Ergodicity breaking, equilibration, and nonthermalization at the many-body energy-level crossing

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This paper provides an analysis of the time evolution of a many-particle system starting out of equilibrium with its control parameter fixed at a value corresponding to a many-body energy-level crossing (degeneracy). We prove theorems concerning ergodicity, equilibration, and thermalization. For certain conditions, the occupancy of symmetrically equivalent basis states has different time-averaged probabilities. This nonergodicity remains in equilibrium. If the symmetrically equivalent states have opposite parity in relation to some physical property, then a left and right particle number imbalance averaged in time is nonzero. This imbalance does not occur for all initial basis states. In addition, the Hilbert space of the system is not fragmented; however, there is a subspace spanned by favored basis states, where the system is most likely to be found. Therefore, our results reveal what appears to be a unique mechanism for a weak eigenstates-thermalization-hypothesis breakdown, where the degenerate eigenstates can work as nonthermal eigenstates. To illustrate these findings, we consider the Hubbard Hamiltonian. In this case, ergodicity breaking produces a left and right magnetization imbalance, where the time-averaged probability of finding a spin-σ electron on one side of the crystal lattice is greater than on the other side. This imbalance is not associated with electrical charge; thus the conductance is preserved. The potential use in technology is discussed.

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I. INTRODUCTION

The existence or lack of degeneracies in the spectrum of a many-body Hamiltonian defines the behavior of several physical properties of the system. An emblematic example involves the statistics of the spacing between consecutive energy levels, which converges to either a Wigner-Dyson distribution (where level crossings are avoided) at the thermodynamic limit and far from the edges of the spectrum, indicating quantum chaos, or Poisson distribution (where level crossings are not prohibited), indicating quantum integrability [1–4]. The presence of quantum chaos implies that the system satisfies the ergodic hypothesis; conversely, the extensive number of conserved quantities in integrable systems breaks down the ergodicity. Furthermore, if the degeneracy occurs as a crossing of the two lowest energy levels under a specific set of parameters for the Hamiltonian, then the system exhibits a quantum phase transition (QPT) [5].

However, in general, a many-body energy-level crossing does not actually occur. Instead, the levels repel each other and an avoided level crossing emerges [6,7]. To understand the conditions under which a crossing takes place, let us consider a Hamiltonian $\hat{H}$ that depends on two parameters:

$$\hat{H} = J\hat{K} + U\hat{V}, \tag{1}$$

where J and U are the parameters of the system, and $\hat{K}$ and $\hat{V}$ are operators. In this way, the dimensionless control parameter (g) can be defined by $g = U/J$. Let $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ be eigenvectors of $\hat{H}$ with eigenenergies E_α and E_β, respectively. Thus a sufficient condition for the levels E_α and E_β to intersect is $[\hat{K}, \hat{V}] = 0$ [5]. If $\hat{K}$ commutes with $\hat{V}$, then $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ are also eigenvectors of both $\hat{K}$ and $\hat{V}$; therefore, $\hat{K}|\psi_{\alpha(\beta)}\rangle = \kappa_{\alpha(\beta)}|\psi_{\alpha(\beta)}\rangle$ and $\hat{V}|\psi_{\alpha(\beta)}\rangle = \upsilon_{\alpha(\beta)}|\psi_{\alpha(\beta)}\rangle$ (where κ and υ are eigenvalues of $\hat{K}$ and $\hat{V}$, respectively). From this we obtain that $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ do not depend on g and that $\hat{H}|\psi_{\alpha(\beta)}\rangle = (J\kappa_{\alpha(\beta)} + U\upsilon_{\alpha(\beta)})|\psi_{\alpha(\beta)}\rangle = E_{\alpha(\beta)}|\psi_{\alpha(\beta)}\rangle$. Hence, the degeneracy $E_\alpha = E_\beta$ arises when the parameters J and U are adjusted such that $g = U/J = (\kappa_\alpha - \kappa_\beta)/(\upsilon_\beta - \upsilon_\alpha)$. Nevertheless, $[\hat{K}, \hat{V}] \neq 0$ usually holds and thus an avoided level crossing is the most probable scenario, as per the von Neumann-Wigner theorem [6,7]. In this case ($[\hat{K}, \hat{V}] \neq 0$), an energy-level crossing can occur if $|\psi_\alpha\rangle$ or $|\psi_\beta\rangle$ satisfies the equation $\langle\psi_{\alpha\ (\text{or}\ \beta)}|[\hat{K}, \hat{V}]|\psi_{\alpha\ (\text{or}\ \beta)}\rangle = 0$; it follows that $|\psi_\alpha\rangle$ or $|\psi_\beta\rangle$ must be a state such that $\hat{K}|\psi_{\alpha\ (\text{or}\ \beta)}\rangle = 0$ or $\hat{V}|\psi_{\alpha\ (\text{or}\ \beta)}\rangle = 0$. Therefore, $|\psi_\alpha\rangle$ or $|\psi_\beta\rangle$ does not depend on g. For $\hat{H}|\psi_\alpha\rangle = J\kappa_\alpha|\psi_\alpha\rangle$ (wherein $\hat{V}|\psi_\alpha\rangle = 0$), for instance, and introducing $\varepsilon_\beta(g) \equiv E_\beta/J$, the energy-level crossing occurs at $g = g_c$ determined by the equation $\varepsilon_\beta(g_c) = \kappa_\alpha$, provided it yields a real solution.

As an example of energy-level crossing associated with $\langle\psi|[\hat{K}, \hat{V}]|\psi\rangle = 0$ (i.e., $\hat{K}|\psi\rangle = 0$ or $\hat{V}|\psi\rangle = 0$), where $|\psi\rangle$ is one of the states involved in the crossing, we have the quantum transition to a saturated ferromagnetic phase of a system of interacting itinerant electrons described by the single-band Hubbard Hamiltonian [8]. This Hamiltonian is expressed as Eq. (1), where

$$\hat{K} = \sum_{\langle r,s\rangle}\sum_{\sigma=\uparrow,\downarrow} \hat{c}^\dagger_{r,\sigma}\hat{c}_{s,\sigma}, \tag{2}$$

and

$$\hat{V} = \sum_r \hat{n}_{r,\uparrow}\hat{n}_{r,\downarrow}. \tag{3}$$

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The creation (annihilation) operator, $\hat{c}^{\dagger}_{r,\sigma}$ ($\hat{c}_{s,\sigma}$), creates (annihilates) one electron with spin σ at site r (s). The sum over $\langle r, s\rangle$ corresponds to nearest-neighbor sites. The particle number operator, $\hat{n}_{r,\sigma} = \hat{c}^{\dagger}_{r,\sigma}\hat{c}_{r,\sigma}$, provides the number of electrons with spin σ at site r ($n_{r,\sigma}$). Thus Eq. (2) describes the hopping between nearest-neighbor sites, and Eq. (3) describes the on-site interaction. The Hubbard Hamiltonian is $SU(2)$ invariant; that is, it commutes with the components of the total spin operator, $\hat{S}_z = (\hbar/2)\sum_r (\hat{n}_{r,\uparrow} - \hat{n}_{r,\downarrow})$, $\hat{S}_x = (\hat{S}^+ + \hat{S}^-)/2$, and $\hat{S}_y = (\hat{S}^+ - \hat{S}^-)/2i$, where $\hat{S}^+ = \hbar\sum_r \hat{c}^{\dagger}_{r,\uparrow}c_{r,\downarrow}$ is the raising operator and $\hat{S}^- = \hbar\sum_r \hat{c}^{\dagger}_{r,\downarrow}c_{r,\uparrow}$ is the lowering operator [9,10]. The commutation with both $\hat{S}_x$ and $\hat{S}_y$ implies the commutation with $\hat{S}^+$ and $\hat{S}^-$. Thus, denoting $|\psi\rangle^{S_{\rm tot}^{(z)}}_{S_{\rm tot}}$ as an eigenstate with total spin quantum number $S_{\rm tot}$ and z component $S^{(z)}_{\rm tot}$, the state $|\psi\rangle^{S_{\rm tot}-\lambda}_{S_{\rm tot}} \equiv (\hat{S}^-)^{\lambda}|\psi\rangle^{S_{\rm tot}}_{S_{\rm tot}}$ (where $\lambda = 1, 2, \ldots, 2S_{\rm tot}$) is also an eigenstate with the same eigenenergy as $|\psi\rangle^{S_{\rm tot}}_{S_{\rm tot}}$. A saturated ferromagnetic transition corresponds to $S_{\rm tot} < S_{\rm max} \to S_{\rm tot} = S_{\rm max}$; therefore, since $S^{(z)}_{\rm tot}$ is conserved, the saturated ferromagnetic state belongs to the Hilbert subspace with $S^{(z)}_{\rm tot} < S_{\rm max}$. For $N \leqslant N_s$ (where N is the number of electrons in the system and N_s is the number of sites in the crystal lattice), a saturated ferromagnetic eigenstate with all spin-up electrons ($S^{(z)}_{\rm tot} = S_{\rm max} = N/2$), $|\psi\rangle^{S_{\rm max}}_{S_{\rm max}}$, has no doublons (double-occupied sites) because of the Pauli exclusion principle. From this and noting that $\hat{S}^-$ does not change the number of electrons at the site, we find that a saturated ferromagnetic eigenstate with $S^{(z)}_{\rm tot} = S_{\rm max} - \lambda$, $|\psi\rangle^{S_{\rm max}-\lambda}_{S_{\rm max}}$, also has no doublons. Hence, given that $\hat{V}$ in Eq. (3) considers the number of doublons, we have $\hat{V}|\psi\rangle^{S_{\rm max}-\lambda}_{S_{\rm max}} = 0$ for all λ. Therefore, a QPT involving an eigenstate $|\psi\rangle^{S_{\rm max}-\lambda}_{S_{\rm max}}$ is associated with an energy-level crossing ($|\psi\rangle^{S_{\rm max}-\lambda}_{S_{\rm max}}$ does not depend on $g = U/J$).

