Exact quench dynamics from algebraic geometry

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We develop a systematic approach to compute physical observables of integrable spin chains with finite length. Our method is based on the Bethe ansatz solution of the integrable spin chain and computational algebraic geometry. The final results are analytic and no longer depend on Bethe roots. The computation is purely algebraic and does not rely on further assumptions or numerics. This method can be applied to compute a broad family of physical quantities in integrable quantum spin chains. We demonstrate the power of the method by computing two important quantities in quench dynamics—the diagonal entropy and the Loschmidt echo—and obtain analytic results.

I. INTRODUCTION

Finding exact solutions of important physical models has been a long-term endeavor. An important class of models that can be solved exactly are integrable models, such as the two-dimensional Ising model and the Heisenberg spin chain. The exact solutions allow us to penetrate more deeply into the properties of the model and lead to different physical intuitions, which in many cases can be generalized to broader models including nonintegrable ones.

A central question in integrable models is computing observables in a pure state $\langle \psi | \mathcal{O}_1 \cdots \mathcal{O}_n | \psi \rangle$ or a thermal state $\text{tr}(\mathcal{O}_1 \cdots \mathcal{O}_n)$ analytically. Here $\mathcal{O}_k$ are certain operators (not necessarily local). A common strategy for computing such quantities is inserting resolution of identity. For example,

$$\langle \psi | \mathcal{O}_1 \mathcal{O}_2 | \psi \rangle = \sum_\alpha \langle \psi | \mathcal{O}_1 | \alpha \rangle \langle \alpha | \mathcal{O}_2 | \psi \rangle,$$
$$\text{tr}(\mathcal{O}_1 \mathcal{O}_2) = \sum_{\alpha, \beta} \langle \alpha | \mathcal{O}_1 | \beta \rangle \langle \beta | \mathcal{O}_2 | \alpha \rangle, \tag{1}$$

where the sums on the right-hand side are over *all* eigenstates of the Hamiltonian. To proceed further, one needs to find out quantities $\langle \psi | \mathcal{O}_i | \alpha \rangle$, $\langle \alpha | \mathcal{O}_i | \beta \rangle$ for all eigenstates and then perform the sum over states. For concreteness, let us now focus on integrable spin chains that can be solved by Bethe ansatz. For such models, the eigenstates $|\alpha\rangle$ can be constructed by Bethe ansatz and are parametrized by Bethe roots. Analytic results for the form factors $\langle \alpha | \mathcal{O} | \beta \rangle$ are known in many cases (see for example [1–3]). On the other hand, performing the sum in (1) is a difficult task in general [4]. For a finite spin chain, so far there is no systematic method to perform such sums.

In this paper, we develop a systematic method for computing such sums analytically. While we focus on the prototypical Heisenberg XXX and XXZ spin chain in the current paper, we emphasize that our method can be generalized to any Bethe ansatz solvable spin chains. In these models, each eigenstate is labeled by a set of Bethe roots, which are solutions of the Bethe equations. Therefore, summing over all eigenstates is essentially equivalent to summing over all physical solutions of Bethe equations. The crucial observation is, while finding a *single* analytic solution of Bethe equation is hard or impossible, summing over *all* solutions can be much simpler. We shall show that the sums in (1) can be performed analytically by using the proper tool—computational algebraic geometry.

The algebro-geometric approach initiated in [5] has been applied to computing partition functions of the six-vertex model on a medium size lattice, both with torus [6] and cylinder geometry [7] at the isotropic point. In this paper, we extend the method to a much wider class of observables as well as to the six-vertex model at generic anisotropy. To demonstrate the power of our method, we consider two important quantities in quench dynamics, which are the diagonal Rényi entropy [8–10] and the Loschmidt echo [11–17].

