Single-energy-measurement integral fluctuation theorem and nonprojective measurements

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We study a Jarzysnki-type equality for work in systems that are monitored using nonprojective unsharp measurements. The information acquired by the observer from the outcome f of an energy measurement and the subsequent conditioned normalized state $\hat{\rho}(t, f)$ evolved up to a final time t are used to define work, as the difference between the final expectation value of the energy and the result f of the measurement. The Jarzynski equality obtained depends on the coherences that the state develops during the process, the characteristics of the meter used to measure the energy, and the noise it induces into the system. We analyze those contributions in some detail to unveil their role. We show that in very particular cases, but not in general, the effect of such noise gives a factor multiplying the result that would be obtained if projective measurements were used instead of nonprojective ones. The unsharp character of the measurements used to monitor the energy of the system, which defines the resolution of the meter, leads to different scenarios of interest. In particular, if the distance between neighboring elements in the energy spectrum is much larger than the resolution of the meter, then a similar result to the projective measurement case is obtained, up to a multiplicative factor that depends on the meter. A more subtle situation arises in the opposite case in which measurements may be noninformative, i.e., they may not contribute to update the information about the system. In this case a correction to the relation obtained in the nonoverlapping case appears. We analyze the conditions in which such a correction becomes negligible. We also study the coherences, in terms of the relative entropy of coherence developed by the evolved post-measurement state. We illustrate the results by analyzing a two-level system monitored by a simple meter.

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I. INTRODUCTION

Fluctuation theorems have been one of the most relevant results of statistical physics in recent decades. The pioneering work of Evans, Morris, and Searles [1,2], Gallavotti and Cohen [3,4], Jarzynski [5,6], and Crooks [7] has been followed by a series of significant results that have highlighted their conceptual depth, as well as its relevance and applicability in many different fields [8–12]. In particular, work fluctuations are one of the issues that have attracted more attention. The most widely used approach to study work fluctuations considers a two-point measurement scheme, in which work is defined as the difference between the outcomes of two projective energy measurements [13]. Specifically, given an initial state, a first projective energy measurement is made, afterwards the system evolves unitarily under the dynamics dictated by a time-dependent Hamiltonian. Then, at a given final time, a second projective energy measurement is performed. Work is defined as the difference between the energies obtained in the two measurements. Although this scheme has been widely explored and has been very fruitful, there are still important open issues that need to be addressed.

After the first energy measurement, the system is projected onto an eigenstate of the initial Hamiltonian. The unitary evolution that follows generally develop coherences in the eigenbasis of the final Hamiltonian. Such coherences disappear once the second energy measurement is performed. In this sense the two-point measurement scheme does not include some important quantum features.

It has been shown that two physically necessary properties of quantum work, namely, respecting the classical limit and obeying the first law of thermodynamics, cannot be simultaneously measured. Considering these results, schemes have been proposed to estimate fluctuating work using global measurements in which the backaction of the measurement can be reduced and thus approximately describe coherent transformations [14].

Another aspect concerns the nature of the measurements performed. In general, if the energy measurement is made using a meter that has a finite resolution, then the result obtained may not correspond to any of the eigenenergies of the measured Hamiltonian [15]. Even so, the measurements may be sufficiently informative to conclude that the post-measurement state will be a given eigenstate of the Hamiltonian. In general, the meter will induce some noise into the system, and therefore work fluctuations will be affected [16–19]. Once the first measurement has been made, knowledge of the conditioned post-measurement state, as well as the unitary operator that dictates the subsequent time evolution provide sufficient information to obtain the expected value of the energy at the final time. In the case of projective measurements, the difference between such energy value and the result obtained in the first measurement has been used as a definition for work in Refs. [20–23] and its extension to the classical treated in Ref. [24]. Remarkably, this definition for work makes possible to derive a modified Jarzynski quantum equality that evidences the role played by the projective measurements and their cost [20]. Notice that the possibility of obtaining work fluctuations from a single measurement was already pointed out in Ref. [25], and for open quantum systems an end-point measurement scheme has been proposed [26]. Also, several measurement protocols have been proposed to obtain the probability distribution of work done on a quantum system using an ancilla or quantum detector. The key point in these schemes is the system-detector interaction considered to couple the system energy operator with an observable of the detector. Then, the information about work statistics is encoded in the quantum state of the detector [27]. More recently, it has been shown that the quantum Jarynski equality arises as a particular case of a general quantum fluctuation theorem for the entropy production of an open quantum system coupled to multiple environments, not necessarily at equilibrium [28].

In the limit of nonprecise measurements, it is interesting to study how nonprojective measurements affect work fluctuations and unveil the role of coherences. Also, the effect of the meter used to measure the energy, the relevance of the initial state, and its average energy in the quantum Jarzynski relation that would emerge, could be analyzed.

In this work, we begin by briefly introducing the two energy measurement approach mostly used in quantum mechanical analysis of work. In particular, we show the Jarsynski equality that arises when all energy measurements are projective. We then focus on a more general setting in which the meter that monitors the system performs unsharp measurements in which the outcomes are not reproducible. We show how these nonprojective measurements can be described. Within this approach, we consider a definition of work based on a single nonprojective energy measurement performed at a given time and the subsequent unitary evolution of the post-measurement state. We obtain the generalized Jarsynski equality that arises in this case, and which makes explicit the role of the measuring apparatus and coherences in work fluctuations. We perform a detailed analysis of the different effects that contribute to such equality. Specifically, we study the coherences, the noise induced by measurement and the information acquired by the observer from measurement outcomes. To illustrate the analysis we consider a two-level system monitored by a simple meter that performs unsharp measurements, characterized by a given conditional probability of outcomes. We study the different scenarios that arise depending on the measurement conditions, defined by the energy resolution of the meter in comparison to the energy spectrum of the system.

Two-point measurement protocol

The definition of work has been widely discussed in quantum mechanical analysis of work fluctuations [11,18,29] as it is known that no Hermitian operator can be assigned to work [30]. The most widely used approach is based on two energy measurements, the first one at the beginning of the process and the second one at the end. Work is defined as the difference of the two energy outcomes [13,31]. There are also some proposals based on a single measure of energy, whereby work can be obtained from a single final generalized energy measurement [25]. We now briefly introduce the two-point protocol to fix the notation.

We consider a system described by a time-dependent Hamiltonian $\hat{H}(\lambda(t)) := \hat{H}(t)$, where $\lambda(t)$ defines the protocol corresponding to an external driving. We assume that at initial time $t = t_0$ the system is in thermal equilibrium with a thermal bath at T temperature. Thus, it is in a Gibbs state given by $\hat{\rho}_{th}(t_0) = e^{-\beta\hat{H}(t_0)}/\mathcal{Z}(t_0)$, with $\beta = (k_B T)^{-1}$ and $\mathcal{Z}(t_0) = \mathrm{Tr}\, e^{-\beta\hat{H}(t_0)}$, where the trace is over the Hilbert space of the system.

