Target control of complex networks: How to save control energy

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Controlling complex networks has received much attention in the past two decades. In order to control complex networks in practice, recent progress is mainly focused on the control energy required to drive the associated system from an initial state to any final state within finite time. However, one of the major challenges when controlling complex networks is that the amount of control energy is usually prohibitively expensive. Previous explorations on reducing the control energy often rely on adding more driver nodes to be controlled directly by external control inputs, or reducing the number of target nodes required to be controlled. Here we show that the required control energy can be reduced exponentially by appropriately setting the initial states of uncontrollable nodes for achieving the target control of complex networks. We further present the energy-optimal initial states and theoretically prove their existence for any structure of network. Moreover, we demonstrate that the control energy could be saved by reducing the distance between the energy-optimal states set and the initial states of uncontrollable nodes. Finally, we propose a strategy to determine the optimal time to inject the control inputs, which may reduce the control energy exponentially. Our conclusions are all verified numerically, and shed light on saving control energy in practical control.

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I. INTRODUCTION

In the past two decades, network science plays a key role in modern science for studying complex systems [1–8]. Typically, the nodes of a network could represent components of the complex system, and the edges describe interactions between the components [9–11]. Recently, the combination of the classical control theory and network science has proved to be a powerful tool for understanding complex systems such as biological systems [12,13], social networks [14,15], brain networks [16–18], and so on [19–23]. A network system is said to be controllable if it can be driven from an arbitrary initial state to any final state with proper external control inputs in finite time [2,24–29].

For a controllable network system, the external inputs to achieve the same control task are not necessarily exclusive. As a result, the control energy measuring how difficult it is to execute the control task has sparked much interest. Theoretical explorations have been carried out to study the internal essence of the minimum control energy on the basis of optimal control theory [25,28–35]. Reference [35] analyzed the control energy in different control directions in state space and concluded that most of the control directions of a controllable network system are energetically inaccessible, which indicates that the required control energy for controlling complex networks is usually tremendous. The upper and lower bounds of the minimum control energy, and the associated scaling laws for static networks are analytically derived for both full and target control [25,28–30,32,33]. For temporal networks, the bounds and scalings of the minimum control energy are also reported [32,36].

While controlling complex networks with large control energy is infeasible in practice, limited methods and strategies are reported on how to save control energy. Reference [36] first compared the control characteristics between static and temporal networks, showing that temporal networks demand orders of magnitude less control energy. And the control energy decays exponentially with the increase of the number of control inputs [35], suggesting that applying additional controllers is an energy-saving strategy. Reference [37] proposed strategies for driver nodes placement that can reduce control energy. For many biological, technological, and social systems, controlling the entire network is neither necessary nor energy saving [38]. Indeed, a more realistic and efficient strategy is to achieve target control, i.e., to control a subset of all nodes (or a subsystem) according to the system's function or task [25,29,33,38]. The nodes that are targeted to achieve our preset goal are referred to as target nodes. Interestingly, the control energy is reported to decay exponentially with the decrease of target nodes [33]. However, little attention is paid to the nontarget nodes, i.e., the complementary set of target nodes. Although achieving the desired final states of target nodes is the most critical element in a control task, nontarget nodes, as an indispensable part of the network system, presumably produce an essential outcome on the control process, such as the control inputs and the associated control energy.

This paper presents a comprehensive study on conserving control energy by specifically focusing on the uncontrollable nodes, namely, the nontarget nodes, in uncontrollable networks under target control. We show that the control energy differs exponentially for different initial states of

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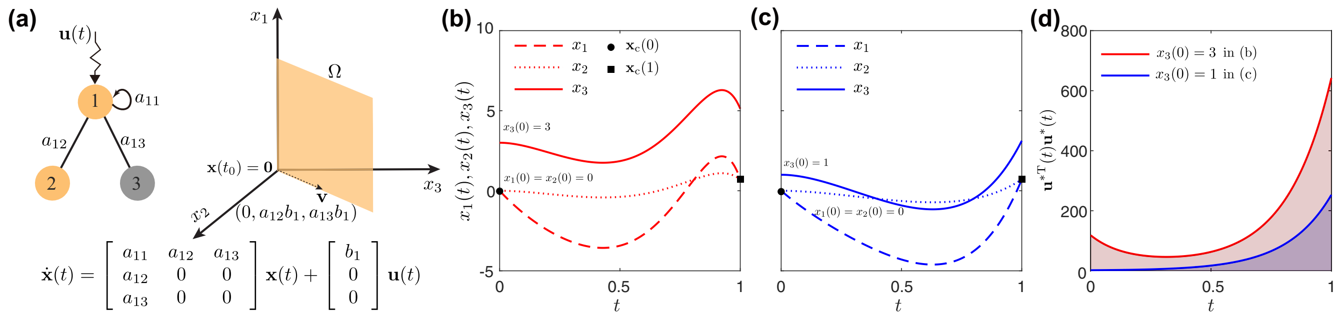

FIG. 1. The initial states of uncontrollable nodes play a key role in saving control energy. (a) A simple three-node network following the dynamics given at the bottom has two-dimensional controllable subspace. The controllable nodes are denoted by 1, 2 without loss of generality, and the yellow region $\boldsymbol{\Omega}$ on the right panel shows the set of final states $\mathbf{x}_\mathrm{f}$ that can be reached from the initial state $\mathbf{x}(t_0)=\mathbf{0}$ in finite time by a proper control input. $\boldsymbol{\Omega}$ is an infinite plane that restricts the evolution of the system states, i.e., $x_3(t_\mathrm{f})=\frac{1}{a_{12}}x_1(t_\mathrm{f})+\frac{a_{13}}{a_{12}}x_2(t_\mathrm{f})$. The vector $\mathbf{v}=(0,a_{12}b_1,a_{13}b_1)$ shown in the dotted arrow is a direction vector of $\boldsymbol{\Omega}\cap\boldsymbol{\Psi}=\{(x_1(0),x_2(0),x_3(0))|x_1(0)=0\}$. (b) The time evolution of the system states to achieve a control task with energy-optimal input $\mathbf{u}^*(t)$ and different settings of $x_3(t_0)$. The network weights $(a_{11},a_{12},a_{13},b_1)$ are set as (1,1,3,1), and the control times are assigned as $t_0=0$ and $t_\mathrm{f}=1$. For the control task, the initial and target states of controllable nodes are set as $\mathbf{x}_\mathrm{c}(0)=[x_1(0),x_2(0)]^\mathrm{T}=\mathbf{0}$ and $\mathbf{x}_\mathrm{c}(1)=[x_1(1),x_2(1)]^\mathrm{T}=[1/\sqrt{2},1/\sqrt{2}]^\mathrm{T}$, respectively. For the state of the uncontrollable node $x_3(0)$, we set $x_3(0)=3$ in (b) with a solid red curve while $x_3(0)=1$ in (c) with a solid blue curve. (d) $\mathbf{u}^{*\mathrm{T}}(t)\mathbf{u}^*(t)$ at different time points under different $x_3(0)$, and $\mathbf{u}^*(t)$ is the optimal control input for each case. The red curve and blue curve present the case of $x_3(0)=3$ and $x_3(0)=1$, respectively. The areas of the shaded part below the curves represent the corresponding optimal control energy.

uncontrollable nodes when a control task is given. Then we give the necessary and sufficient condition for the existence of the energy-optimal states to maximally save energy, and prove that the energy-optimal states always exist. And we further derive the energy-optimal possible states for any network structure analytically. Moreover, we find that the further the distance between the set of energy-optimal states and the initial states of uncontrollable nodes, the more control energy would be required to conduct the control task. Finally, we propose a control strategy to determine when to apply the control inputs into the system. Simulations show that the control strategy can save control energy exponentially.

II. MODEL

We consider the canonical continuous linear time-invariant dynamics

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t). \tag{1}$$

Here $\mathbf{A}\in\mathbb{R}^{n\times n}$ is the adjacency matrix of the undirected network, which represents interactions between the system components. $\mathbf{B}\in\mathbb{R}^{n\times m}$ is the input matrix that has linearly independent columns with a single nonzero element 1. $b_{ij}\neq 0$ means node i is controlled directly by the input j, and node i is thus called the driver node. $\mathbf{x}(t)=[x_1(t),x_2(t),\dots,x_n(t)]^\mathrm{T}\in\mathbb{R}^n$ is an n-dimensional state vector denoting the states of the entire network and $\mathbf{u}(t)=[u_1(t),u_2(t),\dots,u_m(t)]^\mathrm{T}$ is an m-dimensional input vector [Fig. 1(a)].

System (1) is called controllable if there exists a proper control input $\mathbf{u}(t)$ driving the system from any initial state $\mathbf{x}_0=\mathbf{x}(t_0)$ toward any target state $\mathbf{x}_\mathrm{f}=\mathbf{x}(t_\mathrm{f})$ at finite time t_f. According to the Kalman's controllability criterion [39], the controllability matrix of system (1) is

$$\mathbf{K}=[\mathbf{B},\mathbf{A}\mathbf{B},\mathbf{A}^2\mathbf{B},\dots,\mathbf{A}^{n-1}\mathbf{B}],$$

whose column space is the controllable subspace $\boldsymbol{\Omega}$, and its rank is the dimension of $\boldsymbol{\Omega}$ [Fig. 1(a)]. Here we concentrate on those systems that are partially controllable by relaxing the restriction of full controllability, i.e., $\mathrm{rank}(\mathbf{K})=r<n$. And in this case, since the dimension of controllable subspace is r, the states of r nodes can move independently. As a result, there are r nodes controllable, which can be driven toward the desired state in finite time by a proper control input $\mathbf{u}(t)$, while the left $n-r$ nodes are uncontrollable. Therefore, we consider these r controllable nodes to be target nodes. Although pervious studies reveal that the maximum number of the controllable nodes is fixed r, these r controllable nodes are usually not unique [25,38]. Here, without loss of generality, we consider one of many possible combinations, which can be preselected according to the control task. Inspired by prior literature [25], we assume that the first r nodes are controllable, and we consider that the first m nodes are driver nodes, i.e., $\mathbf{x}(t)=[\mathbf{x}_\mathrm{c}^\mathrm{T}(t),\mathbf{x}_\mathrm{nc}^\mathrm{T}(t)]^\mathrm{T}$ and $\mathbf{B}=[\mathbf{I}_{m\times m},\mathbf{0}]^\mathrm{T}\in\mathbb{R}^{n\times m}$, where $\mathbf{x}_\mathrm{c}(t)$ and $\mathbf{x}_\mathrm{nc}(t)$ collect the states of controllable and uncontrollable nodes, respectively. This operation also makes mathematical expressions more convenient for detailed analysis of control energy.

