Quasiprobability distribution of work in the quantum Ising model

Gianluca Francica and Luca Dell'Anna
Dipartimento di Fisica e Astronomia e Sezione INFN, Università di Padova, via Marzolo 8, 35131 Padova, Italy

(Received 23 February 2023; revised 22 April 2023; accepted 15 June 2023; published 6 July 2023)

A complete understanding of the statistics of the work done by quenching a parameter of a quantum many-body system is still lacking in the presence of an initial quantum coherence in the energy basis. In this case, the work can be represented by a class of quasiprobability distributions. Here, we try to clarify the genuinely quantum features of the process by studying the work quasiprobability for an Ising model in a transverse field. We consider both a global and local quench by focusing mainly on the thermodynamic limit. We find that, while for a global quench there is a symmetric noncontextual representation with a Gaussian probability distribution of work, for a local quench we can get quantum contextuality as signaled by a negative fourth moment of the work. Furthermore, we examine the critical features related to a quantum phase transition and the role of the initial quantum coherence as a useful resource.

DOI: 10.1103/PhysRevE.108.014106

I. INTRODUCTION

Out-of-equilibrium processes generated by quenching a parameter of a closed quantum system have been extensively investigated: Outstanding experiments of this kind have been realized with ultracold atoms [1–3] and theoretical problems concerning many-body systems have been examined, such as thermalization and integrability [2,4], the universality of the dynamics across a critical point [5], and the statistics of the work done [6]. In particular, the work statistics can be described in terms of the two-projective measurement scheme [7] if the initial state is incoherent, i.e., there is no initial quantum coherence in the energy basis. In contrast, when the initial state is not incoherent, there may not be a probability distribution for the work done, as proven by a no-go theorem [8]. This is related to the quantum contextuality as discussed in Ref. [9]. In simple terms, the problem is similar to looking for a probability distribution in phase space for a quantum particle in a certain quantum state. Since position and momentum are not compatible observables, in general we get a quasiprobability, e.g., the well-known Wigner quasiprobability [10]. Concerning the work, which in a thermally isolated quantum system is equal to the energy change of the system, the role of position and momentum is played by the initial and final Hamiltonian of the system. Several attempts have been made to describe the work statistics; among these, quasiprobabilities have been defined in terms of full-counting statistics [11] and weak values [12], which can be viewed as particular cases of a more general quasiprobability introduced in Ref. [13]. In general, if some fundamental conditions need to be satisfied, the work will be represented by a class of quasiprobability distributions [14]. Determining the possible representations of the work has a fundamental importance: If there is some quasiprobability that is a non-negative probability, there can be a non-contextual classical representation of the protocol, i.e., the process can be not genuinely quantum.

Here, we focus on the statistics of the work done by quenching a parameter of a many-body system starting from a nonequilibrium state having coherence in the energy basis. Although some investigations on the coherence effects have already been carried out, e.g., in Refs. [15,16], the full-counting statistics and weak value quasiprobabilities have been examined, the work statistics still remains rather uninvestigated, especially in many-body systems. Thus, after discussing the statistics of work and the quantum contextuality, in general, in Sec. II, we focus on an Ising model, which we introduce in Sec. III. Our aim is to derive some general features of global and local quenches present in the thermodynamic limit thanks to the initial coherence. Furthermore, we are interested in clarifying the critical features of the work related to a quantum phase transition: Although several studies have been performed for initial incoherent states (e.g., on the large-deviation of work [17–19] and the Ising model [20–28], just to name a few), the initial coherence also plays a role, as found in Ref. [15], which is not entirely clear. Thus, we focus on a global quench starting from a coherent Gibbs state in Sec. IV, where we show that, unlike a system of finite size, in the thermodynamic limit the symmetric quasiprobability representation of the work tends to be noncontextual; in particular, we get a Gaussian probability distribution, even if there are also other quasiprobabilities that take negative values. In contrast, for a local quench, since the work is not extensive, there are initial states such that all the quasiprobabilities of the class can take negative values as signaled by a negative fourth moment of the work (see Sec. V). Then, these processes also remain genuinely quantum in the thermodynamic limit. Furthermore, we also try to clarify the role of initial quantum coherence as useful resource for the work extraction in Sec. VI, showing that, even when the protocol tends to be noncontextual, the initial coherence still plays an active role. In the end, we summarize and discuss further our results in Sec. VII.

II. WORK STATISTICS

We consider a quantum quench so the system is initially in state ρ_0 and the time evolution is described by the unitary operator $U_{t,0}$ which is generated by the time-dependent Hamiltonian $H(\lambda_t)$, where the control parameter λ_t is changed in the time interval $[0, \tau]$. In detail, $U_{t,0} = \mathcal{T} e^{-i\int_0^t H(\lambda_s)ds}$, where $\mathcal{T}$ is the time order operator and the Hamiltonian can be expressed as $H(\lambda_t) = \sum_k E_k(\lambda_t)|E_k(\lambda_t)\rangle\langle E_k(\lambda_t)|$, where $|E_k(\lambda_t)\rangle$ is the eigenstate with eigenvalue $E_k(\lambda_t)$ at time t. For brevity, we define $E_i = E_i(\lambda_0)$ and $E'_k = E_k(\lambda_\tau)$. The average work $\langle w\rangle$ done on the system in the time interval $[0, \tau]$ can be identified with the average energy change

$$\langle w\rangle = \text{Tr}\{(H^{(H)}(\lambda_\tau) - H(\lambda_0))\rho_0\}, \quad (1)$$