Before diving into technical details, let us first explain the main idea. The Loschmidt echo is defined by $\mathcal{L}_L(t) = |\mathcal{M}_L(it)|^2$, where $\mathcal{M}_L(\omega)$ is given by

$$\mathcal{M}_L(\omega) = \langle \psi | e^{-\omega H} | \psi \rangle = \sum_\alpha |\langle \psi | \alpha \rangle|^2 e^{-\omega E_\alpha}. \tag{2}$$

To evaluate (2), naively we need to first find *all* physical solutions of Bethe equations and then perform the sum. It is precisely at this point that computational algebraic geometry can play an important role. The upshot is that we do not need to solve the Bethe equation. Instead, we construct the *companion matrices* for quantities like $\langle \psi | \alpha \rangle$ and E_α. Performing the sum over states amounts to taking the trace of the companion matrix. The companion matrices are finite dimensional matrices, which can be constructed purely algebraically using the Gröbner basis. Using this method, we obtain the following

result:

$$\mathcal{M}_L(\omega) = \frac{1}{2\pi i} \oint_{\mathcal{C}} F_L(z)\, e^{-\omega z} dz, \qquad (3)$$

where $F_L(z)$ are *rational functions* and the integration contour $\mathcal{C}$ is encircling all the poles counterclockwise. An explicit example will be given in Sec. IV. These rational functions can be worked out easily up to $L = 20$ on a laptop. Compared to the original form (2), the integral representation (3) is much more explicit and no longer depends on Bethe roots. The highly nontrivial job of solving Bethe equations and summing over all solutions has been fully accomplished.

The rest of the paper is structured as follows. In Sec. II, we give a brief review of the Bethe ansatz solution of the XXX spin chain. Section III introduces basic notions of computational algebraic geometry. In Sec. IV, we compute the diagonal entropy and Loschmidt echo using the algebro-geometric approach. We conclude in Sec. V and discuss future directions.

II. BETHE ANSATZ

In this section, we briefly review the Bethe ansatz of XXX spin chain, with a special emphasis on completeness of the Bethe ansatz. Our method can be applied straightforwardly to the XXZ spin chain with generic q, which we present in the Supplemental Material [18]. The Heisenberg XXX spin chain is described by the following Hamiltonian:

$$H = \frac{J}{4} \sum_{n=1}^{L} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z - 1 \right). \qquad (4)$$

We consider the periodic boundary condition $\sigma_{L+1}^\alpha = \sigma_1^\alpha$.

(a) Primary and descendant states. Eigenstates of (4) can be constructed by the Bethe ansatz. An N-magnon ($N = 1, 2, \ldots, L$) state is characterized by N Bethe roots $\mathbf{u}_N = \{u_1, \ldots, u_N\}$. We denote the corresponding eigenstate by $|\mathbf{u}_N\rangle$. The Bethe roots are physical solutions [19] of Bethe equations

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{k \neq j}^{N} \frac{u_j - u_k + i}{u_j - u_k - i}. \qquad (5)$$

We distinguish two types of eigenstates, which are the *primary* and *descendant* states. The primary states have only *finite* Bethe roots, while descendant states contain roots at infinity. Given a primary state $|\mathbf{u}_N\rangle$, its descendant states are obtained by acting operator S^-, where

$$S^\pm = \sum_{n=1}^{L} s_n^\pm, \quad s_n^\pm = \frac{1}{2}\left(\sigma_n^x \pm i \sigma_n^y \right). \qquad (6)$$

In what follows, we will denote the descendant states by

$$(S^-)^n |\mathbf{u}_N\rangle \equiv |\mathbf{u}_N, \infty^n\rangle. \qquad (7)$$

Acting on an S^- operator on a Bethe state amounts to adding a Bethe root at infinity. The primary state $|\mathbf{u}_N\rangle$ and all its descendant states $|\mathbf{u}_N, \infty^n\rangle$ have the same energy, given by

$$E(\mathbf{u}_N) = -\frac{J}{2} \sum_{k=1}^{N} \frac{1}{u_k^2 + \frac{1}{4}}. \qquad (8)$$

It is important to take into account both primary and descendant states when summing over states.