For a nonequilibrium system, an important difference arises when it is driven across a quantum critical point (QCP) associated with an avoided level crossing in comparison with a QCP associated with an energy-level crossing. Let us assume that the system is driven by varying the control parameter; that is, a protocol for $g < g_c \xrightarrow{\text{time}} g > g_c$ is performed. Let $\hat{H}[g(t)]$ be a Hamiltonian in the form of Eq. (1) but with parameter g varying with time t. Now, let $\{|\psi_n(g_k)\rangle\}$ be an orthonormal basis formed by the eigenvectors of $\hat{H}(g_k)$, where g_k is the value of g at time $t = t_k$ ($t_0 < t_1 < t_2 < \cdots$); that is, $g_k = g(t_k)$. The state of the system at time t is denoted by $|\Psi(t)\rangle$. Thus, considering $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ as the states involved in the QPT, and starting from $|\Psi(0)\rangle = |\psi_0(g_0 \ll g_c)\rangle = |\psi_\alpha(g_0)\rangle$ [where $|\psi_0(g)\rangle$ denotes the ground state at g], we have the following scenarios:

(a) *QPT with avoided level crossing.* $|\psi_\alpha(g)\rangle$ has the characteristics of one phase at $g \ll g_c$ and the characteristics of the other phase at $g \gg g_c$; therefore, the QPT is associated with a change (at g_c) in the characteristics of the ground state. If the duration of the process $g = g_0 \ll g_c \to g = g_{k\gg1} \gg g_c$ is infinitely long, then the final state is $|\Psi(t_{k\gg1})\rangle = |\psi_\alpha(g_{k\gg1})\rangle$, as per the adiabatic theorem [7,11,12]. However, if the duration of this process is finite (experimental situation), the final state is $|\Psi(t_{k\gg>1})\rangle = \sum_n c_n(t_{k\gg1})|\psi_n(g_{k\gg1})\rangle$, where $c_n(t_k) = \langle\psi_n(g_k)|\Psi(t_k)\rangle$. Thus the probability for the change of state $|\psi_0(g_0)\rangle = |\psi_\alpha(g_0)\rangle \xrightarrow{\text{time}} |\psi_\beta(g_{k\gg1})\rangle \neq |\psi_0(g_{k\gg1})\rangle$ can be nonzero (for a two-level system this probability is given by the Landau-Zener formula [7,13]). Furthermore, if $|\psi_\alpha(g_{k\gg1})\rangle$ is an ordered state, then the final state $|\Psi(t_{k\gg1})\rangle$ (superposition of stationary states) describes topological defects in the ordered phase of the system, according to the Kibble-Zurek mechanism [14–18].

(b) *QPT with energy-level crossing.* At least one of two eigenstates involved in the transition is not dependent on g (let us say, $|\psi_\beta\rangle$); thus $\langle\psi_\alpha(g_0)|\psi_\beta(g_k)\rangle = \langle\psi_\alpha(g_0)|\psi_\beta(g_0)\rangle = 0$ holds, for all k. Hence, the probability for the change of state $|\psi_0(g_0)\rangle = |\psi_\alpha(g_0)\rangle \xrightarrow{\text{time}} |\psi_\beta(g_{k\gg1})\rangle = |\psi_0(g_{k\gg1})\rangle$ is always zero independently of the duration of the process $g = g_0 \ll g_c \to g = g_{k\gg1} \gg g_c$. Therefore, no phase transition occurs ($|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ have the characteristics of only one phase for all g), and the ground state simply becomes an excited state at $g > g_c$.

A case rarely reported in the literature is one where the control parameter of the system is fixed at a value corresponding to an energy-level crossing and the nonequilibrium system is left to evolve over time [19]. Herein, we analyze this case. An important and well-studied protocol to put a system out of equilibrium is the quantum quench, in which the system is initially prepared in a simple ground state of $\hat{H}(g \neq g_c)$ and then the control parameter is suddenly quenched (at $t = 0$) to the value of interest (in this work, $g = g_c$). Thus $|\psi_0(g \neq g_c)\rangle$ becomes the initial state of the system described by $\hat{H}(g = g_c)$. However, as discussed in (b), a conventional quantum-quench protocol is not suitable for probing an energy-level crossing because the (initial) state $|\psi_\alpha(g)\rangle$ is orthogonal to $|\psi_\beta(g_c)\rangle$ for all g.

In this work, we use as the initial state a basis state that overlaps with the degenerate eigenstates, and then we investigate the ergodicity of the system. For an arbitrary time-independent Hamiltonian that depends on one dimensionless control parameter, we prove that under certain conditions, two basis states that are symmetrically equivalent to each other, where one is transformed into the other by the action of an operator that commutes with the Hamiltonian, are not equally visited by the system (Sec. II). From this result, we address questions about whether a system that does not equally visit symmetrically equivalent states reaches equilibrium and whether the nonergodicity remains in equilibrium. Assuming one two-level crossing, we prove that the answers to both questions are "yes" (Sec. III). The convergence to equilibrium has already been proven by assuming a Hamiltonian with nondegenerate energy gaps [20,21]. However, nondegenerate energy gaps imply the nondegeneracy of the energy levels; therefore, these proofs do not hold in the context observed here.

Another issue we consider is thermalization. An isolated quantum system thermalizes if the time-averaged expectation value of a (few-body) observable coincides with the value predicted by a statistical-mechanical ensemble [22–25]. We prove the possibility of a left and right particle number imbalance, indicating nonthermalization, and discuss the mechanism for the breakdown of the eigenstate thermalization hypothesis (Sec. IV). Finally, we illustrate these findings in the concrete case of the Hubbard Hamiltonian (Sec. V), where the left and right imbalance that emerges is with respect to magnetization. Lastly, technological aspects are analyzed.

II. ERGODICITY

Let $\hat{H}(g)$ be an arbitrary time-independent Hamiltonian that depends on one dimensionless control parameter g. Let us introduce the following terminology:

(1) $\{|\psi_n(g)\rangle\}$ is a complete orthonormal set of eigenvectors of $\hat{H}(g)$. $\{E_n(g)\}$ are the correspondent eigenvalues.

(2) $(\alpha,\beta)_{g_c}$ denotes a degenerate pair at g_c. Thus $|\psi_\alpha(g)\rangle$ and $|\psi_\beta(g)\rangle$ are eigenstates in which $E_\alpha(g_c)=E_\beta(g_c)$.

(3) $\{|\phi_j\rangle\}$ is a complete orthonormal set of basis states.

(4) $\{k,q\}$ denotes a pair of symmetrically equivalent states (SESs). By symmetrically equivalent states we mean that there is an operator, $\hat{F}$, wherein $\hat{F}|\phi_{k(q)}\rangle=|\phi_{q(k)}\rangle$ and $[\hat{H}(g),\hat{F}]=0$.

From this, we prove the following theorem on SESs.

Theorem 1. In the linear combination that results in an eigenvector of $\hat{H}(g)$, the weights of the two basis states constituting a pair of SESs are equal in magnitude.

Proof. From definition (4), we conclude that $\hat{F}(|\phi_k\rangle\pm|\phi_q\rangle)=\pm(|\phi_k\rangle\pm|\phi_q\rangle)$. Therefore, an eigenvector of $\hat{F}$, $|\Phi_\nu\rangle$, written on the basis $\{|\phi_j\rangle\}$, $|\Phi_\nu\rangle=\cdots+f_k^{(\nu)}|\phi_k\rangle+f_q^{(\nu)}|\phi_q\rangle+\cdots$ (where $f_k^{(\nu)}=\langle\phi_k|\Phi_\nu\rangle$), must satisfy the condition $f_q^{(\nu)}=\pm f_k^{(\nu)}$ for any pair $\{k,q\}$. By definition, $\hat{F}$ commutes with the Hamiltonian; thus $\hat{H}|\Phi_\nu\rangle$ is also an eigenvector of $\hat{F}$ and it follows that $\hat{H}$ acting on $|\Phi_\nu\rangle$ does not produce $f_q^{(\nu)}\neq\pm f_k^{(\nu)}$. Considering that $\{|\Phi_\nu\rangle\}$ is a complete orthonormal set and writing $|\psi_n(g)\rangle$ on this basis, we have

$$\begin{aligned}|\psi_n(g)\rangle&=\sum_\nu d_\nu^{(n)}(g)|\Phi_\nu\rangle\\&=\sum_\nu d_\nu^{(n)}(g)[\cdots+f_k^{(\nu)}(|\phi_k\rangle\pm|\phi_q\rangle)+\cdots]\\&=\cdots+\left[\sum_\nu d_\nu^{(n)}(g)f_k^{(\nu)}\right](|\phi_k\rangle\pm|\phi_q\rangle)+\cdots,\end{aligned}$$

where $d_\nu^{(n)}(g)=\langle\Phi_\nu|\psi_n(g)\rangle$. Comparing with $|\psi_n(g)\rangle$ on the basis $\{|\phi_j\rangle\}$, $|\psi_n(g)\rangle=\sum_j a_j^{(n)}(g)|\phi_j\rangle$ (where $a_j^{(n)}(g)=\langle\phi_j|\psi_n(g)\rangle$), we obtain

$$a_q^{(n)}(g)=\pm\sum_\nu d_\nu^{(n)}(g)f_k^{(\nu)}=\pm a_k^{(n)}(g)\ \text{ for all } n. \tag{4}$$

Therefore, we find that

$$|\langle\phi_k|\psi_n(g)\rangle|=|\langle\phi_q|\psi_n(g)\rangle|\ \text{ for all } n. \tag{5}$$

Q.E.D.

