# Numerical evidence of a universal critical behavior of two-dimensional and three-dimensional random quantum clock and Potts models

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The random quantum $q$-state clock and Potts models are studied in two and three dimensions. The existence of Griffiths phases is tested in the two-dimensional case with $q = 6$ by sampling the integrated probability distribution of local susceptibilities of the equivalent McCoy-Wu three-dimensional classical models with Monte Carlo simulations. For the random Potts model, numerical evidence of the existence of Griffiths phases is given and the finite-size effects are analyzed. For the clock model, the data also suggest the existence of a Griffiths phase but with much larger finite-size effects. The critical point of the random quantum clock model is then studied with the Strong-Disorder Renormalization Group. Evidence is given that, at strong enough disorder, this critical behavior is governed by the same infinite-disorder fixed point as the Potts model, for all the number of states $q$ considered. At weak disorder, our renormalization group method becomes unstable and does not allow us to make conclusions.

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## I. INTRODUCTION

Universality is a cornerstone of the theory of critical phenomena. At the vicinity of a continuous phase transition, the critical exponents appearing in the algebraic behavior of thermodynamic quantities are independent from the microscopic details of the system. The renormalization group provides a theoretical foundation to this universality [1]. The symmetry of the Hamiltonian that is spontaneously broken during the phase transition participates in determining the universality class. Universality classes may be drastically affected by the introduction of disorder in the system. The $q$-state Potts model, whose Hamiltonian is invariant under the permutations of the $q$ states, is a well-known example of this principle. In two dimensions, the classical Potts model undergoes a second-order phase transition when $q \leqslant 4$ with critical exponents depending on the number of states $q$ [2]. As shown by Harris, the critical behavior is unchanged in the presence of randomness only if the specific heat of the pure model diverges at most logarithmically [3]. In contrast, when the specific heat exponent $\alpha$ is positive, randomness induces a new critical behavior. This is the case for the two-dimensional (2D) classical Potts model when $2 < q \leqslant 4$ [4–6]. Moreover, discontinuous phase transitions are smoothed by randomness. It was proved that, in two dimensions, an infinitesimal amount of disorder is sufficient to make the transition continuous [7]. Such a situation is observed for the 2D Potts model with $q > 4$. The new critical behavior is again $q$-dependent [8,9].

Surprisingly, a totally different picture emerged in random quantum systems. In the models considered in the literature, the critical behavior turns out to be independent of the spontaneously broken symmetry of the Hamiltonian. The random Ising chain in a transverse field has been extensively studied by Fisher [10–13] using a real-space renormalization group technique, termed the Strong-Disorder Renormalization Group (SDRG) [14]. In contrast to the pure quantum Ising chain in a transverse field, the dynamics is activated, i.e., at the critical point the gap $\Delta E$ vanishes with the lattice size $L$ as $\Delta E \sim e^{-aL^{\psi}}$ instead of the usual law $\Delta E \sim L^{-z}$. The dynamical exponent $z$ is therefore infinite. The exponent $\psi$ takes the value $1/2$ because the logarithm of the renormalized couplings generated along the SDRG flow is uncorrelated random variables. The magnetic critical exponent is predicted to be equal to the golden ratio $(1 + \sqrt{5})/2$. On both sides of the critical point lie two Griffiths phases where the dynamics is dominated by rare macroscopic regions with strong fluctuations of randomness. The dynamical exponent is equal to 1 at the boundary of these phases and diverges as the critical point is approached [15]. The $q$-state random quantum Potts chain was also studied by SDRG [16]. Note that the Ising model in a transverse field is equivalent to the $q = 2$ Potts model. The analysis of the RG flow equations showed that the number of states $q$ is irrelevant. Therefore, the critical behavior of the random Potts chain is governed by the same infinite disorder fixed point (IDFP) as the Ising chain for any finite number of states $q$. The fact that the symmetry group of the Hamiltonian is $q$-dependent does not play any role. DMRG simulations confirmed this result [17]. The $q$-state random quantum clock chain was also considered. In contrast to the Potts model, the Hamiltonian is invariant under circular permutations of the $q$ states. SDRG also predicts that the critical behavior of the random clock chain is described by the same $q$-independent IDFP as the Ising and Potts chains at strong enough disorder [16]. This result was also confirmed by DMRG simulations

 

[18]. Finally, the Ashkin-Teller model, corresponding to two coupled Ising chains, was studied using a numerical implementation of SDRG. Again, along the three transition lines, the critical behavior is governed by the same IDFP as the Ising chain. Only at the tricritical point where these lines merge was a different IDFP identified [19–21].

An interesting question in this context is whether this super-universality is specific to one-dimensional quantum systems or if it also exists in higher dimensions. The random Ising model in a transverse field was studied in dimensions $d = 2, 3$, and 4 using a clever numerical implementation of SDRG [22–24]. It turns out that the critical exponents depend on the dimension $d$. In particular, the exponent $\psi$ takes values smaller than $1/2$, indicating that the logarithms of the renormalized couplings are now correlated. Recently, we applied the same algorithm to the 2D and 3D random Potts models and provided numerical evidence that the critical behavior is the same as the Ising model at the same dimension $d$ for all number of states $q$ considered [25]. Above the upper critical dimension $d_c$, weak randomness is predicted to be irrelevant by the Harris criterion, but the dynamical exponent $z$ is nevertheless expected to diverge [26].

In this work the random $q$-state clock and Potts models are considered. In the first section, the three-dimensional (3D) classical analog of the 2D random quantum six-state clock and Potts models is studied by Monte Carlo simulations. In the second section, the critical behavior of the 2D and 3D random quantum $q$-state clock models is determined by numerically applying the SDRG technique for several numbers of states $q$. It is then compared to the Potts and Ising models. Conclusions follow.

## II. GRIFFITHS PHASES OF THE SIX-STATE POTTS AND CLOCK MODELS

The quantum $q$-state Potts and clock models generalize the quantum Ising model in a transverse field. The Hilbert state is the tensor product of the one-site Hilbert spaces generated by the $q$ states $|\sigma_i\rangle$ with $\sigma_i = 0, \dots, q-1$. The Potts model is defined by the Hamiltonian

$$H = -\sum_{(i,j)} J_{ij} \sum_{\sigma=1}^{q-1} \Omega_i^{\sigma} \Omega_j^{-\sigma} - \sum_i h_i \sum_{\sigma=1}^{q-1} M_i^{\sigma}, \tag{1}$$

where the sum extends over nearest neighbors on the lattice, while for the clock model

$$H = -\sum_{(i,j)} J_{ij} (\Omega_i \Omega_j^{+} + \Omega_j \Omega_i^{+}) - \sum_i h_i (M_i + M_i^{+}). \tag{2}$$

The operator $\Omega_i$ acts only on the spin on site $i$, i.e., $\Omega_i = \mathbb{1}^{\otimes i-1} \otimes \Omega \otimes \mathbb{1}^{\otimes N-i}$ where $N$ is the number of sites of the lattice. The matrix $\Omega$ is a $q \times q$ diagonal matrix whose elements are $\omega^n$ with $\omega = e^{2i\pi/q}$. Similarly, $M_i = \mathbb{1}^{\otimes i-1} \otimes M \otimes \mathbb{1}^{\otimes N-i}$ and $M$ is the $q \times q$ matrix such that $M_i|\sigma_i\rangle = |(\sigma_i + 1) \bmod q\rangle$. The matrices $\Omega$ and $M$ are both unitary and satisfy the relation $\Omega^q = M^q = \mathbb{1}$. In the case $q = 2$, $\Omega = \sigma^z$ and $M = \sigma^x$ so that the Ising model in a transverse field is recovered. In this work, we are interested in the random Potts and clock models which are obtained by choosing the couplings $J_{ij}$ and $h_i$ as positive, random, and uncorrelated.