In the so called *forward protocol*, at time $t = t_0^+$ it is performed a projective measurement of the energy of the system, given by $\hat{H}(t_0) = \sum_\mu \mu|\mu\rangle\langle\mu| \equiv \sum_\mu \mu\, \hat{\Pi}_\mu$. If the result of the measurement is μ, then the post-measurement state becomes

$$\hat{\rho}(t_0^+, \mu) = P_\mu^{-1}\hat{\Pi}_\mu\hat{\rho}_{\mathrm{th}}(0)\hat{\Pi}_\mu, \tag{1}$$

with $P_\mu = \mathrm{Tr}\,\hat{\Pi}_\mu\hat{\rho}_{\mathrm{th}}(0)$ the probability of finding the energy μ as a result of the measurement. After this first energy measurement, the system evolves up to time $t = t_1$ according to the unitary evolution operator $\hat{U}(t_1, t_0)$, which is a solution of the Schrödinger equation

$$i\hbar\partial_t\hat{U}(t_1, t_0) = \hat{H}(t)\hat{U}(t_1, t_0), \tag{2}$$

with the initial condition $\hat{U}(t_0, t_0) = \hat{1}$. Then, the conditional state of the system at time $t = t_1$ is given by

$$\hat{\rho}(t_1, \mu) = \hat{U}(t_1, t_0)\hat{\rho}(t_0^+, \mu)\hat{U}^\dagger(t_1, t_0). \tag{3}$$

At time $t = t_1$ it is performed a second projective measurement of the energy $\hat{H}(t_1) = \sum_\nu \nu|\nu\rangle\langle\nu|$, leading to the outcome ν. Given the energy values obtained in the two projective measurements, one can compose the variable $W = \nu - \mu$. Due to the intrinsic randomness of the measurement process and the fact that the initial state is mixed, W is a random variable. We denote $P_F(W)$ its corresponding probability distribution.

We now consider a second setting, the so-called *backward protocol*, in which at initial time $t = t_1$ the system has the Hamiltonian $\hat{H}(t_1)$ and it is prepared in a thermal state $\hat{\rho}_{th}(t_1) = e^{-\beta\hat{H}(t_1)}/\mathcal{Z}(t_1)$. At this time value a first projective measurement of energy leads to the outcome ν'. Then the system evolves in such a way that after a time interval $t_1 - t_0$ the Hamiltonian becomes $\hat{H}(t_0)$, and a second energy measurement gives the result μ'. The state of the system after this second measurement can be expressed as

$$\hat{\rho}'(t_0) = P^{-1}(\mu', \nu')\hat{\Pi}_{\mu'}\hat{U}(t_0, t_1)\hat{\Pi}_{\nu'}\hat{\rho}_{\mathrm{th}}(t_1)\hat{\Pi}_{\nu'}\hat{U}^\dagger(t_0, t_1)\hat{\Pi}_{\mu'}, \tag{4}$$

with $P(\mu', \nu')$ the joint probability distribution of the outcomes μ' and ν'. As in the forward protocol, the variable $W' = \mu' - \nu'$ is a random variable with probability distribution $P_B(W')$.

The fluctuation theorem elucidates a subtle symmetry between the forward and backward probability distributions, P_F and P_B, namely [7],

$$P_F(W) = e^{-\beta(F(t_1) - F(t_0) - W)}P_B(-W) = e^{-\beta(\Delta F - W)}P_B(-W), \tag{5}$$

where $F = -k_B T \ln \mathcal{Z}$ is the Helmholtz free energy. From such relation it follows the Jarzynski equality [5,6,9]

$$\langle e^{\beta(\Delta F - W)}\rangle = 1. \tag{6}$$

In the above discussion, all energy measurements were projective so the state of the system just after each measurement was an eigenstate of the Hamiltonian. Therefore, according to the projection postulate, if two consecutive measurements were made, then the resulting outcome energies would be the same. In this sense, measurements are reproducible. One may consider a more general setting in which the meter that monitors the system does not perform projective measurements but unsharp ones in which the outcomes are not reproducible, in the sense that two consecutive measurements will give different results [32].

A way to describe nonprecise measurements is to consider the conditional probability $G(f|a)$ of certain outcome record f if the system is in an eigenstate compatible with the eigenvalue a [15,33]. In general, such conditional probability is a function centered around a and presents some width σ that it is interpreted as the measurement error. Notably, there is a fluctuation theorem for those situations [16] that incorporates the details of the measurement through the Fourier transform of the conditional probability $G(f|0)$. In this context, fluctuation theorems for thermodynamics observables and dynamics described by complete positive trace preserving maps have been reported [34].

In both the projective and nonprojective measurement schemes, the work W done on the system is defined as the difference of the outcomes of two energy measurements.

II. MEASUREMENT AND WORK

Recently, definitions of work based on a single generalized energy measurement performed at an specific time value, and the subsequent unitary evolution of the post-measurement state have been proposed [25]. Such definitions are arguably more consistent from a thermodynamic point of view, especially if the cost of such measurement needs to be taken into account. Remarkably, the introduction of a more consistent definition of work during the measuring process leads to a generalized Jarzynski equality [20], latter extended to open systems in Ref. [23].

In this work we consider nonprojective measurements, and make explicit the role of the measuring apparatus and coherences in work fluctuations.

To start with, we consider that the results of measurements when monitoring a system observable $\hat{A}=\sum_a a|a\rangle\langle a|$ can be represented by a continuous real variable f which is located around the spectrum of $\hat{A}$ [15,33,35–38]. Formally, such a variable can be defined in terms of a set of positive operators

$$\hat{G}^{1/2}(f|\hat{A}) \equiv \sum_a \sqrt{G(f|a)}|a\rangle\langle a|, \tag{7}$$

where the functions $G(f|a)$ are assumed to be real and nonnegative in the variable f. We further assume that $G(f|a)=G(f-a)$. If a measurement of $\hat{A}$ on the state $\hat{\rho}(t)$ leads to the result f, then the post-measurement state becomes

$$\hat{\rho}(t^+,f) = P(f)^{-1}\hat{G}^{1/2}(f|\hat{A})\,\hat{\rho}(t)\,\hat{G}^{\dagger 1/2}(f|\hat{A}), \tag{8}$$

where

$$P(f) = \mathrm{Tr}\,\hat{G}^{1/2}(f|\hat{A})\,\hat{\rho}(t)\,\hat{G}^{\dagger 1/2}(f|\hat{A}) \equiv \mathrm{Tr}\,\hat{G}(f|A)\hat{\rho}(t) \tag{9}$$

is the probability density function of obtaining the outcome f. In this sense, we consider unsharp measurements of the observable $\hat{A}$. The operators $\hat{G}^{1/2}(f|\hat{A})$ satisfy $\int df\,\hat{G}^{\dagger 1/2}(f|\hat{A})\hat{G}^{1/2}(f|\hat{A})=\hat{1}$, which ensures that $P(f)$ is normalized to one.