As a consequence of Theorem 1, we have the following.

Corollary. Let $\{k,q\}$ be a pair of SESs. Then, the basis states $|\phi_k\rangle$ and $|\phi_q\rangle$ have the same energy expectation value.

Proof. By expanding $|\phi_k\rangle=\sum_n c_n^{(k)}(g)|\psi_n(g)\rangle$ [where $c_n^{(k)}(g)=\langle\psi_n(g)|\phi_k\rangle$], the energy expectation value for the system in the state $|\phi_k\rangle$ is calculated as $\langle\phi_k|\hat{H}(g)|\phi_k\rangle=\sum_n|c_n^{(k)}|^2E_n(g)$. Similarly, $\langle\phi_q|\hat{H}(g)|\phi_q\rangle=\sum_n|c_n^{(q)}|^2E_n(g)$. As $c_n^{(k)}(g)=a_k^{(n)}(g)^*$, from Eq. (4), we find that $|c_n^{(k)}(g)|=|c_n^{(q)}(g)|$. Therefore, $\langle\phi_k|\hat{H}(g)|\phi_k\rangle=\langle\phi_q|\hat{H}(g)|\phi_q\rangle$. Q.E.D.

If a system is ergodic, then the time-averaged probability is the same for all basis states having the same energy expectation value. Thus, from the Corollary, we can conclude that a system is nonergodic if the states constituting a pair of SESs have different probabilities over an infinitely long timescale. To analyze the ergodicity, we consider the system initially prepared in a basis state, $|\phi_l\rangle\in\{|\phi_j\rangle\}$, and evolving in time according to $\hat{H}(g)$. In this way, the state of the system at time t is given by $|\Psi_{l,g}(t)\rangle=e^{-i\hat{H}(g)t/\hbar}|\phi_l\rangle$. Using $|\phi_l\rangle=\sum_n c_n^{(l)}(g)|\psi_n(g)\rangle$ and $|\psi_n(g)\rangle=\sum_j a_j^{(n)}(g)|\phi_j\rangle$, we obtain the following:

$$|\Psi_{l,g}(t)\rangle=\sum_j b_{l,j}(g,t)|\phi_j\rangle, \tag{6}$$

where

$$\begin{aligned}b_{l,j}(g,t)&=\langle\phi_j|e^{-i\hat{H}(g)t/\hbar}|\phi_l\rangle\\&=\sum_n a_l^{(n)}(g)^*a_j^{(n)}(g)e^{-iE_n(g)t/\hbar}.\end{aligned} \tag{7}$$

From Eq. (7), the probability of finding the system in $|\phi_j\rangle$ at time t after initializing in $|\phi_l\rangle$ is given by

$$\begin{aligned}P_{l,j}(g,t)&=|b_{l,j}(g,t)|^2\\&=\sum_{n,m}a_j^{(n)}(g)^*a_l^{(n)}(g)a_l^{(m)}(g)^*a_j^{(m)}(g)e^{i[E_n(g)-E_m(g)]t/\hbar}.\end{aligned} \tag{8}$$

The time-averaged probability, $\bar{P}_{l,j}(g)$, is expressed as follows:

$$\bar{P}_{l,j}(g)=\lim_{t\to\infty}\frac{1}{t}\int_0^t P_{l,j}(g,t')dt'. \tag{9}$$

Thus substituting Eq. (8) in Eq. (9) yields

$$\bar{P}_{l,j}(g)=\sum_{n,m}a_j^{(n)}(g)^*a_l^{(n)}(g)a_l^{(m)}(g)^*a_j^{(m)}(g)\delta_{E_n,E_m}, \tag{10}$$

where $\delta_{E_n,E_m}=1$ for $E_n=E_m$ (i.e., $n=m$ or degeneracy) and 0 otherwise. For the sum in Eq. (10), all the degenerate pairs at g must be considered. Hence, a quantum critical point and an excited energy-level crossing have the same status.

Theorem 2 (ergodicity breaking). Let $(\alpha,\beta)_{g_c}$ and $\{k,q\}$ be a degenerate pair and a pair of SESs, respectively, of the system evolved by the time-independent Hamiltonian $\hat{H}(g_c)$, and let $|\phi_l\rangle$ be the initial state. Then, $\bar{P}_{l,k}(g_c)\neq\bar{P}_{l,q}(g_c)$ if the following are true:

(i) at g_c, the only degeneracy is that of the pair $(\alpha,\beta)_{g_c}$;

(ii) $\{k,q\}$ satisfies $a_k^{(\alpha)}(g_c)^*a_k^{(\beta)}(g_c)=-a_q^{(\alpha)}(g_c)^*a_q^{(\beta)}(g_c)\neq 0$; and

(iii) $|\phi_l\rangle$ satisfies $a_l^{(\alpha)}(g_c)\neq 0$ and $a_l^{(\beta)}(g_c)\neq 0$.

Proof. From Eq. (10) and condition (i), we find that

$$\begin{aligned}\bar{P}_{l,j}(g_c)&=\sum_n\left|a_l^{(n)}(g_c)\right|^2\left|a_j^{(n)}(g_c)\right|^2\\&\quad+a_j^{(\alpha)}(g_c)^*a_l^{(\alpha)}(g_c)a_l^{(\beta)}(g_c)^*a_j^{(\beta)}(g_c)\\&\quad+a_j^{(\beta)}(g_c)^*a_l^{(\beta)}(g_c)a_l^{(\alpha)}(g_c)^*a_j^{(\alpha)}(g_c)\\&=\sum_n\left|a_l^{(n)}(g_c)\right|^2\left|a_j^{(n)}(g_c)\right|^2\\&\quad+2\mathrm{Re}\left[a_j^{(\alpha)}(g_c)^*a_l^{(\alpha)}(g_c)a_l^{(\beta)}(g_c)^*a_j^{(\beta)}(g_c)\right].\end{aligned} \tag{11}$$

By using $j = k$ and $j = q$ in Eq. (11), we obtain

$$\bar{P}_{l,k}(g_c) = \sum_n \left|a_l^{(n)}(g_c)\right|^2 \left|a_k^{(n)}(g_c)\right|^2 + 2\mathrm{Re}\left[a_k^{(\alpha)}(g_c)^* a_l^{(\alpha)}(g_c) a_l^{(\beta)}(g_c)^* a_k^{(\beta)}(g_c)\right], \quad (12)$$

$$\bar{P}_{l,q}(g_c) = \sum_n \left|a_l^{(n)}(g_c)\right|^2 \left|a_q^{(n)}(g_c)\right|^2 + 2\mathrm{Re}\left[a_q^{(\alpha)}(g_c)^* a_l^{(\alpha)}(g_c) a_l^{(\beta)}(g_c)^* a_q^{(\beta)}(g_c)\right]. \quad (13)$$

Considering condition (ii) and substituting Eq. (4) in Eq. (13), we have

$$\bar{P}_{l,q}(g_c) = \sum_n \left|a_l^{(n)}(g_c)\right|^2 \left|a_k^{(n)}(g_c)\right|^2 - 2\mathrm{Re}\left[a_k^{(n_1)}(g_c)^* a_l^{(n_1)}(g_c) a_l^{(n_2)}(g_c)^* a_k^{(n_2)}(g_c)\right]. \quad (14)$$

As observed from conditions (ii) and (iii), the amplitudes $a_k^{(\alpha)}(g_c)$, $a_k^{(\beta)}(g_c)$, $a_l^{(\alpha)}(g_c)$, and $a_l^{(\beta)}(g_c)$ are all nonzero. Thus, on comparing Eqs. (12) and (14), we conclude that $\bar{P}_{l,k}(g_c) \neq \bar{P}_{l,q}(g_c)$. Q.E.D.

Therefore, the system is nonergodic when conditions (i)–(iii) in Theorem 2 are fulfilled. With regard to these conditions, it is important to analyze each one of them. Condition (iii) is necessary because otherwise the initial state will not be sensitive to degeneracy (see Sec. I). The feasibility of condition (ii) is assured by Theorem 1. However, condition (i) can be subject to two criticisms. First, specifying only one double degeneracy is an excessively restrictive condition, and other degeneracies actually occur. In this case, it is difficult to obtain a precise balance in the sum of Eq. (10) such that $\bar{P}_{l,k}$ equals $\bar{P}_{l,q}$. Indeed, in the unrealistic case where all the eigenvectors of the Hamiltonian have the same energy at g_c, from Eq. (7), we find that $|b_{l,j}(g_c, t)| = |b_{l,j}(g_c, 0)|$, which implies that $P_{l,j}(g_c, t) = P_{l,j}(g_c, 0)$ [see Eq. (8)]. Thus the system stays in the initial state at all times, and even if the initial state belongs to a pair of SESs, the other state of the pair is never visited by the system. Second, only one double degeneracy is an irrelevant aspect for the nonequilibrium quantum dynamics of a large number of particles because the many-body spectrum is highly dense. Nevertheless, quantum many-body scars are examples of cases where few eigenstates can be decisive for the real-time dynamics [26–28]. Moreover, the breakdown of the equal time-averaged probabilities occurs for any pair of SESs that satisfies condition (ii). Thus, even for just one two-level crossing, there may be a large number of pairs of SESs that fulfill this condition. Indeed, in Sec. V, we show that for a general single-band tight-binding Hamiltonian, the number of states belonging to pairs of SESs is greater than otherwise.