where, given an operator $A(t)$, we define the Heisenberg time-evolved operator $A^{(H)}(t) = U^\dagger_{t,0}A(t)U_{t,0}$. In general, the work performed in the quench can be represented by a quasiprobability distribution of work. We recall that if some Gleason-like axioms are satisfied (see Ref. [14] for details), for the events $E \wedge F$ we get the quasiprobability $v(E, F) = \text{ReTr}\{EF\rho_0\}$, but for more than two events, i.e., for $E \wedge F \wedge G \wedge \cdots$, the quasiprobability is not fixed by the axioms. However, we can associate a quasiprobability of the form $\text{ReTr}\{EFG\cdots\rho_0\}$ to each of all possible decompositions of the form $E \wedge F$, $F \wedge G$, $G \wedge \cdots$. By considering this notion of quasiprobability, if we require that (W1) the quasiprobability distribution of work reproduces the two-projective measurement scheme in the case of initial incoherent states (i.e., for states ρ_0 such that $\rho_0 = \Delta(\rho_0)$, where we have defined the dephasing map $\Delta(\rho_0) = \sum_i |E_i\rangle\langle E_i|\rho_0|E_i\rangle\langle E_i|$), (W2) the average calculated with respect to the quasiprobability is equal to Eq. (1), and (W3) the second moment is equal to

$$\langle w^2\rangle = \text{Tr}\{(H^{(H)}(\lambda_\tau) - H(\lambda_0))^2\rho_0\}, \quad (2)$$

the quasiprobability distribution of work belongs to a defined class [13,14], i.e., it takes the form

$$p_q(w) = \sum_{k,j,i} \text{Re}\{\langle E_i|\rho_0|E_j\rangle\langle E_j|U^\dagger_{\tau,0}|E'_k\rangle\langle E'_k|U_{\tau,0}|E_i\rangle\} \times \delta(w - E'_k + qE_i + (1-q)E_j), \quad (3)$$

where q is a real parameter. Our aim is to investigate this quasiprobability for a many-body system. We can focus on the characteristic function which is defined as $\chi_q(u) = \langle e^{iuw}\rangle$ and reads

$$\chi_q(u) = \tfrac{1}{2}(X_q(u) + X_{1-q}(u)),$$

where we have defined

$$X_q(u) = \text{Tr}\{e^{-iuqH(\lambda_0)}\rho_0 e^{-iu(1-q)H(\lambda_0)}e^{iuH^{(H)}(\lambda_\tau)}\}. \quad (4)$$

The moments of work are $\langle w^n\rangle = (-i)^n\partial^n_u\chi_q(0)$, and the higher moments for $n > 2$ depend on the particular representation. In particular, we get

$$\langle w^n\rangle = (-i)^n\partial^n_u\chi_q(0) = \frac{(-i)^n\partial^n_u X_q(0)}{2} + \frac{(-i)^n\partial^n_u X_{1-q}(0)}{2}, \quad (5)$$

where (see Appendix A)

$$(-i)^n\partial^n_u X_q(0) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}q^{n-k-l}(1-q)^l \times \text{Tr}\{\rho_0 H(\lambda_0)^l(H^{(H)}(\lambda_\tau))^k H(\lambda_0)^{n-k-l}\}. \quad (6)$$

We can consider the problem if there is a classical representation, i.e., if there is a noncontextual hidden variable model which satisfies the conditions about the reproduction of the two-projective-measurement scheme, the average, and the second moment. To introduce the concept of contextuality at an operational level (see, e.g., Refs. [9,29]), we consider a set of preparation procedures P and measurement procedures M with outcomes k, so we will observe k with probability $p(k|P, M)$. We aim to reproduce the statistics by using a set of states ζ that are random distributed in set $\mathcal{Z}$ with probability $p(\zeta|P)$ every time the preparation P is performed. If, for a given ζ, we get the outcome k with the probability $p(k|\zeta, M)$, we are able to reproduce the statistics if

$$p(k|P, M) = \int_{\mathcal{Z}} p(\zeta|P)p(k|\zeta, M)d\zeta, \quad (7)$$

and the protocol is called universally noncontextual if $p(\zeta|P)$ is a function of the quantum state alone, i.e., $p(\zeta|P) = p(\zeta|\rho_0)$, and $p(k|\zeta, M)$ depends only on the positive operator-valued measurement element M_k associated to the corresponding outcome of the measurement M, i.e., $p(k|\zeta, M) = p(k|\zeta, M_k)$. In our case, the outcome k corresponds to the work w_k, and if the protocol is noncontextual the work distribution can be expressed as

$$p(w) = \sum_k p(k|P, M)\delta(w - w_k), \quad (8)$$

where $p(k|P, M)$ is given by Eq. (7) with $p(\zeta|P) = p(\zeta|\rho_0)$ and $p(k|\zeta, M) = p(k|\zeta, M_k)$, so for a negative quasiprobability of work we cannot have a noncontextual protocol. Thus, a process that cannot be reproduced within any noncontextual protocol will exhibit genuinely nonclassical features. If all the quasiprobabilities in the class take negative values, the protocol is contextual, whereas if there is a quasiprobability which is non-negative, there can be a noncontextual representation. We recall that for an initial incoherent state $\rho_0 = \Delta(\rho_0)$, we get the two-projective measurement scheme that is noncontextual [9]. In contrast, the presence of initial quantum coherence in the energy basis can lead to a contextual protocol. Let us investigate the effects of the initial quantum coherence by considering a Ising model in a transverse field.

III. MODEL

We consider a chain of L spin 1/2 described by the Ising model in a transverse field with Hamiltonian

$$H(\lambda) = -\lambda\sum_{i=1}^{L}\sigma^z_i - \sum_{i=1}^{L}\sigma^x_i\sigma^x_{i+1}, \quad (9)$$

where we have imposed periodic boundary conditions $\sigma^\alpha_{L+1} = \sigma^\alpha_1$, and σ^α_i with $\alpha = x, y, z$ are the Pauli matrices on the site i. We note that the parity $P = \prod_{i=1}^{L}\sigma^z_i$ is a symmetry of the model, i.e., it commutes with the Hamiltonian. The Hamiltonian can be diagonalized by performing the Jordan-Wigner transformation

$$a_i = \left(\prod_{j<i}\sigma^z_j\right)\sigma^-_i, \quad (10)$$

where the fermionic operators a_i satisfy the anticommutation relations $\{a_i, a_j^\dagger\} = \delta_{i,j}$, $\{a_i, a_j\} = 0$. We get the Hamiltonian of fermions

