Enhancing quantum synchronization through homodyne measurement, noise, and squeezing

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Quantum synchronization has been a central topic in quantum nonlinear dynamics. Despite the rapid development in this field, very few have studied how to efficiently boost synchronization. Homodyne measurement emerges as one of the successful candidates for this task but preferably in the semiclassical regime. In our work, we focus on the phase synchronization of a harmonic-driven quantum Stuart–Landau oscillator and show that the enhancement induced by homodyne measurement persists into the quantum regime. Interestingly, optimal two-photon damping rates exist when the oscillator and driving are at resonance and with a small single-photon damping rate. We also report noise-induced enhancement in quantum synchronization when the single-photon damping rate is sufficiently large. Apart from these results, we discover that adding a squeezing Hamiltonian can further boost synchronization, especially in the semiclassical regime. Furthermore, the addition of squeezing causes the optimal two-photon pumping rates to shift and converge.

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I. INTRODUCTION

Synchronization pervades nature everywhere, from the unison cacophony of fireflies [1] to the rhythmic pulse of heartbeats [2] to the locking march steps of superconducting tunnel junctions [3,4]. We have been captured and mesmerized by the sheer beauty of synchronization.

Quantum analogs of classical synchronization have been studied in quantum models of self-sustained oscillators [5–7]. Various measures of quantum synchronization have also been proposed [8,9]. Genuine quantum effects without classical analog, such as synchronization blockade [10] and nonlinearity-induced synchronization [11], have also been reported. It has also been shown that additional squeezing can produce stronger frequency entrainment than harmonic drive [12]. Recently, the well-studied field of classical noise-induced synchronization [13–16] has also been extended to quantum regime [17]. Experimental works on synchronization, both classical and quantum, have been demonstrated in nanomechanical oscillators [18,19], cold atoms [20,21], quantum dot micropillars [22], trapped ion qubits [23], and so forth.

Homodyne measurement is a fundamental technique developed in quantum optics, yet it plays a pivotal role in quantum information and technology. Several continuous-variable quantum key distribution protocols respond to homodyne detection to extract quadrature information encoded in the signal [24,25], which have been demonstrated experimentally [26,27]. It has also been proposed to improve the sensitivity of quantum sensors [28]. The open quantum system monitored by homodyne measurement can be modeled by a master equation conditioned on the measurement record [29], also known as quantum trajectory theory [30,31].

In a recent work [32], the phase synchronization of a quantum oscillator with an external drive can be enhanced by monitoring the system with continuous homodyne measurement, and quantum fluctuations are reduced by continuous measurement in the semiclassical regime. Here we extend the work in Ref. [32] to the quantum regime, which leads us to discover more interesting phenomenon in quantum synchronization. We find that the dependence of enhancement on the nonlinear damping rate γ_2 is not monotonic, and there is an optimal γ_2 in the semiclassical regime that achieves the greatest enhancement. The most exciting result is the noise-induced synchronization enhancement, where single-photon damping is shown to boost the enhancement. We also report that squeezing can further improve this enhancement, as well as make the optimal γ_2 more robust against noise.

II. MODEL

We study the quantum Stuart–Landau model (also widely accepted as the quantum van der Pol model in the existing literature [7,11,33]) subjected to both a harmonic drive and a two-photon squeeze drive, with continuous homodyne measurement at the output. The stochastic master equation of the system under homodyne measurement in the rotating frame of the drive is given by (with $\hbar = 1$)

$$d\rho = \{-i[\hat{H}, \rho] + \gamma_1 \mathcal{D}[a^\dagger]\rho + \gamma_2 \mathcal{D}[a^2]\rho + \gamma_3 \mathcal{D}[a]\rho\}dt + \sqrt{\eta_d \gamma_3}\mathcal{H}[ae^{-i\theta}]\rho dW, \quad (1)$$

$$\hat{H} = -\Delta a^\dagger a + iE(a - a^\dagger) + i\eta(a^{\dagger 2}e^{2i\phi} - a^2 e^{-2i\phi}), \quad (2)$$