We now consider that at a given time $t=t_0$ the system is in a state $\hat{\rho}(t_0)$. At such time value it is performed an energy measurement which leads to the outcome f_0. Then, our state of knowledge of the system changes abruptly and the post-measurement state $\hat{\rho}(t_0^+,f_0)$ becomes the normalized conditioned state given by Eq. (8). After this energy measurement, the system undergoes a unitary evolution up to time $t=t_1$, dictated by the evolution operator $\hat{U}(t_1,t_0^+)$ corresponding to the Hamiltonian $\hat{H}(t)$. The final state of the system can be expressed as

$$\hat{\rho}(t_1,f_0) = \hat{U}(t_1,t_0^+)\hat{\rho}(t_0^+,f_0)\hat{U}^\dagger(t_1,t_0^+). \tag{10}$$

The outcome f_0 and the post-measurement conditioned state $\hat{\rho}(t_1,f_0)$ provide enough information to define work as the difference between the expectation value of energy at time $t=t_1$ and the measurement outcome f_0 [39]. Thus, the amount of work associated with the unitary evolution from t_0^+ to t_1 under the action of the external driving is defined as

$$W(t_1,t_0^+,f_0) = \mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}(t_1,f_0) - f_0. \tag{11}$$

III. PROBABILITY DENSITY OF WORK AND JARZYNSKI EQUALITY

The work $W(t_1,t_0^+,f)$ has a probability distribution $P_F(W)$ given by

$$P_F(W) = \int df\,P(f)\delta(W - W(t_1,t_0^+,f)), \tag{12}$$

with $\delta(x)$ the Dirac δ distribution.

The average $\langle e^{-\beta W}\rangle$ is formally obtained from $P_F(W)$ as

$$\langle e^{-\beta W}\rangle = \int df\,P(f)e^{-\beta W(t_1,t_0^+,f)}. \tag{13}$$

During the measurement process and the subsequent unitary evolution, the system changes its entropy and it generates coherences. Also, individual energy measurements exhibit fluctuations around the energy expectation value at the initial time. To make explicit these contributions to the fluctuation theorem, one can introduce a reference path of states $\rho_{th}(t) = e^{-\beta H(t)}/\mathcal{Z}(t) = e^{-\beta(H(t)-F(t))}$, where $F(t)$ is the Helmholtz free energy. Then, working on Eq. (13), we reach

$$\langle e^{-\beta W}\rangle = \int df\,P(f)e^{\ln \mathcal{Z}(t_1) - S(\hat{\rho}(t_1,f)||\hat{\rho}_{\mathrm{th}}(t_1)) - S(\hat{\rho}(t_1,f)) + \beta f}, \tag{14}$$

where $S(\hat{X}||\hat{Y}) = \mathrm{Tr}\,\hat{X}\ln\hat{X} - \hat{X}\ln\hat{Y}$ is the quantum relative entropy between the states $\hat{X}$ and $\hat{Y}$ and $S(\hat{X}) = -\mathrm{Tr}\,\hat{X}\ln\hat{X}$ is the von Neumann entropy of the state $\hat{X}$ [40]. If one introduces $\Delta F = F(t_1) - F(t_0)$ as the difference between the Helmholtz free energies of the thermal states $\hat{\rho}_{\mathrm{th}}(t_1)$ and $\hat{\rho}_{\mathrm{th}}(t_0)$, then we arrive to the expression

$$\langle e^{\beta(\Delta F - W)}\rangle = e^{\xi}, \tag{15}$$

with

$$\xi = \ln \left\langle e^{\beta(f-\langle H(t_0)\rangle)} e^{-S(\hat{\rho}(t_1,f))+S(\hat{\rho}(t_0))} \times e^{-S(\hat{\rho}(t_1,f)||\hat{\rho}_{\rm th}(t_1))+S(\hat{\rho}(t_0)||\hat{\rho}_{\rm th}(t_0))}\right\rangle, \quad (16)$$

where the averages are taken over $P(f)$. Different effects such coherences, noise induced by measurements, information gained from measurement outcomes, all contribute to the Jarzynski equality (16). To make their role explicit, we now analyze them separately.

A. Coherences

Coherence is one of the key elements of quantum physics and many of the most fundamental quantum phenomena [41]. It is known that coherences play a role in measurement processes, considered as a change of information between two different basis [42]. Moreover, coherences are involved in the production of entropy in open quantum systems under nonequilibrium conditions as nicely discussed in Ref. [43].

Since the initial state of the system commutes with the initial Hamiltonian, the state immediately following an energy measurement will be diagonal in the eigenbasis of $\hat{H}(t_0)$. However, during the subsequent unitary evolution leading to state $\hat{\rho}(t_1, f)$, coherences will develop in the eigenbasis of the final Hamiltonian $\hat{H}(t_1)$. As we will show below, these coherences play a role in the Jarzynski relation (16).

Given a Hamiltonian $\hat{H}(t) = \sum_\mu |\mu(t)\rangle\mu(t)\langle\mu(t)|$, a dephasing operator relative to the basis $\{|\mu(t)\rangle\}$ and acting on a state $\hat{\rho}$ can be defined as

$$\Delta_{\hat{H}(t)}\hat{\rho} := \hat{\rho}_D = \sum_\mu |\mu(t)\rangle\langle\mu(t)|\hat{\rho}|\mu(t)\rangle\langle\mu(t)|. \quad (17)$$

The action of this operator is to produce a state in the eigenbasis of $H(t)$ and erase all its nondiagonal elements. As a measure of quantum coherence we use the relative entropy of coherence $C_{H(t)}(\hat{\rho})$ referred to $\hat{H}(t)$ [41,44], and given by

$$C_{H(t)}(\hat{\rho}) = S(\hat{\rho}_D) - S(\hat{\rho}). \quad (18)$$

In general coherence is a basis-dependent concept. Here we select the basis that diagonalize the Hamiltonian at the given time and its associated thermal state. In a more extended setting, the role played by coherences in nonequilibrium entropy production has been analyzed in detail in Ref. [43].

It can be seen that the quantum relative entropy $S(\hat{\rho}(t, f)||\hat{\rho}_{\rm th}(t))$ can be decomposed into two contributions, one due to coherences and another associated with the populations. This last contribution measures the observer's ability to realize that after many measurements the populations of the final state are given by the state $\hat{\rho}_D(t, f)$, and not by the thermal state $\hat{\rho}_{th}(t)$ [45]. The coherences are measured by the relative entropy of coherence $C_{H(t)}(\hat{\rho})$, whereas the contribution due to the populations is expressed in terms of the Kullback-Leibler divergence [40]

$$D_{\rm KL}(\hat{\rho}_D(t, f)||\hat{\rho}_{\rm th}(t)) := D_{\rm KL}(\{p_\mu\}||\{q_\mu\}) = \sum_\mu p_\mu \ln\frac{p_\mu}{q_\mu}, \quad (19)$$

with $p_\mu = \langle\mu(t)|\hat{\rho}_D(t, f)|\mu(t)\rangle$ and $q_\mu = \langle\mu(t)|\hat{\rho}_{\rm th}(t)|\mu(t)\rangle$ the probability distributions associated with populations of the states $\hat{\rho}_D(t, f)$ and $\hat{\rho}_{\rm th}(t)$, respectively. Notice that the states $\hat{\rho}_D(t, f)$ and $\hat{\rho}_{\rm th}(t)$ are diagonal in the eigenbasis of $\hat{H}(t)$ so they refer to populations in such basis.