In this section the Griffiths phases are studied by exploiting the quantum-classical mapping of the Potts and clock models onto their classical counterparts. This mapping relies on the equivalence of the imaginary-time evolution operator of the quantum Potts and clock models in dimension $d$ with the transfer matrix of the classical Potts and clock models in dimension $d + 1$. The Hamiltonians of these classical models are

$$-\beta\mathcal{H} = \sum_{(i,j)} K_{ij} \delta_{\sigma_i,\sigma_j}, \quad \sigma_i = 0, \dots, q-1 \tag{3}$$

for the Potts model and

$$-\beta\mathcal{H} = \sum_{(i,j)} K_{ij} \cos\left[\frac{2\pi}{q}(\sigma_i - \sigma_j)\right], \quad \sigma_i = 0, \dots, q-1 \tag{4}$$

for the clock model. The couplings $K$ of these classical models are related to the couplings of the quantum models in the $d$ spatial directions and are constant in the imaginary-time direction. The factor $\beta = 1/k_B T$, where $T$ is the classical temperature, is absorbed into the definition of the couplings. The sum extends over all pairs $(i, j)$ of nearest-neighboring sites on the lattice. In this section the two models are considered in the case $q = 6$ on a cubic lattice whose sizes are denoted $L_\parallel$, $L_\perp$, and $L_\perp$. In the pure case, i.e., $K_{ij} = K$, both models undergo a ferromagnetic-paramagnetic phase transition. The latter is strongly first order for the Potts model and continuous in the universality class of the 3D XY model for the clock model [27–29]. In particular, there is no intermediate critical phase between the ferromagnetic and paramagnetic phases, in contrast to the 2D classical clock model with $q > 4$. The $\mathbb{Z}_q$ symmetry is spontaneously broken in the low-temperature phase. For both the Potts and clock models, the order parameter is the magnetization density $\frac{1}{N}|\sum_i m_i|$ where the local magnetization reads

$$m_i = \frac{q\delta_{\sigma_i,0} - 1}{q - 1} \tag{5}$$

for the Potts model and

$$m_i = e^{\frac{2i\pi}{q}\sigma_i} \tag{6}$$

for the clock model. In the following, we are interested in the random Potts and clock models with a disorder that is infinitely correlated in one direction. The couplings $K_{ij}$ between sites $i$ and $j$ were chosen to depend only on the coordinates $y$ and $z$ of site $i$, i.e., they are translation-invariant in the $x$ direction. The three couplings between site $i$ and its neighbors on the right, at the bottom and at the back, are identical. The couplings are i.i.d random variables with the binary distribution

$$\wp(K_{ij}) = \tfrac{1}{2}[\delta(K_{ij} - K) + \delta(K_{ij} - rK)]. \tag{7}$$

This configuration is one of the possible generalization of the McCoy-Wu model to three dimensions [30]. Since the specific heat exponent of the pure 3D XY model is slightly negative [31–33], or equivalently the correlation length exponent $\nu$ is slightly larger than $2/d$, the pure fixed point of the clock model is expected to be stable upon the introduction of homogeneous disorder [3]. However, $\nu < 2/(d-1) = 1$, so the infinitely correlated disorder that is studied in this paper

is expected to be relevant. For the six-state Potts model, the first-order phase transition of the pure model is expected to be rounded, even with an infinitesimal amount of disorder [34].

For practical reasons, we studied the clock model using the Wolff Monte Carlo algorithm [35] and the Potts model using the Swendsen-Wang algorithm [36]. Both algorithms are equivalent in terms of convergence. In the following, one Monte Carlo step (MCS) corresponds to $N = L_\parallel L_\perp^2$ spin flips on average. We first considered small systems, $L_\perp = 2, 3, 4$ and $L_\parallel = 64$, so that the average over all disorder configurations could be performed exactly. For $L_\perp = 4$, the number of disorder configurations is $2^{L_\perp^2} = 65\,536$. Systems with $L_\perp = 5$ were also considered but with an average over 100 000 disorder configurations only, out of a total number of $2^{25} \simeq 3.3 \times 10^7$. The location of the phase transition can be estimated as the position of the maximum of the average magnetic susceptibility

$$\bar{\chi} = N[\overline{\langle m^2\rangle} - \overline{\langle m\rangle^2}], \tag{8}$$

where the brackets $\langle X\rangle$ denote the average over thermal fluctuations, while the over-line $\overline{X}$ indicates the average over disorder. The system being strongly anisotropic, it is useful to define the two susceptibilities:

$$\begin{aligned} \bar{\chi}_\parallel &= \frac{1}{NL_\parallel}\sum_{y,z}\overline{\left\langle\left|\sum_x m_{(x,y,z)}\right|^2\right\rangle}, \\ \bar{\chi}_\perp &= \frac{1}{NL_\perp^2}\sum_{x}\overline{\left\langle\left|\sum_{y,z} m_{(x,y,z)}\right|^2\right\rangle}, \end{aligned} \tag{9}$$

where $m_{(x,y,z)}$ denotes the local magnetization density on the site whose Cartesian coordinates on the cubic lattice are $(x\ y\ z)$. For both the Potts and clock models, a single peak is observed for both susceptibilities $\bar{\chi}_\parallel$ and $\bar{\chi}_\perp$, indicating the existence of a single phase transition. As can be seen in Fig. 1 for the clock model with $r = 10$, the shift of the location of the maximum of the peak is larger for $\chi_\parallel$ than for $\chi_\perp$. The same picture is qualitatively observed with a weaker disorder $r = 5$, apart from the fact that the peaks of $\chi_\parallel$ and $\chi_\perp$ are almost of same height. However, it can observed in Fig. 1 that the growth of the peaks is much slower for $L_\perp = 5$. This can be attributed to the fact that the average could not be performed over all disorder configurations for this lattice size. Rare events clearly play a dominant role in the divergence of both susceptibilities, and it is essential to average over all disorder configurations, in contrast to the case of homogeneous disorder. As a consequence, the critical behavior at the transition cannot be studied by Monte Carlo simulations. In the next section, the SDRG will be applied to the quantum equivalent of the clock and Potts models. Nevertheless, the existence of Griffiths phases around the critical point can be investigated using an analysis of the probability distribution of the susceptibilities rather than their averages.