For a given network system, the permutation matrix

$$\mathbf{P}=[\mathbf{I}_{\mathrm{c}_1},\mathbf{I}_{\mathrm{c}_2},\dots,\mathbf{I}_{\mathrm{c}_m},\dots,\mathbf{I}_{\mathrm{c}_r},\mathbf{I}_{\mathrm{nc}_1},\dots,\mathbf{I}_{\mathrm{nc}_{n-r}}]^\mathrm{T}$$

can conveniently transfer the system states as we have assumed, where $\mathbf{I}_i$ is a unit vector with the ith element being 1 [25]. Nodes labeled as $\mathrm{c}_1,\dots,\mathrm{c}_m$ are the driver nodes, and $\mathrm{c}_1,\dots,\mathrm{c}_r$ are the preselected target nodes of the given network, and the left nodes labeled as $\mathrm{nc}_1,\dots,\mathrm{nc}_{n-r}$ are the uncontrollable nodes. In fact, the m driver nodes are part of the controllable nodes, as they directly receive independent control inputs to achieve the full control of other controllable nodes. In the following part of the paper, the network system has been transformed by $\mathbf{P}$ in advance to ensure that our assumption holds before further detailed analysis. We aim to drive the initial states of target nodes from $\mathbf{x}_{\mathrm{c}0}=\mathbf{x}_\mathrm{c}(t_0)$ to $\mathbf{x}_\mathrm{cf}=\mathbf{x}_\mathrm{c}(t_\mathrm{f})$ with the minimum control energy under the control time $t_\mathrm{f} - t_0$, which is defined as

$$E = \int_{t_0}^{t_\mathrm{f}} \mathbf{u}^\mathrm{T}(t)\mathbf{u}(t)dt.$$

Note that the states of uncontrollable nodes $\mathbf{x}_\mathrm{nc}(t)$ and their final destinations $\mathbf{x}_\mathrm{ncf} = \mathbf{x}_\mathrm{nc}(t_\mathrm{f})$ are not considered in advance to achieve the control task in previous explorations, which means that only the states of controllable nodes are taken into consideration. However, in the process of executing control tasks, it is necessary to have knowledge of the initial states of uncontrollable nodes $\mathbf{x}_\mathrm{nc0} = \mathbf{x}_\mathrm{nc}(t_0)$ because it is one of the key factors for the system's control trajectory, which is vital for conducting the practical control [Fig. 1(b)]. As a result, the minimum control energy would be greatly influenced by $\mathbf{x}_\mathrm{nc0}$, whose derivation and the corresponding control input are presented in Appendix A.

In order to further analyze the impact of the initial states of uncontrollable nodes on the control energy, here we employ the state equation decomposition related to controllability [40], which always transforms system (1) into the controllable and uncontrollable parts with block-diagonal structure. In this way, further analysis can be conducted based on the two parts.

Applying QR factorization into matrix $\mathbf{K}$ yields $\mathbf{K} = \mathbf{Q}^\mathrm{T}\mathbf{R}$; then the orthogonal matrix $\mathbf{Q}^\mathrm{T}$ can be obtained (whose first r columns are the orthogonal basis of $\mathbf{K}$'s column space), and the left $n - r$ columns are its orthogonal complement [41]. By the state variable transformation

$$\bar{\mathbf{x}}(t) = \mathbf{Q}\mathbf{x}(t), \tag{2}$$

system (1) is equivalent to

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}\mathbf{A}\mathbf{Q}^\mathrm{T}\bar{\mathbf{x}}(t) + \mathbf{Q}\mathbf{B}\mathbf{u}(t),$$

which has the following specific form [25,29,40]:

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_\mathrm{c}(t) \\ \dot{\bar{\mathbf{x}}}_\mathrm{nc}(t) \end{bmatrix} = \dot{\bar{\mathbf{x}}}(t) = \begin{bmatrix} \mathbf{A}_\mathrm{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_\mathrm{nc} \end{bmatrix}\bar{\mathbf{x}}(t) + \begin{bmatrix} \mathbf{B}_\mathrm{c} \\ \mathbf{0} \end{bmatrix}\mathbf{u}(t),$$

where $\bar{\mathbf{x}}_\mathrm{c}(t) \in \mathbb{R}^r$, $\mathbf{A}_\mathrm{c} \in \mathbb{R}^{r\times r}$, and $\mathbf{B}_\mathrm{c} = [\mathbf{I}_{m\times m}, \mathbf{0}]^\mathrm{T} \in \mathbb{R}^{r\times m}$. Accordingly, the system can be decomposed into the controllable part

$$\dot{\bar{\mathbf{x}}}_\mathrm{c}(t) = \mathbf{A}_\mathrm{c}\bar{\mathbf{x}}_\mathrm{c}(t) + \mathbf{B}_\mathrm{c}\mathbf{u}(t), \tag{3}$$

and the corresponding uncontrollable part

$$\dot{\bar{\mathbf{x}}}_\mathrm{nc}(t) = \mathbf{A}_\mathrm{nc}\bar{\mathbf{x}}_\mathrm{nc}(t). \tag{4}$$

By this means, the control input $\mathbf{u}$ applied to (1) can be regarded as that applied to (3), and so is the control energy. Denote $\mathbf{Q} = [\mathbf{Q}_\mathrm{c}^\mathrm{T}, \mathbf{Q}_\mathrm{nc}^\mathrm{T}]^\mathrm{T}$ with $\mathbf{Q}_\mathrm{c} = [\mathbf{Q}_1, \mathbf{Q}_2]$ and $\mathbf{Q}_\mathrm{nc} = [\mathbf{Q}_3, \mathbf{Q}_4]$, where $\mathbf{Q}_1 \in \mathbb{R}^{r\times r}$ and $\mathbf{Q}_4 \in \mathbb{R}^{(n-r)\times(n-r)}$. Then we have $\bar{\mathbf{x}}_\mathrm{c}(t) = \mathbf{Q}_\mathrm{c}[\mathbf{x}_\mathrm{c}^\mathrm{T}(t), \mathbf{x}_\mathrm{nc}^\mathrm{T}(t)]^\mathrm{T}$ and $\bar{\mathbf{x}}_\mathrm{nc}(t) = \mathbf{Q}_\mathrm{nc}[\mathbf{x}_\mathrm{c}^\mathrm{T}(t), \mathbf{x}_\mathrm{nc}^\mathrm{T}(t)]^\mathrm{T}$, which implies that the state of the controllable part is a linear combination of the states of both the controllable nodes and the uncontrollable nodes of (1). Similarly, the state of the uncontrollable part is also a linear combination of these nodes' states. The matrices $\mathbf{Q}_1$ and $\mathbf{Q}_4$ are all invertible since the first r nodes are controllable and the last $n - r$ nodes are uncontrollable.

It is also worth noting that we specially select the orthogonal matrix $\mathbf{Q}$ derived by applying QR factorization into matrix $\mathbf{K}$ since the state variable transformation by such $\mathbf{Q}$ does not change the symmetry of the system matrix $\mathbf{Q}\mathbf{A}\mathbf{Q}^\mathrm{T}$, although the forms of the variable transformation matrix decomposing system (1) into the two parts are not unique as stated in [40]. For one thing, $\mathbf{A}_\mathrm{c}$ and $\mathbf{A}_\mathrm{nc}$ are both symmetric when $\mathbf{A}$ is symmetric, and thus the properties of a symmetric matrix that have potential advantages for further calculations are preserved. For another, the controllable part (3) and the uncontrollable part (4) are formally decoupled since the submatrices of $\mathbf{Q}\mathbf{A}\mathbf{Q}^\mathrm{T}$ in the secondary diagonal line are zero matrices with proper size. In addition, the input matrix of the controllable part (3) is obtained as $\mathbf{B}_\mathrm{c} = [\mathbf{I}_{m\times m}, \mathbf{0}]^\mathrm{T} \in \mathbb{R}^{r\times m}$ since the first m nodes of system (1) are driver nodes.

The control input under the minimum control energy for controlling the controllable part (3) can be theoretically derived (see Appendix A), which is

$$\mathbf{u}^*(t) = \mathbf{B}_\mathrm{c}^\mathrm{T} e^{\mathbf{A}_\mathrm{c}^\mathrm{T}(t_\mathrm{f}-t)}\mathbf{W}_\mathrm{c}^{-1}(t_0, t_\mathrm{f})\boldsymbol{\xi}(t_\mathrm{f} - t_0),$$

where

$$\boldsymbol{\xi}(t_\mathrm{f} - t_0) = e^{\mathbf{A}_\mathrm{c}(t_\mathrm{f}-t_0)}\bar{\mathbf{x}}_\mathrm{c}(t_0) - \bar{\mathbf{x}}_\mathrm{c}(t_\mathrm{f}),$$

and

$$\mathbf{W}_\mathrm{c}(t_0, t_\mathrm{f}) = \int_{t_0}^{t_\mathrm{f}} e^{\mathbf{A}_\mathrm{c}(t_\mathrm{f}-\tau)}\mathbf{B}_\mathrm{c}\mathbf{B}_\mathrm{c}^\mathrm{T} e^{\mathbf{A}_\mathrm{c}^\mathrm{T}(t_\mathrm{f}-\tau)}d\tau$$

is called controllability Gramian matrix of controllable part (3).