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where $\mathcal{D}[L]\rho = L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$, $\mathcal{H}[L]\rho = L\rho + \rho L^\dagger - \mathrm{Tr}[(L + L^\dagger)\rho]\rho$, and $\mathcal{H}[ae^{-i\theta}]$ characterizes the measurement on the quadrature $ae^{-i\theta} + a^\dagger e^{i\theta}$, with γ_1, γ_2, and γ_3 corresponding to negative damping, nonlinear damping, and linear damping, respectively. Without loss of generality we assume a perfect detector, and its detection efficiency is set to $\eta_d = 1$ throughout this paper. We denote $\Delta = \omega_d - \omega_0$ as the amount of initial detuning between the frequency of the drive ω_d and the natural frequency of the oscillator ω_0. E denotes the amplitude of the harmonic drive, with a the annihilation operator and $a^\dagger$ the creation operator. η is the squeezing parameter, and ϕ represents the phase of the squeezing. W represents the Wiener process, where $\mathbb{E}[dW] = 0$ and $\mathbb{E}[dW^2] = dt$, and the measurement record $dY = \sqrt{\eta_d\gamma_3}\ \mathrm{Tr}[(ae^{-i\theta} + a^\dagger e^{i\theta})\rho]dt + dW$. We can recover the model described in Ref. [32] without feedback control by setting $\eta = 0$. For simplicity we scale every parameter in the unit of $\gamma_1 = 1$.

As a measure of synchronization, the phase coherence is frequently used in the literature [10,32], defined as

$$S = |S|e^{i\phi_{\mathrm{avg}}} = \frac{\mathrm{Tr}[a\rho]}{\sqrt{\mathrm{Tr}[a^\dagger a\rho]}}, \tag{3}$$

where $|S|$ measures the degree of phase coherence with a range of $0 \leqslant |S| \leqslant 1$. ϕ_{avg} represents the average phase of the oscillator. The phase coherence quantifies the statistic fluctuation in the phase distribution and therefore the tendency for the quantum oscillator to lock phase with the external drive.

The enhancement of phase coherence through homodyne measurement is calculated as $\mathcal{F} = |S_{HD}|/|S_0|$, where S_{HD} is the average phase coherence over N_{traj} trajectories, defined by

$$S_{HD} = \frac{1}{N_{\mathrm{traj}}}\Sigma_{k=1}^{N_{\mathrm{traj}}}\frac{\mathrm{Tr}[a\rho_k]}{\sqrt{\mathrm{Tr}[a^\dagger a\rho_k]}}, \tag{4}$$

and S_0 is the phase coherence obtained from unconditioned master equation without homodyne measurements. We will refer to $\mathcal{F}$ as the enhancement factor. In our simulations we ensure the finite Hilbert space truncation by examining the Fock-space distribution in the steady states, and we fixed the number of trajectories at $N_{\mathrm{traj}} = 300$. This allows us to efficiently simulate without utilizing too many computation resources, and further increase the number of trajectories does not change the result qualitatively.

III. SYNCHRONIZATION ENHANCEMENT IN QUANTUM REGIME

We first study the phase synchronization of a quantum Stuart–Landau oscillator with coherent drive but without squeezing Hamiltonian (by simply set $\eta = 0$). Quantum synchronization (i.e., phase locking) has been shown to improve when the system in the semiclassical regime (the definition of different regimes will be provided later in the section) is continuously monitored by homodyne measurement [32]. Here we investigate the enhancing effect of homodyne measurement in a wider parameter regime and under different driving amplitude. In our simulations, we set zero initial detuning between the oscillator and drive, i.e., $\Delta = 0$. This might seem unusual in the context of classical synchronization, where two systems are definitely synchronized without detuning. But here we are interested in the phase synchronization particularly, which is not guaranteed by zero initial detuning. This is due to the presence of quantum fluctuations, which prevent the phase-space portrait of the steady state from concentrating onto one fixed point, and consequently cause diffusion around it. Another reason for choosing zero initial detuning is that changing γ_2 also changes the optimal phase θ of the measurement when the detuning is nonzero, whereas with zero detuning, $\theta = \pi/2$ is optimal for all values of γ_2 (see Appendix A).