An interesting choice is the initial state belonging to a pair of SESs, $\{l, \ell\}$. Using Eqs. (12) and (14), we obtain the following:

$$\bar{P}_{l,l}(g_c) = \sum_n \left|a_l^{(n)}(g_c)\right|^4 + 2\left|a_l^{(\alpha)}(g_c)\right|^2 \left|a_l^{(\beta)}(g_c)\right|^2, \quad (15)$$

$$\bar{P}_{l,\ell}(g_c) = \sum_n \left|a_l^{(n)}(g_c)\right|^4 - 2\left|a_l^{(\alpha)}(g_c)\right|^2 \left|a_l^{(\beta)}(g_c)\right|^2. \quad (16)$$

The quantity $\bar{P}_{l,l}(g_c)$ can be interpreted as the time-averaged Loschmidt echo. Therefore, the time-averaged probability of the system returning to its initial state is shifted up by an amount $2\left|a_l^{(\alpha)}(g_c)\right|^2 \left|a_l^{(\beta)}(g_c)\right|^2$ as a result of the degeneracy [Eq. (15)]. Thus, if the initial state is a SES, then the degeneracy works as a mechanism by which the system memorizes its initial state.

III. EQUILIBRATION

An important question raised by Theorem 2 is whether a system that does not equally visit SESs reaches equilibrium. An observable, O, with corresponding operator $\hat{O}$ relaxes to an equilibrium value if its expectation value, $\langle O\rangle_t = \langle\Psi(t)|\hat{O}|\Psi(t)\rangle$, remains close to the equilibrium value, $\langle O\rangle_{\mathrm{eq}}$, for almost all time, that is, $\overline{[\langle O\rangle_t - \langle O\rangle_{\mathrm{eq}}]^2} \ll \langle O\rangle_{\mathrm{eq}}^2$, where $\overline{(\cdots)} = \lim_{t\to\infty} t^{-1}\int_0^t (\cdots)dt'$ [29,30]. Using the density matrix, $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$, we have $\langle O\rangle_t = \mathrm{Tr}[\rho(t)\hat{O}]$, and thus $\langle O\rangle_{\mathrm{eq}} = \mathrm{Tr}(\rho_{\mathrm{eq}}\hat{O})$ is the value averaged over an equilibrium ensemble, ρ_{eq}. It is reasonable to assume that if the expectation value of a particular observable relaxes, then it must be to its time-averaged value, i.e., $\rho_{\mathrm{eq}} = \overline{\rho(t)}$.

Now, inspired by Ref. [20], we prove the following theorem.

Theorem 3 (equilibration). Let $H(g_c)$ be a time-independent Hamiltonian with only one degenerate pair and nondegenerate energy gaps apart from the trivial degeneracy with relation to the two levels that intersect at g_c. Consider O to be an observable representing an experimental device with a finite range (Δ) of possible outcomes for a measurement. Then, for the system evolving under $H(g_c)$ after a measurement of O, i.e., starting from nonstationary initial state $|\phi_l\rangle$ (an eigenvector of $\hat{O}$), the difference between the expectation value of O at time t and its time-averaged value is such that the following is observed:

$$\overline{[\langle O\rangle_t(l, g_c) - \overline{\langle O\rangle}(l, g_c)]^2} \leqslant 5\Delta^2 \max_n p_n^{(l)}(g_c), \quad (17)$$

where $p_n^l(g_c)$ is the probabilistic weight of the nth eigenvector of $H(g_c)$ in the linear combination that results in $|\phi_l\rangle$.

Proof. Taking as basis states the eigenvectors of $\hat{O}$, i.e., $\hat{O}|\phi_j\rangle = O_j|\phi_j\rangle$, from Eqs. (7) and (8), we have

$$\langle O\rangle_t(l, g) = \langle\Psi_{l,g}(t)|\hat{O}|\Psi_{l,g}(t)\rangle = \sum_j P_{l,j}(g, t)O_j. \quad (18)$$

By using Eqs. (9) and (10), it follows that

$$\overline{\langle O\rangle}_t(l, g) = \sum_j \overline{P}_{l,j}(g)O_j. \quad (19)$$

Rewriting Eq. (8) as $P_{l,j}(g, t) = \sum_{n,m} b_{l,j}^{n,m}(g)\, e^{i[E_n(g)-E_m(g)]t/\hbar}$ and Eq. (10) as $\bar{P}_{l,j}(g) = \sum_{n,m} b_{l,j}^{n,m}(g)\, \delta_{E_m,E_n}$, where $b_{l,j}^{n,m}(g) \equiv a_j^{(n)}(g)^* a_l^{(n)}(g) a_l^{(m)}(g)^* a_j^{(m)}(g)$, we can define $\tilde{P}_{l,j}(g, t) \equiv P_{l,j}(g, t) - \bar{P}_{l,j}(g) = \sum_{n,m} \tilde{b}_{l,j}^{n,m}(g) e^{i[E_n(g)-E_m(g)]t/\hbar}$, where $\tilde{b}_{l,j}^{n,m}(g) \equiv b_{l,j}^{n,m}(g)(1 - \delta_{E_n,E_m})$. With this, and as is assumed only one two-level crossing $[E_\alpha(g_c) = E_\beta(g_c)]$, we have $\tilde{b}_{l,j}^{n,n}(g_c) = 0$ and

$\tilde{b}^{\alpha,\beta}_{l,j}(g_c) = \tilde{b}^{\beta,\alpha}_{l,j}(g_c) = 0$. Thus

$$\overline{[\langle O\rangle_t(l,g_c) - \overline{\langle O\rangle_t}(l,g_c)]^2} = \sum_{j,j'} \overline{\tilde{P}_{l,j}(g_c,t)\tilde{P}_{l,j'}(g_c,t)} O_j O_{j'}$$
$$= \sum_{j,j'} \sum_{n_1,m_1,n_2,m_2} \tilde{b}^{n_1,m_1}_{l,j}(g_c) O_j \tilde{b}^{n_2,m_2}_{l,j'}(g_c) O_{j'} \delta_{E_{n_1}(g_c)-E_{m_1}(g_c)+E_{n_2}(g_c)-E_{m_2}(g_c),0}. \tag{20}$$

Since $a^{(n)}_j(g) = \langle\phi_j|\psi_n(g)\rangle$, and using completeness relations $\sum_n |\psi_n(g)\rangle\langle\psi_n(g)| = 1$ and $\sum_j |\phi_j\rangle\langle\phi_j| = 1$, we calculate the following:

$$\sum_j P_{l,j}(g_c,t) = \sum_j \sum_{n,m} b^{n,m}_{l,j}(g_c) e^{i[E_n(g_c)-E_m(g_c)]t/\hbar}$$
$$= \sum_{n,m} \langle\phi_l|\psi_n(g_c)\rangle \underbrace{\left[\sum_j \langle\psi_n(g_c)|\phi_j\rangle\langle\phi_j|\psi_m(g_c)\rangle\right]}_{=\delta_{n,m}} \langle\psi_m(g_c)|\phi_l\rangle e^{i[E_n(g_c)-E_m(g_c)]t/\hbar} = 1,$$

and

$$\sum_j \bar{P}_{l,j}(g_c) = \sum_j \sum_{n,m} b^{n,m}_{l,j}(g_c)\delta_{E_n,E_m} = \sum_j \sum_n b^{n,n}_{l,j}(g_c) + \sum_j b^{\alpha,\beta}_{l,j}(g_c) + \sum_j b^{\beta,\alpha}_{l,j}(g_c)$$
$$= \sum_n \langle\phi_l|\psi_n(g_c)\rangle \underbrace{\left[\sum_j \langle\psi_n(g_c)|\phi_j\rangle\langle\phi_j|\psi_n(g_c)\rangle\right]}_{=1} \langle\psi_n(g_c)|\phi_l\rangle$$
$$+ \langle\phi_l|\psi_\alpha(g_c)\rangle \underbrace{\left[\sum_j \langle\psi_\alpha(g_c)|\phi_j\rangle\langle\phi_j|\psi_\beta(g_c)\rangle\right]}_{=0} \langle\psi_\beta(g_c)|\phi_l\rangle$$
$$+ \langle\phi_l|\psi_\beta(g_c)\rangle \underbrace{\left[\sum_j \langle\psi_\beta(g_c)|\phi_j\rangle\langle\phi_j|\psi_\alpha(g_c)\rangle\right]}_{=0} \langle\psi_\alpha(g_c)|\phi_l\rangle = 1.$$

Therefore, $\sum_j \tilde{P}_{l,j}(g_c,t) = \sum_j [P_{l,j}(g_c,t) - \bar{P}_{l,j}(g_c)] = 0$. From this, and considering $O_{\min} \leqslant O_j \leqslant O_{\max}$ (for all j), where it is possible to define a non-negative auxiliary quantity, $\tilde{O}_j \equiv O_j - O_{\min}$ ($0 \leqslant \tilde{O}_j \leqslant \Delta$ and $\Delta \equiv O_{\max} - O_{\min}$), we obtain

$$\overline{[\langle\tilde{O}\rangle_t(l,g_c) - \overline{\langle\tilde{O}\rangle_t}(l,g_c)]^2} = \sum_{j,j'} \overline{\tilde{P}_{l,j}(g_c,t)\tilde{P}_{l,j'}(g_c,t)}\tilde{O}_j\tilde{O}_{j'} = \sum_{j,j'} \overline{\tilde{P}_{l,j}(g_c,t)\tilde{P}_{l,j'}(g_c,t)} O_j O_{j'} = \overline{[\langle O\rangle_t(l,g_c) - \overline{\langle O\rangle_t}(l,g_c)]^2}. \tag{21}$$