$$H(\lambda) = -\lambda \sum_{i=1}^{L} (2a_i^\dagger a_i - 1) - \sum_{i=1}^{L-1} (a_i^\dagger - a_i)(a_{i+1} + a_{i+1}^\dagger) + P(a_L^\dagger - a_L)(a_1 + a_1^\dagger), \tag{11}$$

where the parity reads $P = e^{i\pi N}$ and $N = \sum_{i=1}^{L} a_i^\dagger a_i$ is the number operator. We consider the projector $P_\pm$ on the sector with parity $P = \pm 1$, then the Hamiltonian reads

$$H(\lambda) = P_+ H_+(\lambda) P_+ + P_- H_-(\lambda) P_-. \tag{12}$$

For the sector with odd parity $P = -1$, we get the Kitaev chain

$$H_-(\lambda) = -\lambda \sum_{i=1}^{L} (2a_i^\dagger a_i - 1) - \sum_{i=1}^{L} (a_i^\dagger - a_i)(a_{i+1} + a_{i+1}^\dagger), \tag{13}$$

with periodic boundary conditions $a_{L+1} = a_1$. We perform a Fourier transform $a_j = 1/\sqrt{L} \sum_k e^{-ikj} a_k$, where $k = 2\pi n/L$ with $n = -(L-1)/2, \dots, (L-1)/2$ for L odd and $n = -L/2 + 1, \dots, L/2$ for L even. Thus, the Hamiltonian reads

$$H_-(\lambda) = \sum_k \Psi_k^\dagger [-(\lambda + \cos k)\sigma^z + \sin k \sigma^y] \Psi_k, \tag{14}$$

where σ^α with $\alpha = x, y, z$ are the Pauli matrices and we have defined the Nambu spinor $\Psi_k = (a_k, a_{-k}^\dagger)^T$. In particular,the Hamiltonian can be written as $H_-(\lambda) = \sum_k \Psi_k^\dagger \vec{d}_k \cdot \vec{\sigma} \Psi_k$, which, in the diagonal form, reads

$$H_-(\lambda) = \sum_k \epsilon_k \left(\alpha_k^\dagger \alpha_k - \frac{1}{2} \right) = \sum_k \epsilon_k \alpha_k^\dagger \alpha_k + E_-, \tag{15}$$

where $E_- = -\sum_k \epsilon_k/2$. In detail, we have performed a rotation with respect to the x axis with an angle θ_k between $\vec{d}_k$ and the z axis, corresponding to the Bogoliubov transformation $\alpha_k = \cos(\theta_k/2) a_k - i \sin(\theta_k/2) a_{-k}^\dagger$, where $\epsilon_k = 2||\vec{d}_k||$, or more explicitly,

$$\epsilon_k = 2\sqrt{(\lambda + \cos k)^2 + \sin^2 k}. \tag{16}$$

For the sector with even parity $P = 1$, we get the Hamiltonian $H_+(\lambda)$, which is equal to the one in Eq. (13) with antiperiodic boundary conditions $a_{L+1} = -a_1$, thus the only difference is in the momenta k which are $k = 2\pi(n - 1/2)/L$. Of course, not all eigenstates of the Hamiltonians $H_\pm$ are eigenstates of the Hamiltonian H, and their parity needs to be discussed. Let us consider L even. Thus, in the even parity sector, $k \in K_+$, for each k there is $-k$, and the eigenstates of the Hamiltonian are the states

$$\alpha_{k_1}^\dagger \cdots \alpha_{k_{2m}}^\dagger |\tilde{0}_+\rangle, \tag{17}$$

where m is an integer, $k_i \in K_+$ and $|\tilde{0}_+\rangle$ is the vacuum state of α_k with $k \in K_+$. In contrast, in the odd parity sector, $k \in K_-$, for each k there is $-k$ except for $k = 0$ and π. For $\lambda < -1$, we get $\alpha_0 = a_0$ and $\alpha_\pi = a_\pi$, for $\lambda > 1$ we get $\alpha_0 = a_0^\dagger$ and $\alpha_\pi = a_\pi^\dagger$, and for $|\lambda| < 1$ we get $\alpha_0 = a_0^\dagger$ and $\alpha_\pi = a_\pi$. Then, for $|\lambda| > 1$, the vacuum state $|\tilde{0}_-\rangle$ of α_k with $k \in K_-$ has even parity and the eigenstates of the Hamiltonian are the states

$$\alpha_{k_1}^\dagger \cdots \alpha_{k_{2m+1}}^\dagger |\tilde{0}_-\rangle, \tag{18}$$

with $k_i \in K_-$. Conversely, for $|\lambda| < 1$ the vacuum state $|\tilde{0}_-\rangle$ of α_k with $k \in K_-$ has odd parity since has the fermion a_0 but not a_π, and the eigenstates of the Hamiltonian are the states

$$\alpha_{k_1}^\dagger \cdots \alpha_{k_{2m}}^\dagger |\tilde{0}_-\rangle, \tag{19}$$

with $k_i \in K_-$. Then, for $|\lambda| < 1$ both states $|\tilde{0}_+\rangle$ and $|\tilde{0}_-\rangle$ are eigenstates of the Hamiltonian with energies E_+ and E_-, so the ground state is twofold degenerate in the thermodynamic limit. Thus, at the points $\lambda = \pm 1$ we get a second-order quantum phase transition.