(b) Completeness of the Bethe ansatz. An important question about the Bethe ansatz concerns its completeness: does the Bethe ansatz give *all* eigenstates of the Hamiltonian? This is a subtle question for generic integrable models. Fortunately, for XXX spin chain, the completeness problem has been studied extensively in the literature; see for example [20–22]. The main conclusions are as follows. *(i)* The number of *physical solutions* of the Bethe equation (5) is $\mathcal{N}_{L,N}^{\text{phys}} = \binom{L}{N} - \binom{L}{N-1}$ for $N \leqslant [L/2]$, where $[L/2]$ is the integer part of $[L/2]$. *(ii)* Eigenstates with $N > [L/2]$ correspond to dual solutions of the Bethe equation. They can be obtained from the ones with $N \leqslant [L/2]$ by flipping all the spins. From these, it is easy to verify that *(iii)* the Bethe ansatz is complete.

(c) Rational Q system. The original form of the Bethe equations (5) has solutions that are not physical. To select only the physical solutions, one can impose additional constraints or, more elegantly, reformulate Bethe equations. One nice reformulation of such kind is the rational Q system [23–25]. For the XXX spin chain, this is an alternative incarnation of the Wronskian relation of the Baxter's TQ relation. The rational Q system gives only physical solutions of the Bethe equation and is easier to solve. Therefore, in the algebro-geometric computations we will work with the rational Q system instead of the Bethe equations. For more details and examples of the rational Q system, we refer to [6,26].

The rational Q system gives a set of algebraic equations for the coefficients of the Q function, which is defined by

$$Q(u) = \prod_{j=1}^{N} (u - u_j) = u^N + \sum_{k=0}^{N-1} (-1)^k s_k u^k, \qquad (9)$$

where $\{u_1, \ldots, u_N\}$ are Bethe roots. The rational Q system leads to a set of algebraic equations for $\{s_0, s_1, \ldots, s_{N-1}\}$.

III. ALGEBRAIC GEOMETRY

In this section, we introduce basic notions of computational algebraic geometry which we need in what follows. A detailed and pedagogical introduction to these notions in the context of the Bethe ansatz can be found in [5]. Here we only highlight the main ideas.

Computational algebraic geometry is a modern tool to deal with complicated algebraic varieties [27], with a broad application in physics. We introduce this tool by posing a concrete question, which will be solved by computational algebraic geometry.

Consider a set of N-variable algebraic equations:

$$F_1(z_1, \ldots, z_N) = \cdots = F_n(z_1, \ldots, z_N) = 0. \qquad (10)$$

We assume there are $\mathcal{N}$ solutions. We consider another polynomial $P(z_1, \ldots, z_N)$ and want to compute the following sum analytically:

$$\mathcal{S}[P] \equiv \sum_{\text{sol}} P(z_1, \ldots, z_N), \qquad (11)$$

where we sum over the solutions of (10). A brute force computation of this sum by numeric solutions is cumbersome and suffers from numeric errors. Here we introduce the ingredients

of applying computational algebraic geometry to get the sum analytically.

(a) Gröbner basis. The key point for this computation is to reduce the given polynomial towards the equations (10). However, the remainder is not unique. To have a well-defined remainder, we need to transform the equations to a *Gröbner basis.* Let $I = \langle F_1 \ldots F_n \rangle$ be the *ideal* generated by $F_1, \ldots, F_n$. The Gröbner basis is another set of polynomials $G_1, \ldots, G_m$ such that $I = \langle G_1 \ldots G_m \rangle$, with the additional property that the remainder of the polynomial reduction for any polynomial P is unique with respect to $\{G_j\}$. The Gröbner basis can be understood as a nonlinear analog of Gaussian elimination and is computed by standard algorithms [27,28], which we review in the Supplemental Material [18].

(b) Quotient ring. On the solution set of an equation system, the value of a test function P is well defined modulo the ideal I. Therefore, we consider the quotient ring $A = \mathbb{Q}[z_1, \ldots, z_N]/I$. Over $\mathbb{Q}$, A is a finite dimensional linear space with the dimension

$$\dim_{\mathbb{Q}} A = \mathcal{N}. \tag{12}$$

The linear basis of A is naturally the list of all monomials $\{m_1 \ldots m_{\mathcal{N}}\}$ which are not divided by any leading terms of $\{G_1 \ldots G_m\}$.