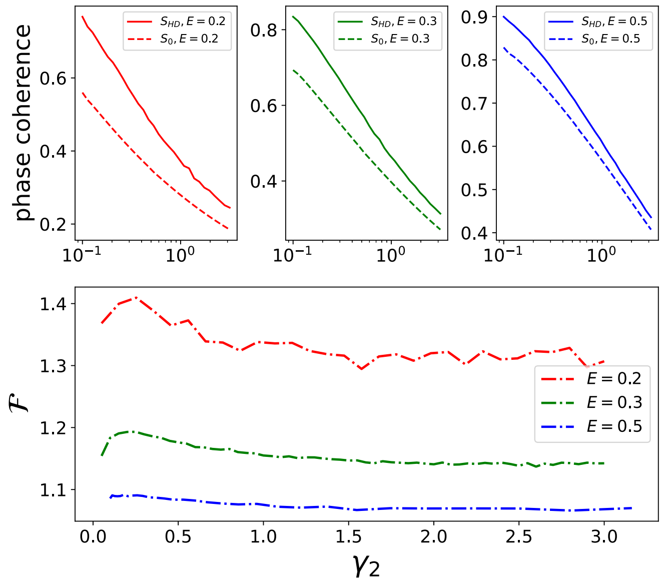

FIG. 1. Phase coherence with and without homodyne detection (top) and enhancement factor $\mathcal{F}$ (bottom) plotted against nonlinear damping rate $\gamma_2 \in [0.5, 3]$, under different driving amplitudes E. Fixed parameters: $\Delta = 0$, $\gamma_3 = 0.1$, $\theta = \pi/2$ (optimized for the highest enhancement factor). Take note that the fluctuations in the curves are consequences of a finite number of trajectories averaged in the simulations (also see Appendix B for simulation errors). The same applies for other plots below.

In the limit of large mean photon number, the oscillator can be statistically described by a set of classical equation of motion [7]. This limit is referred to as the semiclassical regime. On the other hand, in the regime with low mean photon number, where the classical model breaks down, the oscillator is considered in the quantum regime. In this regime, quantum noise comes into play and genuine quantum phenomena arise [11,34]. Different synchronization regimes of quantum Stuart–Landau oscillator can be characterized in terms of the nonlinear two-photon damping rate γ_2. By increasing this rate, the limit cycle of oscillator shrinks and its mean photon number decreases, signifying a transition from semiclassical regime ($\gamma_2 < 1$) to quantum regime ($\gamma_2 \gg 1$). In order to access different regimes, we assume the parameter γ_2 can be tuned freely. States in the quantum regime will be more prone to diffusion through quantum fluctuations, causing them to lose phase synchronization with the driving force. This is shown in the top row panels of Fig. 1, where phase coherence drops with increasing γ_2, regardless of the driving amplitude.

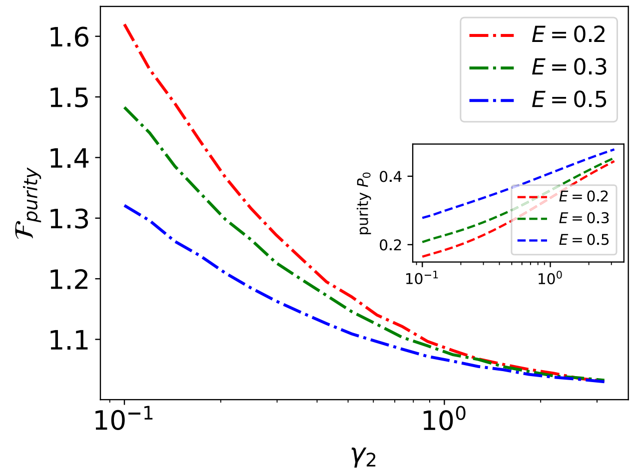

FIG. 2. Enhancement in purity of steady state due to homodyne measurement is always present but decays with increasing γ_2. (Inset) Purity scales approximately with $\gamma_2^{1/3}$. Fixed parameters: $\Delta = 0, \gamma_3 = 0.1, \theta = \pi/2$.

