Diversity-enhanced stability

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We give compelling evidence that diversity, represented by a quenched disorder, can produce a resonant collective transition between two unsteady states in a network of coupled oscillators. The stability of a metastable state is optimized and the mean first-passage time maximized at an intermediate value of diversity. This finding shows that a system can benefit from inherent heterogeneity by allowing it to maximize the transition time from one state to another at the appropriate degree of heterogeneity.

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I. INTRODUCTION

Many fields, from physics and chemistry to engineering and biology, are interested in first-passage phenomena [1]. Following Kramers' fundamental publication on kinetic reactions [2], much effort has been given to evaluate first-passage processes, or the related narrow-escape problems [3] in a variety of scenarios, utilizing both theoretical modeling and numerical analysis [4]. The primary question is to determine the mean first-passage time (MFPT), which is the time required for a particle (atom, molecule, cell, animal, or signal) to reach a certain state for the first time [1–4].

The MFPT is an essential metric for characterizing the escape process, which has been extensively investigated theoretically and quantitatively over the last few decades [5–9]. The average escape duration from metastable states in fluctuating potentials exhibits nonmonotonic behavior as a function of the noise intensity, with the existence of a clear-cut maximum. This is the noise-enhanced stability (NES) phenomenon: the stability of metastable states can be improved, and the metastable state's average life span rises nonmonotonically with noise strength. The NES has been evidenced in financial markets [10], ecological systems [11], magnetic systems [12], electrical circuits [13], chemical systems [14], and Josephson junctions [15]. Fiasconaro *et al.* [16–18] investigated the MFPT of a Brownian particle from an initial unsteady state in a metastable underdamped system and discovered the usual NES effect via MFPT with an evident hump (or resonantlike) structure or divergent behavior. In contrast to Fiasconaro *et al.* [16–18], the present research aims to study the enhancement of stability in the absence of dynamical noise in a network of coupled deterministic but heterogeneous oscillators.

In the absence of dynamical noise, intrinsic demographic noise in chemical and population models [19,20] as well as parameter diversity (or heterogeneity) [21,22] are sufficient to induce the resonant amplification of system response. In a biological network, diversity can induce an optimal oscillatory performance [23] or, can provoke the emergence of global oscillations from individually quiescent elements [24]. In electrochemical-oscillator experiments performed on a multielectrode array network [25], the dynamical stability optimally increases for intermediate levels of parameter heterogeneity (quenched disorder) even in the absence of dynamical noise. Motivated by the rather intriguing beneficial effect of heterogeneity [19–25], the present study investigates whether diversity (or quenched disorder) can induce a phenomenon *qualitatively similar* to NES in a *noiseless* system of coupled oscillators.

This paper demonstrates that network diversity (in the absence of dynamical noise) unexpectedly induces an increase of MFPT at intermediate disorder level in a system of coupled oscillators, similar to the NES shown in *single isolated* oscillators with external Brownian's noise [16–18]. In particular, we find a nonmonotonic behavior of the MFPT with a clear-cut maximum at a finite and optimal level of diversity. This might help an extended system increase the lifetime of a metastable state.

II. MODEL DESCRIPTION

The model is a network of N globally coupled units submitted to a cubic metastable potential energy [26,27]

$$\dot{x}_i = x_i^2 - \alpha_i^2 + \frac{C}{N}\sum_{j=1}^{N}(x_j - x_i), \tag{1}$$

where $x_i(t)$, $i = 1 \cdots N$, is the position of the ith unit at time t, and C is the coupling strength. The location and relative stability of the fixed points of the dynamics of an isolated unit i are modified by the parameter α_i. We assume that α_i takes fixed and independent values distributed according to $\alpha_i = \mu + \sigma U_i$, μ is a constant, and U_i is randomly and uniformly sampled in the interval $[-\varepsilon, +\varepsilon]$ for the ith unit. The parameter σ will be referred to as the *diversity* level.