Thus, the quantum relative entropy can be written as

$$S(\hat{\rho}(t, f)||\hat{\rho}_{\rm th}(t)) = C_{H(t)}(\hat{\rho}(t, f)) + D_{\rm KL}(\hat{\rho}_D(t, f)||\hat{\rho}_{\rm th}(t)), \quad (20)$$

which makes explicit the role of coherences in the Jarzynski equality (15). In particular, at the initial time

$$S(\hat{\rho}(t_0)||\hat{\rho}_{\rm th}(t_0)) = C_{H(t_0)}(\hat{\rho}(t_0)) + D_{\rm KL}(\hat{\rho}_D(t_0)||\hat{\rho}_{\rm th}(t_0)), \quad (21)$$

which becomes zero in the case of an initial thermal state $\hat{\rho}(t_0) = \hat{\rho}_{\rm th}(t_0)$.

B. Information gained by the observer in a measurement

The term corresponding to the difference of von Neumann entropies $\Delta S = S(\hat{\rho}(t, f)) - S(\hat{\rho}(t_0))$ is related to the amount of information involved in the measurement process. We start by considering the average of ΔS over the probability distribution $P(f)$, such that

$$P(f)\Delta S = P(f)\ln P(f) - {\rm Tr}\,\hat{G}(f|\hat{H}(t_0))\hat{\rho}(t_0)\ln\hat{G}(f|\hat{H}(t_0)) - {\rm Tr}\,(\hat{G}(f|\hat{H}(t_0)) - P(f))\hat{\rho}(t_0)\ln\hat{\rho}(t_0). \quad (22)$$

For simplicity we have assumed that the initial state commutes with $H(t_0)$.

Then, we introduce the Shannon entropy of a distribution $g(f)$ [40],

$$H_S(g(f)) = -\int df\, g(f)\ln g(f), \quad (23)$$

and integrate Eq. (22) over all possible outcomes f. The resulting average entropy increase is given by

$$\langle\Delta S\rangle = -H_S(P(f)) + H_S(G(f|0)), \quad (24)$$

with $0 \leqslant \langle\Delta S\rangle \leqslant S(\rho(t_0))$.

It is natural that the quantity $\langle\Delta S\rangle$ is bounded by the entropy of the initial state when the observer collects its measurements. In the case that the post-measurement state is a pure state, such that $\langle\Delta S\rangle = S(\rho(t_0))$, the observer can infer the initial state from a sufficient large set of measurements. However, if $\langle\Delta S\rangle \leqslant S(\rho(t_0))$, then the observer may get outcomes that do not add information about the state after the measurement.

Thus, the term ΔS is related to the difference of two Shannon entropies for the probability distributions $P(f)$ and $G(f|0)$, which suggest its relation to the net amount of information involved in the measuring process. A more detailed analysis, taking into account the accuracy of the meter, can be done to give meaning to Eq. (24) in terms of information [46].

Notice that this last term is relevant in the case of nonprojective measurements, and it contains the fact that some measurements can be more informative that others.

C. Jarzynski equality

Once we have analyzed the different terms involved in the Jarzynski identity (16), it can be expressed in a more appealing form as

$$\xi = \ln\left\langle e^{\beta(f-\langle H(t_0)\rangle)} e^{-\Delta S} e^{-\Delta C} e^{-\Delta D_{\rm KL}}\right\rangle, \quad (25)$$

where

$$\Delta C = C_{H(t_1)}(\hat{\rho}(t_1, f)) - C_{H(t_0)}(\hat{\rho}(t_0)) \tag{26}$$

gives the change in the amount of coherences during the measurement process with respect to coherences of the known initial state, and

$$\Delta D_{\mathrm{KL}} = D_{\mathrm{KL}}(\hat{\rho}_D(t_1, f)||\hat{\rho}_{\mathrm{th}}(t_1)) - D_{\mathrm{KL}}(\hat{\rho}_D(t_0)||\hat{\rho}_{\mathrm{th}}(t_0)) \tag{27}$$

is the difference of the final and initial Kullback-Leibler divergences. As an initial hypothesis, the thermal state $\hat{\rho}_{\mathrm{th}}(t)$ can be used by the observer as a basis for predicting the probability of observing a given measurement result. However, it may well be that a different state $\hat{\rho}$ leads to better predictions. The Kullback-Leibler divergence measures whether such initial hypothesis is wrong. Specifically, if after a sufficiently large number n of measurements it follows that $nD_{\mathrm{KL}}(\hat{\rho}||\hat{\rho}_{\mathrm{th}}) \ll 1$, then the observer can conclude that the state $\hat{\rho}_{th}(t)$ is not a good representation of the observed statistics. Thus, Eq. (27) measures how wrong the observer will be if the outcome statistics are assumed to be given by thermal states.

Finally, the Jarzynski identity (16) establishes a bound on the average work done on the system. Taking into account the Jensen inequality [47] it follows

$$\beta\langle W\rangle \geqslant \beta\Delta F - \xi, \tag{28}$$

which can be considered as a generalization of the second law [9]. This last expression provides a generalization of an analogous relation obtained for projective measurements [20] to the case of nonorthogonal measurements, insofar as ξ includes information regarding the resolution of the apparatus that monitors the energy of the system. We analyze this issue in more detail in the illustrative example presented below.

IV. TWO-LEVEL SYSTEM

We now consider a nontrivial example to illustrate the essential elements discussed in the previous sections. Specifically, we consider a system of two energy levels, which is described by the time-dependent Hamiltonian

$$\hat{H}(t) = \sum_{i=1,2} |\mu_i(t)\rangle\mu_i(t)\langle\mu_i(t)|, \tag{29}$$

in terms of which we can write

$$\hat{G}^{1/2}(f|\hat{H}(t_0)) = \sum_{i=1,2} \sqrt{G(f|\mu_i(t_0))}|\mu_i(t_0)\rangle\langle\mu_i(t_0)|. \tag{30}$$

We remind that the functions $G(f|\mu_i(t_0)) = G(f - \mu_i(t_0))$. We also assume that such functions exhibit a characteristic width of order σ around $\mu_i(t_0)$, and tend to zero as $|f - \mu_i(t_0)|$ becomes large enough. Then, the probability density function Eq. (9) for an initial state that commutes with the initial Hamiltonian, becomes

$$\begin{aligned} P(f) &= \sum_{i=1,2} G(f - \mu_i(t_0))\langle\mu_i(t_0)|\hat{\rho}(t_0)|\mu_i(t_0)\rangle \\ &\equiv \sum_{i=1,2} G(f - \mu_i(t_0))p_i. \end{aligned} \tag{31}$$

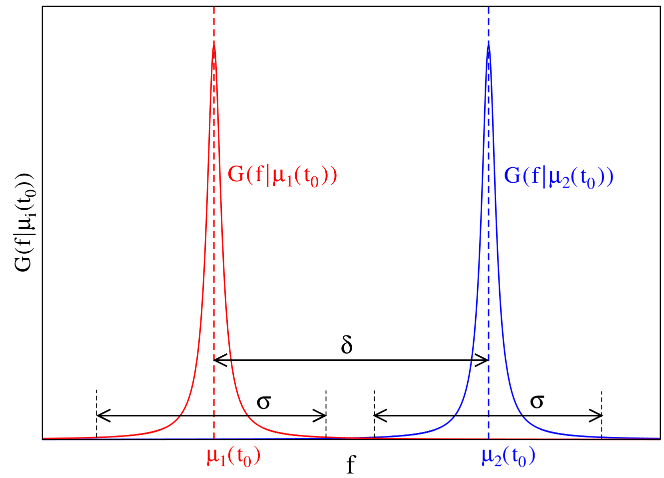

FIG. 1. A scheme of a possible distribution of $G(f|\mu_1(t_0))$ and $G(f|\mu_2(t_0))$, as functions of the outcome of the measurement f.