In a random system, large local fluctuations of the couplings allow for the existence of macroscopic ordered (resp. unordered) regions while the rest of the system is still in the paramagnetic (resp. ferromagnetic) phase [37]. The average magnetization is a nonanalytic function of the magnetic field in both the ordered and disordered Griffiths phases that

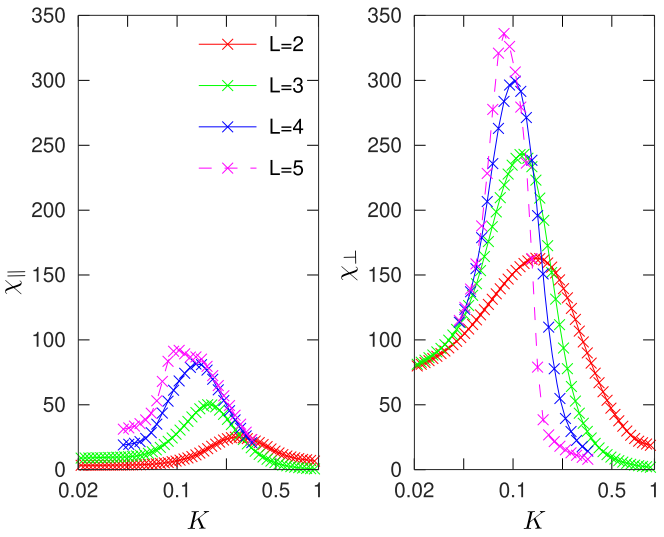


FIG. 1. Longitudinal and transverse susceptibilities $\chi_\parallel$ (left) and $\chi_\perp$ (right) of the 3D six-state random clock model vs the weakest exchange coupling $K$. The different curves correspond to different transverse lattice sizes $L_\perp$ ($L_\perp = 2$ for the smallest peak and $L_\perp = 5$ for the highest one). The longitudinal lattice size is $L_\parallel = 64$.

surround the critical point. As a consequence, the magnetic susceptibility diverges in the whole Griffiths phases. These singularities are too weak to be observed in classical systems with homogeneous disorder but not in the McCoy-Wu model, for which the couplings are infinitely correlated in one direction. Due to the limitation to small lattice sizes, the divergence of the susceptibility in a finite range of couplings is not observed in Fig. 1. However, the Griffiths phases also manifest themselves by an algebraic decay of the probability distribution of local linear and nonlinear susceptibilities [39]. The advantage is that the method does not require an exact average over all disorder configurations. The local susceptibility is defined as

$$\chi_{\rm loc} = L_\parallel\left[\langle m_1^2\rangle - \langle m_1\rangle^2\right] \tag{10}$$

and the nonlinear susceptibility (or Binder cumulant) as

$$U_{\rm loc} = L_\parallel^2\left[\langle m_1^4\rangle - 4\langle m_1\rangle\langle m_1^3\rangle + 12\langle m_1\rangle^2\langle m_1^2\rangle - 3\langle m_1^2\rangle^2 - 6\langle m_1\rangle^4\right], \tag{11}$$

where the magnetization density $m_1 = \frac{1}{L_\parallel}\sum_x m_{(x,1,1)}$ is computed over the first row of the lattice only. The local susceptibility $\chi_{\rm loc}$ is related to the correlation length in the longitudinal direction:

$$\begin{aligned} \chi_{\rm loc} &= \frac{1}{L_\parallel}\sum_{x,x'}[\langle m_{(x,1,1)}m_{(x',1,1)}\rangle - \langle m_{(x,1,1)}\rangle^2]\rangle \\ &\simeq \frac{1}{L_\parallel}\int_0^{L_\parallel} e^{-|x-x'|/\xi_\parallel}\,dx\,dx' \\ &\simeq 2\xi_\parallel(\xi_\parallel \ll L_\parallel). \end{aligned} \tag{12}$$

The disordered Griffiths phase is due to clusters of strong couplings $rK$ that order before the rest of the system. The probability of such a cluster of characteristic size $\ell$ decays

exponentially fast as

$$\wp(\ell^{d_\perp}) \sim p^{\ell^{d_\perp}}, \tag{13}$$

where $p$ is the probability of a strong coupling (chosen to be $p = 1/2$ in this work) and $d_\perp = d - 1$. When the temperature $T$ is between the critical temperature of the strong and weak couplings, this cluster tends to order ferromagnetically because the free energy gain in volume compensates the loss at its boundaries with the paramagnetic phase:

$$\Delta F = -\ell^{d_\perp} L_\parallel [f_\mathrm{o} - f_\mathrm{d}] + \ell^{d_\perp - 1} L_\parallel \sigma_{\mathrm{o,d}} > 0, \tag{14}$$

where $f_\mathrm{o}(rK)$ and $f_\mathrm{d}(K)$ are the free energy densities of the ordered and disordered homogeneous phases for strong and weak couplings, respectively. $\sigma_{o,d}(K, rK)$ is the surface tension between ordered and disordered phases. It follows from Eq. (14) that there exists a minimal size $\ell_\mathrm{min} = \sigma_{o,d}/[f_\mathrm{o} - f_\mathrm{d}]$ for a ferromagnetic cluster to be stable. Moreover, the long-range ordering of the cluster in the longitudinal direction may be reduced by the coexistence of different ferromagnetic phases separated by domain walls whose free energy cost is

$$\Delta F_\mathrm{DW} = \sigma_{o,o} \ell^{d_\perp} \tag{15}$$

per domain wall, where $\sigma_{o,o}$ is the surface tension between two ordered phases. The average size of a magnetic domain is therefore related to the size of the cluster as

$$\xi_\parallel \sim e^{\beta \sigma_{o,o} \ell^{d_\perp}}. \tag{16}$$

Since $\xi_\parallel \simeq \chi_\mathrm{loc}$ according to Eq. (12), and since the probability of a cluster of strong couplings of characteristic size $\ell$ decays algebraically with $\ell_\perp^d$ as Eq. (13), it follows that $\wp(\ln \chi_\mathrm{loc}) \simeq \wp(\ln \xi_\parallel) \sim \wp(\ell^d)$ and therefore [39]

$$\ln \wp(\ln \chi_\mathrm{loc}) = \mathrm{Cst} - \frac{d_\perp}{z} \ln \chi_\mathrm{loc}, \tag{17}$$

where $d_\perp/z = -\ln p \frac{k_B T}{\sigma_{o,o}}$. The same behavior is expected for the Binder cumulant but with a coefficient $d_\perp/3z$ instead of $d_\perp/z$. This prediction has been exploited to estimate the dynamical exponent $z$ of the transverse-field Ising spin glass [38,39] and of the random Ising ferromagnet [40]. Note finally that the average susceptibility $\overline{\chi_\mathrm{loc}} = \int \chi_\mathrm{loc} \wp(\chi_\mathrm{loc})\, d\chi_\mathrm{loc}$ diverges when $z > 1/d_\perp$.

We performed Monte Carlo simulations of the random clock and Potts models with lattice sizes $L_\perp = 12$ and 16 and $L_\parallel = 128, 256, 1024$ and 4096. The system was thermalized with 1000 MC iterations, and thermal averages were then computed over 10 000 MC iterations. Ten thousand random configurations were sampled for $L_\parallel \leqslant 1024$ and 1500 for $L_\parallel = 4096$. We computed the integrated probability distributions $1 - F(\chi_\mathrm{loc})$ and $F(U_\mathrm{loc})$ that are expected to display the same algebraic decay in Griffiths phases as $\wp(\ln \chi_\mathrm{loc})$ and $\wp(\ln U_\mathrm{loc})$. Examples of integrated probability distributions of both $\chi_\mathrm{loc}$ and $U_\mathrm{loc}$ are plotted in Figs. 2–5.