(c) Companion matrix. The values of a polynomial P on the solution set can be represented as the *companion matrix* $\mathbf{M}_P$. In the quotient ring A, by the polynomial division towards the Gröbner basis, we have

$$P m_i = \sum_{j=1}^{\mathcal{N}} a_{ij} m_j, \quad a_{ij} \in \mathbb{Q}. \tag{13}$$

We denote $\mathbf{M}_P$ as an $\mathcal{N} \times \mathcal{N}$ matrix with the entries a'_{ij}s. By the evaluation of the equation above, the value of P on a solution corresponds to an eigenvalue of $\mathbf{M}_P$. Therefore, we have the central formula for our purpose,

$$\mathcal{S}[P] = \operatorname{Tr} \mathbf{M}_P. \tag{14}$$

Consider two polynomials P_1 and P_2 and their corresponding companion matrices $\mathbf{M}_{P_1}$ and $\mathbf{M}_{P_2}$. The companion matrix satisfies the following properties:

$$\begin{aligned} \mathbf{M}_{P_1 \pm P_2} &= \mathbf{M}_{P_1} \pm \mathbf{M}_{P_2}, \\ \mathbf{M}_{P_1 P_2} &= \mathbf{M}_{P_1} \cdot \mathbf{M}_{P_2}, \\ \mathbf{M}_{P_1/P_2} &= \mathbf{M}_{P_1} \cdot \mathbf{M}_{P_2}^{-1}. \end{aligned} \tag{15}$$

In particular, using the last property in (15), we can generalize the sum computation (14) for a rational function,

$$\mathcal{S}[P_1/P_2] = \operatorname{Tr}\left(\mathbf{M}_{P_1} \cdot \mathbf{M}_{P_2}^{-1}\right). \tag{16}$$

The construction of the Gröbner basis and the companion matrix is purely arithmetic. Therefore, it can be readily applied to algebraic equations with parameters, such as the Bethe equation of the XXZ model which depends on the parameter q. This is a huge advantage compared to the usual numerical approach which typically requires fixing the parameter to a certain value.

The method discussed here has been applied to the computation of partition functions of the six-vertex model [6,7]. In those cases, the function P in (11) is the eigenvalue of the transfer matrix $T(u_1, \ldots, u_N)$, which is a *rational function* of the Bethe roots. Therefore, the sum over solutions can be computed using (16).

(d) Beyond rational functions. In more general situations, we cannot restrict $P(u_1, \ldots, u_N)$ to be rational functions. One simple example is the thermal partition function of the XXX spin chain

$$Z_L(\beta) = \operatorname{tr} e^{-\beta H} = \sum_{\alpha} \langle \alpha | e^{-\beta H} | \alpha \rangle, \tag{17}$$

which can be computed by the Bethe ansatz. We first decompose the Hilbert space into different sectors of fixed magnon numbers. In each sector, the calculation boils down to computing the sum of the following type:

$$\sum_{\text{sol}} e^{-\beta E_N(\mathbf{u})}, \tag{18}$$

where the sum is over all physical solutions of the Bethe equation. From (8), it is clear that $e^{-\beta E_N(\mathbf{u})}$ is not a rational function of $\mathbf{u}$ and our method does not apply directly. To write down an explicit analytic result, we seek an alternative representation of $Z_L(\beta)$.

To incorporate this more general situation into our method, let us consider the following sum:

$$\sum_{\text{sol}} p(\mathbf{u}) F(q(\mathbf{u})), \tag{19}$$

where $p(\mathbf{u})$ and $q(\mathbf{u})$ are rational functions of $\{u_1, \ldots, u_N\}$ and $F(z)$ can be any function that does not have singularities at $z = q(\mathbf{u})$. We can rewrite the sum (19) as

$$\sum_{\text{sol}} p(\mathbf{u}) F(q(\mathbf{u})) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} F(z) \sum_{\text{sol}} \frac{p(\mathbf{u})}{z - q(\mathbf{u})}, \tag{20}$$

where the contour encircles all possible values of $q(\mathbf{u})$. Denoting the companion matrices of $p(\mathbf{u})$ and $q(\mathbf{u})$ by $\mathbf{M}_p$ and $\mathbf{M}_q$, respectively, we have

$$\sum_{\text{sol}} p(\mathbf{u}) F(q(\mathbf{u})) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} F(z) \operatorname{tr}[\mathbf{M}_p (z - \mathbf{M}_q)^{-1}]. \tag{21}$$

It is straightforward to write down similar contour integral representations for more complicated sums.