To further understand this key point intuitively, here we consider a network system with three nodes in Fig. 1. The network structure is shown in Fig. 1(a), where the edges are consistent with nonzero elements in the system matrix $\mathbf{A}$. We also derive its controllable region $\boldsymbol{\Omega}$ with the initial state $\mathbf{x}_0 = \mathbf{0}$, which visualizes the restriction on the final state of the network system. In this case, the controllable region $\boldsymbol{\Omega}$ is a two-dimensional plane restricting the evolution of the system states, which indicates that the number of controllable nodes is two, i.e., nodes 1 and 2 [yellow in Fig. 1(a)] are controllable while node 3 [gray in Fig. 1(a)] is uncontrollable. The controllable region passes through the origin $\mathbf{x}_0 = \mathbf{0}$, and it can be determined by vector $\mathbf{v}$. $\mathbf{v}$ is a direction vector of $\boldsymbol{\Omega} \cap \boldsymbol{\Psi}$, where the plane $\boldsymbol{\Psi} = \{(x_1(0), x_2(0), x_3(0))|x_1(0) = 0\}$. In Figs. 1(b) and 1(c), we obtain the time evolution of the states and the corresponding inputs $\mathbf{u}^*(t)$ by energy-optimal control for different values of $x_3(0)$. To obtain detailed derivation of the energy-optimal input, one may refer to Appendix A. In Fig. 1(b), although the states of target nodes are all driven from $\mathbf{x}_\mathrm{c0} = \mathbf{0}$ to $\mathbf{x}_\mathrm{cf} = [1/\sqrt{2}, 1/\sqrt{2}]^\mathrm{T}$, evolutions of the system states under $x_3(0) = 3$ (with solid red curve) and $x_3(0) = 1$ (with solid blue curve) differ significantly. In Fig. 1(c), the areas of the shaded region below the red and blue curves represent the corresponding minimum control energy for $x_3(0) = 3$ and $x_3(0) = 1$, respectively. It is obvious that the minimum control energy for $x_3(0) = 1$ is significantly less than that for $x_3(0) = 3$.

III. RESULTS

A. How the states of uncontrollable nodes affect the control energy

Since the state of the controllable part is a linear combination of the states of both the controllable and uncontrollable nodes, the states of uncontrollable nodes are also needed to obtain the minimum control energy. For the uncontrollable part (4), we have

$$\bar{\mathbf{x}}_{\rm nc}(t_{\rm f}) = e^{\mathbf{A}_{\rm nc}(t_{\rm f}-t_0)}\bar{\mathbf{x}}_{\rm nc}(t_0)=\mathbf{Q}_{\rm nc}\mathbf{x}_{\rm f},$$

which further yields

$$\mathbf{x}_{\rm ncf} = \mathbf{Q}_4^{-1}\left(e^{\mathbf{A}_{\rm nc}(t_{\rm f}-t_0)}\mathbf{Q}_{\rm nc}\mathbf{x}_0 - \mathbf{Q}_3\mathbf{x}_{\rm cf}\right). \tag{5}$$

For a detailed derivation of Eq. (5), please see Appendix A. Therefore, controlling the state of controllable nodes $\mathbf{x}_0$ to the desired target state $\mathbf{x}_{\rm cf}$ is equivalent to controlling the controllable part (3) from the initial state

$$\bar{\mathbf{x}}_{\rm c}(t_0) = \mathbf{Q}_{\rm c}\left[\mathbf{x}_{\rm c0}^{\rm T}, \mathbf{x}_{\rm nc0}^{\rm T}\right]^{\rm T}$$

to the target state

$$\bar{\mathbf{x}}_{\rm c}(t_{\rm f}) = \mathbf{Q}_{\rm c}\left[\mathbf{x}_{\rm cf}^{\rm T}, \mathbf{x}_{\rm ncf}^{\rm T}\right]^{\rm T}.$$

Note that although the control energy is solely defined in terms of $\mathbf{u}(t)$ that is only applied to the controllable part (3), the minimum control energy would be effected by the state of the uncontrollable part because it appears in the initial and target states of (3). For the Gramian matrix $\mathbf{W}_{\rm c}(t_0, t_{\rm f})$, by some variable substitutions (see Appendix B) we have $\mathbf{W}_{\rm c}(t_0, t_{\rm f}) = \mathbf{W}_{\rm c}(0, t_{\rm f} - t_0)$, thus the corresponding minimum control energy is

$$\tilde{E} = \boldsymbol{\xi}^{\rm T}(t_{\rm f} - t_0)\mathbf{W}_{\rm c}^{-1}(0, t_{\rm f} - t_0)\boldsymbol{\xi}(t_{\rm f} - t_0).$$

In the following analysis, we will omit the time variable $t_{\rm f} - t_0$ of matrices, vectors, and constant for simplifying equations. The control energy analyzed here refers to the minimum control energy.

B. Energy-optimal states of uncontrollable nodes

As shown in the previous part (Fig. 1), the initial states of uncontrollable nodes $\mathbf{x}_{\rm nc0}$ has a remarkable influence on control energy. Although the states of uncontrollable nodes $\mathbf{x}_{\rm nc}(t)$ is not controllable, the control energy would be different for different locations of $\mathbf{x}_{\rm nc}(t)$. Inspired by this, we would infer that a proper $\mathbf{x}_{\rm nc0}$ may result in less control energy under the same control task, which makes it possible to make use of $\mathbf{x}_{\rm nc0}$ to save control energy to the largest extent. In the following, we aim to find an optimal initial state $\mathbf{x}_{\rm nc0}^*$ to save the control energy $\tilde{E}$ further.

The optimal $\mathbf{x}_{\rm nc0}^*$ minimizing the control energy $\tilde{E}$ can be derived by solving the optimal problem

$$\min_{\mathbf{x}_{\rm nc0}} \tilde{E} = \mathbf{x}_{\rm nc0}^{\rm T}\mathbf{G}\mathbf{x}_{\rm nc0} + \boldsymbol{\zeta}\mathbf{x}_{\rm nc0} + c$$

$$\text{subject to}\begin{cases}\mathbf{H} = \mathbf{Q}_1^{\rm T}\mathbf{W}_{\rm c}\mathbf{Q}_1\\ \mathbf{G} = \mathbf{U}^{\rm T}\mathbf{H}^{-1}\mathbf{U}\\ c = \boldsymbol{\eta}^{\rm T}\mathbf{H}^{-1}\boldsymbol{\eta}\\ \boldsymbol{\zeta} = 2\boldsymbol{\eta}^{\rm T}\mathbf{H}^{-1}\mathbf{U}\\ \boldsymbol{\eta} = \mathbf{R}\mathbf{x}_{\rm c0} - \mathbf{x}_{\rm cf}\\ \mathbf{R} = \mathbf{Q}_1^{\rm T}e^{\mathbf{A}_{\rm c}(t_{\rm f}-t_0)}\mathbf{Q}_1 + \mathbf{Q}_3^{\rm T}e^{\mathbf{A}_{\rm nc}(t_{\rm f}-t_0)}\mathbf{Q}_3\\ \mathbf{U} = \mathbf{Q}_1^{\rm T}e^{\mathbf{A}_{\rm c}(t_{\rm f}-t_0)}\mathbf{Q}_2 + \mathbf{Q}_3^{\rm T}e^{\mathbf{A}_{\rm nc}(t_{\rm f}-t_0)}\mathbf{Q}_4.\end{cases} \tag{6}$$

Here, $\mathbf{H}$ is a submatrix of system (1)'s controllability Gramian, whose first r columns and rows are the corresponding elements in $\mathbf{H}$. The positive semidefinite $\mathbf{G}$ and vector $\boldsymbol{\zeta}$ capture the effects of quadratic and linear terms of $\mathbf{x}_{\rm nc0}$ on the control energy, respectively. $\mathbf{R}$ is related to the self-evolutions of the system, and $\boldsymbol{\eta}$ describes the control distance of the control task. For different structures of the network system, the forms of $\mathbf{U}$ are usually different, which is also a key factor determining the strength of the influence of $\mathbf{x}_{\rm nc0}$ on the control energy; for example, when a network system consists of two disjoint subnetworks, one of which is fully controllable, and the other is uncontrollable. And this leads to $\mathbf{U} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$, and $\boldsymbol{\zeta} = \mathbf{0}$, where the uncontrollable initial state $\mathbf{x}_{\rm nc0}$ has no effect on the control energy. For a detailed derivation of the problem (6), please refer to Appendix C. For the network system (1), when the control duration $t_{\rm f} - t_0$, the initial state $\mathbf{x}_{\rm c0}$, and the desired final state $\mathbf{x}_{\rm cf}$ of the target nodes are fixed, the control energy $\tilde{E}$ is a univalent function of the initial states of uncontrollable nodes $\mathbf{x}_{\rm nc0}$.