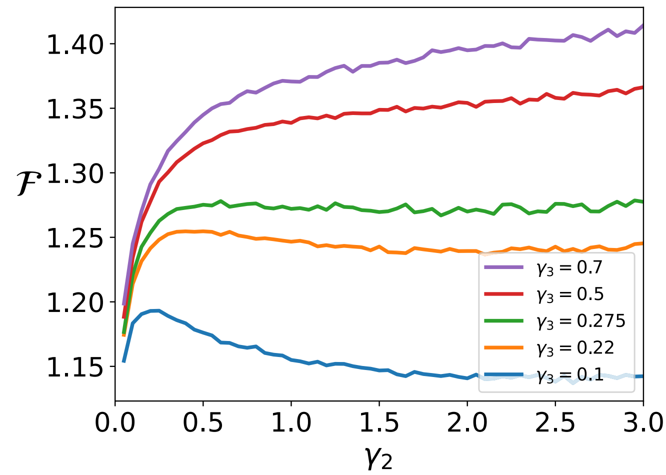

FIG. 3. Effect of γ_2 on enhancement factor $\mathcal{F}$ at different single-photon dissipation γ_3 with fixed parameters $\Delta = 0, E = 0.3, \theta = \pi/2$. When single-photon dissipation γ_3 is small, the enhancement factor $\mathcal{F}$ drops after reaching the maximum. However, the enhancement factor starts to rise even in the quantum regime when a large single-photon dissipation is present. Note that the $\gamma_3 = 0.1$ curve corresponds to the $E = 0.3$ curve in Fig. 1.

In Fig. 1 we also show the enhancing effect of homodyne measurement on quantum synchronization, quantified by the enhancement factor $\mathcal{F}$. The presence of homodyne measurement always enhances the phase coherence, even in the quantum regime, provided that the phase θ of the measurement is optimized. Larger enhancement is observed when driving amplitude E is small. Notice in Fig. 1, for small γ_2 there is an optimal ratio for the enhancement factor to peak at, which only appears at zero initial detuning between the oscillator and driving force (see Appendix A for nonzero detuning cases). After that the enhancement factor drops with γ_2 until asymptotically reaching a ratio above unity. In addition, later in Sec. V we will show that this optimal ratio is sensitive to the dissipative noise and squeezing.

It has been shown that this enhancement $\mathcal{F}$ is a consequence of the increase in the purity of states, defined as $P = \text{Tr}[\rho^2]$, as the effective phase-space diffusion is inversely proportional to the purity of states [31]. We note that on average, homodyne measurement increases the purity. To this end, we look at the ratio of the purity of the steady state P_{HD} at $\eta_d = 1$, corresponding to homodyne measurement, to the purity P_0 at $\eta_d = 0$ where homodyne measurement is turned off, i.e., $\mathcal{F}_{\text{purity}} = |P_{HD}|/|P_0|$. Figure 2 shows this ratio with γ_2. We see that this ratio is always greater than 1, indicating that homodyne measurement enhances the purity of states. However, with increasing γ_2 this ratio tends to unity, regardless of the amplitude E.

IV. NOISE-INDUCED SYNCHRONIZATION ENHANCEMENT

It has been established that classically chaotic systems exhibit noise-induced synchronization in the presence of common Gaussian noise [13,16], and this synchronization can be enhanced, for example, by additional dichotomic noise [15]. In the quantum domain, noise is also reported to boost quantum synchronization, such as inducing frequency entrainment in quantum Stuart–Landau oscillators [34] and helping to develop correlation and entanglement between two ends of a quantum spin chain [17]. Here we show another example of such noise-induced phase synchronization of a driven quantum Stuart–Landau oscillator.