IV. GLOBAL QUENCH

We start to focus on a sudden global quench of the transverse field λ, i.e., λ is suddenly changed from the value λ_0 to λ_τ, so $\tau \to 0$ and $U_{\tau,0} = I$. To investigate the role of initial quantum coherence, we focus on a coherent Gibbs state

$$|\Psi_G(\beta)\rangle = \frac{1}{\sqrt{Z}} \sum_j e^{-\beta E_j/2 + i\varphi_j} |E_j\rangle, \tag{20}$$

where E_j are the eigenenergies of $H(\lambda_0)$, φ_j is a phase, $Z = Z(\lambda_0)$, and $Z(\lambda)$ is the partition function defined as $Z(\lambda) = \mathrm{Tr}\{e^{-\beta H(\lambda)}\}$. Of course, the incoherent part of the state $|\Psi_G(\beta)\rangle$ is $\Delta(|\Psi_G(\beta)\rangle\langle\Psi_G(\beta)|) = \rho_G(\beta)$, where $\rho_G(\beta)$ is the Gibbs state $\rho_G(\beta) = e^{-\beta H(\lambda_0)}/Z$. With the aim to calculate the characteristic function for an arbitrary size L, from Eq. (4), by using the relations $\sum_s P_s = I$, $P_s^2 = P_s$, $[P_s, H(\lambda)] = 0$ and $[P_s, H_\pm(\lambda)] = 0$, where $s = \pm$, it is easy to see that

$$X_q(u) = \sum_s \mathrm{Tr}\left\{e^{-iuqH_s(\lambda_0)} P_s \rho_0 P_s e^{-iu(1-q)H_s(\lambda_0)} e^{iuH_s^{(H)}(\lambda_\tau)}\right\}. \tag{21}$$

We get $P_s \rho_0 P_s = P_s \rho_0^s$, where for the Gibbs state $\rho_0^s = e^{-\beta H_s(\lambda_0)}/Z$ and for the coherent Gibbs state $\rho_0^s = |\Psi_G^s\rangle\langle\Psi_G^s|$. In particular, we get

$$\left|\Psi_G^s\right\rangle = \frac{1}{\sqrt{Z}} \otimes_{k \in K_s} \left(e^{\frac{\beta\epsilon_k}{4}} |\tilde{0}_k\rangle + e^{-\frac{\beta\epsilon_k}{4} + i\phi_k} |\tilde{1}_k\rangle \right), \tag{22}$$

where we consider a phase such that $\phi_{-k} = \phi_k$, with $|\tilde{n}_k\rangle = (\alpha_k^\dagger)^{n_k} |\tilde{0}_k\rangle$, where $\epsilon_k = \epsilon_k(\lambda_0)$, $\alpha_k = \alpha_k(\lambda_0)$, and $|\tilde{0}_k\rangle$ is the vacuum state for the fermion α_k. As shown in Appendix B, we get

$$X_q(u) = \frac{1}{2} \sum_s X_q^s(u) + \eta_s X_q'^s(u), \tag{23}$$

where we have defined $\eta_s = s\langle\tilde{0}_s|e^{i\pi N}|\tilde{0}_s\rangle$, which is $\eta_+ = 1$ and $\eta_- = -1$ for $|\lambda_0| > 1$ and $\eta_- = 1$ for $|\lambda_0| < 1$, and

$$X_q^s(u) = \frac{1}{Z} \prod_{k \in K_s : k \geqslant 0} X_q^{(k)}(u). \tag{24}$$

In detail, for $k > 0$ and $k \neq \pi$, we get

$$X_q^{(k)}(u) = X_q^{(k),th}(u) + X_q^{(k),\mathrm{coh}}(u), \tag{25}$$

where $X_q^{(k),th}(u)$ is the incoherent contribution, which reads

$$X_q^{(k),th}(u) = 2(\cos((u - i\beta)\epsilon_k)\cos(u\epsilon'_k) + \sin((u - i\beta)\epsilon_k) \times \sin(u\epsilon'_k)\hat{d}_k \cdot \hat{d}'_k + 1), \quad (26)$$

and $X_q^{(k),\mathrm{coh}}(u)$ is the coherent contribution, which reads

$$X_q^{(k),\mathrm{coh}}(u) = -2i\sin(u\epsilon'_k)\sin(u(2q-1)\epsilon_k - 2\phi_k)(\hat{d}_k \times \hat{d}'_k)_x, \quad (27)$$

where, for brevity we have defined $\epsilon'_k = \epsilon_k(\lambda_\tau)$, $\vec{d}_k = \vec{d}_k(\lambda_0)$, and $\vec{d}'_k = \vec{d}_k(\lambda_\tau)$. Furthermore, we have

$$X'^{s}_q(u) = \frac{1}{Z}\prod_{k\in K_s: k\geqslant 0} X'^{(k)}_q(u), \quad (28)$$

with

$$X'^{(k)}_q(u) = X_q^{(k)}(u) - 4. \quad (29)$$

In contrast, for $k = 0$ and $k = \pi$, we get

$$X_q^{(0,\pi)}(u) = 2\cosh\left(\frac{\beta\epsilon_{0,\pi} - iu(s_{0,\pi}\epsilon'_{0,\pi} - \epsilon_{0,\pi})}{2}\right), \quad (30)$$

$$X'^{(0,\pi)}_q(u) = 2\sinh\left(\frac{\beta\epsilon_{0,\pi} - iu(s_{0,\pi}\epsilon'_{0,\pi} - \epsilon_{0,\pi})}{2}\right), \quad (31)$$

where $s_\pi = -1$ if $|\lambda_0| < 1$ and $\lambda_\tau > 1$ or $|\lambda_\tau| < 1$ and $\lambda_0 > 1$, otherwise $s_\pi = 1$, and $s_0 = -1$ if $|\lambda_0| < 1$ and $\lambda_\tau < -1$ or $|\lambda_\tau| < 1$ and $\lambda_0 < -1$, otherwise $s_0 = 1$, while the partition function is

$$Z = \frac{1}{2}\sum_s \prod_{k\in K_s} 2\cosh(\beta\epsilon_k/2) + \eta_s \prod_{k\in K_s} 2\sinh(\beta\epsilon_k/2). \quad (32)$$