For nondegenerate energy gaps apart from the trivial degeneracy with relation to levels E_α and E_β, in Eq. (20), excluding the cases where $\tilde{b}^{n_1,m_1}_{l,j}(g_c) = 0$ or $\tilde{b}^{n_2,m_2}_{l,j'}(g_c) = 0$, the condition $E_{n_1}(g_c) - E_{m_1}(g_c) + E_{n_2}(g_c) - E_{m_2}(g_c) = 0$ implies that $n_1 = m_2$ and $m_1 = n_2$; or $n_1 = m_2 \neq \alpha, \beta$, $m_1 = \alpha(\beta)$, and $n_2 = \beta(\alpha)$; or $m_1 = n_2 \neq \alpha, \beta$, $n_1 = \alpha(\beta)$, and $m_2 = \beta(\alpha)$. Hence, Eq. (21) is modified as follows:

$$\overline{[\langle O\rangle_t(l,g_c) - \overline{\langle O\rangle_t}(l,g_c)]^2} = \sum_{j,j'} \tilde{O}_j\tilde{O}_{j'} \Bigg\{ \sum_{\substack{n,\,m \\ [n\neq m,\ n(m)\neq\alpha(\beta),\ n(m)\neq\beta(\alpha)]}} \tilde{b}^{n,m}_{l,j}(g_c)\tilde{b}^{m,n}_{l,j'}(g_c)$$
$$+ \sum_{\substack{n \\ (n\neq\alpha,\,\beta)}} \left[\tilde{b}^{n,\alpha}_{l,j}(g_c)\tilde{b}^{\beta,n}_{l,j'}(g_c) + \tilde{b}^{n,\beta}_{l,j}(g_c)\tilde{b}^{\alpha,n}_{l,j'}(g_c)\right]$$
$$+ \sum_{\substack{m \\ (m\neq\alpha,\,\beta)}} \left[\tilde{b}^{\alpha,m}_{l,j}(g_c)\tilde{b}^{m,\beta}_{l,j'}(g_c) + \tilde{b}^{\beta,m}_{l,j}(g_c)\tilde{b}^{m,\alpha}_{l,j'}(g_c)\right] \Bigg\}.$$

Since $\tilde{b}^{n,n}_{l,j}(g_c) = \tilde{b}^{\alpha,\beta}_{l,j}(g_c) = 0$, the restrictions on the sums can be dropped, and we find that

$$\overline{[\langle O\rangle_t(l,g_c) - \overline{\langle O\rangle_t}(l,g_c)]^2} = \sum_{j,j'} \tilde{O}_j \tilde{O}_{j'} \left\{ \sum_{n,m} \tilde{b}^{n,m}_{l,j}(g_c)\tilde{b}^{m,n}_{l,j'}(g_c) + 2\sum_{n} \left[\tilde{b}^{\alpha,n}_{l,j}(g_c)\tilde{b}^{n,\beta}_{l,j'}(g_c) + \tilde{b}^{\beta,n}_{l,j}(g_c)\tilde{b}^{n,\alpha}_{l,j'}(g_c)\right] \right\}.$$

By using $\tilde{b}^{m,n}_{l,j}(g) = \tilde{b}^{n,m}_{l,j}(g)^*$, $0 \leqslant \tilde{O}_j \leqslant \Delta$, and $|\tilde{b}^{n,m}_{l,j}(g)| \leqslant |b^{n,m}_{l,j}(g)|$, we can conclude that

$$\begin{aligned}\overline{[\langle O\rangle_t(l,g_c) - \overline{\langle O\rangle_t}(l,g_c)]^2} &= \sum_{n,m} \left| \sum_j \tilde{b}^{n,m}_{l,j}(g_c)\tilde{O}_j \right|^2 + 4\mathrm{Re}\sum_n \sum_{j,j'} \tilde{b}^{\alpha,n}_{l,j}(g_c)\tilde{b}^{n,\beta}_{l,j'}(g_c)\tilde{O}_j\tilde{O}_{j'} \\ &\leqslant \Delta^2 \sum_{n,m}\sum_j \left|b^{n,m}_{l,j}(g_c)\right|^2 + 4\left| \sum_n \sum_{j,j'} \tilde{b}^{\alpha,n}_{l,j}(g_c)\tilde{b}^{n,\beta}_{l,j'}(g_c)\tilde{O}_j\tilde{O}_{j'} \right|. \end{aligned} \tag{22}$$

Analyzing the first term on the right-hand side of Eq. (22), we have

$$\begin{aligned}\sum_{n,m,j} \left|b^{n,m}_{l,j}(g_c)\right|^2 &= \sum_{n,m,j} \left|a^{(n)}_j(g_c)^* a^{(n)}_l(g_c) a^{(m)}_l(g_c)^* a^{(m)}_j(g_c)\right|^2 \\ &= \sum_{n,m,j} \left|a^{(n)}_j(g_c)\right|^2 \left|a^{(n)}_l(g_c)\right|^2 \left|a^{(m)}_l(g_c)\right|^2 \left|a^{(m)}_j(g_c)\right|^2 \\ &\leqslant \max_n \left|a^{(n)}_l(g_c)\right|^2 \sum_m \left|a^{(m)}_l(g_c)\right|^2 \sum_j \left|a^{(m)}_j(g_c)\right|^2 \sum_n \left|a^{(n)}_j(g_c)\right|^2 \leqslant \max_n \left|a^{(n)}_l(g_c)\right|^2. \end{aligned} \tag{23}$$

For the second term on the right-hand side of Eq. (22), it is important to note the following:

$$\begin{aligned}|\langle\psi_n(g_c)|\hat{\tilde{O}}|\psi_m(g_c)\rangle| &\leqslant \sqrt{\langle\psi_n(g_c)|\psi_n(g_c)\rangle}\sqrt{\langle\psi_m(g_c)|\hat{\tilde{O}}^\dagger\hat{\tilde{O}}|\psi_m(g_c)\rangle} \\ &\leqslant \sqrt{\langle\psi_m(g_c)|\hat{\tilde{O}}^2|\psi_m(g_c)\rangle} \\ &\leqslant \sqrt{\sum_j \left|a^{(m)}_j(g_c)\right|^2 \tilde{O}_j^2} \\ &\leqslant \sqrt{\Delta^2 \sum_j \left|a^{(m)}_j(g_c)\right|^2} \\ &\leqslant \Delta, \end{aligned}$$

where in the first line we use the Cauchy-Schwarz inequality, and the second line is a consequence of $\hat{\tilde{O}} \equiv \hat{O} - O_{\min}\hat{I}$ ($\hat{I}$ denotes the identity operator) being a Hermitian operator. With this we obtain

$$\begin{aligned}\left| \sum_n \sum_{j,j'} \tilde{b}^{\alpha,n}_{l,j}(g_c)\tilde{b}^{n,\beta}_{l,j'}(g_c)\tilde{O}_j\tilde{O}_{j'} \right| &= \left| \sum_{\substack{n \\ (n\neq\alpha,\,\beta)}} \sum_{j,j'} b^{\alpha,n}_{l,j}(g_c) b^{n,\beta}_{l,j'}(g_c)\tilde{O}_j\tilde{O}_{j'} \right| \\ &= \left| a^{(\alpha)}_l(g_c) a^{(\beta)}_l(g_c)^* \sum_{\substack{n \\ (n\neq p,\,q)}} \left|a^{(n)}_l(g_c)\right|^2 \underbrace{\sum_j a^{(\alpha)}_j(g_c)^* a^{(n)}_j(g_c)\tilde{O}_j}_{=\langle\psi_\alpha(g_c)|\hat{\tilde{O}}|\psi_n(g_c)\rangle} \underbrace{\sum_{j'} a^{(n)}_{j'}(g_c)^* a^{(\beta)}_{j'}(g_c)\tilde{O}_{j'}}_{=\langle\psi_n(g_c)|\hat{\tilde{O}}|\psi_\beta(g_c)\rangle} \right| \\ &\leqslant \left|a^{(\alpha)}_l(g_c)\right|\left|a^{(\beta)}_l(g_c)\right| \sum_{\substack{n \\ (n\neq p,\,q)}} \left|a^{(n)}_l(g_c)\right|^2 |\langle\psi_\alpha(g_c)|\hat{\tilde{O}}|\psi_n(g_c)\rangle||\langle\psi_n(g_c)|\hat{\tilde{O}}|\psi_\beta(g_c)\rangle| \\ &\leqslant \left|a^{(\alpha)}_l(g_c)\right|\left|a^{(\beta)}_l(g_c)\right| \Delta^2 \sum_n \left|a^{(n)}_l(g_c)\right|^2 \leqslant \Delta^2 \max_n \left|a^{(n)}_l(g_c)\right|^2. \end{aligned} \tag{24}$$

Defining $p_n^{(l)}(g) \equiv |a_l^{(n)}(g)|^2$ as the probabilistic weight of eigenvector $|\psi_n(g)\rangle$ in the linear combination that results in the initial state $|\phi_l\rangle$, and substituting Eqs. (23) and (24) in Eq. (22), we finally arrive at Eq. (17). Q.E.D.