Congruent transformation is commonly used to simplify the expression of quadratic forms, which facilitates further analysis. Here, in order to further analyze the optimal problem (6) theoretically, by congruent transformation [41] we have

$$\boldsymbol{\Delta}^{\rm T}\mathbf{G}\boldsymbol{\Delta} = \boldsymbol{\Lambda} = \begin{pmatrix}\boldsymbol{\Lambda}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},$$

where $q = \text{rank}(\mathbf{G})$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Lambda}_q$ are nonsingular, and $\boldsymbol{\Lambda}_q$ is a q-dimensional diagonal matrix. Since $\mathbf{G}$ is positive semidefinite, $\boldsymbol{\Lambda}_q'$s diagonal elements are positive by congruent transformation. By denoting $\boldsymbol{\delta}^{\rm T} = \boldsymbol{\zeta}\boldsymbol{\Delta}$, we know that the solution of problem (6) exists if and only if $\text{rank}(\boldsymbol{\Lambda}) = \text{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$ (see Appendix D 1 for proof). It presents the necessary and sufficient condition of the existence of the energy-optimal initial states of uncontrollable nodes. In order to solve it, the first step is to determine whether there exists an energy-optimal solution. We further find that the solution always exists, i.e., $\text{rank}(\boldsymbol{\Lambda}) \equiv \text{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$ (see Appendix D 2 for proof). Let $\mathbf{x}_{\rm nc0} = \boldsymbol{\Delta}\mathbf{y}$, and then we have the control energy

$$\tilde{E} = \mathbf{x}_{\rm nc0}^{\rm T}\mathbf{G}\mathbf{x}_{\rm nc0} + \boldsymbol{\zeta}\mathbf{x}_{\rm nc0} + c$$
$$= \sum_{i=1}^{l}\lambda_i\left(y_i + \frac{\delta_i}{2\lambda_i}\right)^2 + \sum_{i=1+l}^{q}\lambda_i y_i^2 + c - \sum_{i=1}^{q}\frac{\delta_i^2}{4\lambda_i},$$

where l is the location of the last nonzero element of vector $\boldsymbol{\delta}$, λ_i is the ith diagonal element of $\boldsymbol{\Lambda}_q$, and δ_i is the ith element of

δ. Since $\lambda_i > 0$, ensuring that the first two quadratic terms are zero guarantees that $\tilde{E}$ attains its minimum value. As a result, we summarize that the problem (6) always has the energy-optimal solution, and the solution is

$$\mathbf{x}_{\text{nc0}}^{*} = \mathbf{\Delta}\left(-\frac{\delta_1}{2\lambda_1}, \ldots, -\frac{\delta_l}{2\lambda_l}, \underbrace{0, \ldots, 0}_{q-l}, \sigma_1, \ldots, \sigma_{n-r-q}\right)^{\mathrm{T}} \quad (7)$$

where $\sigma_1, \sigma_2, \ldots, \sigma_{n-r-q}$ are $n-r-q$ arbitrary parameters and the number of zero element is $q-l$. The corresponding optimal control energy is

$$\tilde{E}^{*} = c - \frac{1}{4}\sum_{i=1}^{q}\frac{\delta_i^2}{\lambda_i}.$$

For the solution of problem (6), there are distinct $n-r-q$ arbitrary parameters $\sigma_1, \sigma_2, \ldots, \sigma_{n-r-q}$ indicating that the number of $\mathbf{x}_{\text{nc0}}^{*}$ can be 1 (when $\mathbf{G}$ has full rank, i.e., $n-r=q$) or countless (when $\mathbf{G}$ is singular). Specifically, when matrix $\mathbf{G}$ has full rank, the only solution is $\mathbf{x}_{\text{nc0}}^{*} = -0.5\mathbf{G}^{-1}\boldsymbol{\zeta}^{\mathrm{T}}$ with the corresponding optimal control energy $\tilde{E}^{*} = c - 0.25\boldsymbol{\zeta}\mathbf{G}^{-1}\boldsymbol{\zeta}^{\mathrm{T}}$.

Here we take a six-dimensional network system as an example to present the corresponding results in Fig. 2 intuitively. In Fig. 2(b), one of the optimal initial states $\mathbf{x}_{\text{nc0}}^{*}$ is found by Eq. (7) with $t_{\mathrm{f}} = 1$, which is plotted with a black square. We randomly select $\mathbf{x}_{\text{nc0}}^{\text{ran}}$ with the same norm as $\mathbf{x}_{\text{nc0}}^{*}$, and calculate the corresponding control input $\mathbf{u}^{*}(t)$ in Fig. 2(d). For different control time t_{f}, we derive the $\mathbf{x}_{\text{nc0}}^{*}$, and generate 100 different $\mathbf{x}_{\text{nc0}}$ uniformly at the same norm as $\mathbf{x}_{\text{nc0}}^{*}$. In Fig. 2(c), the control energy with $\mathbf{x}_{\text{nc0}}^{*}$ is plotted in blue ($\tilde{E}^{*}$), and the average energy with 100 different $\mathbf{x}_{\text{nc0}}$ is plotted in red ($\langle\tilde{E}^{*}\rangle$), whose range is the pink shadows. Figure 2(c) shows that control energy varies exponentially when $\mathbf{x}_{\text{nc0}}$ are in different directions but of the same distance to the origin.

Therefore, the control energy can be exponentially different with different $\mathbf{x}_{\text{nc0}}$. In most previous studies, the control energy for target control is believed to be determined by the states of controllable nodes. Hence, this is a surprising finding since the control energy can be drastically affected by the states of the uncontrollable nodes other than the controllable nodes, which is counterintuitive. This finding also provides a novel approach to exponentially saving the control energy for target control by taking advantage of the derived energy-optimal solution, Eq. (7). For example, by selecting appropriate $\mathbf{x}_{\text{nc0}}$ the control energy can be largely saved. In what follows, we propose a possible control strategy to save the control energy for target control based on the derived energy-optimal solution of problem (6).

C. Strategy for determining the energy-optimal time to apply control inputs

Here we further analyze the control energy $\tilde{E}$ with respect to the states of uncontrollable nodes. Specifically, we study the energy-optimal time $t_0^{\mathrm{I}*}$ to apply the control inputs when the final state of target nodes is determined. We denote the set of $\mathbf{x}_{\text{nc0}}^{*}$ as $\mathbf{X}$. In Fig. 3(a), we generate 200×200 initial states of uncontrollable nodes and calculate the corresponding control energy. By Eq. (7), we analytically derive the set $\mathbf{X}$, which is

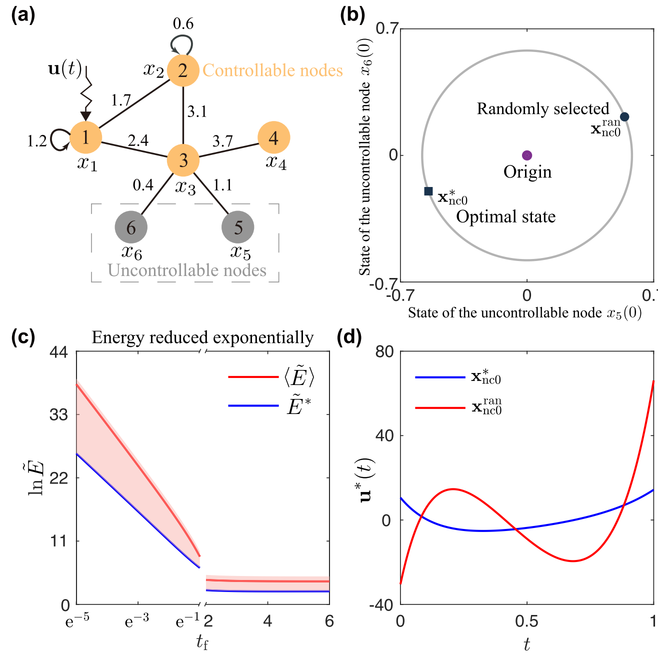

FIG. 2. Controlling a network with different initial states of uncontrollable nodes. (a) A partially controllable network with four controllable nodes in yellow and two uncontrollable nodes in gray. (b) The optimal initial states of uncontrollable nodes is denoted by $\mathbf{x}_{\text{nc0}}^{*}$ for the control time $t_0 = 0$ and $t_{\mathrm{f}} = 1$, where $\mathbf{x}_{\text{nc0}}^{\text{ran}}$ indicates the randomly selected initial state at the same distance to the origin. For controllable nodes, the initial state is set as $\mathbf{x}_0 = \mathbf{0}$, and the target state is $\mathbf{x}_{\text{cf}} = [0.71, 0.11, 0.32, 0.62]^{\mathrm{T}}$, which is randomly selected on a unit hypersphere. (c) The control energy $\tilde{E}^{*}$ with the energy-optimal initial states of uncontrollable nodes $\mathbf{x}_{\text{nc0}}^{*}$ (blue line) and the average control energy $\langle\tilde{E}\rangle$ of 100 different $\mathbf{x}_{\text{nc0}}^{\text{ran}}$ (red line) for different control time t_{f}. These $\mathbf{x}_{\text{nc0}}^{\text{ran}}$ are generated uniformly in a circle with radius being $\|\mathbf{x}_{\text{nc0}}^{*}\|_2$, as (b) shows the case for $t_{\mathrm{f}} = 1$. Here the control energy is exponentially reduced by the energy-optimal $\mathbf{x}_{\text{nc0}}^{*}$. (d) Optimal control inputs with the optimal $\mathbf{x}_{\text{nc0}}^{*}$ and a randomly selected $\mathbf{x}_{\text{nc0}}^{\text{ran}}$ on the circle in (b).

a blue line with its equation being $2.75x_5(0) + x_6(0) = 28.48$. We can see the control energy tends to be larger when the distance between states of uncontrollable nodes and the blue line tends to be longer, which inspires us to explore the relationship between the control energy and the distance between states of uncontrollable nodes $\mathbf{x}_{\text{nc0}}$ and $\mathbf{X}$. Hence we define the geometric distance between them by

$$d_{\min} = \min_{\mathbf{x}_{\text{nc0}}^{*} \in \mathbf{X}} \|\mathbf{x}_{\text{nc0}} - \mathbf{x}_{\text{nc0}}^{*}\|^2. \quad (8)$$

The geometric distance of these 200×200 states and the corresponding control energy are shown in Fig. 3(b). We can see the control energy is a monotonically increasing function of $d_{\min}$ for the six-dimensional network system. For large networks with different degree distributions and other statistical characteristics, the energy-optimal $\mathbf{x}_{\text{nc0}}^{*}$ can also be obtained analytically by our method. In Figs. 3(c) and 3(d), we generate one Barabási-Albert (BA) network and one Erdös-Rényi (ER) network with the same scale $n = 20$ and same average degree $\langle k\rangle = 5.5$. The control time is set as $t_0 = 0$ and $t_{\mathrm{f}} = 1$, and the initial state is set as 0.2357 while the target is 2 for each 18 controllable nodes ($r = 18$). Then we calculate the control