If the initial quantum coherence does not contribute, i.e., $X_q^{(k),\mathrm{coh}}(u) = 0$, we get $X_q^{(k)}(u) = X_q^{(k),th}(u)$ and the characteristic function is the one of the initial Gibbs state $\rho_G(\beta)$. We get $X_q^{(k),\mathrm{coh}}(u) = 0$ for $q = 1/2$ and $\phi_k = n\pi/2$, and in this case the quasiprobability is non-negative; in particular, it is equivalent to the two-projective-measurement scheme which is noncontextual. For $q = 1/2$, the initial quantum coherence contributes only for $\phi_k \neq n\pi/2$ with n integer. In this case, the quasiprobability can take negative values. However, in the thermodynamic limit the negativity of the quasiprobability is always subdominant for $q = 1/2$, and we get a Gaussian probability distribution of work. To prove it, we note that in the thermodynamic limit we get $Z = \prod_{k\in K_+} Z_k$ with $Z_k = 2\cosh(\beta\epsilon_k/2)$, then

$$X_q(u) = \prod_{k\in K_+: k>0} \frac{X_q^{(k)}(u)}{Z_k^2}. \quad (33)$$

Basically, in the thermodynamic limit the model is equivalent to the system of fermions with Hamiltonian H_+. Thus, we can write

$$X_q(u) = e^{Lg_q(u)}, \quad (34)$$

where $g_q(u)$ is intensive, so the work is extensive, i.e., $\langle w^n\rangle \sim L^n$. In particular, for the initial coherent Gibbs state under consideration, $g_q(u)$ explicitly reads

$$g_q(u) = \frac{1}{2\pi}\int_0^\pi \ln\left(\frac{X_q^{(k)}(u)}{Z_k^2}\right)dk. \quad (35)$$

Then, if Eq. (34) holds, regardless of the explicit form of the intensive function $g_q(u)$, as $L \to \infty$ we can consider

$$X_q(u) \sim e^{L\left(\partial_u g_q(0)u + \frac{1}{2}\partial_u^2 g_q(0)u^2\right)}, \quad (36)$$

since in the calculation of the Fourier transform of $X_q(u)$ the dominant contribution of the integral is near $u = 0$, so that we can expand $g_q(u)$ in Taylor series about $u = 0$, and thus the neglected terms in Eq. (36) do not contribute in the asymptotic formula of the quasiprobability $p_q(w)$. We note that, although the characteristic function $\chi_q(u)$ depends on q, the first two moments do not depend on q. In particular, we note that the relative fluctuations of work scale as $\sigma_w/\langle w\rangle \sim 1/\sqrt{L}$, where we have defined the variance $\sigma_w^2 = \langle w^2\rangle - \langle w\rangle^2$. By noting that $\partial_u g_q(0)$ does not depend on q and $\partial_u^2 g_{1-q}(0) = \partial_u^2 g_q^*(0)$, we get the quasiprobability of work

$$p_q(w) \sim \frac{1}{\sqrt{2\pi}}\mathrm{Re}\left(\frac{e^{-\frac{(w-\bar{w})^2}{2v_q}}}{\sqrt{v_q}}\right), \quad (37)$$

where $\bar{w} = -i\partial_u g_q(0)L$ and $v_q = -\partial_u^2 g_q(0)L$. In particular, the average work is $\langle w\rangle = \bar{w}$ and the variance σ_w^2 is the real part of v_q, i.e., $v_q = \sigma_w^2 + ir_q$. As shown in Fig. 1, for $q \neq 1/2$ the asymptotic formula of the quasiprobability can take negative values due to the presence of the imaginary part r_q. In contrast, for $q = 1/2$, we get $\chi_{1/2}(u) = X_{1/2}(u)$, from which $\sigma_w^2 = -\partial_u^2 g_{1/2}(0)L$, i.e., $r_{1/2} = 0$ and thus we get the Gaussian probability distribution:

$$p_{1/2}(w) \sim \frac{e^{-\frac{(w-\bar{w})^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w}. \quad (38)$$

It is worth noting that the protocol tends to be noncontextual. To prove it, we consider the operator $\Delta H = H^{(H)}(\lambda_\tau) - H(\lambda_0)$ and the probability distribution

$$p(\Delta E) = \sum_\mu \langle\Delta E_\mu|\rho_0|\Delta E_\mu\rangle\delta(\Delta E - \Delta E_\mu), \quad (39)$$

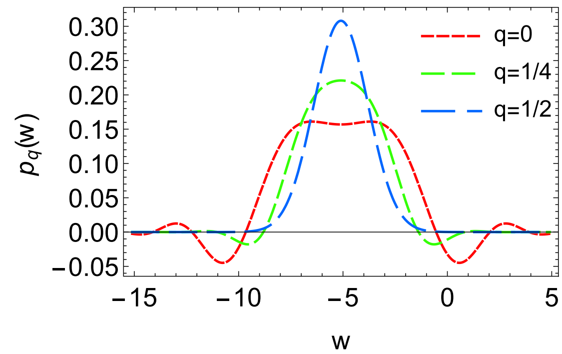

FIG. 1. The quasiprobability of work in Eq. (37) for different values of q. We put $L = 50$, $\beta = 1$, $\lambda_0 = 0.9$, $\lambda_\tau = 1.1$, and $\phi_k = 0$.

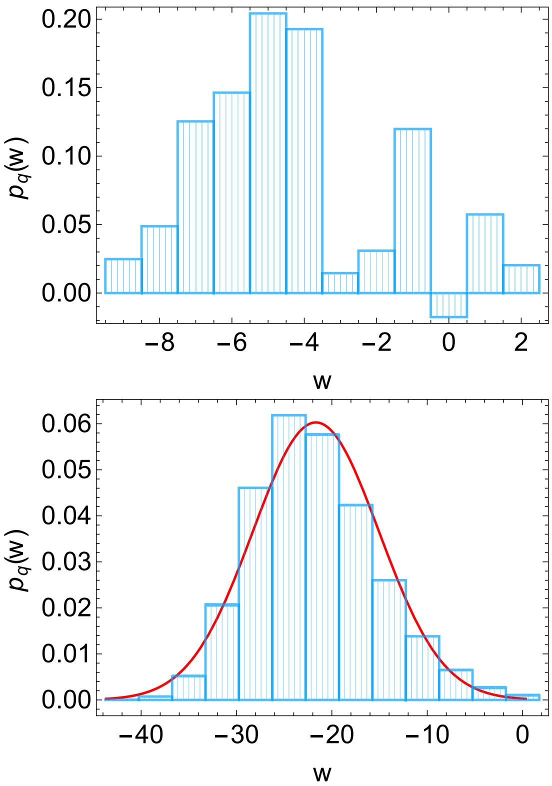

FIG. 2. The histogram of the work distribution. We put $L = 10$ in the top panel, $L = 50$ in the bottom panel, $q = 1/2$, $\beta = 1$, $\lambda_\tau = 1.5$, $\lambda_0 = 0.5$, and $\phi_k = \pi/4$. The red line corresponds to the Gaussian distribution probability in Eq. (38). We note that for $L = 50$ there is still some skewness. The histograms are calculated by using the characteristic function of Eq. (23).