At $K = 0.1156$ (Fig. 2), both the integrated probability distributions $1 - F(\chi_\mathrm{loc})$ and $F(U_\mathrm{loc})$ of the Potts model display an algebraic decay over several decades. The slopes of the curves in a log-log scale give estimates of the dynamical exponent around $z \simeq 0.6$–$0.8$. The latter being larger than $1/d_\perp = 1/2$, the system is in a Griffiths phase. At smaller couplings, closer to the critical point, the dynamical exponent

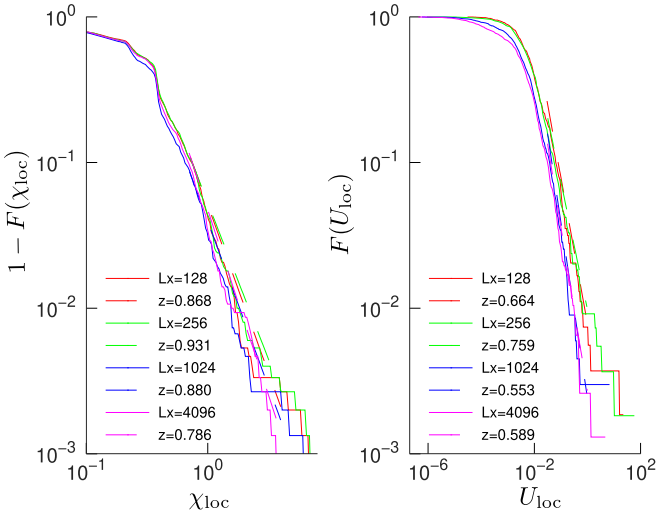


FIG. 2. Integrated probability distribution of the local susceptibility $\chi_\mathrm{loc}$ (left) and of the Binder cumulant $U_\mathrm{loc}$ (right) of the six-state random Potts model at $K = 0.1156$ and $r = 10$. The different curves correspond to different longitudinal lattice sizes $L_\parallel$. The transverse lattice size is $L_\perp = 16$.

is expected to be larger. However, an algebraic decay is still observed in Figs. 3 ($K = 0.1037$) and 4 ($K = 0.0951$) but over a smaller range of susceptibilities. Various finite-size effects can be invoked to explain the shape of the curves. A first regime, where the decay is slow and not algebraic, is observed at small susceptibilities $\chi_\mathrm{loc}$ and $U_\mathrm{loc}$. This regime is due to clusters that are too small to satisfy Eq. (14), i.e., $\ell < \ell_\mathrm{min}$. At intermediate susceptibilities, an algebraic decay is observed but with a strong dependence on the longitudinal lattice size $L_\parallel$. This dependence can be attributed to Eq. (12). For large

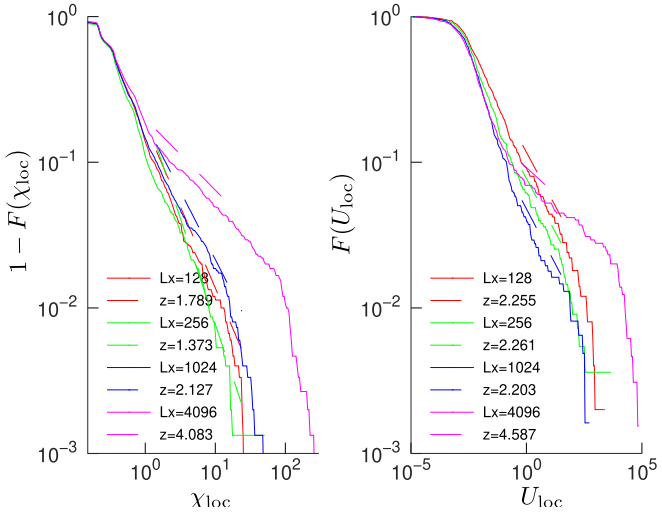


FIG. 3. Integrated probability distribution of the local susceptibility $\chi_\mathrm{loc}$ (left) and of the Binder cumulant $U_\mathrm{loc}$ (right) of the six-state random Potts model at $K = 0.1037$ and $r = 10$. The different curves correspond to different longitudinal lattice sizes $L_\parallel$. The transverse lattice size is $L_\perp = 16$.

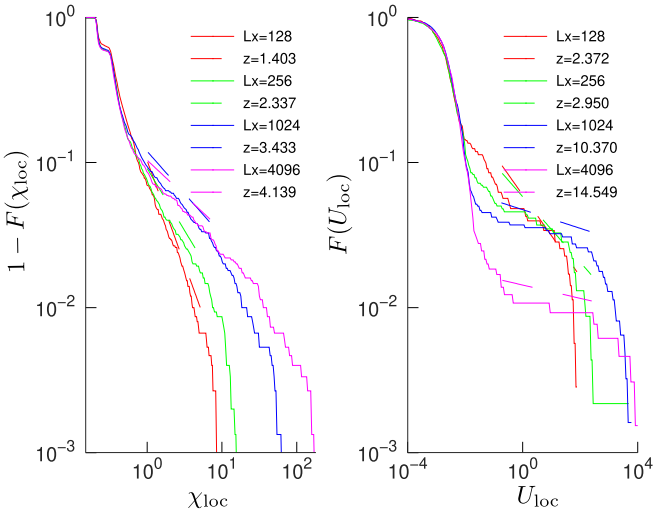


FIG. 4. Integrated probability distribution of the local susceptibility $\chi_{\rm loc}$ (left) and of the Binder cumulant $U_{\rm loc}$ (right) of the six-state random Potts model at $K = 0.0951$ and $r = 10$. The different curves correspond to different longitudinal lattice sizes $L_\parallel$. The transverse lattice size is $L_\perp = 16$.

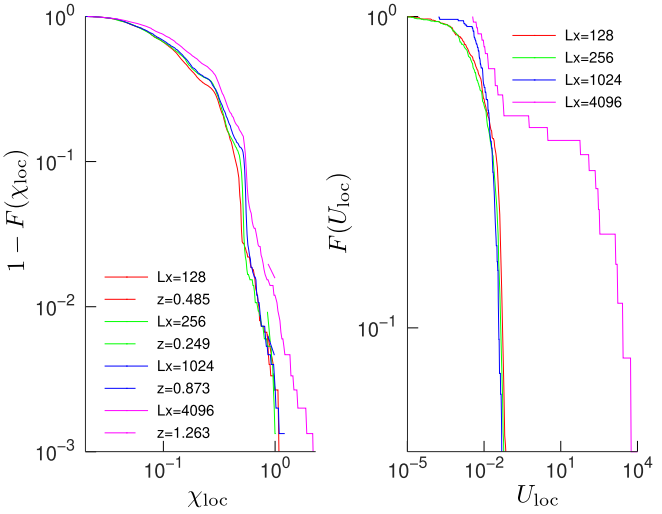


FIG. 5. Integrated probability distribution of the local susceptibility $\chi_{\rm loc}$ (left) and of the Binder cumulant $U_{\rm loc}$ (right) of the six-state random clock model at $K = 0.1233$ ($\chi_{\rm loc}$) and $K = 0.1131$ ($U_{\rm loc}$) for a disorder strength $r = 10$. The different curves correspond to different longitudinal lattice sizes $L_\parallel$. The transverse lattice size is $L_\perp = 16$.

correlation lengths $\xi_\parallel$, one should replace Eq. (12) with (with periodic boundary conditions)

$$\chi_{\rm loc} \simeq 2\xi_\parallel(1 - e^{-L_\parallel/\xi_\parallel}), \tag{18}$$

which leads to a probability distribution $\wp(\ln \chi_{\rm loc}) = \wp(\ln \xi_\parallel)\frac{d\ln \xi_\parallel}{d\ln \chi_{\rm loc}}$ that depends on $L_\parallel$, in contrast to Eq. (17). Finally, at large susceptibilities, the probability distributions fall much faster than a power law. This is due to the fact that the size $\ell$ of the clusters is bounded by the transverse lattice size $L_\perp$. As a consequence, the susceptibility $\chi_{\rm loc}$ cannot be larger than $e^{\beta\sigma_{o,o}L_\perp^{d_\perp}}$. In conclusion, we infer that the random Potts model is also in a Griffiths phase for $K = 0.0951$ and $0.1037$ but that finite-size effects are strong and explain why the algebraic decay is observed only over a smaller range of susceptibilities. The dynamical exponents that can be estimated using Eq. (17) are, as expected, large. However, they strongly depend on $L_\parallel$. These estimates should be considered with caution because a more accurate estimate would require use of the exact $L_\parallel$-dependent expression of $\wp(\ln \chi_{\rm loc})$ and not Eq. (17), which is valid only in the limit $\xi_\parallel \ll L_\parallel$.