The conditional state Eq. (8), just after performing an energy measurement with outcome f at time t_0, is given by

$$\hat{\rho}(t_0^+|f) = P^{-1}(f)\sum_{i=1,2} p_i G(f - \mu_i(t_0))|\mu_i(t_0)\rangle\langle\mu_i(t_0)|. \tag{32}$$

Whereas, according to Eq. (10), the state after the unitary evolution up to the final time t_1 can be written as

$$\begin{aligned} \hat{\rho}(t_1|f) &= \sum_{i=1,2} p_i \frac{G(f - \mu_i(t_0))}{P(f)}\hat{U}(t_1, t_0^+)|\mu_i(t_0)\rangle\langle\mu_i(t_0)| \\ &\quad\times \hat{U}^\dagger(t_1, t_0^+) \\ &\equiv \sum_{i=1,2} p_i \frac{G(f - \mu_i(t_0))}{P(f)}\hat{\rho}_i(t_1) \\ &\equiv \sum_{i=1,2} p_i\tilde{G}(f - \mu_i(t_0))\hat{\rho}_i(t_1). \end{aligned} \tag{33}$$

Two different scenarios arise depending on whether the two functions $G(f - \mu_1(t_0))$ and $G(f - \mu_2(t_0))$ overlap or not; see Fig. 1. On the one hand, if $|\mu_1(t_0) - \mu_2(t_0)| \gg \sigma$, then the functions $G(f - \mu_1(t_0))$ and $G(f - \mu_2(t_0))$ do not overlap. In this case, if the result of an energy measurement at time t_0 is f, then the observer finds out that immediately after such measurement the system is in the state $|\mu_1(t_0)\rangle\langle\mu_1(t_0)|$ if f is located within the domain of $G(f - \mu_1(t_0))$ or in the state $|\mu_2(t_0)\rangle\langle\mu_2(t_0)|$ otherwise. In this sense we say that the measurement is informative since the outcome f reveals which of the two possible energy states the system is in. On the other hand, if f is within the domain where there is significant overlap of both G-functions, then the post-measurement state will be a linear combination of the states $\{|\mu_i(t_0)\rangle\langle\mu_i(t_0)|\}$ $(i = 1, 2)$. In this case we refer to noninformative measurements. The two possible scenarios are analyzed in detail below.

A. Nonoverlapping case

In this case all measurements are informative, and the state just after the measurement will be either $|\mu_1(t_0)\rangle\langle\mu_1(t_0)|$ or

$|\mu_2(t_0)\rangle\langle\mu_2(t_0)|$. If J_i is the domain in which $G(f-\mu_i(t_0))$ is appreciably different from zero, and $J_1\cap J_2=\emptyset$, then

$$\langle e^{-\beta W}\rangle=\sum_{i=1,2}\int_{J_i}df\, p_i G(f-\mu_i(t_0))e^{-\beta \mathrm{Tr}\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f}. \quad (34)$$

Notice that the extension of the above expression to a system with a larger number of energy levels is straightforward. It can be rewritten as

$$e^{\xi}=\bar{G}(i\beta)\sum_{i=1,2}p_i\mathcal{Z}(0)e^{\beta\mu_i(t_0)}e^{-S(\hat{\rho}_i(t_1)||\hat{\rho}_{\mathrm{th}}(t_1))}, \quad (35)$$

where we have introduced the characteristics of the measurement procedure, which are given by the Fourier transforms of the $G(f)$ functions, i.e., $\bar{G}(u)=\int df\, e^{-iuf}G(f)$. Equations (15) and (35) extend the fluctuation theorem derived in Ref. [20] to the generalized measurements discussed in previous sections and initial states commuting with the initial Hamiltonian. In the particular case of thermal initial states, the above expression reduces to

$$e^{\xi}=\bar{G}(i\beta)\sum_{i=1,2}e^{-S(\hat{\rho}_i(t_1)||\hat{\rho}_{\mathrm{th}}(t_1))}. \quad (36)$$

The fluctuation relation obtained in this case is the one that would be obtained in projective measurements, multiplied by the function $\bar{G}(i\beta)$, which characterizes the noise induced by nonprojective measurements. In a two-point measurement protocol such factor takes the form $|\bar{G}(i\beta)|^2$ due to the two measurements performed [16]. Here, coherences play no role in the resulting Jarzynski equality.

B. Overlapping case

A more subtle situation arises in the case that $|\mu_1(t_0)-\mu_2(t_0)|\ll\sigma$, as there may be measurements in which the post-measurement state is a linear combination of $|\mu_1(t_0)\rangle\langle\mu_1(t_0)|$ and $|\mu_2(t_0)\rangle\langle\mu_2(t_0)|$. Here we denote by I_1 the domain in which the function $G(f-\mu_1(t_0))$ has values different from zero and the function $G(f-\mu_2(t_0))$ is zero or almost zero. The domain I_2 is defined in a similar way. Thus, in such domains there is no appreciable overlap between the two G-functions. We call the domain in which such overlap takes place I_{ov}. Then, we can write

$$\langle e^{-\beta W}\rangle=\left(\int_{I_1}df+\int_{I_2}df+\int_{I_{\mathrm{ov}}}df\right)P(f)e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}(t_1,f)+\beta f}. \quad (37)$$

If the outcome f of the measurement is located within a nonoverlapping domain I_i, with $i=1$ or 2, then the resultant state will be $|\mu_i(t_0)\rangle\langle\mu_i(t_0)|$, and the corresponding value of the probability density becomes $P(f)=p_iG(f-\mu_i(t_0))$. Whereas, a value of f located within the overlapping region I_{ov} leads to a post-measurement state given by Eq. (33). Thus, Eq. (37) can be rewritten as

$$\langle e^{-\beta W}\rangle=\sum_{i=1,2}\int_{I_i}df\, p_i\, G(f-\mu_i(t_0))e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f} + \int_{I_{\mathrm{ov}}}df\, P(f)e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}(t_1,f)+\beta f}. \quad (38)$$

Considering that

$$\int_{I_i\cup I_{\mathrm{ov}}}df\, p_i\, G(f-\mu_i(t_0))e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f} = \int_{J_i}df\, p_i\, G(f-\mu_i(t_0))e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f}, \quad (39)$$

it follows

$$\langle e^{-\beta W}\rangle=\sum_{i=1,2}\int_{J_i}df\, p_i\, G(f-\mu_i(t_0))e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f} + \int_{I_{\mathrm{ov}}}df\left(P(f)e^{-\beta\sum_i p_i\tilde{G}(f-\mu_i(t_0))\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f} - \sum_{i=1,2}p_i\, G(f-\mu_i(t_0))e^{-\beta\mathrm{Tr}\,\hat{H}(t_1)\hat{\rho}_i(t_1)+\beta f}\right), \quad (40)$$

where we have used Eq. (33). The first element in this expression leads to the fluctuation expression obtained in the nonoverlapping case. The second part gives the contribution due to the fact that measurements may be noninformative. Introducing the function

$$\bar{G}_i(i\beta)=\int_{I_{\mathrm{ov}}}df\, G(f-\mu_i(t_0))e^{\beta(f-\mu_i(t_0))} \quad (41)$$

into Eq. (40), it follows that

$$e^{\xi}=\sum_{i=1,2}p_i\mathcal{Z}(0)e^{\beta\mu_i(t_0)}(\bar{G}(i\beta)-\bar{G}_i(i\beta))e^{-S(\hat{\rho}_i(t_1)||\hat{\rho}_{\mathrm{th}}(t_1))} + \int_{I_{ov}}df\left(\sum_{i=1,2}p_i\mathcal{Z}(0)G(f-\mu_i(t_0))e^{\beta f}\right) \times e^{-\sum_i p_i\tilde{G}(f-\mu_i(t_0))S(\hat{\rho}_i(t_1)||\hat{\rho}_{\mathrm{th}}(t_1))}. \quad (42)$$