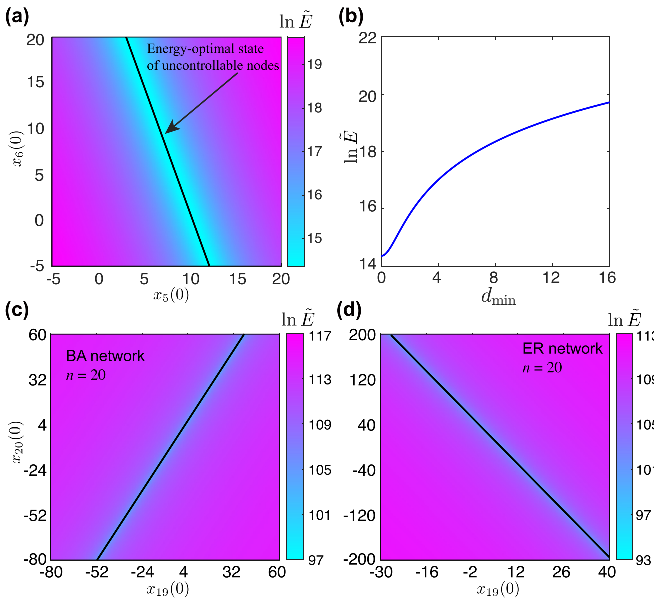

FIG. 3. Control energy with different states of uncontrollable nodes for the network system shown in Fig. 2(a). For the region of $\mathbf{x}_{\text{nc}0}$ in panel (a), 200×200 states of uncontrollable nodes $\mathbf{x}_{\text{nc}0}$ are uniformly selected. The initial state is set as $\mathbf{x}_{\text{c}0} = [1, 1, 1, 1]^{\text{T}}$ for controllable nodes, and the target state is $\mathbf{x}_{\text{cf}} = [2, 3, 4, 5]^{\text{T}}$ with the control time $t_0 = 0$ and $t_{\text{f}} = e^{-2}$. The control energy of these 200×200 states of $\mathbf{x}_{\text{nc}0}$ is calculated and presented in different colors. The energy-optimal initial states of uncontrollable nodes $\mathbf{x}^*_{\text{nc}0}$ are plotted in blue, which is aligned with its equation being $2.75x_5(0) + x_6(0) = 28.48$. (b) The control energy with respect to the geometric distance $d_{\min}$ between $\mathbf{x}_{\text{nc}0}$ [200×200 selected point in (a)] and the corresponding energy-optimal initial states. The control energy increases as the geometric distance $d_{\min}$ grows. For the BA network and ER network with $n = 20$, $r = 18$ and same average degree $\langle k \rangle = 5.5$, the control energy with the region of $\mathbf{x}_{\text{nc}0}$ in (c) and (d) is calculated. The 200×200 initial states of uncontrollable nodes are also uniformly selected. Here the control time is set as $t_0 = 0$ and $t_{\text{f}} = 1$. For each controllable node, the initial state is set as 0.2357 and the target is 2.

energy with 200×200 initial states of uncontrollable nodes in the region of Figs. 3(c) and 3(d), respectively.

Therefore, we can make full use of $\mathbf{x}^*_{\text{nc}0}$ to save control energy to the largest extent. When the initial states of uncontrolled nodes can be adjusted in advance, one can set it according to the energy-optimal solution denoted by $\mathbf{x}^*_{\text{nc}0}$ to minimize the control energy. In the case of fixed initial state $\mathbf{x}_{\text{nc}0}$, determining an optimal time to implement external inputs is a reasonable alternative to save the control energy when the original system evolves autonomously. When one is unable to adjust the initial state $\mathbf{x}_{\text{nc}0}$ in advance, the energy-optimal time $t_0^{\text{I}*}$ to apply the control inputs can also be forecasted according to $\mathbf{x}^*_{\text{nc}0}$. This is because both the state evolutions of target nodes $\mathbf{x}_{\text{c}}(t)$ and that of uncontrolled nodes $\mathbf{x}_{\text{nc}}(t)$ can be obtained when the network dynamics is given, which makes it possible to find the time t_0^{I} to implement control inputs when the former final states of uncontrolled nodes $\mathbf{x}_{\text{nc}}(t_0^{\text{I}})$ are relatively close to $\mathbf{X}$. That is, the geometric distance $d_{\min}$ may be relatively small when we can save the control energy cost.

Here we propose a strategy to determine the energy-optimal time $t_0^{\text{I}*}$ to apply the control inputs for the six-dimensional network system. For the control task with the initial state $\mathbf{x}_0$, the target state $\mathbf{x}_{\text{cf}}$ and control duration $t_{\text{f}}^{\text{I}} - t_0^{\text{I}}$ are given, and the control input is $\mathbf{u}(t) = \mathbf{0}$, i.e., that system is originally an autonomous system. We solve the differential equation, Eq. (1), at first, yielding $\{\mathbf{x}(t) | t \in [t_0, t_{\text{e}})\}$ where t_{e} is a relatively large real number. Then, we can calculate the energy-optimal $\mathbf{x}^*_{\text{nc}0}(t)$ by Eq. (7) and the corresponding distance $d_{\min}(t)$ when $t \in [t_0, t_{\text{e}})$ via Eq. (8) simultaneously. When $d_{\min}$ reaches a minimum value, the energy-optimal time $t_0^{\text{I}*}$ would be obtained, i.e., $t_0^{\text{I}*} = \arg\min\,[d_{\min}(t)]$, $t \in [t_0, t_{\text{e}})$.

To demonstrate an implementation of our strategy, we set the control duration as $t_{\text{f}}^{\text{I}} - t_0^{\text{I}} = e^{-2}$, aiming to drive the system from $\mathbf{x}_0 = [1, 1, 1, 1, 0, 0]^{\text{T}}$ to $\mathbf{x}_{\text{cf}} = [2, 3, 4, 5]^{\text{T}}$. By applying the proposed method above, we adopt $t_{\text{e}} = 0.3$ and derive the evolution of the system with zero input, i.e., $\mathbf{u}(t) = \mathbf{0}$. The optimal control input $\mathbf{u}^*(t)$ for each $t_0^{\text{I}} \in [0, 0.3)$ with step 0.001 is calculated. Simultaneously the corresponding control energy $\tilde{E}$ (red curve) and the distance $d_{\min}$ (blue curve) are obtained for $t_0^{\text{I}} \in [0, 0.3)$ in Fig. 4(a). Obviously, we can derive the energy-optimal time $t_0^{\text{I}*} = 0.173$ with $\tilde{E}^* = e^{11.85}$, where the distance $d_{\min}$ and control energy $\tilde{E}$ reach the minimum value concurrently. In Figs. 4(b)–4(d), the evolutions of each state are presented with different time t_0^{I} to implement the control inputs. The target state is all achieved in the case of $t_0^{\text{I}1} = 0.1$, $t_0^{\text{I}*} = 0.173$, and $t_0^{\text{I}2} = 0.2$. However, the control energy is tremendously different, and the control energy can be exponentially smaller than others when $t_0^{\text{I}} = t_0^{\text{I}*}$. These simulations prove that the proposed strategy is effective and efficient for saving control energy.

IV. DISCUSSION

In the past two decades, much attention has been paid to controlling complex networks. One of the major challenges to perform control tasks is how to save control energy. Here, our results first show that, surprisingly the initial states of uncontrollable nodes has a significant impact on saving control energy. The necessary and sufficient condition for the existence of the energy-optimal states of uncontrollable nodes is given, and we prove that they always exist for any network structure. Furthermore, the possible optimal solutions are all given analytically. We also point out that a large distance between the energy-optimal solution and the states of uncontrollable nodes results in more control energy needed to conduct a control task. In addition, in order to save control energy, we propose a strategy to determine the appropriate time to implement external control inputs. Simulations show that exponential control energy can be saved by applying our strategy, which suggests that large networks can be controlled by a relatively small number of inputs when the control energy is limited.

Although the nonlinear dynamics and network temporality often appear in natural complex systems [42], the internal interactions of the system's components are usually difficult to detect. In addition, the nonlinear dynamics varies significantly between different types of complex systems, which makes it hard to derive the general results. To deal with the nonlinearity, one general strategy is to explore the linearized