From Chebyshev's inequality with $\langle O\rangle_t(l,g)$ being a random variable with a mean value given by Eq. (19), variance given by Eq. (17), and standard deviation (χ) bounded by $\chi \leqslant \Delta\eta\sqrt{5\eta}$ [where $\eta^3 \equiv \max_n p_n^{(l)}(g_c)$] [20,31], we obtain the following:

$$P[|\langle O\rangle_t(l,g_c) - \overline{\langle O\rangle_t}(l,g_c)| > \Delta\eta] \leqslant 5\eta, \qquad (25)$$

where $P(x)$ denotes the probability that a random variable takes on value x.

Let us now analyze the magnitude of η and $\Delta\eta$. Consider $d_l \equiv \sum_n (1 - \delta_{\langle\psi_n|\phi_l\rangle,0})$ as the effective dimension of the initial state $|\phi_l\rangle$, that is, the number of states $|\psi_n\rangle$ such that $\langle\psi_n|\phi_l\rangle \neq 0$. The information entropy of $|\phi_l\rangle$ (S_l), which is the average level of uncertainty about $|\phi_l\rangle$ when a $|\psi_n\rangle$ is measured, can be calculated by $S_l = \sum_n p_n^{(l)} \ln(1/p_n^{(l)})$ [31–33]. Thus S_l becomes larger as distribution $p_n^{(l)}$ becomes more uniform. In this case, $p_n^{(l)} = (1 - \delta_{\langle\psi_n|\phi_l\rangle,0})/d_l$ and $S_l = \ln(d_l)$; it follows that S_l increases with d_l and vice versa. As the entropy of a state is associated with the number of different ways wherein the system arrives at it according to the dynamics given by the Hamiltonian, we conclude that d_l increases with $w_l \equiv \sum_j (1 - \delta_{\langle\phi_l|\hat{H}|\phi_j\rangle,0})$. Considering, for example, nearest-neighbor tight-binding Hamiltonians on a bipartite lattice, such as the Hubbard Hamiltonian [Eqs. (1)–(3)], the state with $N^\uparrow = N^\downarrow = N_s/2$ (where N^σ is the number of spin-σ electrons of the system) and an antiferromagnetic configuration has $w_l = z^{N_s}$ (where z is the coordination number of the lattice). In addition, $w_l = z^{N_s/2}$ for the state with $N^\uparrow = N^\downarrow = N_s/4$ and a single-occupied site as the nearest neighbor to empty sites (independently of the spin configuration). Therefore, for macroscopic systems (N_s and N on the order of 10^{23}) we have $d_l \sim z^{\mathcal{O}(N_s)}$, where $\mathcal{O}$ denotes the order of magnitude. This example illustrates that the value of $p_n^{(l)} \sim 1/d_l$ is typically extremely small. For generic many-body systems with f degrees of freedom, we can estimate that $\eta = [\max_n p_n^{(l)}(g_c)]^{1/3} \sim 1/10^{\mathcal{O}(f)}$ [20,21,34]. From this, we have that $\Delta\eta \sim \Delta/10^{\mathcal{O}(f)}$. Calling δO the resolution limit of an experimental device measuring O, since Δ is the (finite) range of possible outcomes of a measurement of O, then, even for $\Delta \sim \mathcal{O}(f)$, there is a sufficiently large number of particles (typically $\ll 10^{23}$) where $\Delta\eta \sim \mathcal{O}(f)/10^{\mathcal{O}(f)} < \delta O$. Therefore, from Eq. (25), we conclude that for the overwhelming majority of times $\langle O\rangle_t(l,g_c)$ is excessively close to $\overline{\langle O\rangle_t}(l,g_c)$ such that the accuracy of the measurement using any experimental device cannot capture the difference between them. This is an apparent equilibration because Theorem 3 does not in fact ensure that $\langle O\rangle_t(l,g_c)$ converges to a certain value ($\langle O\rangle_t$ can even be periodic [34]). Finally, from Eq. (19), we find that the equilibrium ensemble of basis states is $\rho_{\rm eq} = \sum_j \overline{P}_{l,j}(g_c)|\phi_j\rangle\langle\phi_j|$. The presence of $\overline{P}_{l,j}(g_c)$ in $\rho_{\rm eq}$ means that the breakdown of ergodicity stated by Theorem 2 occurs in equilibrium.

IV. THERMALIZATION

Theorems 2 and 3 imply that a system in equilibrium at an energy-level crossing point is nonergodic, at least for the cases where the assumptions of these theorems are fulfilled. Thus thermalization obviously does not occur. However, it is important to understand the mechanism that leads to the breakdown of the eigenstate thermalization hypothesis (ETH).

The ETH is a sufficient condition for thermalization and asserts that $\langle\psi_n|\hat{O}|\psi_n\rangle = \langle O\rangle_{\rm mc}(E_n)$, where $\langle\cdots\rangle_{\rm mc}(E)$ denotes the microcanonical average on an energy window of width δE around energy E [3,22,24,35,36]. Since a window δE defines a Hilbert subspace spanned by the neighboring eigenstates $|\psi_m\rangle$ such that $E_m \in (E_n - \delta E/2, E_n + \delta E/2)$, the conjecture underlying the ETH is that $\langle\psi_m|\hat{O}|\psi_m\rangle$ fluctuates very little between these eigenstates. Thus, for instance, if there is no degeneracy, then $\overline{\langle O\rangle_t} = \sum_n |c_n^{(l)}|^2 \langle\psi_n|\hat{O}|\psi_n\rangle \approx \sum_{|E_m - E| < \delta E/2} |c_m^{(l)}|^2 \langle O\rangle_{\rm mc}(E) = \langle O\rangle_{\rm mc}(E)$ (the ergodic hypothesis holds), where E is defined by the initial state (i.e., $E = \langle\phi_l|\hat{H}|\phi_l\rangle$) and the coefficients $|c_m^{(l)}|^2$ are assumed to be strongly clustered around n such that $\langle\psi_n|\hat{H}|\psi_n\rangle = E$. Therefore, $\overline{\langle O\rangle_t}$ does not depend on the choice of the initial state and $\overline{\langle O\rangle_t} = \langle O\rangle_{\rm mc}(E)$ holds for all initial states that are in the microcanonical energy window.

Systems of large enough size with rare nonthermal eigenstates, comprising a vanishing fraction of the Hilbert space, can still thermalize [37]. However, even for these systems, if the initial state strongly overlaps with the nonthermal eigenstates, then thermalization is prevented from occurring [38]. For example, initial states that strongly overlap with some quantum many-body scars give rise to nonergodic dynamics [26–28]. A fundamental theoretical aspect of models with quantum many-body scars is that the Hilbert subspace spanned by the nonthermal eigenstates (scars) is disconnected from the thermalizing subspace [28]. This is the mechanism whereby the ergodicity breaks. Dramatically, there are models where the Hilbert space for each symmetry sector fragments into many exponentially disconnected subspaces [39–41]. This phenomenon is called Hilbert space fragmentation, and, in contrast to quantum many-body scars, this fragmentation occurs in relation to some local basis (for instance, product states). The fragmentation occurs because an interplay among the terms of the Hamiltonian imposes constraints on the dynamics of the particles, which results in conserved quantities (without any underlying symmetry). In the following, we shall show, by analyzing the particle number imbalance in the system, that the origin of nonthermalization of the system at an energy-level crossing point is the existence of both favored *and* disfavored Hilbert subspaces, where the system is most *and* less likely to be found, respectively. Since breakdown of ergodicity only occurs for initial states that overlap with the degenerate eigenstates (see Theorem 2), this appears to be a unique mechanism of weak ETH breakdown, where the degenerate eigenstates can work as nonthermal eigenstates.

The imbalance in the distribution in real space of some property of the particles (single-particle energy, electrical charge, spin, etc.) is adequate to identify a system that is nonthermal [42–47]. In other words, if a system starts from an out-of-equilibrium state with some asymmetry in real space,

and if it evolves such that a nonzero imbalance remains, then this system retains a memory of the initial state and therefore does not undergo thermalization. Let us consider that N^X is the number of particles with value X for some physical property and that basis states $\{|\phi_j\rangle\}$ are eigenvectors of $\hat{N}^X$. Thus, for N^X particles having an asymmetric distribution between the left and right sides of the system, the left ($N^{X,L}$) and right ($N^{X,R}$) particle number imbalance (I^X) can be quantified by

$$I^X_{l,g}(t) = \frac{1}{N}\big[N^{X,L}_{l,g}(t) - N^{X,R}_{l,g}(t)\big], \tag{26}$$

where $N^{X,Y}_{l,g}(t) = \langle\Psi_{l,g}(t)|\hat{N}^{X,Y}|\Psi_{l,g}(t)\rangle$ and $Y = L, R$. The separation of the system into left and right sides is not unique, but this does not affect the results obtained here. It is important to mention that the ETH may be valid for operators acting on a subsystem of up to half of the total system size (i.e., few-body observables) [48]. Using $\hat{N}^{X,Y}|\phi_j\rangle = N^{X,Y}_j|\phi_j\rangle$ (because $[\hat{N}^X, \hat{N}^{X,Y}] = 0$) and Eqs. (6) and (10), we find that the time-averaged imbalance is as follows:

$$\overline{I^X_{l,g}} = \frac{1}{N}\sum_j \bar{P}_{l,j}(g)\big[N^{X,L}_j - N^{X,R}_j\big]. \tag{27}$$

Theorem 4 (nonthermalization). If conditions (i)–(iii) of Theorem 2 are satisfied and $\hat{F}\hat{N}^{X,L(R)} = \hat{N}^{X,R(L)}\hat{F}$, then the time-averaged imbalance is nonzero at an energy-level crossing point.