We remind that $\tilde{G}(f-\mu_i(t_0))=G(f-\mu_i(t_0))/P(f)$. As mentioned above, $p_i\mathcal{Z}(0)e^{\beta\mu_i(t_0)}=1$ for an initial thermal sate.

The last expression establishes the fluctuation relation for energy measurements in which it is not possible to conclude that the post-measurement state is an eigenstate of the Hamiltonian being monitored. In contrast to the nonoverlapping case, the fluctuation relation can now not be expressed as the product of a term corresponding to projective measurements and one term involving the characteristics of the measurement procedure. In this sense, the information concerning the system and the noise induced by measurements appear mixed in the fluctuation relation for the overlapping case. For further discussion of this issue, details concerning the measurement procedure, such as the $G(f)$ function, should be analyzed to provide a more explicit expression for the last term in Eq. (42).

V. ILLUSTRATIVE EXAMPLE

In general, the conditional probability $G(f|a)$ is a function with appreciable nonzero values in the variable f within a domain of size of order σ around the value a. In the case that G is a Gaussian function, σ would correspond to several times its width.

To illustrate the analysis shown in the previous sections we consider the following form of the function G:

$$G(f|a) = \begin{cases} \sigma^{-1} & f \in (a-\sigma/2, a+\sigma/2), \\ 0 & \text{otherwise}, \end{cases} \tag{43}$$

and a two-level system with initial and final Hamiltonians $\hat{H}(t_0)$ and $\hat{H}(t)$, respectively; see Eq. (29). The initial state is taken to be $\hat{\rho}_{th}(t_0) = \mathcal{Z}^{-1}(t_0)\sum_{i=1,2} e^{-\beta\mu_i(t_0)}|\mu_i(t_0)\rangle\langle\mu_i(t_0)|$. The probability density $P(f)$ of measurement outcome f is given by

$$\begin{aligned} P(f) &= \mathcal{Z}^{-1}(t_0)(e^{-\beta\mu_1(t_0)}G(f|\mu_1(t_0)) + e^{-\beta\mu_2(t_0)}G(f|\mu_2(t_0))) \\ &= p_1 G(f|\mu_1(t_0)) + p_2 G(f|\mu_2(t_0)). \end{aligned} \tag{44}$$

As mentioned above, two interesting cases arise. In the case of nonoverlapping, in which $\sigma < |\mu_2(t_0) - \mu_1(t_0)|$, a measurement corresponding to a given outcome f updates the observer state of knowledge. Here the normalized post-measurement state will be one of the eigenstates of the observable measured $\hat{H}(0)$. Whereas if overlap occurs, with $\sigma > |\mu_2(t_0) - \mu_1(t_0)|$, those measurement outcomes that lie in the region of overlap between $G(f|\mu_1(t_0))$ and $G(f|\mu_2(t_0))$ do not provide a gain of information. In this case the state of knowledge of the observer will not be updated and neither will be the quantum state. We analyze both cases in more detail below.

A. Nonoverlapping case

The post-measurement state is given by a two-piecewise density matrix, defined within the intervals $I_1 = \{f|\mu_1(t_0) - \sigma/2 < f < \mu_1(t_0) + \sigma/2\}$ and $I_2 = \{f|\mu_2(t_0) - \sigma/2 < f < \mu_2(t_0) + \sigma/2\}$, i.e.,

$$\begin{aligned} \hat{\rho}(t_0^+, I_1) &= |\mu_1(t_0)\rangle\langle\mu_1(t_0)| \quad \text{and} \\ \hat{\rho}(t_0^+, I_2) &= |\mu_2(t_0)\rangle\langle\mu_2(t_0)|. \end{aligned} \tag{45}$$

The probability of an outcome f is given by Eq. (44), and the state at time t_1 is given by

$$\hat{\rho}(t_1, I_i) = \hat{U}(t_1, t_0^+)\hat{\rho}(t_0^+, I_i)\hat{U}^\dagger(t_1, t_0^+), \quad i \in \{1, 2\}. \tag{46}$$

From the state of the system at final time, the probability distribution $P(f)$ and Eq. (16) it follows that

$$e^{\xi} = \frac{2}{\beta\sigma}\sinh(\beta\sigma/2)\sum_{i=1,2} e^{-S(\hat{\rho}(t_1,I_i)||\hat{\rho}_{\rm th}(t_1))}. \tag{47}$$

The first factor shows the influence of the measuring device on the work fluctuations. Considering $\bar{G}(u) = \int df\, e^{-iuf}G(f|0)$, such factor is just $\bar{G}(i\beta)$. The second factor corresponds to ideal von Neumann measurements [20]. So we can conclude that the fluctuation theorem for work, as defined in Eq. (11), when extended to nonprojective measurements, see Eq. (7), is modified by a prefactor containing the effect of measurements.

B. Overlapping case

A more intriguing situation arises in the overlapping case. Here there is an overlapping region for possible measurement outcomes, $I_{\rm ov} = \{f|\mu_2(t_0) - \sigma/2 < f < \mu_1(t_0) + \sigma/2\}$, in which no information is gained by the observer. If $I_1 = \{f|\mu_1(t_0) - \sigma/2 < f < \mu_2(t_0) - \sigma/2\}$ and $I_2 = \{f|\mu_1(t_0) + \sigma/2 < f < \mu_2(t_0) + \sigma/2\}$, then the corresponding normalized post-measurement states with outcomes within each of those regions are given by

$$\begin{aligned} &\hat{\rho}(t_0^+, I_1) = |\mu_1(t_0)\rangle\langle\mu_1(t_0)|,\ \hat{\rho}(t_0^+, I_2) = |\mu_2(t_0)\rangle\langle\mu_2(t_0)| \text{ and} \\ &\hat{\rho}(t_0^+, I_{\rm ov}) = \hat{\rho}_{\rm th}(t_0), \end{aligned} \tag{48}$$

respectively. The state after the unitary evolution up to time t_1 takes the form

$$\hat{\rho}(t_1, I_i) = \hat{U}(t_1, t_0^+)\hat{\rho}(t_0^+, I_i)\hat{U}^\dagger(t_1, t_0^+), \tag{49}$$

with $i \in \{1, 2, \text{ov}\}$.