IV. EXACT QUENCH DYNAMICS

In this section, we compute two important quantities in quench dynamics as concrete examples of our general method outlined in the previous sections. The two quantities are defined in the following.

(a) Diagonal Rényi entropy. We define the diagonal ensemble by the following density matrix:

$$\rho_d = \sum_m O_m |m\rangle \langle m|, \tag{22}$$

where the states $|m\rangle$ are eigenstates of the XXX spin chain and the coefficients O_m are the overlaps with the initial state

$$O_m = |\langle \Psi_0 | m \rangle|^2. \tag{23}$$

The diagonal Rényi entropy is defined by

$$S_d^{(\alpha)} \equiv \frac{1}{1-\alpha} \ln \mathrm{Tr}\, \rho_d^\alpha = \frac{1}{1-\alpha} \ln \sum_m O_m^\alpha. \qquad (24)$$

For fixed and integer values of α, the overlap in (24) is a rational function of rapidities and the sum can be calculated analytically.

(b) Loschmidt echo. The Loschmidt amplitude has been defined in (2) which we quote here:

$$\mathcal{M}_L(\omega) = \langle \Psi_0 | e^{-\omega H} | \Psi_0 \rangle = \sum_m O_m\, e^{-\omega E_m}, \qquad (25)$$

where O_m has been defined in (23) and E_m is the energy of state $|m\rangle$. The Loschmidt echo is given by $\mathcal{L}(t) = |\mathcal{M}_L(it)|^2$. Similar to the thermal partition function, the analytic result of the sum in the Loschmidt amplitude (25) cannot be written down directly. We will give the result in the contour integral representation (21).

(c) Integrable quench. We consider the integrable quench where the initial state $|\Psi_0\rangle$ is an integrable initial state [29]. Such states have a number of nice properties which makes them especially suitable for analytical studies. In particular, the overlap $O_m = |\langle \Psi_0 | m \rangle|^2$, which is an important ingredient for both the diagonal entropy (24) and the Loschmidt echo (25), can be written down explicitly as a rational function of rapidities [30–34].

For a Bethe state with N rapidities (assuming N is even for simplicity), the overlap $\langle \Psi_0 | \mathbf{u}_N \rangle$ is nonzero only if the Bethe roots are paired, namely

$$\mathbf{u}_N = \{u_1, -u_1, u_2, -u_2, \ldots, u_{\frac{N}{2}}, -u_{\frac{N}{2}}\}. \qquad (26)$$

A simple example of an integrable initial state is the Néel state

$$|\Psi_0\rangle = \tfrac{1}{2}(|\uparrow\downarrow\rangle^{\otimes L/2} + |\downarrow\uparrow\rangle^{\otimes L/2}). \qquad (27)$$

Due to magnon number conservation, the states which have nonzero overlaps with Néel state (27) are $L/2$-magnon states, which consist of $|\mathbf{u}_N, \infty^{L/2-N}\rangle$ ($N = 0, 1, \ldots, L/2$). The overlap is given by [30]

$$\frac{\langle \Psi_0 | \mathbf{u}_N \rangle}{\sqrt{\langle \mathbf{u}_N | \mathbf{u}_N \rangle}} = \frac{\sqrt{2}(L/2 - N)!}{\sqrt{(L - 2N)!}} \prod_{j=1}^{N/2} \frac{\sqrt{u_j^2 + \frac{1}{4}}}{4u_j} \sqrt{\frac{\det G^+}{\det G^-}}, \quad (28)$$

where $|\mathbf{u}_N\rangle \equiv |\mathbf{u}_N, \infty^{L/2-N}\rangle$ and $G^\pm$ is the Gaudin matrix

$$G_{jk}^\pm = \delta_{jk}\left(L K_{1/2}(u_j) - \sum_{l=1}^{M/2} K_1^+(u_j, u_l)\right) + K_1^\pm(u_j, u_k),$$