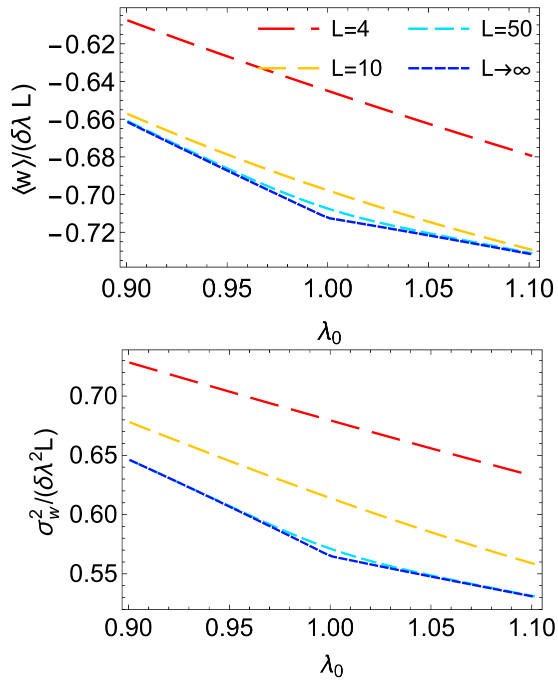

FIG. 3. The average work $\bar{w}$ and the variance σ_w^2 in the function of λ_0 for different values of L. We put $\beta = 1$, $\delta\lambda = \lambda_\tau - \lambda_0 = 0.1$, and $\phi_k = \pi/4$. The values for finite sizes L are calculated by using the characteristic function of Eq. (23).

where $|\Delta E_\mu\rangle$ is the eigenstate of ΔH with eigenvalue ΔE_μ. Of course, $p(\Delta E)$ is noncontextual, and it is easy to see that $p_{1/2}(w) \sim p(w)$ as $L \to \infty$. Thus, the work tends to be an observable with respect to the noncontextual symmetric representation, differently from finite sizes, where it is not, as was originally noted for incoherent initial states in Ref. [7]. In particular, for the quench considered, we have $\Delta H = (\lambda_\tau - \lambda_0)S_z$, where $S_z = \sum_{j=1}^{L} \sigma_j^z$, so the symmetric representation for $q = 1/2$ tends to be equivalent to the distribution probability of the transverse magnetization S_z. We emphasize that for small sizes L, the quasiprobability at $q = 1/2$ can take negative values, but for large L it is well described by the Gaussian probability distribution in Eq. (38) (see Fig. 2). We note that for an arbitrary initial state, the representation for $q = 1/2$ is still noncontextual (see Appendix C). In general, the negativity of the quasiprobability $p_q(w)$ can be characterized by the integral

$$\mathcal{N} \equiv \int |p_q(w)| dw, \tag{40}$$

which is equal to one if $p_q(w) \geqslant 0$. In our case, $\mathcal{N} \sim \frac{(\sigma_w^4 + r_q^2)^{\frac{1}{4}}}{\sigma_w}$, so $\mathcal{N} = 1$ implies that $r_q = 0$ and thus $p_q(w) \geqslant 0$. We note that $\mathcal{N} = 1$ implies, in general, that $p_q(w) \geqslant 0$ (see Appendix D). In the end, we note that the effects related to the negativity of the quasiprobability start to affect the statistics from the fourth moment, which reads $\langle w^4\rangle \sim \bar{w}^4 + 6\bar{w}^2\sigma_w^2 + 3\sigma_w^4 - 3r_q^2$. In contrast, the first three moments do not depend on r_q; explicitly, they read $\langle w\rangle = \bar{w}$, $\langle w^2\rangle = \bar{w}^2 + \sigma_w^2$, and $\langle w^3\rangle \sim \bar{w}^3 + 3\bar{w}\sigma_w^2$. In particular, the kurtosis is Kurt $\equiv \langle (w - \langle w\rangle)^4\rangle/\sigma_w^4 \sim 3 - 3r_q^2/\sigma_w^4$ which is always smaller than 3 if $r_q \neq 0$, i.e., the distribution is more flat than the normal one. We note that if $\bar{w} \neq 0$, since $\bar{w} \sim L$ and $\sigma_w^2 \sim L$, the fourth moment is always positive. On the other hand, for $\bar{w} = 0$, the fourth moment reads $\langle w^4\rangle \sim 3\sigma_w^4 - 3r_q^2$ and becomes negative for $r_q > \sigma_w^2$, so in this regime the negativity for $q \neq 1/2$ will be strong. To conclude our investigation concerning the global quench, we note that the average work reads

$$\bar{w} = \frac{(\lambda_0 - \lambda_\tau)L}{\pi} \int_0^\pi \frac{(\lambda_0 + \cos k)\sinh(\beta\epsilon_k) + \sin k \sin(2\phi_k)}{\epsilon_k \cosh^2(\beta\epsilon_k/2)} dk \tag{41}$$

and the variance reads

$$\sigma_w^2 = \frac{(\lambda_0 - \lambda_\tau)^2 L}{\pi} \int_0^\pi \frac{1}{\cosh^4(\beta\epsilon_k/2)} \times \left(\cosh^2(\beta\epsilon_k/2)\cosh(\beta\epsilon_k) - \frac{2}{\epsilon_k^2}(\sin k \sin(2\phi_k) + (\lambda_0 + \cos k)\sinh(\beta\epsilon_k))^2\right) dk. \tag{42}$$