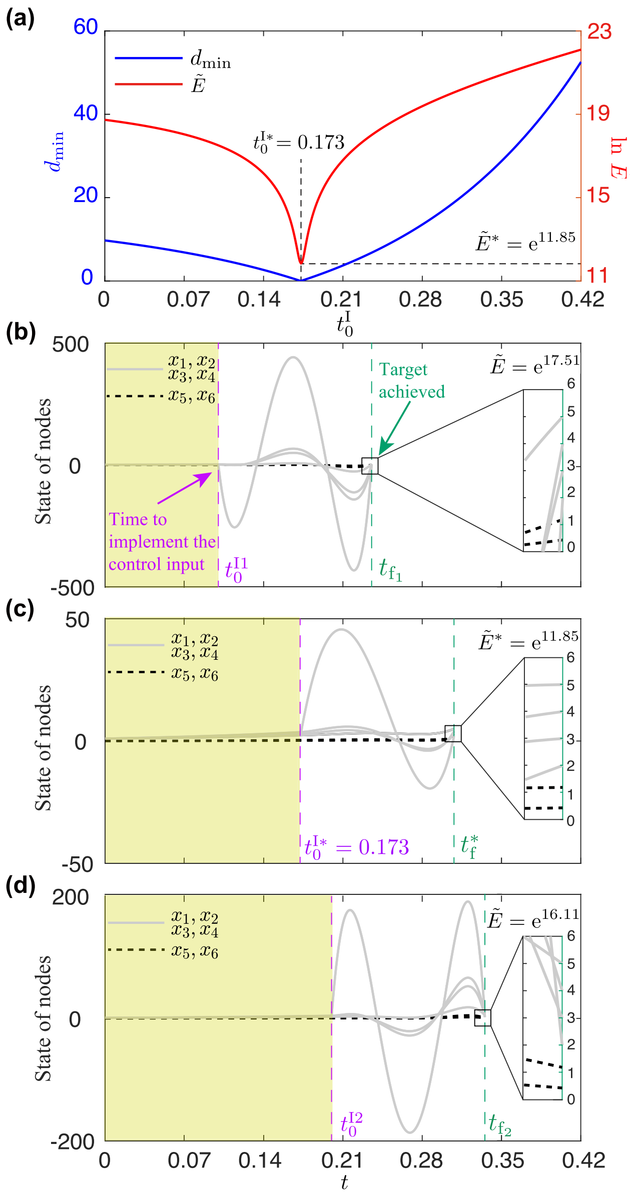

FIG. 4. Control strategy determining the energy-optimal time to inject the external control inputs. Here we apply the proposed strategy to the network system presented in Fig. 2(a) with a control time $t_f^I - t_0^I = e^{-2}$, where t_0^I indicates the time when we inject the external control input. The initial state is set as $\mathbf{x}_0 = [1, 1, 1, 1, 0, 0]^T$ at initial time $t_0 = 0$, and the target state is $\mathbf{x}_{cf} = [2, 3, 4, 5]^T$ for controllable nodes. (a) We derive the system's state evolutions with zero input for $t \in [0, t_0^I)$ before we inject the external control input, where t_0^I is uniformly selected from [0,0.3] with step 0.001. In this way, we obtain 300 different initial states to apply the external control inputs. Then we derive the optimal control input for each initial state and calculate the corresponding control energy $\tilde{E}$ in red and the geometric distance $d_{\min}$ in blue. Manifestly the distance $d_{\min}$ and the control energy $\tilde{E}$ share the same monotonicity, where the optimal time $t_0^{I*} = 0.173$ for minimizing the control energy to $\tilde{E}^* = e^{11.85}$. [(b)–(d)] For different time to implement control input $t_0^{I1} = 0.1$, $t_0^{I*} = 0.173$, and $t_0^{I2} = 0.2$, the state time evolutions are shown separately. Yellow regions represent the self-evolution of the system without control, after which the control inputs are injected. Although the target is achieved in all cases, the control energy is different, and we show $\tilde{E}_1 = e^{17.51}$, $\tilde{E}^* = e^{11.85}$, and $\tilde{E}_2 = e^{16.11}$, respectively.

version of the nonlinear system [43]. For temporal networks, which are reported universal in many real complex networks, our framework of static networks also provides effective approaches to analyze each snapshot. To save more control energy when controlling the temporal networks, one can adopt target control for each snapshot, and our methods can further analyze the appropriate time to inject control inputs into each snapshot.

In practical control of real complex systems, the definitions of control energy are diverse. In this paper, the control energy is defined by the integration of control inputs, which only describes the external efforts to steer a complex network. In future work, internal efforts such as the trajectory of the system states could be considered when considering the general control energy [44]. Network structure and properties, such as degree distribution, centrality, betweenness, and multilayerness, are diverse in applications of practical network systems [45–48]. These properties can affect the control energy by Eq. (7) as we have shown in this paper, which is worth exploring systematically. Here, the methods proposed to save control energy may serve as preliminary results and provide feasible methods for subsequent explorations in these aspects. In addition, there are high levels of uncertainty in some complex systems, such as biological systems, whose edge weights are often drawn from a given distribution rather than being exactly known [34,49]. For temporal networks whose network evolution is only probabilistically known, the required minimum control energy can be orders of magnitude more than the deterministic temporal networks [34]. Future investigations could analyze the minimum control energy for these types of nondeterministic networks, as well as the energy-saving control strategies, as we have done for static networks.

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APPENDIX A: DERIVING THE MINIMUM CONTROL ENERGY

In this paper, we consider the linear time-invariant dynamics (1), assuming there are r preselected controllable nodes labeled as $c_1, c_2, \dots, c_r$, and $n - r$ uncontrollable nodes labeled as $nc_1, nc_2, \dots, nc_{n-r}$. Our control task is to drive the states of r controllable nodes from $\mathbf{x}_{c0}$ to $\mathbf{x}_{cf}$ with the minimum control energy in fixed control duration $t_f - t_0$.

In the following, we theoretically derive the optimal control input minimizing the control energy. Then we present the analytic relation of control energy and the initial states of uncontrollable nodes, which makes it possible to analyze the impact of the initial states of uncontrollable nodes on control energy.

In order to drive the states of r controllable nodes from $\mathbf{x}_{c0}$ to $\mathbf{x}_{cf}$ with the minimum control energy, we first perform controllable decomposition on system (1), which yields the dynamics of the controllable part (3) and uncontrollable part (4).

Then from Eqs. (2) and (4) we have

$$\begin{aligned}\bar{\mathbf{x}}_{\mathrm{nc}}(t_{\mathrm{f}}) &= \mathbf{Q}_{\mathrm{nc}}\mathbf{x}_{\mathrm{f}}\\ &= \mathbf{Q}_3\mathbf{x}_{\mathrm{cf}} + \mathbf{Q}_4\mathbf{x}_{\mathrm{ncf}}\end{aligned}$$

and

$$\bar{\mathbf{x}}_{\mathrm{nc}}(t_{\mathrm{f}}) = e^{\mathbf{A}_{\mathrm{nc}}(t_{\mathrm{f}}-t_0)}\bar{\mathbf{x}}_{\mathrm{nc}}(t_0),$$

respectively. Thus we obtain

$$\mathbf{Q}_3\mathbf{x}_{\mathrm{cf}} + \mathbf{Q}_4\mathbf{x}_{\mathrm{ncf}} = e^{\mathbf{A}_{\mathrm{nc}}(t_{\mathrm{f}}-t_0)}\bar{\mathbf{x}}_{\mathrm{nc}}(t_0),$$

which yields

$$\mathbf{x}_{\mathrm{ncf}} = \mathbf{Q}_4^{-1}\left(e^{\mathbf{A}_{\mathrm{nc}}(t_{\mathrm{f}}-t_0)}\mathbf{Q}_{\mathrm{nc}}\mathbf{x}_0 - \mathbf{Q}_3\mathbf{x}_{\mathrm{cf}}\right). \tag{A1}$$

Therefore, controlling the initial states of controllable nodes $\mathbf{x}_0$ to the desired target state $\mathbf{x}_{\mathrm{cf}}$ is equivalent to controlling the controllable part (3) from the initial state $\bar{\mathbf{x}}_{\mathrm{c}}(t_0) = \mathbf{Q}_{\mathrm{c}}\mathbf{x}_0$ to the target state $\bar{\mathbf{x}}_{\mathrm{c}}(t_{\mathrm{f}}) = \mathbf{Q}_1\mathbf{x}_{\mathrm{cf}} + \mathbf{Q}_2\mathbf{x}_{\mathrm{ncf}}$.

Here we present how to derive the minimum control energy as well as the optimal control input under this control task. The problem can be summarized as follows:

$$\min_{\mathbf{u}(t)} E = \int_{t_0}^{t_{\mathrm{f}}} \mathbf{u}^{\mathrm{T}}(t)\mathbf{u}(t)dt$$

$$\text{subject to}\begin{cases}\dot{\bar{\mathbf{x}}}_{\mathrm{c}}(t) = \mathbf{A}_{\mathrm{c}}\bar{\mathbf{x}}_{\mathrm{c}}(t) + \mathbf{B}_{\mathrm{c}}\mathbf{u}(t)\\ \bar{\mathbf{x}}_{\mathrm{c}}(t_0) = \bar{\mathbf{x}}_{\mathrm{c}0}\\ \bar{\mathbf{x}}_{\mathrm{c}}(t_{\mathrm{f}}) = \bar{\mathbf{x}}_{\mathrm{cf}}.\end{cases}$$

The optimal solution to the minimization problem can be found according to the optimal control energy theory. We use the Hamiltonian function

$$\mathbf{H}[\bar{\mathbf{x}}(t), \bar{\mathbf{v}}(t), \bar{\mathbf{u}}(t)] = \mathbf{u}^{\mathrm{T}}(t)\mathbf{u}(t) + \mathbf{v}^{\mathrm{T}}(t)[\mathbf{A}_{\mathrm{c}}\bar{\mathbf{x}}(t) + \mathbf{B}_{\mathrm{c}}\mathbf{u}(t)],$$

and the following dynamical relations can be determined:

$$\text{State Equation}: \dot{\bar{\mathbf{x}}}_{\mathrm{c}}(t) = \frac{\partial \mathbf{H}}{\partial \mathbf{v}} = \mathbf{A}_{\mathrm{c}}\bar{\mathbf{x}}_{\mathrm{c}}(t) + \mathbf{B}_{\mathrm{c}}\mathbf{u}(t),$$

$$\text{Costate Equation}: \dot{\mathbf{v}}(t) = -\frac{\partial \mathbf{H}}{\partial \bar{\mathbf{x}}} = -\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}\mathbf{v}(t),$$

$$\text{Stationary Equation}: \mathbf{0} = \frac{\partial \mathbf{H}}{\partial \mathbf{u}(t)} = \mathbf{u}(t) + \mathbf{B}_{\mathrm{c}}^{\mathrm{T}}\mathbf{v}(t).$$

By solving the Stationary Equation, we derive the optimal input as

$$\mathbf{u}^*(t) = -\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}\mathbf{v}(t). \tag{A2}$$

Furthermore, from the Costate Equation we get

$$\mathbf{v}(t) = e^{-\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t-t_{\mathrm{f}})}\mathbf{v}(t_{\mathrm{f}}). \tag{A3}$$

By Eqs. (A2) and (A3), we have

$$\mathbf{u}^*(t) = -\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{-\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t-t_{\mathrm{f}})}\mathbf{v}(t_{\mathrm{f}}). \tag{A4}$$

Therefore the State Equation can be solved with the input Eq. (A4), that is,

$$\bar{\mathbf{x}}(t) = e^{\mathbf{A}_{\mathrm{c}}(t-t_0)}\bar{\mathbf{x}}_{\mathrm{c}0} - \int_{t_0}^{t_{\mathrm{f}}} e^{\mathbf{A}_{\mathrm{c}}(t-\tau)}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-\tau)}d\tau\, \mathbf{v}(t_{\mathrm{f}}).$$