$$K_\alpha^\pm(u, v) = K_\alpha(u - v) \pm K_\alpha(u + v),$$

$$K_\alpha(u) = \frac{2\alpha}{x^2 + \alpha^2}. \qquad (29)$$

(d) Diagonal Rényi entropy. For the computation of diagonal Rényi, the crucial quantity is the sum $\sum_m O_m^\alpha$. For a Néel state $|\Psi_0\rangle$, the nonzero overlaps come from Bethe states $|\mathbf{u}_N, \infty^{L/2-N}\rangle$, with $N = 0, 2, \ldots, 2\lfloor L/4 \rfloor$. If α is an integer, O_m^α is a rational function of Bethe roots. The sum can thus be performed by the algebro-geometric method directly. We denote the companion matrix of

$$O_N(\mathbf{u}_N) = \frac{|\langle \mathbf{u}_N, \infty^{L/2-N} | \Psi_0 \rangle|^2}{\langle \mathbf{u}_N, \infty^{L/2-N} | \mathbf{u}_N, \infty^{L/2-N} \rangle} \qquad (30)$$

by $\mathbf{M}_{O_N}$. Using the property (15), the diagonal Rényi entropy can be written as

$$S_d^\alpha(L) = \frac{1}{1-\alpha} \ln \sum_{N=0}^{2\lfloor L/4 \rfloor} \mathrm{Tr}\,(\mathbf{M}_{O_N})^\alpha. \qquad (31)$$

We study an explicit example for $L = 8$ as an illustration. In this case only Bethe states $|\mathbf{u}_N, \infty^{4-N}\rangle$ with $N = 0, 2, 4$ contribute. We discuss the $N = 4$ sector in detail. The other sectors are simpler and can be computed in a similar way.

In the $M = 4$ sector, the paired Bethe roots take the form $\mathbf{u}_4 = \{u_1, -u_1, u_2, -u_2\}$. The overlap square $O_4(u_1, u_2)$ is a symmetric rational function in the rapidities u_1, u_2. The ideal consists of the algebraic equations from a rational Q system (see the Supplemental Material [18] for more details), together with the nonsingular condition [35]

$$w(u_1^2 + 1)(u_2^2 + 1) + 1 = 0. \qquad (32)$$

Following the standard algorithm, we can compute a Gröbner basis of this ideal and construct the companion matrix of O_4, which reads

$$\mathbf{M}_{O_4} = \begin{bmatrix} \frac{19}{156} & \frac{3}{13} & -\frac{109}{1872} \\ -\frac{7}{6240} & \frac{19}{1560} & \frac{9}{8320} \\ -\frac{5}{26} & -\frac{14}{13} & \frac{83}{312} \end{bmatrix}. \qquad (33)$$

The companion matrices of O_2, O_0 can be computed in a similar fashion. Then diagonal Rényi entropy can be computed straightforwardly. We list results for $\alpha = 2, 6, 10$:

$$S_d^2(8) = \ln\left(\frac{143325}{49009}\right),$$

$$S_d^6(8) = \frac{1}{5} \ln\left(\frac{38274471591890625}{512161566111913}\right),$$

$$S_d^{10}(8) = \frac{1}{9} \ln\left(\frac{1123194749225856453808593
75}{75385210067492164108951}\right). \qquad (34)$$

For any noninteger $\alpha > 0$, using the representation (21) we have

$$S_d^\alpha = \frac{1}{1-\alpha} \ln\left(\oint_{\mathcal{C}} \frac{dz}{2\pi i} z^\alpha \sum_{N=0}^{2\lfloor L/4 \rfloor} \mathrm{Tr}\,(z - \mathbf{M}_{O_N})^{-1}\right)$$

$$= \frac{1}{1-\alpha} \ln\left[\oint_{\mathcal{C}} \frac{dz}{2\pi i} z^\alpha \left(\frac{21(1323z^2 - 504z + 20)}{9261z^3 - 5292z^2 + 420z - 8} + \frac{39(1755z^2 - 468z + 16)}{22815z^3 - 9126z^2 + 624z - 8} + \frac{35}{35z - 1}\right)\right]. \qquad (35)$$

The diagonal Rényi entropy has been computed in the thermodynamic limit in [9,10] using the quench action approach. Here we offer a general method for finite length spin chains which gives exact analytic results. Our approach is complimentary to the works [9,10] in the thermodynamic limit.