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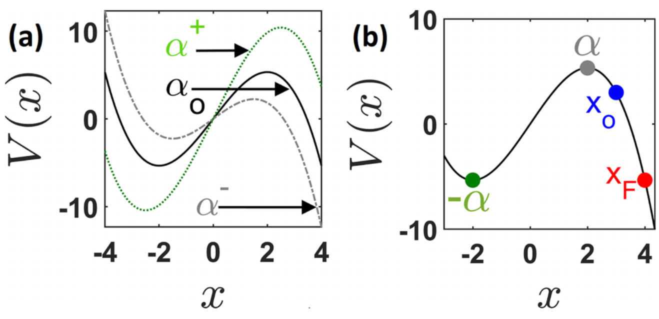

FIG. 1. Comparison of different shapes of the potential energy of uncoupled oscillators for $\alpha^+ = \mu + \sigma\varepsilon$, $\alpha^- = \mu - \sigma\varepsilon$, and $\alpha_0 = \mu$ (a). In (b) the initial (x_0) and the final state (x_F) are represented in comparison to the position of the equilibria $\pm\alpha$. The parameters used are $\sigma = 0.5$, $\mu = 2$, and $\varepsilon = 1$.

The metastable potential $V(x_i) = -x_i^3/3 + \alpha_i^2 x_i$ in Eq. (1) was used in Refs. [16–18,26,27] to study the escape time from the potential. The cubic potential with a metastable state has been employed as an archetypal model in the study of the escape dynamics of particles from a metastable state, since a metastable state with a smooth potential, in the neighborhood of a critical point, can be characterized by a cubic potential in most circumstances [16–18,26,27]. This cubic potential is shown in Fig. 1(a). Its profile has a local stable state at $x_i = -\alpha_i$, and an unstable state at $x_i = +\alpha_i$.

III. SIMULATION RESULTS

Following Refs. [16–18], after fixing a given target position x_F^i beyond the unstable equilibrium, the first-passage time $\tau_i(x_0^i, x_F^i)$, the time for the ith oscillator starting from an initial position x_0^i to reach x_F^i [Fig. 1(b)] is estimated through the fifth-order Runge-Kutta algorithm [28], with a fixed time step $\Delta t = 10^{-3}$. The initial conditions are $x_i(t = 0) = x_0^i = \kappa_1\alpha_i$. The final states are fixed as $x_F^i = \kappa_2\alpha_i$, with $\kappa_1 < \kappa_2$, and $\kappa_{1,2} > 1.0$. The initial conditions and the final states are therefore different depending on the value of the diversity level σ. Having set the initial conditions and final states, the average of times of the oscillators is estimated as the mean value of the passage times of the network's units: $N^{-1}\sum_{i=1}^{N}\tau_i(x_0^i, x_F^i)$. The MFPT ($\tau$) is subsequently estimated by averaging this time over 10^4 different independent realizations with independent random sets of the parameters α_i.

In Fig. 2, we plot the MFPT τ versus the diversity σ, for different values of the coupling constant C (as sketched in the legend). Strikingly, the mean escape time exhibits a *resonance structure*; viz., there exists a diversity σ_{cr} for which first passage proceeds slower than for all other diversity strengths. Two different regimes for the MFPT process can be observed, depending on the coupling strength C (Fig. 2). When $C = C_{cr} \simeq 1$, the resonancelike structure appears. Upon lowering $C < C_{cr}$ we notice a substantial rise of the passage time. For $C > C_{cr}$ the graph exhibits only a moderately growing MFPT followed by a decrease with higher diversity σ. In general, for substantially lower (or higher) coupling compared to C_{cr}, the passage time does not resonate but increases (or decreases) monotonically with increasing σ. In this sense C_{cr} represents indeed the optimal coupling strength for which