Proof. Since $\hat{F}|\phi_{k(q)}\rangle = |\phi_{q(k)}\rangle$ [definition (4) in Sec. II] and $\hat{N}^{X,Y}|\phi_j\rangle = N^{X,Y}_j|\phi_j\rangle$, from assumption $\hat{F}\hat{N}^{X,L(R)} = \hat{N}^{X,R(L)}\hat{F}$, we calculate the following:

$$\begin{aligned}
\hat{F}\hat{N}^{X,L(R)}|\phi_k\rangle &= \hat{N}^{X,R(L)}\hat{F}|\phi_k\rangle,\\
\hat{F}\big(N^{X,L(R)}_k|\phi_k\rangle\big) &= \hat{N}^{X,R(L)}|\phi_q\rangle,\\
N^{X,L(R)}_k\hat{F}|\phi_k\rangle &= N^{X,R(L)}_{q(k)}|\phi_{q(k)}\rangle,\\
N^{X,L(R)}_k|\phi_q\rangle &= N^{X,R(L)}_q|\phi_q\rangle,\\
\Rightarrow N^{X,L(R)}_k &= N^{X,R(L)}_q.
\end{aligned} \tag{28}$$

Therefore, the states of every pair of SESs, $\{k, q\}$, have opposite parity in relation to X, and the action of $\hat{F}$ corresponds to a parity inversion. Based on this, it is convenient to write $B = B_{L>R} \cup B_{L<R} \cup B_{L=R}$, where B is the complete set of basis states, $B_{L>R}$ is the subset in which $N^{X,L}_j > N^{X,R}_j$, $B_{L<R}$ is the subset in which $N^{X,L}_j < N^{X,R}_j$, and $B_{L=R}$ is the subset in which $N^{X,L}_j = N^{X,R}_j$ (in this case $j \notin \{k, q\}$). Thus $\hat{F}$ transforms a state of $B_{L>R}$ into a state of $B_{L<R}$, and vice versa. Then, for any pair $\{k, q\}$, we have $k \in B_{L>R}$ and $q \in B_{L<R}$. For pairs of SESs that satisfy condition (ii) of Theorem 2, we have $\bar{P}_{l,k}(g_c) \neq \bar{P}_{l,q}(g_c)$. Based on symmetry, we conclude that the states favored by the degeneracy are all in either $B_{L>R}$ or $B_{L<R}$. Thus, and using Eq. (28), we obtain the following:

$$\begin{aligned}
\sum_{j\in B}\bar{P}_{l,j}(g_c)\big[N^{X,L}_j - N^{X,R}_j\big] &= \sum_{j\in B_{L>R}}\bar{P}_{l,j}(g_c)\big[N^{X,L}_j - N^{X,R}_j\big]\\
&\quad + \sum_{j\in B_{L<R}}\bar{P}_{l,j}(g_c)\big[N^{X,L}_j - N^{X,R}_j\big] + \sum_{j\in B_{L=R}}\bar{P}_{l,j}(g_c)\underbrace{\big[N^{X,L}_j - N^{X,R}_j\big]}_{=0}\\
&= \sum_{\substack{\{k,q\}\\ k\in B_{L>R},\, q\in B_{L<R}}}\big\{\bar{P}_{l,k}(g_c)\big[N^{X,L}_k - N^{X,R}_k\big] + \bar{P}_{l,q}(g_c)\big[N^{X,L}_q - N^{X,R}_q\big]\big\}\\
&= \sum_{\substack{\{k,q\}\\ k\in B_{L>R},\, q\in B_{L<R}}}\big\{\bar{P}_{l,k}(g_c)\big[N^{X,L}_k - N^{X,R}_k\big] + \bar{P}_{l,q}(g_c)\big[N^{X,R}_k - N^{X,L}_k\big]\big\}\\
&= \sum_{\substack{\{k,q\}\\ k\in B_{L>R},\, q\in B_{L<R}}}[\bar{P}_{l,k}(g_c) - \bar{P}_{l,q}(g_c)]\underbrace{\big[N^{X,L}_k - N^{X,R}_k\big]}_{>0}.
\end{aligned} \tag{29}$$

Therefore, if $\bar{P}_{l,k\in B_{L>R}}(g_c) \geqslant \bar{P}_{l,q\in B_{L<R}}(g_c)$ for every pair of SESs [the equality holds for pairs that do not satisfy condition (ii) of Theorem 2], then $\sum_{j\in B}\bar{P}_{l,j}(g_c)[N^{X,L}_j - N^{X,R}_j] > 0$. Conversely, if $\bar{P}_{l,k\in B_{L>R}}(g_c) \leqslant \bar{P}_{l,q\in B_{L<R}}(g_c)$ for every $\{k, q\}$, then $\sum_{j\in B}\bar{P}_{l,j}(g_c)[N^{X,L}_j - N^{X,R}_j] < 0$. In any case, we have $\overline{I^X_{l,g_c}} \neq 0$ [see Eq. (27)]. Q.E.D.

In addition to Theorem 3 for equilibration, Theorem 4 does not avoid the possibility that $\overline{I^X_{l,g_c}}$ will have a nonzero value that is not detectable by any experimental device. In this case, there is an apparent thermalization. Furthermore, note that the nonthermalization at g_c is not because of disconnected Hilbert subspaces. Since $\hat{F}$ commutes with $\hat{H}$, and the nonthermalization is associated with an occupancy imbalance between basis states of $B_{L>R}$ and $B_{L<R}$, which are exchanged by an action of $\hat{F}$, then the time evolution operator, $\exp[-i\hat{H}(g_c)t/\hbar]$, is able to drive the system from one subspace (spanned by $B_{L>R}$, $B_{L<R}$, or $B_{L=R}$) to another. The degeneracy favors the occupancy of a Hilbert subspace, but no decoupling occurs.

V. MODEL

In order to illustrate the results found in the previous sections, let us consider a system composed of interacting itinerant electrons on an arbitrary lattice with dynamics

expressed by the following general single-band tight-binding Hamiltonian:

$$\hat{H} = \hat{H}_{\rm hop} + \hat{H}_{\rm int}, \tag{30}$$

where

$$\hat{H}_{\rm hop} = \sum_{r,s=1}^{N_s} \sum_{\sigma=\uparrow,\downarrow} t_{r,s} \hat{c}^{\dagger}_{r,\sigma} \hat{c}_{s,\sigma}, \tag{31}$$

$$\hat{H}_{\rm int} = \frac{1}{2} \sum_{r,s=1}^{N_s} \sum_{\sigma,\lambda=\uparrow,\downarrow} V(|r-s|) \hat{n}_{r,\sigma} \hat{n}_{s,\lambda}. \tag{32}$$

$t_{r,s}$ is the hopping amplitude between the sites (Wannier orbitals) r and s, and we assume that it is real and isotropic (i.e., $t_{r,s} = t_{s,r}$). $V(|r-s|)$ is the electrostatic interaction between electrons; therefore, it is real and only depends on the distance $|r-s|$ between the electrons. For the case where $r = s$, the Pauli exclusion principle imposes that $\lambda = -\sigma$ and thus V coincides with the on-site Hubbard interaction U.

Let us adopt as basis states the eigenvectors of the particle number operator, $\hat{N} = \hat{N}^{\uparrow} + \hat{N}^{\downarrow}$ (where $\hat{N}^{\sigma} = \sum_{r=1}^{N_s} \hat{n}_{r,\sigma}$), that is, $|\phi_j\rangle = |n^{(j)}_{1,\uparrow} n^{(j)}_{1,\downarrow},\ n^{(j)}_{2,\uparrow} n^{(j)}_{2,\downarrow},\ \ldots,\ n^{(j)}_{N_s,\uparrow} n^{(j)}_{N_s,\downarrow}\rangle$. For an isolated system, $\hat{N}|\phi_j\rangle = N|\phi_j\rangle$ for all the basis states (N is conserved). The states $\{|\phi_j\rangle\}$ are also eigenvectors of $\hat{H}_{\rm int}$ [Eq. (32)], but not of $\hat{H}_{\rm hop}$ [Eq. (31)]. However, $\hat{H}_{\rm hop}$ can be expressed as $\sum_{\sigma} \hat{H}_{{\rm hop},\sigma} = \hat{H}_{{\rm hop},\uparrow} + \hat{H}_{{\rm hop},\downarrow}$, where $H_{{\rm hop},\sigma} = \sum_{r,s} t_{r,s} \hat{c}^{\dagger}_{r,\sigma} \hat{c}_{s,\sigma}$ [see Eq. (31)]. Thus the itinerant dynamics for spin-up and spin-down electrons are decoupled. This implies that N^{σ} is conserved and states with different values of $N^{\uparrow}$ or $N^{\downarrow}$ belong to different Hilbert subspaces.