For the clock model, the integrated probability distributions of both $\chi_{\rm loc}$ and $U_{\rm loc}$ decay much faster than a power of law for all the considered couplings $K$ when $L_\perp = 12$. At the larger lattice size $L_\perp = 16$ and for $L_\parallel = 4096$, two couplings lead to behaviors similar to that of the Potts model in the Griffiths phase. As can be seen in Fig. 5, the probability distribution of the local susceptibility $\chi_{\rm loc}$ displays an algebraic decay at $K = 0.1233$. For the Binder cumulant, an algebraic plateau is observed at $K = 0.1131$. This suggests that a Griffiths phase may also exist for the clock model but that the finite-size effects are much stronger.

## III. CRITICAL BEHAVIOR OF 2D AND 3D RANDOM QUANTUM CLOCK MODELS

In this section the critical behavior of the 2D and 3D random quantum $q$-state clock model is studied with SDRG. Note that the 2D and 3D random quantum $q$-state Potts models have been considered in Ref. [25]. The random couplings are initially distributed according to

$$\begin{aligned} P_0(J_{ij}) &\sim J_{ij}^{1/\Delta-1}, \quad (0 < J_{ij} < 1), \\ Q_0(h_i) &\sim h_i^{1/\Delta-1}, \quad (0 < h_i < h_{\rm max}), \end{aligned} \tag{19}$$

where $\Delta$ is a free parameter controlling the broadness of the distributions and therefore the strength of the distribution. $\theta = \ln h_{\rm max}$ plays the role of the control parameter of the quantum phase transition. At large (resp. small) $\theta$, the system is expected to be in the paramagnetic phase (resp. ferromagnetic phase).

### A. SDRG decimation rules and RG flow

The SDRG decimation rules are easily derived for the $q$-state clock model [16]. When the largest coupling is $\Omega = J_{ij}$, the Hamiltonian (2) is projected onto the ground state of the local Hamiltonian $-J_{ij}(\Omega_i\Omega_j^+ + \Omega_j\Omega_i^+)$. Since the latter is $q$-fold degenerated, the two spins on sites $i$ and $j$ are replaced by an effective $q$-state macrospin. Second-order perturbation theory shows that this macrospin is coupled to an effective transverse field

$$\tilde{h} = \frac{h_i h_j}{\kappa J_{ij}}, \tag{20}$$

where $\kappa = 1 - \cos 2\pi/q$ for $q > 2$ and to all the neighbors of sites $i$ and $j$ by a coupling $J_{ik} + J_{jk}$. When the largest coupling is $\Omega = h_i$, the spin is frozen in the ground state of the local Hamiltonian $-h_i(M_i + M_i^+)$. An effective coupling between

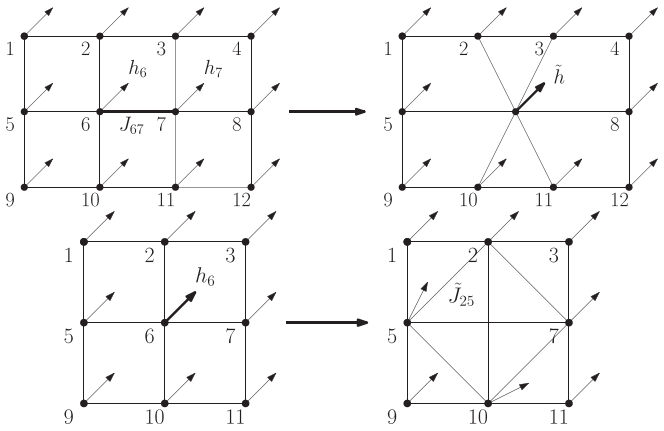


FIG. 6. SDRG decimation rules for the Potts and clock models. The black disks represent the lattice sites, the lines connecting them the coupling $J_{ij}$, and the arrows the transverse fields $h_i$. Above the strongest coupling is $\Omega = J_{67}$. According to Eq. (20), sites 6 and 7 are merged, and the macrospin on the new site is coupled to an effective transverse field $\tilde{h}$. Below, the strongest coupling is $\Omega = h_6$. According to Eq. (21), site 6 is decimated, and new couplings $\tilde{J}$ are induced between its neighbors. In both cases the topology of the lattice has changed.

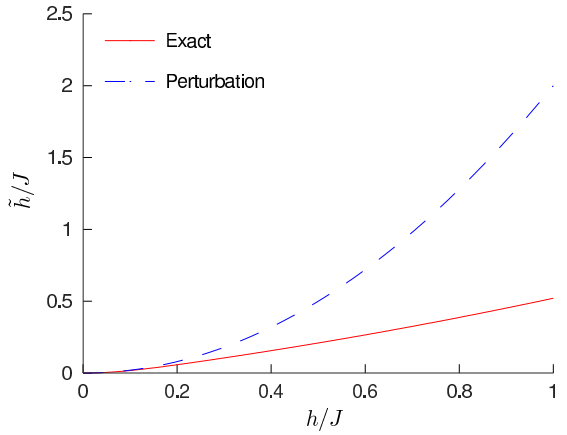


FIG. 7. Effective transverse field $\tilde{h}$ vs $h/J$ ($h_1 = h_2 = h$) computed either by second-order perturbation theory (20) or by exact diagonalization of the two-site Hamiltonian $-J(\Omega_1\Omega_2^+ + \Omega_2\Omega_1)^+ - h(M_1 + M_1^+) - h(M_2 + M_2^+)$ for the six-state clock model. In the second case, the effective transverse field is extracted from the gap $\Delta E$ between the first excited state and the ground state as $\tilde{h} = \Delta E/2\kappa$.

any pairs $(k, l)$ of neighboring sites coupled to $i$ is induced at second-order perturbation theory:

$$\tilde{J}_{kl} = J_{kl} + \frac{J_{ki}J_{il}}{\kappa h_i}. \tag{21}$$

These decimation rules are represented in Fig. 6. For the $q$-state Potts model, the same decimation rules are obtained but with $\kappa = q/2$. This small difference, however, has important consequences. For the Potts model, the effective coupling is always smaller than $\Omega$. It is not always the case for the clock model since $\kappa < 1$ for $q > 4$. The same phenomena is also present in the SDRG flow of various other models [41–43]. According to Eq. (21), when a bond $\Omega = J_{ij}$ is decimated, an effective transverse field $\tilde{h}$ larger than $\Omega$ can be generated when the two transverse fields $h_i$ and $h_j$ are large, more precisely when $h_i h_j > \kappa\Omega$ although $h_i, h_j < \Omega$. This large induced transverse field $\tilde{h}$ is not physical and comes from the fact that the renormalized couplings are computed using second-order perturbation theory. At higher orders, the renormalized couplings are expected to remain smaller than $\Omega$. As can be seen in Fig. 7, the comparison with an exact diagonalization of the two-site Hamiltonian shows that the use of second-order perturbation theory is justified when $h_1, h_2 \lesssim 0.2J_{ij}$ for the six-state clock model, while the effective transverse field becomes larger than $\Omega = J_{ij}$ when $h_1, h_2 \simeq 0.7J_{ij}$. The consequences of this are severe only far from the IDFP. As the IDFP is approached, the probability distribution of the couplings becomes broader and broader. As a consequence, the probability that $h_i h_j > \kappa\Omega$, and therefore that an effective transverse field $\tilde{h} > \Omega$ be induced by the decimation of $J_{ij}$, becomes smaller and smaller. However, numerical simulations are limited to finite systems and therefore to a finite number of RG steps. For small systems, the simulation may end up still relatively far from the IDFP. We observed that renormalized couplings $\tilde{h}, \tilde{J} > \Omega$ slow the convergence of the RG flow towards the IDFP. In some situations, the RG flow can even look chaotic. Note that the renormalized couplings are proportional to $1/\kappa$ so the effect is stronger for clock models with large numbers of states $q$.