(e) Loschmidt echo. We apply the contour integral representation to compute the Loschmidt echo. The Loschmidt amplitude can be written as

$$\mathcal{M}_L(\omega) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \widetilde{\mathcal{M}}_L(z) e^{-\omega z}, \qquad (36)$$

where

$$\widetilde{\mathcal{M}}_L(z) := \sum_{k=0}^{[L/4]} \widetilde{\mathcal{M}}_L^{(2[L/4]-2k)}(z),$$

$$\widetilde{\mathcal{M}}_L^{(N)}(z) := \mathrm{Tr}\,[\mathbf{M}_{O_N} \cdot (z\mathbf{1} - \mathbf{M}_{E_N})^{-1}]. \qquad (37)$$

Apart from the overlap matrix $\mathbf{M}_{O_N}$, we also need the companion matrix of the energy $E_N = E(\mathbf{u}_N)$. For the $L = 8$ example, only the states with $N = 4, 2, 0$ contribute. In the $N = 4$ sector, the energy is (taking $J = 1$) given by

$$E_4 = -\frac{16s_1 + 8}{4s_1 + 16s_3 + 1}. \qquad (38)$$

The companion matrix reads

$$\mathbf{M}_{E_4} = \begin{bmatrix} -\frac{27}{8} & -\frac{5}{2} & \frac{65}{96} \\ \frac{1}{64} & -\frac{25}{16} & -\frac{11}{768} \\ \frac{9}{4} & 15 & -\frac{81}{16} \end{bmatrix}. \qquad (39)$$

Using (37), we obtain

$$\widetilde{\mathcal{M}}_8^{(4)}(z) = \frac{2(3z^2 + 15z + 17)}{15(z^3 + 10z^2 + 29z + 25)}. \qquad (40)$$

The contributions of $M = 0, 2$ can be calculated in the same way. Combining them, we get

$$\widetilde{\mathcal{M}}_8(z) = \frac{z^6 + 13z^5 + 63z^4 + 143z^3 + 153z^2 + 65z + 5}{z(z^3 + 10z^2 + 29z + 25)(z^3 + 7z^2 + 14z + 7)}. \qquad (41)$$

The Loschmidt amplitude then takes the contour integral form

$$\mathcal{M}_8(\omega) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \widetilde{\mathcal{M}}_8(z) e^{-\omega z}. \qquad (42)$$

The integral can be evaluated by various methods straightforwardly. We give more results in the Supplemental Material [18].

In [16,17], the analytical results of the Loschmidt echo are obtained in the thermodynamic limit using the quantum transfer matrix method. Here we give a systematic method to compute it analytically for any finite length spin chain.

V. CONCLUSIONS

In this paper, we present a systematic method to compute a large family of physical observables for the finite length XXX spin chain. Our method is based on the Bethe ansatz solution of the spin chain and computational algebraic geometry. We applied this method to compute the diagonal Rényi entropy and the Loschmidt echo, obtaining analytic results for both quantities in (34) and (41). Our method is completely algebraic and avoids the need for solving Bethe equations. Apparently there are many interesting directions to explore based on our current results. For the XXX spin chain, it would be interesting to compute other important quantities such as correlation functions of local spin operators. These quantities are considerably more involved than the Loschmidt echo and diagonal entropy. To compute such quantities for relatively long spin chains ($L \sim 20$), a more efficient implementation of our method is needed. The general strategy in this paper clearly generalizes to other types of spin chains, such as the XXZ spin chain and chains solvable by the nested Bethe ansatz. Naively the Bethe equations of the XXZ spin chain involve hyperbolic functions and our approach, which works for polynomial functions, does not apply directly. However, by a change of variable to the multiplicative variables, the Bethe equation can be brought to a polynomial form. The rational Q system for the XXZ spin chain has been studied in [24]. Finally, it is worth noting that the Gröbner basis is a general tool which works for any polynomial equations. It is therefore desirable to find alternative ways to construct the companion matrices, which are more fine-tuned for integrable models.

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