Let $\hat{F}$ be an operator that changes the spin direction of electrons in singly occupied sites. Thus $\hat{F}|\phi_k\rangle = |\phi_q\rangle$ if $|\phi_k\rangle$ and $|\phi_q\rangle$ are two states with the same spatial configuration and opposite spins, such as $\hat{F}|\uparrow,\ \uparrow,\ 0,\ \downarrow,\ 0,\ \uparrow\downarrow, \cdots\rangle = |\downarrow,\ \downarrow,\ 0,\ \uparrow,\ 0,\ \uparrow\downarrow, \cdots\rangle$. The following expression can be written for this operator:

$$\hat{F} = \prod_{r} \left\{ \hat{\sigma}^{+}_{r} + \hat{\sigma}^{-}_{r} + \left[1 - \left(\hat{\sigma}^{z}_{r}\right)^2\right] \right\}, \tag{33}$$

where $\hat{\sigma}^{+}_{r} = \hat{c}^{\dagger}_{r,\uparrow} \hat{c}_{r,\downarrow}$, $\hat{\sigma}^{-}_{r} = \hat{c}^{\dagger}_{r,\downarrow} \hat{c}_{r,\uparrow}$, and $\hat{\sigma}^{z}_{r} = \hat{n}_{r,\uparrow} - \hat{n}_{r,\downarrow}$.

Now let us focus on subspace $N^{\uparrow} = N^{\downarrow}$. In this case, we have that $\hat{F}\hat{H}_{{\rm hop},\sigma}|\phi_j\rangle = \hat{H}_{{\rm hop},-\sigma}\hat{F}|\phi_j\rangle$ because the hopping amplitude $t_{r,s}$ does not depend on the spin. Therefore, the sum over σ results in $\hat{F}\hat{H}_{\rm hop}|\phi_j\rangle = \hat{H}_{\rm hop}\hat{F}|\phi_j\rangle \Rightarrow [\hat{F}, \hat{H}_{\rm hop}] = 0$. Furthermore, we also have that $\hat{F}\hat{H}_{\rm int}|\phi_j\rangle = \hat{H}_{\rm int}\hat{F}|\phi_j\rangle \Rightarrow [\hat{F}, \hat{H}_{\rm int}] = 0$ because $|\phi_j\rangle$ is an eigenvector of $\hat{H}_{\rm int}$ and the interaction $V(|r-s|)$ does not depend on the spin. Hence, $\hat{F}$ commutes with the Hamiltonian of Eq. (30), and thus two states with the same spatial configuration and opposite spins constitute a pair of SESs, independent of the values of parameters $\{t_{r,s}\}$ and $\{V(r-s)\}$. Each basis state having at least one singlon (singly occupied site) belongs to a pair of SESs. By evaluating the total number of basis states (N_B), $N_B = \{N_s!/[N^{\sigma}!(N_s - N^{\sigma})!]\}^2$, and the number of basis states having no singlon (in which $\hat{F}|\phi_j\rangle = |\phi_j\rangle$), $\sqrt{N_B}$, we conclude that there are $N_B - \sqrt{N_B}$ states belonging to pairs of SESs. Thus, in the thermodynamic limit, the fraction of nonsymmetrically equivalent states tends to zero. This reveals that SESs are typical states; however, the SESs that strongly overlap with degenerate eigenstates can be rare.

However, it is difficult to know whether the spectrum of the Hamiltonian (30)–(32) presents any energy-level crossing. Nevertheless, for $V(r-s) = U\delta_{r,s}$, this Hamiltonian reduces to the standard Hubbard Hamiltonian [8] [Eqs. (1)–(3) are obtained if $t_{r,s} = J\ (= 0)$ for r and s denoting (non-) nearest-neighbor sites]. Because the Hubbard Hamiltonian is $SU(2)$ invariant, a saturated ferromagnetic transition, if it exists, is associated with an energy-level crossing, as discussed in Sec. I. Fortunately, Tasaki proved that certain Hubbard models exhibit saturated ferromagnetism at zero temperature and sufficiently large (but finite) interaction U [49]. Thus, at the ferromagnetic quantum critical point of a Hubbard model, the ergodicity is expected to be broken for a subset of basis states satisfying condition (ii) of Theorem 2. On the other hand, Theorem 4 states that an imbalance will occur over a long timescale if $\hat{F}\hat{N}^{X,L(R)} = \hat{N}^{X,R(L)}\hat{F}$. Hence the choice $\hat{N}^{X} = \hat{N}^{\sigma}$ is not valid because, for example, $\hat{F}|\uparrow,\downarrow\rangle_L|0,0\rangle_R$ equals $|\downarrow,\uparrow\rangle_L|0,0\rangle_R$ and not $|0,0\rangle_L|\uparrow,\downarrow\rangle_R$, therefore, $\hat{F}\hat{N}^{\sigma,L(R)} \neq \hat{N}^{\sigma,R(L)}\hat{F}$. Thus we use $\hat{N}^{X} = \hat{M} \equiv \hat{N}^{\uparrow} - \hat{N}^{\downarrow}$, where M is the total magnetic moment of the system; therefore, Eq. (26) becomes $I^{M}_{l,g}(t) = [M^{L}_{l,g}(t) - M^{R}_{l,g}(t)]/N$ which denotes a left and right magnetization imbalance. In this way, $\hat{F}\hat{M}^{L(R)} = \hat{M}^{R(L)}\hat{F}$ is valid. Consequently, Theorem 4 holds, and a left and right magnetization imbalance must persist. In other words, although a long-time electrical charge density is not spatially asymmetric (the electrical conductance is preserved), the time average of the spin-σ electron numbers for different regions of the crystal lattice is imbalanced (if the initial state overlaps with the degenerate eigenstates). This is in agreement with the results reported in Ref. [19] for Hubbard clusters solved by numerical exact diagonalization. Additionally, nonergodicity has been observed in tilted Hubbard chains; however, in this case, the underlying mechanism is the Hilbert space fragmentation due to the kinetic constraints imposed by a large tilt energy [47].

Strictly speaking, Theorem 4 is valid only at g_c. However, experimental measurements are obtained over a finite timescale. Then, for g close to g_c (where $E_{\alpha} - E_{\beta}$ is very small), the period of the oscillation produced by $\exp\{i[E_{\alpha}(g) - E_{\beta}(g)]t/\hbar\}$ in Eq. (8) is excessively large, such that $\overline{I^{M}_{l,g}} \neq 0$ close to an energy-level crossing point. This aspect can be relevant for technological applications such as quantum memories [50–52]. Nevertheless, a fundamental question related to technology is whether $\overline{I^{M}_{l,g_c}}$ is sufficiently large to facilitate measurement. By noting Eq. (29), we find that $\overline{I^{M}_{l,g_c}}$ depends on $\Delta\bar{P}_{k,q}(g_c) \equiv \bar{P}_{l,k\in B_{L>R}}(g_c) - \bar{P}_{l,q\in B_{L<R}}(g_c)$ and, from Eqs. (12) and (14), we have that $\Delta\bar{P}_{k,q}(g_c) = 4{\rm Re}[a^{(\alpha)}_{k}(g_c)^{*} a^{(\alpha)}_{l}(g_c) a^{(\beta)}_{l}(g_c)^{*} a^{(\beta)}_{k}(g_c)]$. Thus, for a uniform distribution of the coefficients, i.e., $|a^{(n)}_{j}(g)|^2 = 1/N_B$, we obtain $\Delta\bar{P}_{k,q}(g_c) \leqslant 4/N_B^2$ and $\overline{I^{M}_{l,g_c}} = N^{-1} \sum_{\{k,q\}} \Delta\bar{P}_{k,q}(g_c)[M^{L}_{k} - M^{R}_{k}] \leqslant 4N_P/N_B^2$, where N_P is the number of pairs of SESs. Since $N_P = (N_B - \sqrt{N_B})/2$, we find that $\overline{I^{M}_{l,g_c}} < 2/N_B$. For $N^{\uparrow} = N^{\downarrow} = N_s/2$, for example, N_B scales asymptotically as 2^{N_s}; thus $\overline{I^{M}_{l,g_c}} < 2/N_B \sim 2^{1-N_s}$. Therefore, the technological use of the ergodicity breaking

should be possible only for small systems, such as magnetic molecules [19,53–56].

VI. CONCLUSION

We investigated the effect of degeneracy resulting from a many-body energy-level crossing driven by a control parameter in a time-independent Hamiltonian that depends on one dimensionless control parameter. A ferromagnetic quantum phase transition described by the Hubbard model is an example of this scenario.

At the energy-level crossing point, starting from a nonequilibrium basis state that overlaps with the degenerate eigenstates, the evolution in time of the system is such that it visits one subset of basis states more than the other, even in a case where the states in one are symmetrically equivalent to those in the other. This ergodicity breaking implies that the time-averaged probability of finding the system in a certain basis state of the favored subspace is higher than that of finding it in the symmetrically equivalent state of the disfavored subspace. Symmetrically equivalent states are those where one state is transformed into the other by the action of an operator that commutes with the Hamiltonian. No additional requirements are imposed on the Hamiltonian beyond a two-level crossing; in this sense, the Hamiltonian is general and arbitrary. The favored subspace is defined by choosing the initial state. After a long time, the system reaches equilibrium, and in the overwhelming majority of times, the difference between the expectation value of an observable and its time-averaged value is below the resolution limit of any realistic experimental device. Therefore, nonergodicity is a feature of the equilibrium state, and thus the system is not subjected to thermalization. Since this nonthermalization is not true for all initial basis states and the system Hilbert space is not fragmented, our results seemingly reveal a unique mechanism for a weak ETH breakdown, where the degenerate eigenstates can work as nonthermal eigenstates.

Considering a Hubbard Hamiltonian with the number of spin-up electrons equal to the number of spin-down electrons, the nonergodicity at a degenerate point (and neighboring point) corresponds to a higher time-averaged probability of finding spin-up electrons on one side and spin-down electrons on the opposite side of the crystal lattice than the reverse. Thus a left and right magnetization imbalance remains over a long timescale. The use of this phenomenon in technologies is apparently possible, at least for small systems.

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