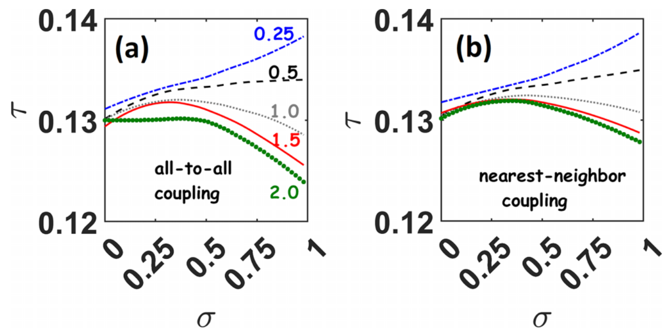

FIG. 2. Mean first-passage time τ from $x_0^i = 1.5\alpha_i$ to $x_F^i = 2.0\alpha_i$ evaluated from Eq. (1) versus the diversity level σ, and for different values of the coupling C (see the colored legends). The DES effect is observed for C around 1. The parameters used are $\mu = 2$, $\varepsilon = 1$, and $N = 100$. In (a) the structure is all-to-all global coupling, while in (b) the network has a ring structure with nearest network coupling and periodic boundary conditions.

resonancelike MFPT occurs. When the network structure (Fig. 2), and size (Fig. 3) are varied, the appearance of such a clear-cut optimum value of the diversity for maximum time duration is still observed. This strongly resembles the NES effect, ubiquitously found in different physical systems with time-dependent noise [10–18,26,27]. Since the system is deterministic, the phenomenon is diversity induced instead of noise induced; thus a diversity-enhanced stability (DES).

IV. MEAN-FIELD ANALYSIS

To study the effects of diversity analytically, the mean-field approach is applied by introducing the global variable $\chi(t) = N^{-1}\sum_{i=1}^{N} x_i$. Equation (1) can be rewritten as

$$\dot{x}_i = x_i^2 - \alpha_i^2 + C(\chi - x_i). \tag{2}$$

After averaging Eq. (2) over N oscillators,

$$\dot{\chi} = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\sum_{i=1}^{N}\alpha_i^2. \tag{3}$$

Following Refs. [21,22,29,30], we then define δ_i as the difference between x_i and χ, i.e., $x_i = \chi + \delta_i$. The

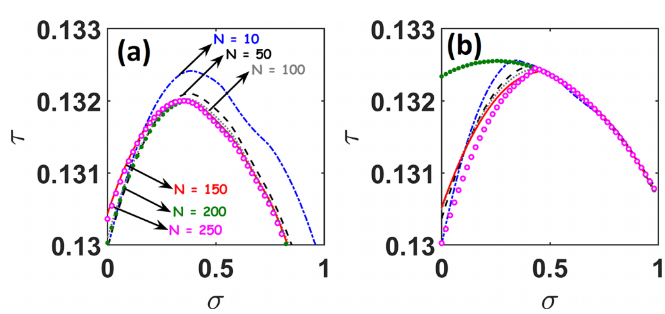

FIG. 3. Graphical representation of the mean first-passage time (τ) when the network diversity varies, with different network size (N) and structures. In (a) the structure is an all-to-all global coupling, while in (b) the network has a ring structure with nearest network coupling and periodic boundary conditions. For all the simulations, $x_0^i = 1.5\alpha_i$, $x_F^i = 2.0\alpha_i$, and $C = 1$.

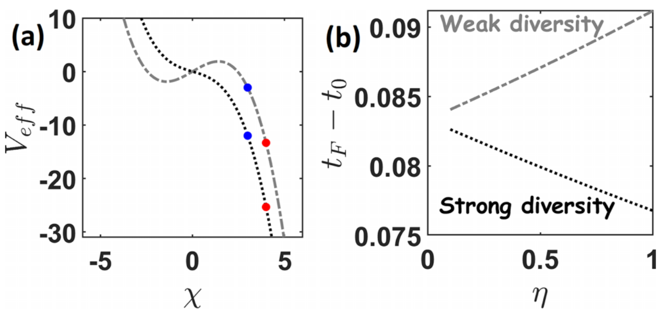

FIG. 4. (a) Graphical representation of the effective potential from Eq. (2), $V_{\text{eff}}(\chi) = -\chi^3/3 - (\xi - a)\chi$ for $\xi - a < 0$ (gray line) ($\xi - a = -2$) and $\xi - a > 0$ (black line) ($\xi - a = 1$). The blue and red dots are the starting and arrival points, respectively. (b) The mean first-passage time is evaluated from Eq. (3) versus η. The gray and black lines are obtained from the first and second analytical expressions of t_F, respectively, in Eq. (5). The parameters used are $\chi_0 = 1.5\mu$, $\chi_F = 2.0\mu$, and $\mu = 2$.