We show that more single-photon damping can effectively raise the enhancement through homodyne measurement. In Eq. (1), γ_3 corresponds to the rate of single-photon damping, acting as a dissipative noise for the oscillator. At the same time, it characterizes the coupling between the measurement device and the monitored system, since all dissipated photons will be captured by the detector ($\eta_d = 1$). The damping rate γ_3 also scales the backaction of homodyne detection [last term in Eq. (1)]. Therefore it is not surprising that increasing γ_3 results in a more potent homodyne measurement and thereby improves the enhancement factor. This is shown in Fig. 3, where the enhancement factor is plotted against the nonlinear damping rate γ_2 at different single-photon damping γ_3. Dissipative noise increases the enhancement factor across all values of γ_2. Especially in the quantum regime where $\gamma_2 > 1$, the enhancement factor receives a greater boost compared to the semiclassical regime. As a consequence, the enhancement factor increases with nonlinear damping rate γ_2, given a moderate single-photon damping ($\gamma_3 \approx 0.3$) is present. We refer to this as noise-induced synchronization enhancement through homodyne measurement. This is a significant result; compared to Fig. 1, we seem to reverse the effect of increasing γ_2 from detrimental to enhancing. And this is achieved by simply adding linear damping.

It has also been reported that without any measurements, dissipative noise can produce an increase in off-diagonal density matrix elements in the deep quantum regime [34], which contributes to greater phase coherence. But such an increase is marginal in amplitude, as seen in our results in Fig. 4.

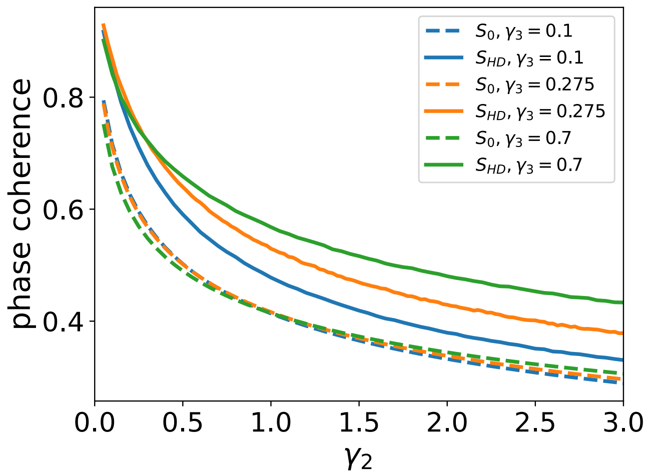

FIG. 4. Phase coherence generally vanishes with increasing γ_2, with fixed parameters $\Delta = 0, E = 0.3, \theta = \pi/2$. However, in the quantum regime, the presence of single-photon dissipation γ_3 can boost phase coherence. Homodyne measurement greatly amplifies this boost in the quantum regime.

V. SQUEEZING FURTHER ENHANCES SYNCHRONIZATION

Another useful technique to improve quantum synchronization is squeezing [12]. However, the measure of phase coherence is no longer appropriate for capturing quantum phase synchronization when large squeezing is present. The calculation of phase coherence is valid when the density matrix has only first off-diagonal coherences [10]. Therefore, to study the effects of squeezing, we applied only a small amount of squeezing while ensuring that the second and higher off-diagonal elements remain negligible. It is also known that, under squeezing, the Wigner function of such system will undergo a pitchfork bifurcation [12], i.e., the steady-state Wigner function has two peaks. In the presence of two peaks, the use of phase coherence as a measure is generally small, due to some form of averaging. Such cases are beyond our scope for this study; therefore we limit the amplitude of squeezing below $\eta = 0.1$ in the $\phi = 0$ direction, so that the peak in the Wigner function is not split into two or more peaks.