For a given disorder configuration, we observed that the series of the largest couplings $\Omega_i$ along the RG flow are very sensitive to the control parameter $\theta$. When comparing the series at two different control parameters $\theta$ and $\theta + \varepsilon$ with $\varepsilon \ll 1$, an intermediate region of fast increase of the difference $\Omega_i(\theta + \varepsilon) - \Omega_i(\theta)$ can be seen. This has a direct impact on the behavior of the average magnetic moment $\bar{\mu}$ of the last cluster at the end of the RG procedure. The latter is expected to be a smooth function of $\theta$. However, we observed that it is not the case for weak initial disorder or large number of states $q$. Instead of a smooth behavior with $\theta$, the magnetic moment displays larger and larger fluctuations as the disorder strength is decreased or the number of states increased. For weak disorder, the magnetic moment looks like a noisy signal without any correlation between successive data points. These correlations can be qualitatively brought out by means of a recurrence plot [44–46]. To construct such a plot, we applied the RG procedure to the same disorder realization for $n$ values $\theta_1 < \theta_2 < \cdots < \theta_n$ of the control parameter $\theta$ and measured the magnetic moment $\mu_i = \mu(\theta_i)$ of the last macrospin. A $n \times n$ recurrence matrix $R$ is constructed as $R_{ij} = 1$ if $\mu_i = \mu_j$ and $R_{ij} = 0$ otherwise. The matrix is plotted in Fig. 8 for four different numbers of states $q$ of the clock model. For $q = 2$, the matrix is block diagonal because the magnetic moment is a monotonous step function of the control parameter $\theta$. Two control parameters $\theta_i$ and $\theta_j$ lead therefore to the same magnetic moment if they belong to the same plateau. In contrast, for $q = 10$, the magnetic moment oscillates widely with $\theta$ like a noisy signal. There does not seem to exist any correlation between two successive control parameters $\theta$ and $\theta_{i+1}$, which is seen by the diluted aspect of the recurrence plot. Figure 8 shows that the noisy behavior increases with the number of states $q$. We have also considered the recurrence matrices for a

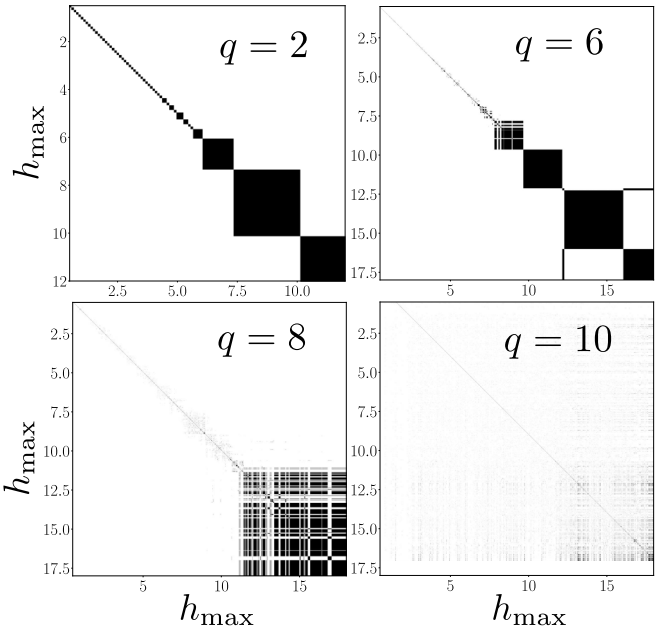


FIG. 8. Recurrence matrices for a $32 \times 32$ clock model with uniform distributions $P_0(J_{ij})$ and $Q_0(h_i)$ of the initial couplings. The number of states of the clock model is two (top left), six (top right), eight (bottom left), and ten (bottom right).

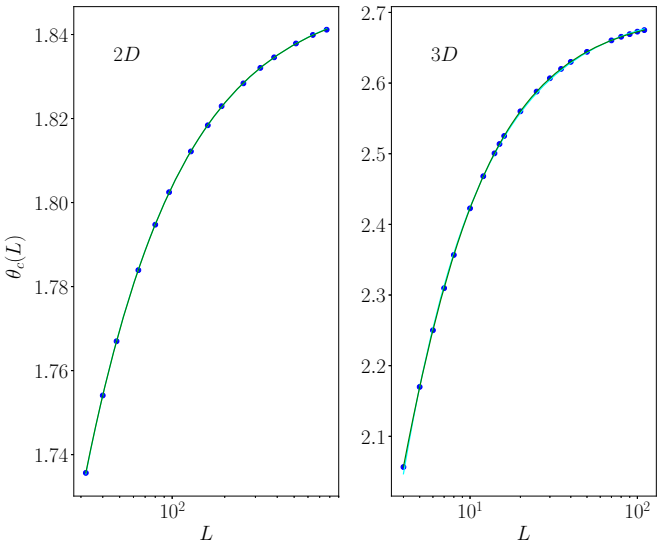


FIG. 9. Average control parameter $\overline{\theta_c(L)}$ at the pseudocritical critical point vs the lattice $L$ for the 2D (left) and 3D (right) random ten-state clock model with $\Delta = 5$. The blue curve corresponds to a fit to Eq. (22), while scaling corrections are taken into account in the fit plotted in green. As can be seen, the two curves can hardly be distinguished.

fixed number of states $q$ and an increasing strength of disorder $\Delta$. As mentioned earlier, the chaotic-like behavior decreases with $\Delta$.

As discussed above, the chaotic-like behavior of the RG flow is the consequence of renormalized couplings larger than the initial ones due to the use of second-order perturbation theory. These couplings prevent the RG flow to approach the fixed point. One way to overcome the problem posed by these large renormalized couplings is to modify the decimation rules, either by including higher-orders in perturbation or simply by using *ad hoc* effective couplings equal to (20) and (21) in the limit $\tilde{h}, \tilde{J} \ll \Omega$. Another possibility is to start the simulation with broad initial distributions of the couplings $h_i$, $J_{ij}$, close to those expected at the IDFP. We will avoid large effective couplings by starting the simulations with strong enough disorder strengths. Moreover, we used a SDRG algorithm similar to the one employed for the Potts model [25] but with a decimation of the global maximum and not of the local maximum. This algorithm has the advantage of limiting drastically the number of new lattice bonds generated during the RG flow by the use of maximum rule, but it requires the effective couplings to be of the form (20) and (21).