Both $\bar{w}$ and σ_w^2 are not regular at $|\lambda_0| = 1$ for $\phi_k = \phi \neq n\pi/2$ due to the presence of a quantum phase transition (see Fig. 3). Furthermore, concerning the negativity of the quasiprobability of work, we have

$$r_q = \frac{2(1-2q)(\lambda_\tau - \lambda_0)L}{\pi} \int_0^\pi \frac{\sin k \cos(2\phi_k)}{\cosh^2(\beta\epsilon_k/2)} dk, \quad (43)$$

which is regular. We deduce that the protocol admits a noncontextual description, i.e., $r_q = 0$, for any q and $\phi_k = (2n+1)\pi/4$ or for $q = 1/2$. In the end, to investigate the critical features of the work which can be related to the presence of the quantum phase transition, we introduce the energy scale J such the Hamiltonian reads

$$H_J(\lambda) = -J\lambda \sum_{i=1}^L \sigma_i^z - J \sum_{i=1}^L \sigma_i^x \sigma_{i+1}^x. \quad (44)$$

We focus on $\lambda_0 \approx 1$ and we start to consider the average work given by Eq. (41) multiplied by J. Then, we change variable $k' = \pi - k$ in the integral and we define $\kappa = k'/a$, and the renormalized couplings $J = c/(2a)$ and $\lambda_0 = 1 - mca$. In the scaling limit $a \to 0$, we get

$$\bar{w} \sim \frac{J(\lambda_0 - \lambda_\tau)aL}{2\pi} \int_0^{\frac{\pi}{a}} \frac{\kappa \sin(2\phi_\pi) - cm \sinh(\beta c\omega_\kappa)}{\omega_\kappa \cosh^2(\beta c\omega_\kappa/2)} d\kappa, \quad (45)$$

where $\omega_\kappa = \sqrt{\kappa^2 + c^2m^2}$. We note that the integral extended to the interval $[0, \infty)$ does not converge. Thus the integral is not determined only by small κ, and the behavior is not universal. Similarly, concerning the variance σ_w^2, the integral extended to the interval $[0, \infty)$ does not converge, so it is not universal. The coherent contribution to the average work is defined as

$$\bar{w}_{coh} = \bar{w} - \bar{w}_{th}, \quad (46)$$

where $\bar{w}_{th}$ is the average work corresponding to the initial state $\rho_0 = \rho_G(\beta)$. Then, the coherent contribution is given by the term proportional to $\sin(2\phi_\pi)$ in Eq. (45), i.e.,

$$\bar{w}_{coh} \sim \frac{J(\lambda_0 - \lambda_\tau)aL}{2\pi} \int_0^{\frac{\pi}{a}} \frac{\kappa \sin(2\phi_\pi)}{\omega_\kappa \cosh^2(\beta c\omega_\kappa/2)} d\kappa. \quad (47)$$

In this case, we can extend the integral to the interval $[0, \infty)$, so the coherent contribution $\bar{w}_{\rm coh}$ is described by the continuum model; in this sense it is a universal feature. From Eq. (47), by noting that

$$\int_0^\infty \frac{y}{\sqrt{1+y^2}\cosh^2(x\sqrt{1+y^2}/2)} dy = \frac{4}{(1+e^{|x|})|x|}, \quad (48)$$

the coherent contribution to the average work can be expressed as

$$\bar{w}_{\rm coh} \sim \frac{(\lambda_0 - \lambda_\tau)\sin(2\phi_\pi)L}{\pi\beta} g_{FD}(\beta mc^2), \quad (49)$$

where we have defined the Fermi-Dirac distribution $g_{\rm FD}(x) = 1/(1+e^{|x|})$ and $mc^2 = 2J(1-\lambda_0)$. In the end, let us consider the limit of high temperatures $\beta \to 0$, so we get

$$g_q(u) = \frac{1}{2\pi} \int_0^\pi \ln \frac{1}{2}(\cos(u\epsilon_k)\cos(u\epsilon'_k) + \sin(u\epsilon_k)\sin(u\epsilon'_k) \times \hat{d}_k \cdot \hat{d}'_k + 1 - i\sin(u\epsilon'_k)\sin(u(2q-1)\epsilon_k - 2\phi_k) \times (\hat{d}_k \times \hat{d}'_k)_x) dk. \quad (50)$$

For $\phi_k = \phi$, we get the closed form of the derivatives

$$\partial_u g_q(0) = -\frac{i(\lambda_\tau - \lambda_0)}{2\pi|\lambda_0|} \sin(2\phi)(1 + |\lambda_0| - |1 - |\lambda_0||), \quad (51)$$

$$\partial_u^2 g_q(0) = -(\lambda_\tau - \lambda_0)^2 \Big(1 - \frac{1}{8\lambda_0^2}\big(1 + \lambda_0^2 - (1+|\lambda_0|)|1-|\lambda_0||\big)\sin^2(2\phi)\Big) - \frac{4i}{\pi}(\lambda_\tau - \lambda_0)(1-2q)\cos(2\phi), \quad (52)$$

from which it is evident that the work statistics is not regular at $|\lambda_0| = 1$ for $\phi \neq n\pi/2$. Of course, in this limit we can extract the work $W_{\rm ex} = -\langle w \rangle$, equal to

$$W_{\rm ex} = \frac{(\lambda_\tau - \lambda_0)L}{2\pi|\lambda_0|} \sin(2\phi)(1 + |\lambda_0| - |1 - |\lambda_0||), \quad (53)$$

only because of the presence of the initial coherence; otherwise, for an initial Gibbs state we will get $\langle w \rangle = 0$.