In this case it follows that

$$S(\hat{\rho}(t_1, I_{\rm ov})||\hat{\rho}_{\rm th}(t)) = -S(\rho(t_1, I_{\rm ov})) + p_1 S_{\rm re}(1) + p_2 S_{\rm re}(2). \tag{50}$$

To shorten the expressions we have introduced the following notation, $S_{\rm re}(i) = S(\hat{\rho}_i(t_1)||\hat{\rho}_{\rm th}(t))$ and $p_i = \mathcal{Z}^{-1}(t_0)e^{-\beta\mu_i(t_0)}$, for $i \in \{1, 2\}$.

Here, following the same steps as in the previous section, it can be concluded that

$$\begin{aligned} e^{\xi} &= \frac{2}{\beta\sigma}\sinh(\beta\sigma/2)\sum_{i=1,2} e^{-S_{\rm re}(i)} \\ &\quad + \frac{2}{\beta\sigma}(\sinh(\beta(\mu_2(t_0) - \mu_1(t_0)) - \beta\sigma/2) - \sinh(\beta\sigma/2)) \\ &\quad \times \left(p_1 e^{-S_{\rm re}(1)} + p_2 e^{-S_{\rm re}(2)} - e^{-p_1 S_{\rm re}(1) - p_2 S_{\rm re}(2)}\right). \end{aligned} \tag{51}$$

In this expression one can identify the contribution due to the nonoverlapping case, and the correction that arises due to uninformative measurement records. In contrast to what happens in the nonoverlapping case, it is striking that here the effect of the measuring device is not a global factor that multiplies a term that depends only on the system. In this situation, in which the observer is confronted with measurement outcomes that do not contribute to update her/his state of knowledge about the system, Jarzynski's equality exhibits a mixture of information concerning the system and that corresponding to the measuring apparatus. The last term in Eq. (51) contains the fluctuations of the relative entropies in the regions I_1 and I_2 and is a correction to fluctuation relation (36). Indeed, the functions $S_{\rm re}(i)$ are constant within the intervals I_i so we can write

$$p_1 e^{-S_{\rm re}(1)} + p_2 e^{-S_{\rm re}(2)} - e^{-p_1 S_{\rm re}(1) - p_2 S_{\rm re}(2)} := \left\langle e^{-S_{\rm re}}\right\rangle - e^{-\langle S_{\rm re}\rangle}. \tag{52}$$

It is worth analyzing the measurement regime, in which $\sigma \gg |\mu_2(t_0) - \mu_1(t_0)|$, the expression for e^{ξ} factorizes in a term that depends only on the meter characteristics and the temperature, and another term that does not contain any information about the meter. In such regime

$$e^{\xi} \approx \frac{2}{\beta\sigma}\sinh(\beta\sigma/2)\left(\sum_{i=1,2} e^{-S_{\rm re}(i)} + 2(\langle e^{-S_{\rm re}}\rangle - e^{-\langle S_{\rm re}\rangle})\right). \tag{53}$$

However, in the low-temperature regime the probabilities $p_1 \approx 1$ ($p_2 \approx 0$), then the quantity (52) approaches to zero and expression (47) is recovered.

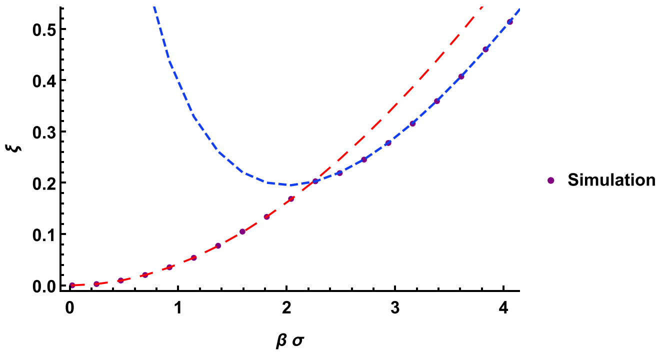

FIG. 2. The evolution of ξ vs $\beta\sigma$ in a measurement process with characteristic function (43), in a two-level system described by the Hamiltonian (54). The dots are the result of the numerical simulation. The blue dashed line corresponds to the theoretical expressions, (36) and (47), for the nonoverlapping case, and the red long-dashed line to the expressions, (42) and (51), for the overlapping case. The system parameters, $\omega_q = 2\pi \times 6.541 \times 10^9$Hz and $\Omega_R = 2\pi \times 10^6$Hz, have been taken from Ref. [48]. The temperature is $T = 0.14$K, the final time $t_f = 2\pi/3\omega_q$, and $\psi = \pi/4$.

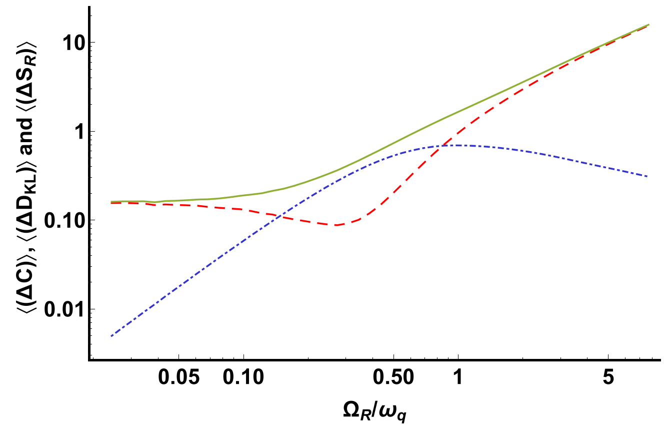

FIG. 3. Numerical results for the mean relative entropy of coherence $\langle \Delta C\rangle = \langle C_{H(t)}(\hat\rho(t,f)) - C_{H(t_0)}(\hat\rho(t_0))\rangle$ (blue dashed dotted line), the mean increment of Kullback-Leibler divergence $\langle \Delta D_{\rm KL}\rangle = \langle D_{\rm KL}(\hat\rho_D(t,f)||\hat\rho_{\rm th}(t)) - D_{\rm KL}(\hat\rho_D(t_0)||\hat\rho_{\rm th}(t_0))\rangle$ (red dashed line), and mean increment of the relative entropy $\langle \Delta S_R\rangle = \langle S(\hat\rho(t,f)||\hat\rho_{\rm th}(t_0)) - S(\hat\rho(t_0)||\hat\rho_{\rm th}(t_0))\rangle$ (green full line) vs the frequency ratio Ω_R/ω_q. Notice how the increment of the relative entropy of coherence saturates at a value ln 2 (see text). The resonant frequency has been set at $\omega_q = 2\pi \times 6.541 \times 10^9$Hz. The temperature is $T = 0.14$K, the final time $t_f = 2\pi/3\omega_q$, and $\psi = 0$ and $\sigma = 2\hbar\omega_q$. The averages $\langle * \rangle$ are taken over 10^5 realizations.

This result is an illustration of Eq. (16) as incorporates the contribution of noninformative measurements to Jarzynski equality derived in Ref. [20].

C. Numerical simulations

To numerically simulate the measurement process we have considered the characteristic function (43) and the dynamics of a two-level system described by the Hamiltonian

$$H(t) = \hbar\omega_q \frac{\sigma_z}{2} + \hbar\Omega_R\sigma_y \cos(\omega_q t + \psi), \tag{54}$$

with σ_α ($\alpha = y, z$) the Pauli matrices, ω_q the resonant frequency, ψ a phase, and Ω_R the Rabi frequency associated with the driving. This system has been implemented in a nice experiment [48] to study individual quantum state trajectories and its properties. Specifically, it has been shown that it is possible to study quantum thermodynamic properties for individual quantum trajectories, and that the results obtained from the master equation approach are consistent with the two-projective-measurement scheme.