quantity $\delta_i = x_i - \chi$ is introduced to represent the trajectory deviation between x_i and the average activity of the system χ [21,22,29,30]. After development, $N^{-1}\sum_{i=1}^{N} x_i^2 = \chi^2 + 2M_1\chi + \xi$, with $M_1 = N^{-1}\sum_{i=1}^{N}\delta_i$, $\xi = N^{-1}\sum_{i=1}^{N}\delta_i^2$. Due to the effect of the diversity, it is assumed that $M_1 \simeq 0$ [21,22,29,30]. The parameter ξ will increase when diversity increases [21,22,29,30]. After setting $N^{-1}\sum_{j=1}^{N}\alpha_i^2 = a$, the ensemble dynamics can be therefore translated into

$$\dot{\chi} = \chi^2 + \xi - a. \tag{4}$$

The MFPT t_F obtained after integration of Eq. (4) is

$$\frac{-1}{\sqrt{|\eta|}}\left[\text{arctanh}\left(\frac{\chi_F}{\sqrt{|\eta|}}\right) - \text{arctanh}\left(\frac{\chi_0}{\sqrt{|\eta|}}\right)\right] + t_0, \quad \text{if } \xi < a,$$
$$\frac{1}{\sqrt{\eta}}\left[\arctan\left(\frac{\chi_F}{\sqrt{\eta}}\right) - \arctan\left(\frac{\chi_0}{\sqrt{\eta}}\right)\right] + t_0, \quad \text{if } \xi > a, \tag{5}$$

$\eta = \xi - a$. The variables χ_F and χ_0 are the average positions at the final (t_F) and initial time (t_0), respectively. It can be easily verified that t_F increases or decreases monotonically depending on the values of η [Fig. 4(b)]. Thus it highlights increasing (decreasing) MFPT for weaker (higher) diversity, suggesting the existence of a critical value ξ_{cr} (or σ_{cr}) separating the two monotonic regimes of MFPT variation with the increasing diversity. Despite the rough approximation [Eq. (4)], this nonmonotonic dependence on the diversity is in good qualitative agreement with the results of the numerical simulations of Eq. (1).

For $\sigma = 0$, ξ vanishes after an initial transient to wash out the effect of the possibly different initial conditions for the x_i's. For weaker diversity values of ξ ($\xi < a$), the global variable χ will move from the initial to the final point slowly. As the diversity increases, ξ increases and reaches higher values ($\xi > a$); the time spent, rather, decreases as the diversity increases. We then predict a resonance effect for intermediate values of the diversity for which τ will be maximum. A more satisfying theory explicitly linking ξ_{cr} (or σ_{cr}) and C for DES is left as an open question in this paper.

The energy barrier from the departure and arrival points [Fig. 4(a)] $V_{\text{eff}}(\chi_F) - V_{\text{eff}}(\chi_0)$ is considerably high in the lower diversity case compared to the higher diversity case. Therefore, for very low diversity ($\xi - a < 0$), the averaged state is subjected to a high potential barrier, as a consequence of the divergence of the MFPT in the limit $\sigma \to 0$. For increasing diversity ($\xi - a > 0$), the particle can escape out more easily, and the MFPT decreases. As the diversity reaches a value σ_{cr}, the concavity of the MFPT curves changes. Close to such a diversity, the MFPT process of the pseudoparticle is optimally slowed down.