Due to $\bar{\mathbf{x}}_{\mathrm{c}}(t_{\mathrm{f}}) = \bar{\mathbf{x}}_{\mathrm{cf}}$, we obtain

$$\mathbf{v}(t_{\mathrm{f}}) = -\mathbf{W}_{\mathrm{c}}^{-1}(t_0, t_{\mathrm{f}})\left(\bar{\mathbf{x}}_{\mathrm{cf}} - e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-t_0)}\bar{\mathbf{x}}_{\mathrm{c}0}\right),$$

where $\mathbf{W}_{\mathrm{c}}(t_0, t_{\mathrm{f}}) = \int_{t_0}^{t_{\mathrm{f}}} e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-\tau)}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-\tau)}d\tau$ is nonsingular because of the controllable system (3). Thus we derive the optimal control input

$$\mathbf{u}^*(t) = \mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-t)}\mathbf{W}_{\mathrm{c}}^{-1}(t_0, t_{\mathrm{f}})\boldsymbol{\xi}(t_{\mathrm{f}} - t_0)$$

and the corresponding minimum control energy

$$\tilde{E} = \boldsymbol{\xi}^{\mathrm{T}}(t_{\mathrm{f}} - t_0)\mathbf{W}_{\mathrm{c}}^{-1}(t_0, t_{\mathrm{f}})\boldsymbol{\xi}(t_{\mathrm{f}} - t_0), \tag{A5}$$

where $\boldsymbol{\xi}(t_{\mathrm{f}} - t_0) = e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-t_0)}\bar{\mathbf{x}}_{\mathrm{c}0} - \bar{\mathbf{x}}_{\mathrm{cf}}$.

APPENDIX B: PROOF OF $\mathbf{W}_{\mathrm{c}}(t_0, t_{\mathrm{f}}) = \mathbf{W}_{\mathrm{c}}(0, t_{\mathrm{f}} - t_0)$

For the controllability Gramian matrix

$$\mathbf{W}_{\mathrm{c}}(t_0, t_{\mathrm{f}}) = \int_{t_0}^{t_{\mathrm{f}}} e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-\tau)}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-\tau)}d\tau,$$

by variable substitution $t = t_{\mathrm{f}} - \tau$ we have

$$\tau \in [t_0, t_{\mathrm{f}}] \rightarrow t \in [t_{\mathrm{f}} - t_0, 0] \quad \text{and} \quad dt = -d\tau,$$

which yields

$$\begin{aligned}\mathbf{W}_{\mathrm{c}}(t_0, t_{\mathrm{f}}) &= \int_{t_0}^{t_{\mathrm{f}}} e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-\tau)}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-\tau)}d\tau\\ &= -\int_{t_{\mathrm{f}}-t_0}^{0} e^{\mathbf{A}_{\mathrm{c}}t}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}t}dt\\ &= \int_{0}^{t_{\mathrm{f}}-t_0} e^{\mathbf{A}_{\mathrm{c}}t}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}t}dt.\end{aligned} \tag{B1}$$

Similarly, for

$$\mathbf{W}_{\mathrm{c}}(0, t_{\mathrm{f}} - t_0) = \int_{0}^{t_{\mathrm{f}}-t_0} e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-t_0-\tau)}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-t_0-\tau)}d\tau,$$

by variable substitution $h = t_{\mathrm{f}} - t_0 - \tau$ we obtain

$$\tau \in [0, t_{\mathrm{f}} - t_0] \rightarrow h \in [t_{\mathrm{f}} - t_0, 0] \quad \text{and} \quad dh = -d\tau,$$

which yields

$$\begin{aligned}\mathbf{W}_{\mathrm{c}}(0, t_{\mathrm{f}} - t_0) &= \int_{0}^{t_{\mathrm{f}}-t_0} e^{\mathbf{A}_{\mathrm{c}}(t_{\mathrm{f}}-t_0-\tau)}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}(t_{\mathrm{f}}-t_0-\tau)}d\tau\\ &= -\int_{t_{\mathrm{f}}-t_0}^{0} e^{\mathbf{A}_{\mathrm{c}}h}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}h}dh\\ &= \int_{0}^{t_{\mathrm{f}}-t_0} e^{\mathbf{A}_{\mathrm{c}}t}\mathbf{B}_{\mathrm{c}}\mathbf{B}_{\mathrm{c}}^{\mathrm{T}}e^{\mathbf{A}_{\mathrm{c}}^{\mathrm{T}}t}dt.\end{aligned} \tag{B2}$$

By Eqs. (B1) and (B2), we know $\mathbf{W}_{\mathrm{c}}(t_0, t_{\mathrm{f}}) = \mathbf{W}_{\mathrm{c}}(0, t_{\mathrm{f}} - t_0)$.

APPENDIX C: ANALYTIC RELATION OF CONTROL ENERGY AND INITIAL STATES OF UNCONTROLLABLE NODES

In this section, we obtain the analytic relation of control energy and the initial states of uncontrollable nodes for the convenience of analyzing the effect of the initial states on saving control energy.

For $\boldsymbol{\xi}(t_f - t_0)$, we have

$$\begin{aligned}\boldsymbol{\xi}(t_f - t_0) &= e^{\mathbf{A}_c(t_f-t_0)}\bar{\mathbf{x}}_{c0} - \bar{\mathbf{x}}_{cf}\\ &= e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_c\mathbf{x}_0 - \mathbf{Q}_c\mathbf{x}_f\\ &= e^{\mathbf{A}_c(t_f-t_0)}(\mathbf{Q}_1\mathbf{x}_{c0} + \mathbf{Q}_2\mathbf{x}_{nc0}) - \mathbf{Q}_1\mathbf{x}_{cf} - \mathbf{Q}_2\mathbf{x}_{ncf}.\end{aligned} \tag{C1}$$

Substituting Eq. (A1) into Eq. (C1), we derive

$$\begin{aligned}\boldsymbol{\xi}(t_f - t_0) &= e^{\mathbf{A}_c(t_f-t_0)}(\mathbf{Q}_1\mathbf{x}_{c0} + \mathbf{Q}_2\mathbf{x}_{nc0}) - \mathbf{Q}_1\mathbf{x}_{cf} - \mathbf{Q}_2\mathbf{x}_{ncf}\\ &= e^{\mathbf{A}_c(t_f-t_0)}(\mathbf{Q}_1\mathbf{x}_{c0} + \mathbf{Q}_2\mathbf{x}_{nc0}) - \mathbf{Q}_1\mathbf{x}_{cf}\\ &\quad - \mathbf{Q}_2\mathbf{Q}_4^{-1}\left[e^{\mathbf{A}_{nc}(t_f-t_0)}(\mathbf{Q}_3\mathbf{x}_{c0} + \mathbf{Q}_4\mathbf{x}_{nc0}) - \mathbf{Q}_3\mathbf{x}_{cf}\right]\\ &= \left(\mathbf{Q}_2\mathbf{Q}_4^{-1}\mathbf{Q}_3 - \mathbf{Q}_1\right)\mathbf{x}_{cf} + \left(e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2\right.\\ &\quad \left.- \mathbf{Q}_2\mathbf{Q}_4^{-1}e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0} + \left(e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1\right.\\ &\quad \left.- \mathbf{Q}_2\mathbf{Q}_4^{-1}e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right)\mathbf{x}_{c0}.\end{aligned}$$

For the orthogonal matrix $\mathbf{Q}$, we have

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\mathbf{Q}_1^T & \mathbf{Q}_3^T\\ \mathbf{Q}_2^T & \mathbf{Q}_4^T\end{pmatrix}\begin{pmatrix}\mathbf{Q}_1 & \mathbf{Q}_2\\ \mathbf{Q}_3 & \mathbf{Q}_4\end{pmatrix} = \begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix},$$

and thus

$$\mathbf{Q}_1^T\mathbf{Q}_1 + \mathbf{Q}_3^T\mathbf{Q}_3 = \mathbf{I}_r, \tag{C2}$$

$$\mathbf{Q}_1^T\mathbf{Q}_2 + \mathbf{Q}_3^T\mathbf{Q}_4 = \mathbf{0}. \tag{C3}$$

By Eq. (C3) we have

$$\mathbf{Q}_3^T = -\mathbf{Q}_1^T\mathbf{Q}_2\mathbf{Q}_4^{-1}. \tag{C4}$$

Substituting Eq. (C4) into Eq. (C2), we get

$$\left(\mathbf{Q}_1^T\right)^{-1} = \mathbf{Q}_1 - \mathbf{Q}_2\mathbf{Q}_4{}^{-1}\mathbf{Q}_3.$$

So we obtain

$$\begin{aligned}\boldsymbol{\xi}(t_f - t_0) &= \left(\mathbf{Q}_2\mathbf{Q}_4^{-1}\mathbf{Q}_3 - \mathbf{Q}_1\right)\mathbf{x}_{cf} + \left(e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2\right.\\ &\quad \left.- \mathbf{Q}_2\mathbf{Q}_4^{-1}e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0} + \left(e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1\right.\\ &\quad \left.- \mathbf{Q}_2\mathbf{Q}_4^{-1}e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right)\mathbf{x}_{c0}\\ &= \left(\mathbf{Q}_1^T\right)^{-1}\left\{-\mathbf{x}_{cf} + \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2\right.\right.\\ &\quad \left.+ \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0} + \left[\left(\mathbf{Q}_1^T\right)e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1\right.\\ &\quad \left.\left.+ \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right]\mathbf{x}_{c0}\right\},\end{aligned}$$

which yields

$$\begin{aligned}\tilde{E} &= \boldsymbol{\xi}^T\mathbf{W}_c^{-1}\boldsymbol{\xi}\\ &= \left[-\mathbf{x}_{cf} + \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0}\right.\\ &\quad \left.+ \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right)\mathbf{x}_{c0}\right]^T\mathbf{Q}_1^{-1}\mathbf{W}_c^{-1}\\ &\quad \times \left(\mathbf{Q}_1^{-1}\right)^T\left[-\mathbf{x}_{cf} + \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0}\right.\\ &\quad \left.+ \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right)\mathbf{x}_{c0}\right].\end{aligned}$$

