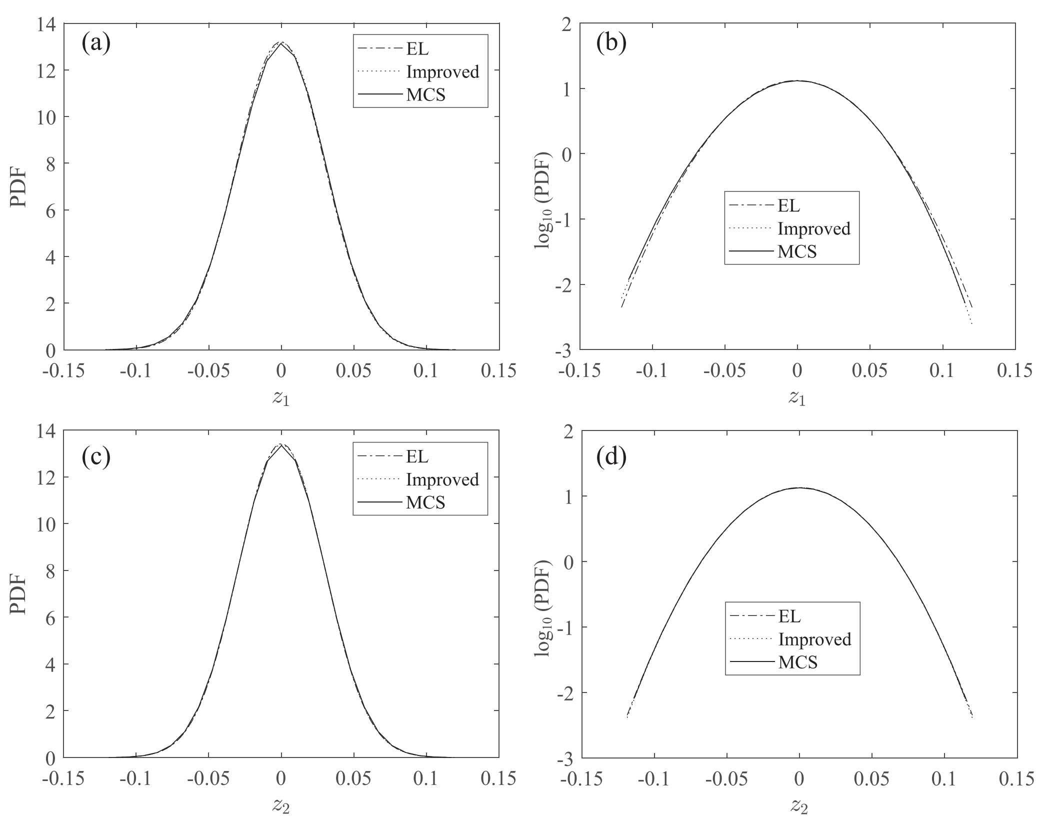

FIG. 21. PDFs and $\log_{10}$(PDFs) of z_1 and z_2 at $t = 40$ s.

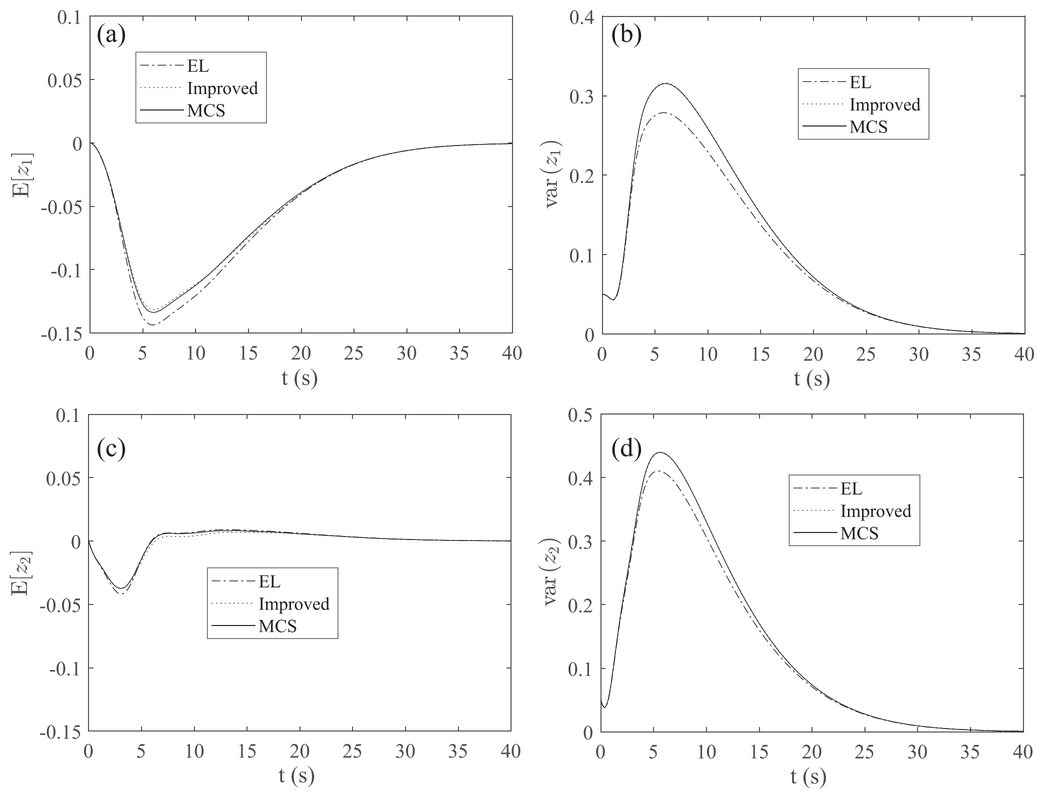

FIG. 22. Evolutionary (a), (c) mean values and (b), (d) variances of z_1 and z_2 acquired by various approaches in example 3.

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