### B. Critical exponents at the IDFP

We considered quantum $q$-state clock models with $q = 5$, 8, and 10. We added the cases $q = 2$ and 3 that are equivalent to the Potts models and were already studied [25]. Lattice sizes up to $L = 768$ (1536 in some cases) have been reached in two dimensions and $L = 90$ (120 in some cases) in three dimensions. Averages were performed over at least 8000 disorder realizations.

The control parameter $\theta_c(L)$ at the critical point is estimated using the doubling method for each random sample [22,23,47]. The SDRG procedure is applied to two replicas of the same system with the same disorder configuration coupled together at their boundaries. The pseudocritical point $\theta_c(L)$ is localized iteratively by using the fact that the magnetic moment of the last cluster of the replicated system is expected to be twice the magnetization of a single replica in the ferromagnetic phase but the same in the paramagnetic phase. For large lattice sizes $L$, the average pseudocritical control parameter $\overline{\theta_c(L)}$ displays the power-law behavior

$$|\overline{\theta_c(L)} - \theta_c| \sim L^{-1/\nu}. \tag{22}$$

The data are shown in Fig. 9 in the case of the 2D and 3D random ten-state clock model. The critical exponent $\nu$ has been estimated by a nonlinear fit with the law (22). However, although small, the existence of scaling corrections can be seen on the data, especially in three dimensions. To take them into account, we restricted the fit to lattice sizes $L > L_{\min}$ and repeated the operation with increasing sizes $L_{\min}$. The effective exponents $\nu(L_{\min})$ given by these fits are plotted versus $1/L_{\min}$ in Fig. 10. Most of the points are compatible within error bars, and no indication of a dependence on the number of states $q$ is observed. Extrapolating to $L_{\min} \to +\infty$, the exponent can be estimated to be $\nu \simeq 1.24(3)$ in two dimensions and 0.99(3) in three dimensions. These values are in agreement with those obtained for the Ising model [22,23] [$\nu \simeq 1.24(2)$ in the 2D case and 0.99(2) in the 3D] and the Potts model for various numbers of states [25] [$\nu \simeq 1.25(6)$ in the 2D case and 1.01(5) in the 3D]. The strong dependence of the effective exponent on $L_{\min}$ in 3D suggests the existence of $q$-dependent scaling corrections. We have also performed fits with the scaling law $|\overline{\theta_c(L)} - \theta_c| \sim L^{-1/\nu}(1 + aL^{-\omega})$ including the first scaling correction. However, the fit becomes

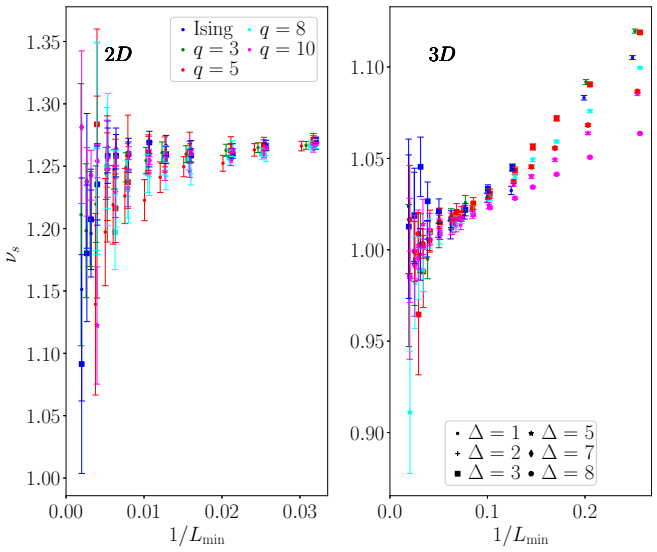


FIG. 10. Critical exponent $\nu$ extracted from the finite-size behavior of the shift of the average control parameter $\overline{\theta_c(L)}$ at the pseudocritical critical point vs the inverse $1/L_{\min}$ of the smallest lattice size $L_{\min}$ taken into account in the fit. The left figure corresponds to the 2D random $q$-state clock model, and the right one to the 3D model. The color of the symbols is related to the number of states $q$, and their shape to the strength of the initial disorder.

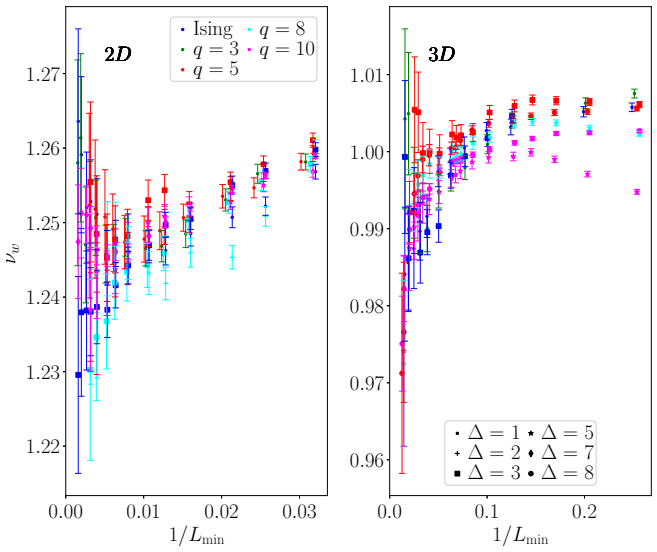


FIG. 11. Critical exponent $\nu$ extracted from the finite-size behavior of the standard deviation $\Delta\theta_c(L)$ of the control parameter at the pseudocritical critical point vs the inverse $1/L_{\min}$ of the smallest lattice size $L_{\min}$ taken into account in the fit. The left figure corresponds to the 2D random $q$-state clock model, and the right one to the 3D model. The color of the symbols is related to the number of states $q$, and their shape to the strength of the initial disorder.

unstable, and the error bars on $\nu$ are large in the 2D case, while it gives compatible estimates of $\nu$ for 3D clock models.

The critical exponent $\nu$ can also be estimated from the finite-size scaling of the standard deviation of the pseudocritical points $\Delta\theta_c(L) = \{\overline{[\theta_c(L)]^2} - \overline{\theta_c(L)}^2\}^{1/2}$. The same procedure as above is applied to take into account scaling corrections. The effective exponents $\nu(L_{\min})$ are presented in Fig. 11 in the case of the ten-state random clock model. In the limit $L_{\min} \to +\infty$, the critical exponents are estimated to be $\nu \simeq 1.24(2)$ in the 2D case and 0.99(1) in 3D. These values are in agreement with those obtained for the Ising model [22,23] [$\nu \simeq 1.24(2)$ in the 2D case and 0.97(5) in the 3D] and the Potts model [25] [$\nu \simeq 1.25(3)$ in the 2D case and 0.985(10) in the 3D].