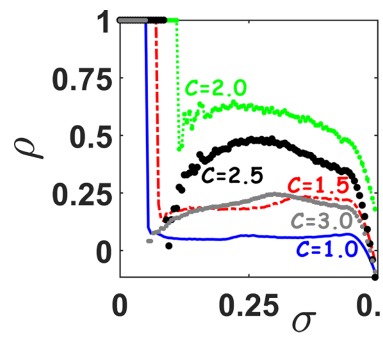

FIG. 5. Synchronization coefficient ρ of the MFTP (τ_i) all-to-all globally coupled oscillators i with diversity level σ. The size of the network is fixed at $N = 50$. The value of the coupling C used for each case is specified in the colored legends.

V. SYNCHRONIZATION ANALYSIS

To reveal a possible connection between the maximum MFPT and the network synchronization, the degree of spatial synchronization of the units is used as order parameter. To quantify in a systematic way the synchronization capabilities of the network, we computed a synchronization coefficient adapted from [31]

$$\rho = \frac{\langle \overline{\tau_i^2} \rangle - \langle \overline{\tau}_i \rangle^2}{\langle \tau_i^2 \rangle - \langle \tau_i \rangle^2}, \tag{6}$$

where $\overline{\cdots}$ is the average over nodes (oscillators) and $\langle \cdots \rangle$ is the average over the different realizations. When ρ tends to 1, the individual oscillators i have closer MFPT (τ_i) (synchronized passage time) whereas ρ reaching 0 means that the oscillators MFPTs are not synchronized. The synchronization quantifier is plotted in Fig. 5 as a function of σ. For weak diversity levels ($\sigma \simeq 0$), the starting (x_0^i) and final (x_F^i) state values as well as α_i are very close, therefore τ_i are identical leading to ρ closer to 1. When the diversity becomes different to 0 the synchronization becomes closer to 0. For intermediate values of σ where the DES occurs, the synchronization coefficient becomes optimal again. Beyond this critical value of σ, the coefficient ρ falls again toward 0. More interestingly, the optimal value of ρ does not increase linearly with the coupling C. When the coupling gradually increases, the synchronization increases, which is naturally expected. However, after a critical value of $C \simeq 2$, the maximal value deceases again. This suggests that during the occurrence of the DES, the diverse oscillator moves between the two states in a nearly closer time ($\rho \simeq 0.6$).

VI. DISCUSSION

Diversity is widely established to play a positive function in many coupled systems [19–25,29,32,33]. It can enhance collective behaviors such as synchronization, control, resonance, or (de)coherence [19–22,25,29,32,33]. While the original inspiring work of Fiasconaro *et al.* [16–18] on stochastic systems showed that noise can enhance stability, our results suggest that diversity can also enhance stability.

The present paper builds on many previous papers that have already identified (pure) diversity-induced effects [21–25]. Here, another example of such an effect, analogous to the NES effect [16–18], is provided. In contrast to the literature [21–25], this paper studies the MFPT, which is a microscopic quantity, rather than a quantity measured indirectly in a network of coupled oscillators, e.g., synchronization, coherence, activation, or amplification.

Using experimental systems consisting of coupled electromechanical, power grid, or electrochemical oscillators, the stability of the frequency-synchronization state is stabilized by the random oscillator heterogeneity [25,32,33]. Though the present study focused on the first-passage time from one state to another, it would be interesting to investigate it in a practical physical setup [25,32,33].

The investigation of the beneficial effects of diversity on synchronization was evidenced by several authors [25,32–34]. The present study shows that it is possible to synchronize the transition time of coupled oscillators with the right amount of quenched disorder.

The DES phenomenon's mechanism is very simple: depending on the starting state, a percentage of the units is able to slow down the system's evolution toward the final state; such units, via the coupling terms, are able to pull the others in the direction of the force via a drive-over effect. Units with short (start to arrival) distances cannot outnumber units with long distances when there is too much diversity. This type of resonancelike mechanism is expected to emerge in many fields.

VII. CONCLUSION

In conclusion, we showed that heterogeneity, in the form of quenched noise, may enhance the stability of metastable systems without the help of external noise. We expect the same effect to occur in other more complex coupled systems.

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