In the numerical simulations we have considered as initial state the one corresponding to the canonical thermal state of the Hamiltonian $H(0)$, and a measurement process with characteristic function (43). Figure 2 shows the agreement between the numerical results and those obtained from the analytical expressions derived in the previous sections. In particular, the transition between the nonoverlapping and overlapping cases can be clearly observed. Note that this transition would be smoother if $G(f|0)$ were a smooth function, such as a Gaussian. In Fig. 2, the difference between the blue and red lines evidences the contribution of the region of overlap, given by the second term of Eq. (51).

Due to the initial state chosen in the measurement process, the initial relative entropy, the relative entropy of coherence and the Kullback-Leibler divergence are all zero. However, once the measurement occurs and the subsequent time evolution takes place the system will develop coherences due to the external driving, and will reach the final time with nonzero Kullback-Leibler divergence.

The relative entropy of coherence is expected to grow as the Rabi frequency increases. However, for large values of $\beta\Omega_R/\omega_q$ the coherences decrease until they decay to zero. In this limit the initial and final thermal states, the post-measurement state and the evolution operator all commute and no coherences develop. At the strict limit of no driving, in which $\Omega_R/\omega_q \to 0$, the increment of the entropy equals the von Neumann entropy of the initial state; see Fig. 3. Additionally, the range of σ values analyzed is such that small values of Ω_R/ω_q correspond to the overlapping case. Therefore, the relative entropy is given by $(p_1 S_{\rm re}(1) + p_2 S_{\rm re}(2))/\sigma$. While in the case of large values of $\beta\Omega_R/\omega_q$ the mean increment of the relative entropy behaves as $\Delta S_R \approx \Delta D_{\rm KL} \approx 2\beta\hbar\Omega_R \cos(\omega_q t_f + \psi)$.

To conclude, it is interesting to analyze the expected profile of the probability distribution of the work, according to the measurement conditions. In the TPM protocol applied to a two-level system, the work values are given by all possible differences between the eigenenergies of the final and initial Hamiltonians with some broadering due to measurements [16,18,49]. However, in the OPM protocol, there are only two possible values of work determined by Eq. (11). In a recent study [50] it has been established a detailed fluctuation theorem, showing that under an appropriate choice of the final Hamiltonian, based on the knowledge of the initial measurement and the subsequent dynamics of the system, the TPM protocol is equivalent to the OPM protocol.

In a two-level system, there are two potential values for the work variable, however, measurements introduce a highly

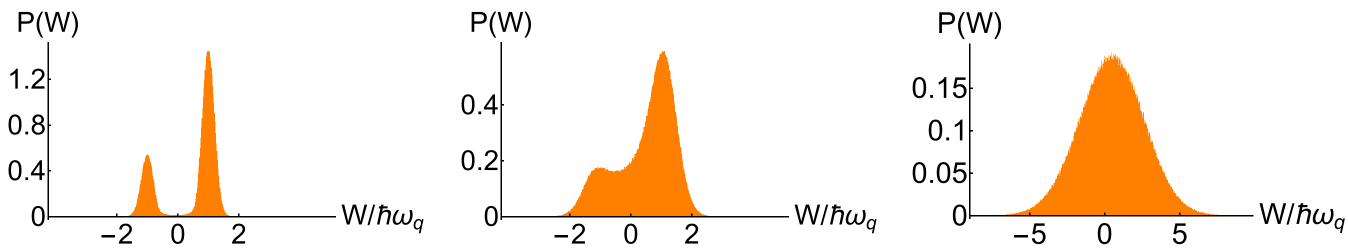

FIG. 4. Numerical results for the work distribution. The mean work for the two-level system with Hamiltonian (54), with energies $\{-\mu_1, \mu_1\}$ and initial thermal state with populations $\{p_1, p_2\}$ is given by $\langle W\rangle = (p_1 - p_2)(\langle\mu_1(t_0)|\hat{U}^\dagger(t_1, t_0^+)H(t_1)\hat{U}(t_1, t_0^+)|\mu_1(t_0)\rangle\rangle - \mu_1)$, which is independent on the precision of measurements (see text). In the example illustrated, the resonant frequency has been set at $\omega_q = 2\pi \times 6.541 \times 10^9$Hz. The temperature is $T = 1$K, the final time $t_f = 2\pi/3\omega_q$, $\psi = 0$ and $\sigma/\hbar\omega_q \in \{0.2, 0.45, 2\}$ from left to right. The histograms are build including 10^6 realizations.

localized or a broader distribution depending on whether the measurements overlap or not. This distinction determines the shape of the probability distribution of the work, resulting in either two distinct peaks in the non overlapping case or a more Gaussian-like distribution in the extreme case of measurements with very large overlap.

The two extreme cases arise when there is minimal overlap in the measurements, as well as when the measurements heavily overlap and offer limited information. In the former scenario, the ideal measurement scheme is replicated, resulting in slightly broadened δ peaks at the possible work values due to the nature of the measurements. In the latter case, the work distribution tends to assume a Gaussian-like shape centered around the mean work, with a significantly larger variance that coincides with the variance of the meter when $\sigma^2 \gg 1$. In the intermediate regime, the work distribution function exhibits a rich structure that can be derived from the knowledge of $P(f)$ and $W(t_0, t_1, f)$, given by Eqs. (9) and (11), respectively.

Figure 4 illustrates several examples of the work distributions in the system studied within these three regimes of interest.

VI. CONCLUSIONS

In this work we have analyzed the Jarzynski equality based on a protocol of a single-energy measurement proposed in Ref. [20] for nonprojective measurements. In this scheme the coherences developed in the system during the post-measurement time evolution play an important role. A comparative study of the resolution of the measurement device with respect to the distance between neighboring energy eigenvalues of the system allows different scenarios to be introduced.

If the energy resolution of the meter can resolve the individual elements of the energy spectrum, then the resulting Jarzynski equality is that which would be obtained in projective measurements except for a multiplicative factor that depends on the meter. In this case the resulting signal can be convoluted to extract the one corresponding to the Jarzynski equality. However, if the meter cannot resolve a part of the energy spectrum, then the effect of the measurement is not a factor that multiplies the Jarzynski relation.

We have analyzed a simple model of a two-level system in which, although informative measurements appear due to the bad energy resolution of the meter, in the weak measurement limit it is possible to extract a new Jarzynski equality containing an additional term related to the nonprojective measurements used.

We have also studied the behavior of coherences in a driven two-level system by means of the relative entropy coherence and the relative entropy change during a single-measurement protocol. Numerical results show that the coherences grow with the Rabi frequency until they reach a maximum, and then decay to zero as this frequency becomes larger. Also, the change in the relative entropy of the post-measurement state at final time and the reference thermal state increases linearly with the Rabi frequency. Additionally, while the average work remains unaffected by the measurement resolution, the shape of the work distribution is greatly influenced by it.

To conclude, we have shown that there are realistic systems in which it is possible to observe the corrections to the Jarzynski equality based on a protocol of a single-energy measurement.

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