The average magnetic moment of the last macrospin at the end of the RG process is expected to scale as

$$\bar{\mu} \sim L^{d_f}. \tag{23}$$

Effective fractal dimensions $d_f(L_{\min})$ are computed by a simple power-law fit. The estimates are presented in Fig. 12 versus the smallest lattice size $L_{\min}$ considered in the fit. In the 2D case, the effective exponents for different numbers of states $q$ get closer when $L_{\min}$ is increased, but they are still incompatible in the limit $L_{\min} \to +\infty$ when considering the error bars. We note, however, that, when the disorder strength $\Delta$ is increased, the effective fractal dimensions $d_f(L_{\min})$ of the Ising model ($q = 2$) are relatively stable. The effective fractal dimensions for $q = 8$, 10, and 5 are systematically lower than the Ising values, but the difference is much smaller at strong disorder than at weak disorder. It is therefore plausible that the fractal dimensions eventually become compatible for all $q$-state clock models at stronger disorders. The Ising estimates being the more stable, the fractal dimension would be $d_f \simeq 1.018$, in agreement with the previous estimate 1.018(16) [22]. Note that it was shown that the fractal dimensions of the 2D $q$-state random Potts model is compatible with this value [1.021(5)] for all number of states $q$ [25]. A strong dependence of the scaling corrections on the disorder strength $\Delta$ is

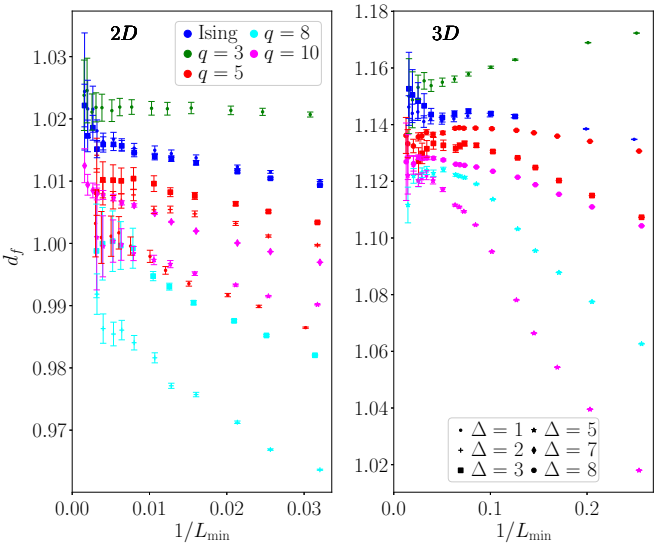


FIG. 12. Magnetic fractal dimension $d_f$ extracted from the finite-size behavior of the average magnetic moment $\bar{\mu}$ of the last cluster vs the inverse $1/L_{\min}$ of the smallest lattice size $L_{\min}$ taken into account in the fit. The left figure corresponds to the 2D random $q$-state clock model, and the right one to the 3D model. The color of the symbols is related to the number of states $q$, and their shape to the strength of the initial disorder.

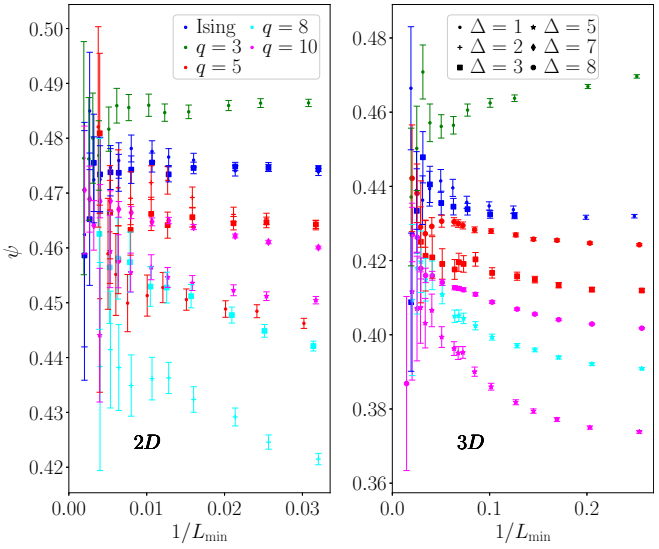


FIG. 13. Critical exponent $\psi$ extracted from the finite-size behavior of the logarithm of the energy gap $\overline{\ln \Delta E}$ of the last cluster vs the inverse $1/L_{\min}$ of the smallest lattice size $L_{\min}$ taken into account in the fit. The left figure corresponds to the 2D random $q$-state clock model, and the right one to the 3D model. The color of the symbols is related to the number of states $q$, and their shape to the strength of the initial disorder.

also observed in the 3D case. The effect is particularly important for the $q = 10$ clock model: The slope of the effective fractal dimensions $d_f(L_{\min})$ depends strongly on the disorder strength. Nevertheless, for the strongest disorder $\Delta = 8$, the fractal dimensions of the $q = 5$- and ten-clock models are compatible within errors bars with the value $d_f \simeq 1.132(6)$. The latter is close, although not compatible within error bars, with the estimates 1.161(15) for the 3D random Ising model [23] and 1.155(8) for the 3D random $q$-state Potts model [25].

The dynamical exponent being infinite at the IDFP, the energy gap scales with the lattice size as

$$\Delta E \sim a e^{-b L^{\psi}}, \tag{24}$$

where $a$ and $b$ are two constants. An effective exponent $\psi(L_{\min})$ is estimated by a nonlinear fit of the average $\overline{\ln \Delta E}$ of the logarithm of the energy gap as $\ln a + b L^{\psi}$. The estimates of this exponent are presented in Fig. 13. The error bars are much larger than for the fractal dimensions $d_f$ and the exponents $\nu$. Again, the scaling corrections are observed to depend both on the number of states $q$ and on the disorder strength $\Delta$. In the 2D case, when considering only the disorder strengths $\Delta \geqslant 3$, the effective critical exponents in the limit $L_{\min} \to +\infty$ are compatible within error bars with the value $\psi \simeq 0.467(10)$. They are also compatible with the estimates 0.46(2) obtained for the 2D random quantum Ising model [22] and 0.48(2) for the Potts model [25]. In the 3D case, the error bars are even larger. For disorder strength $\Delta \geqslant 3$, the critical exponents are compatible with $\psi \simeq 0.43(2)$. For the strongest disorder considered $\Delta = 8$, the $q = 5$ and 10 clock models are compatible with the value $\psi \simeq 0.425(5)$. The estimates 0.48(2) for the 3D random Ising model [23] and 0.46(4) for the Potts model [25] have been reported.

## IV. CONCLUSIONS

We have analyzed the phase transition undergone by the 2D and 3D random quantum clock and Potts models. By studying the analog of the classical McCoy-Wu model, a single phase transition is observed as a peak of the magnetic susceptibility. It is, however, not possible with our data to decide whether the susceptibility diverges only at one critical temperature or over a finite range of temperatures as expected in Griffiths phases. The lattice sizes for which an exact average over disorder can be performed are too small. We then analyzed the integrated probability distribution of the logarithm of the local susceptibilities, which is expected to decay algebraically in Griffiths phases [39]. An algebraic decay is clearly observed over a wide range of several decades for the Potts model. However, close to the critical point, this range is reduced by strong finite-size effects. These effects could be interpreted in light of the physical mechanism proposed in Ref. [39]. In contrast, for the clock model, a behavior similar to that of the Potts model in the Griffiths phase is observed for two couplings and at the largest lattice sizes only. Finite-size effects seem to be much stronger for the clock model. Further investigations would require larger lattice sizes but also a larger number of disorder configurations because rare events are increasingly difficult to sample for larger lattice sizes.

In the second section of this paper, we studied the critical behavior of the 2D and 3D random quantum clock models with the Strong Disorder Renormalization Group (SDRG) and compared the estimated exponents with the known values for the Ising and Potts models. We have observed that the use of second-order perturbation theory in the determination of the SDRG rules is responsible for a chaotic-like behavior of the RG flow at weak disorder. Nevertheless, at strong enough disorder, our estimates of the critical exponents are compatible, or at least close in some cases, to the Ising and Potts values leading to the conclusion that the super-universality class of the random Ising and Potts models encompasses the clock model for the numbers of states $q$ considered.

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