Indeed, squeezing is shown to further improve quantum phase synchronization. In Fig. 5 we show that squeezing is more likely to boost synchronization in the semiclassical regime, while in the quantum regime, squeezing has negligible effects. The parameter region where squeezing offers the most observable benefits is when both (γ_2, γ_3) are small (bottom-left corner of the contour plots). Moving on to the large (γ_2, γ_3), the enhancing effect of squeezing vanishes, as illustrated in Fig. 5(c). Squeezing and nonlinear damping appear to be more closely related to each other, as they are two-photon processes. This is evident from the three-dimensional plots in Fig. 5, where increasing squeezing parameter pushes the landscape along γ_2 rather than γ_3.

In addition to this additional enhancement caused by squeezing itself, adding squeezing also modifies the enhancement produced by homodyne measurement. Interestingly, the optimal values of γ_2 for the highest enhancement factor under various γ_3 converge to a narrow region around $\gamma_2 = 0.2$, with a small squeezing up to $\eta = 0.1$. This is shown in Fig. 6(a), where we numerically plot the optimal γ_2 as a function of the squeezing parameter η. Another interpretation of the effect is that additional squeezing makes the optimal point stronger against dissipative noise γ_3. As shown in Fig. 6(b), the optimal points γ_2 move closer to the left vertical axis when the pressure increases and eventually become independent of γ_3 when $\eta = 0.1$. This effect is interesting, as it provides some insights into possible stochastic resonance in the dynamics.

VI. CONCLUSION

To conclude, our study has made the following observations in quantum synchronization: First, in the presence of homodyne measurement, enhancement in phase synchronization persists to the quantum regime. Next, optimal two-photon nonlinear damping rates (γ_2) exist in which the enhancement factors are maximum, with small single-photon damping rates (γ_3) when the oscillator is driven at resonance ($\Delta = 0$). This phenomenon is unusual, as it appears only with zero initial detuning. However, moderate single-photon damping rates (γ_3) allow higher enhancement factors to be achieved even in the quantum regime, despite acting as a source of dissipative noise. Additionally, adding a small amount of squeezing can further enhance quantum phase synchronization, especially in the semiclassical regime. More strikingly, with additional

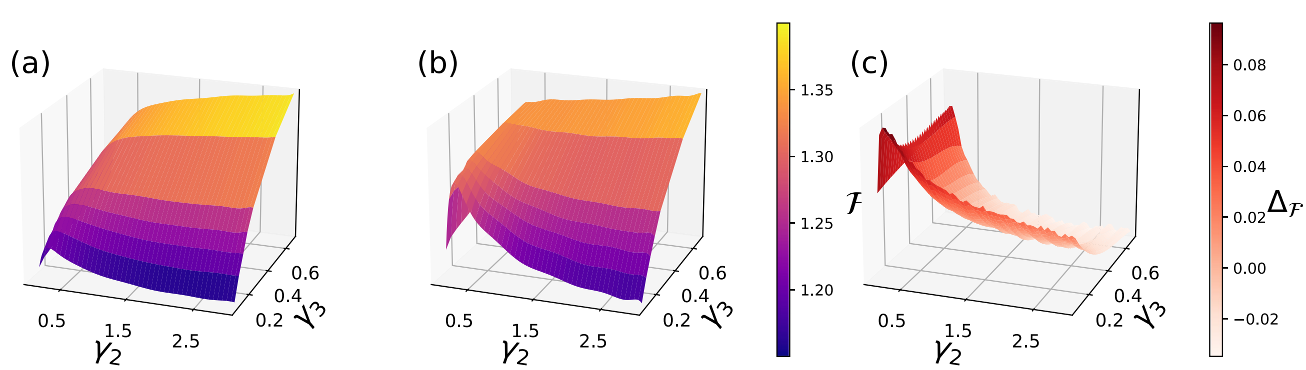

FIG. 5. Contour plot of enhancement factor $\mathcal{F}$ against γ_2 and γ_3 with (a) $\eta = 0$ and (b) $\eta = 0.1$. (c) Differences between enhancement factor $\mathcal{F}$ of (a) and (b).