For simplification, denote $\mathbf{H}(t_0, t_f) = \mathbf{Q}_1^T\mathbf{W}_c(t_0, t_f)\mathbf{Q}_1$, $\mathbf{U}(t_f - t_0) = \mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4$, $\mathbf{R}(t_f - t_0) = \mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3$, $\mathbf{G}(t_0, t_f) = \mathbf{U}^T(t_f - t_0)\mathbf{H}^{-1}\mathbf{U}(t_f - t_0)$, $\boldsymbol{\eta}(t_f - t_0) = \mathbf{R}(t_f - t_0)\mathbf{x}_{c0} - \mathbf{x}_{cf}$, $\boldsymbol{\zeta}(t_0, t_f) = 2\boldsymbol{\eta}^T\mathbf{H}^{-1}(t_0, t_f)\mathbf{U}(t_f - t_0)$, and $c(t_0, t_f) = \boldsymbol{\eta}^T(t_f - t_0)\mathbf{H}^{-1}(t_0, t_f)\boldsymbol{\eta}(t_f - t_0)$. Since $\mathbf{W}_c(t_0, t_f) = \mathbf{W}_c(0, t_f - t_0)$, we have $\mathbf{G}(t_0, t_f) = \mathbf{G}(0, t_f - t_0)$, $\boldsymbol{\zeta}(t_0, t_f) = \boldsymbol{\zeta}(0, t_f - t_0)$, and $c(t_0, t_f) = c(0, t_f - t_0)$. By omitting the time variable $t_f - t_0$ of these matrices, vectors, and constant, we have

$$\begin{aligned}\tilde{E} &= \left[-\mathbf{x}_{cf} + \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0}\right.\\ &\quad \left.+ \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right)\mathbf{x}_{c0}\right]^T\mathbf{Q}_1^{-1}\mathbf{W}_c^{-1}\\ &\quad \times \left(\mathbf{Q}_1^{-1}\right)^T\left[-\mathbf{x}_{cf} + \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_2 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_4\right)\mathbf{x}_{nc0}\right.\\ &\quad \left.+ \left(\mathbf{Q}_1^T e^{\mathbf{A}_c(t_f-t_0)}\mathbf{Q}_1 + \mathbf{Q}_3^T e^{\mathbf{A}_{nc}(t_f-t_0)}\mathbf{Q}_3\right)\mathbf{x}_{c0}\right]\\ &= (\mathbf{U}\mathbf{x}_{nc0} + \mathbf{R}\mathbf{x}_{c0} - \mathbf{x}_{cf})^T\mathbf{H}^{-1}(\mathbf{U}\mathbf{x}_{nc0} + \mathbf{R}\mathbf{x}_{c0} - \mathbf{x}_{cf})\\ &= (\mathbf{U}\mathbf{x}_{nc0} + \boldsymbol{\eta})^T\mathbf{H}^{-1}(\mathbf{U}\mathbf{x}_{nc0} + \boldsymbol{\eta})\\ &= \mathbf{x}_{nc0}^T\mathbf{U}^T\mathbf{H}^{-1}\mathbf{U}\mathbf{x}_{nc0} + 2\boldsymbol{\eta}^T\mathbf{H}^{-1}\mathbf{U}\mathbf{x}_{nc0} + \boldsymbol{\eta}^T\mathbf{H}^{-1}\boldsymbol{\eta}\\ &= \mathbf{x}_{nc0}^T\mathbf{G}\mathbf{x}_{nc0} + \boldsymbol{\zeta}\mathbf{x}_{nc0} + c.\end{aligned}$$

In this way, we can separate the initial states of uncontrollable nodes $\mathbf{x}_{nc0}$ in Eq. (A5) as

$$\tilde{E} = \mathbf{x}_{nc0}^T\mathbf{G}\mathbf{x}_{nc0} + \boldsymbol{\zeta}\mathbf{x}_{nc0} + c,$$

which is an analytic form about control energy and the initial states of uncontrollable nodes.

APPENDIX D: PROOF OF THE ENERGY-OPTIMAL SOLUTION AND ITS EXISTENCE

1. The solution of problem (6) exists if and only if $\mathrm{rank}(\boldsymbol{\Lambda}) = \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$

Proof. Necessary: Suppose that $\mathrm{rank}(\boldsymbol{\Lambda}) < \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$; by denoting $\mathbf{x}_{nc0} = \boldsymbol{\Delta}\mathbf{y}$, we have

$$\begin{aligned}\tilde{E} &= \mathbf{x}_{nc0}^T\mathbf{G}\mathbf{x}_{nc0} + \boldsymbol{\zeta}\mathbf{x}_{nc0} + c\\ &= \mathbf{y}^T\boldsymbol{\Delta}^T\mathbf{G}\boldsymbol{\Delta}\mathbf{y} + \boldsymbol{\zeta}\boldsymbol{\Delta}\mathbf{y} + c\\ &= \sum_{i=1}^{q}\lambda_i y_i^2 + \sum_{i=1}^{l}\delta_i y_i + c,\end{aligned}$$

where l is the location of the last nonzero element of vector $\boldsymbol{\delta}$, λ_i is the ith diagonal element of $\boldsymbol{\Lambda}_q$, and δ_i is the ith element of $\boldsymbol{\delta}$. Since $\mathrm{rank}(\boldsymbol{\Lambda}) < \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$ as supposed, we have $l > q$. Then we get

$$\begin{aligned}\tilde{E} &= \sum_{i=1}^{q}\lambda_i y_i^2 + \sum_{i=1}^{l}\delta_i y_i + c\\ &= \sum_{i=1}^{q}\lambda_i\left(y_i + \frac{\delta_i}{2\lambda_i}\right)^2 + \sum_{i=1+q}^{l}\delta_i y_i + c - \sum_{i=1}^{q}\frac{\delta_i^2}{4\lambda_i}.\end{aligned}$$

There is a first-order term of the above equation, which is $\delta_i y_i$ for $i = 1 + q, \ldots, l$. Since $\mathbf{x}_{nc0} = \boldsymbol{\Delta}\mathbf{y}$ and $\boldsymbol{\Delta}$ is nonsingular, for any y_i there always exists a suitable $\mathbf{x}_{nc0}$, then we have $\tilde{E} \in (-\infty, +\infty)$, which means that the optimal problem (6) does not have a solution. This is a contradiction.

Sufficiency: As $\mathrm{rank}(\boldsymbol{\Lambda}) = \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$, we have $l \leqslant q$. Hence we obtain

$$\begin{aligned}\tilde{E} &= \sum_{i=1}^{q} \lambda_i y_i^2 + \sum_{i=1}^{l} \delta_i y_i + c \\ &= \sum_{i=1}^{l} \lambda_i \left(y_i + \frac{\delta_i}{2\lambda_i} \right)^2 + \sum_{i=1+l}^{q} \lambda_i y_i^2 + c - \sum_{i=1}^{q} \frac{\delta_i^2}{4\lambda_i}. \quad \text{(D1)}\end{aligned}$$

Let

$$\mathbf{y}^* = \left(-\frac{\delta_1}{2\lambda_1}, \dots, -\frac{\delta_l}{2\lambda_l}, \underbrace{0, \dots, 0}_{q-l}, \sigma_1, \dots, \sigma_{n-r-q} \right)^{\mathrm{T}},$$

where $\sigma_1, \sigma_2, \dots, \sigma_{n-r-q}$ are $n - q$ arbitrary parameters and the number of zero element is $q - l$. Since λ_i is positive for $i = 1, \dots, q$, it is obvious that the first l elements and the $q - l$ zero elements of $\mathbf{y}^*$ results in $\sum_{i=1}^{l} \lambda_i (y_i + \frac{\delta_i}{2\lambda_i})^2 = 0$ and $\sum_{i=1+l}^{q} \lambda_i y_i^2 = 0$ in Eq. (D1), respectively. The terms containing variable y_i all reach the minimum value 0 in Eq. (D1). Therefore, if $\mathrm{rank}(\boldsymbol{\Lambda}) = \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$, the optimal problem has the solution $\mathbf{x}_{\mathrm{nc}0}^* = \boldsymbol{\Delta}\mathbf{y}^*$. The corresponding optimal control energy is also obtained as

$$\tilde{E}^* = c - \frac{1}{4} \sum_{i=1}^{q} \frac{\delta_i^2}{\lambda_i}.$$

■

2. The solution of problem (6) always exists, i.e., rank(Λ) ≡ rank(Λ, δ)

Proof. If $\mathrm{rank}(\boldsymbol{\Lambda}) < \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$, we have

$$\tilde{E} = \sum_{i=1}^{q} \lambda_i \left(y_i + \frac{\delta_i}{2\lambda_i} \right)^2 + \sum_{i=1+q}^{l} \delta_i y_i + c - \sum_{i=1}^{q} \frac{\delta_i^2}{4\lambda_i}.$$

Since $\mathbf{x}_{\mathrm{nc}0} = \boldsymbol{\Delta}\mathbf{y}$ and $\boldsymbol{\Delta}$ is nonsingular, y_i can be an arbitrary value by adjusting $\mathbf{x}_{\mathrm{nc}0}$. Let

$$y_{q+1} = \frac{\sum_{i=1}^{q} \frac{\delta_i^2}{4\lambda_i} - c - \sum_{i=1}^{q} \lambda_i \left(y_i + \frac{\delta_i}{2\lambda_i} \right)^2 - 1}{\delta_{1+q}},$$

and $y_j = 0, j = q+2, \dots, l$, which yields $\tilde{E} = -1 < 0$. This is in direct contradiction with $\tilde{E} \geqslant 0$. So we derive $\mathrm{rank}(\boldsymbol{\Lambda}) = \mathrm{rank}(\boldsymbol{\Lambda}, \boldsymbol{\delta})$, and problem (6) always has the solution. ■

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