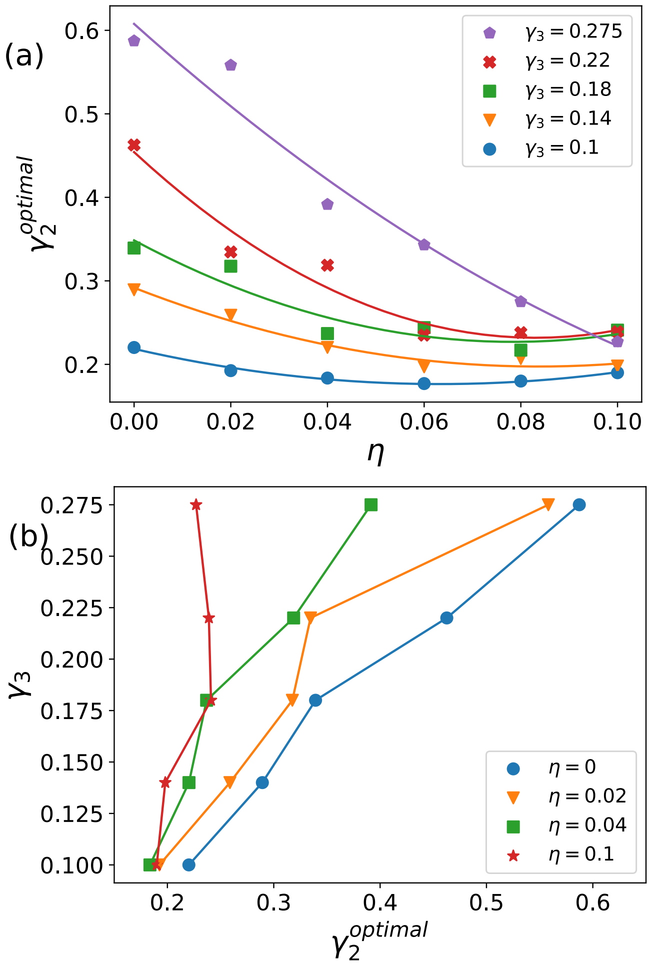

FIG. 6. (a) The convergence of optimal γ_2 with increasing squeezing η. (b) Top-down view of the optimal γ_2 positions in the transition from Figs. 5(a) and 5(b).

squeeze, optimal nonlinear damping rates (γ_2) become insensitive to dissipative noise (γ_3).

The Stuart–Landau model used in this work is implementable using the state-of-the-art superconducting circuits, where the two-photon dissipation and squeezing can be engineered using a parametric conversion process in Josephson junctions [35,36]. As a future direction, it would be interesting to explore the effects of homodyne measurement and squeezing on a true van der Pol oscillator [33], where a modified quantum Stuart–Landau oscillator provides a phase-space plot that closely resembles the classical diamondlike phase-space plot of a van der Pol oscillator.

ACKNOWLEDGMENTS

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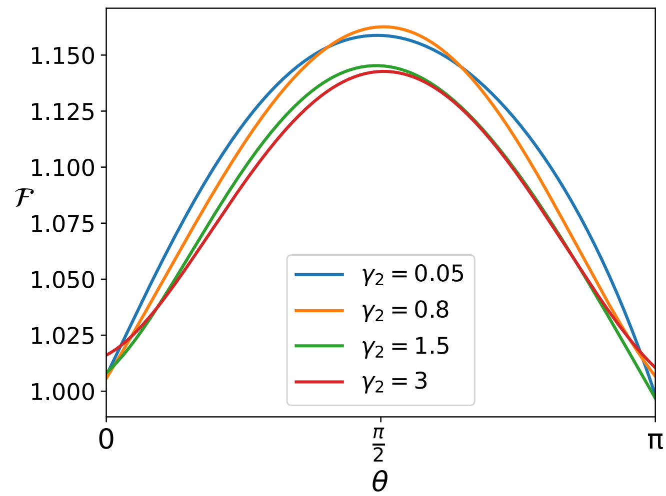
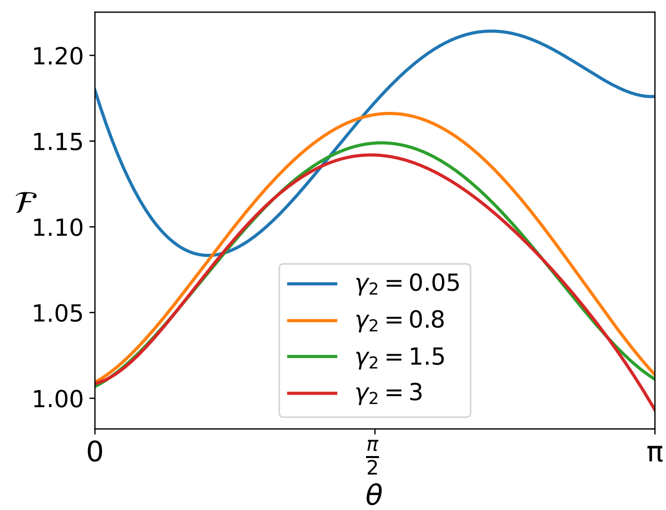


(a) $\Delta = 0$



(b) $\Delta = 0.05$

FIG. 7. Enhancement factor at different measurement angles θ: (a) zero detuning and (b) nonvanishing detuning $\Delta = 0.05$. The curves are smoothed using polynomial interpolation.

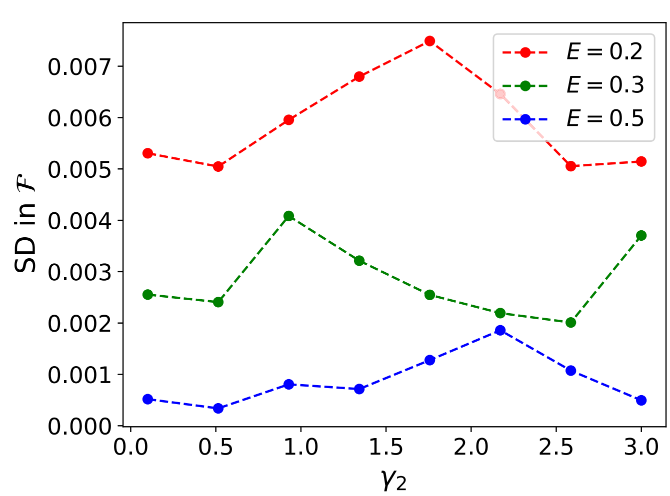


FIG. 8. Standard deviation in the enhancement factor $\mathcal{F}$ of Fig. 1.

APPENDIX A: ENHANCEMENT UNDER NONVANISHING DETUNING

In this section we present the simulation results for nonvanishing detuning, as a supplement reference to the claims we are making. As shown in Fig. 7(a), with zero detuning, the optimal enhancement factors are obtained at $\theta = \pi/2$ and the corresponding measurement angle do not depend on γ_2. One can also see the optimal enhancement factors do not scale with γ_2 monotonically, i.e., the highest points of the curves rise and then drop, in the range of $\gamma_2 = 0.05$–3.

In Fig. 7(b), with nonvanishing detuning, the optimal measurement angles shift away from $\pi/2$. Additionally, the highest enhancement factors are monotonically decreasing with respect to increasing γ_2.

APPENDIX B: SIMULATION ERROR ANALYSIS

We numerically simulated the error (standard deviation) in the data of Fig. 1, with 100 sample runs and number of trajectories $N_{\text{traj}} = 300$. According to the quantum Monte Carlo simulation method [39], the standard deviation is on the order of $N_{\text{traj}}^{-1/2}$. In Fig. 8, we show that the largest standard deviation is below 1% of the data in Fig. 1.

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