V. LOCAL QUENCH

Things change drastically when the work is nonextensive, e.g., for a local quench. We focus on the case of a sudden quench in the transverse field, i.e., the initial Hamiltonian is $H = H(\lambda_0)$ and we perform a sudden quench of the transverse field in a site l, so the final Hamiltonian is $H' = H(\lambda_0) - \epsilon\sigma_l^z$. Since we are interested only in large sizes L, we describe the model with the corresponding fermionic Hamiltonian H_+. Here we are interested in investigating how contextuality can emerge in a local quench, thus we focus on the states $|\Psi_1(\beta)\rangle$ and $|\Psi_2(\beta)\rangle$, which are defined as

$$|\Psi_1(\beta)\rangle = \frac{e^{\frac{\beta}{4}\sum_k \epsilon_k}}{\sqrt{Z_1}} \exp\left(\sum_k e^{-\frac{\beta\epsilon_k}{2} + i\phi_k} \alpha_k^\dagger\right) |\tilde{0}_+\rangle \quad (54)$$

and

$$|\Psi_2(\beta)\rangle = \frac{e^{\frac{\beta}{4}\sum_k \epsilon_k}}{\sqrt{Z_2}} \Big(1 + \sum_k e^{-\frac{\beta\epsilon_k}{2} + i\phi_k}\alpha_k^\dagger + \frac{1}{2}\sum_{k,k'} s_{k,k'} e^{-\frac{\beta(\epsilon_k + \epsilon_{k'})}{2} + i(\phi_k + \phi_{k'})} \alpha_k^\dagger \alpha_{k'}^\dagger\Big) |\tilde{0}_+\rangle, \quad (55)$$

where $s_{k,k'} = 1$ if $k > k'$, $s_{k,k'} = -1$ if $k < k'$ and $s_{k,k} = 0$, and Z_1 and Z_2 are normalization factors such that $Z \sim Z_2 \sim Z_1$ as $\beta \to \infty$. Indeed, $|\Psi_G(\beta)\rangle \sim |\Psi_2(\beta)\rangle \sim |\Psi_1(\beta)\rangle$ as $\beta \to \infty$. In general, for these initial states, the function $X_q(u)$ can be calculated with the help of Grassmann variables (see Appendix E). While for the initial state $|\Psi_1(\beta)\rangle$, we find that the fourth moment of work is positive; for the initial state $|\Psi_2(\beta)\rangle$, we find that the fourth moment of work can be negative for β small enough (see Fig. 4). This suggests that to get a contextual protocol with a negative fourth moment, we need to start from an initial state which involves at least couples of quasiparticles, such as $|\Psi_2(\beta)\rangle$. This result is corroborated by considering states like $|\Psi_1(\beta)\rangle$ but with random coefficients

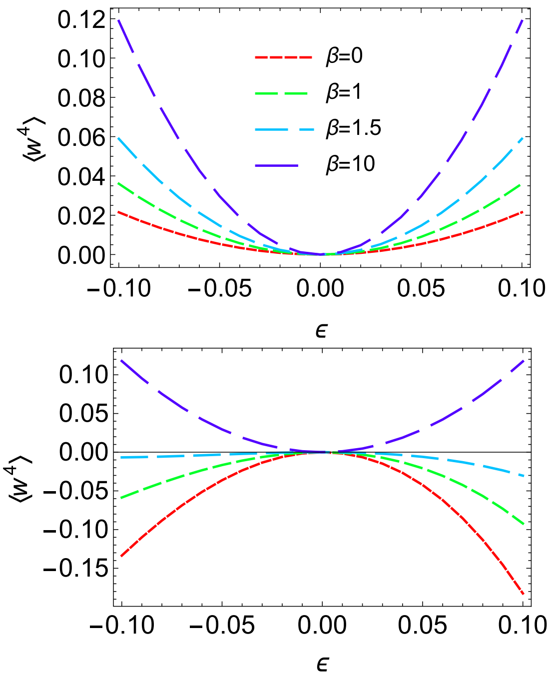

FIG. 4. The fourth moment of work $\langle w^4\rangle$ in the function of the local field ϵ for $q = 1/2$ for the states $|\Psi_1(\beta)\rangle$ (top panel) and $|\Psi_2(\beta)\rangle$ (bottom panel). The curves for other values of $q \in [0, 1/2]$ are not distinguishable by eye from the one for $q = 1/2$. We put $L = 50$, $\lambda_0 = 1$, $\phi_k = \phi_{-k} = \pi$, and $\phi_k = \phi_{-k} = 0$ for n odd and even, where $k = 2\pi(n - 1/2)/L$.

instead of $e^{-\frac{\beta\epsilon_k}{2}+i\phi_k}$, for which we get a non-negative fourth moment for the local quench.

VI. INITIAL QUANTUM COHERENCE

To conclude, we investigate further the role of initial coherence by focusing on an initial state ρ_0 with a thermal incoherent part, i.e., $\Delta(\rho_0) = \rho_G(\beta)$. In general, we have the equality (see Ref. [13])

$$\langle e^{-\beta w - C}\rangle = e^{-\beta\Delta F}, \tag{56}$$

where $\Delta F = F(\lambda_\tau) - F(\lambda_0)$ is the change in the equilibrium free energy, where $F(\lambda) = -\beta^{-1}\ln Z(\lambda)$, and C is the random quantum coherence that has the probability distribution

$$p_c(C) = \sum_{i,n} R_n |\langle E_i|R_n\rangle|^2 \delta(C + \ln\langle E_i|\rho_0|E_i\rangle - \ln R_n), \tag{57}$$

where we have considered the decomposition $\rho_0 = \sum_n R_n|R_n\rangle\langle R_n|$. In detail, the average of C is the relative entropy of coherence $\langle C\rangle = S(\Delta(\rho_0)) - S(\rho_0)$, where $S(\rho)$ is the von Neumann entropy defined as $S(\rho) = -\mathrm{Tr}\{\rho\ln\rho\}$, and we have the equality $\langle e^{-C}\rangle = 1$. In particular, from Eq. (56), we get the inequality $\langle w\rangle \geqslant \Delta F - \beta^{-1}\langle C\rangle$, and we note that Eq. (56) reduces to the Jarzynski equality [30] $\langle e^{-\beta w}\rangle = e^{-\beta\Delta F}$ when $\rho_0 = \rho_G(\beta)$. From Eq. (56), we get

$$\Delta F = \beta^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\kappa_n(s), \tag{58}$$

where $\kappa_n(s)$ is the nth cumulant of $s = \beta w + C$ which, of course, can be expressed in terms of expectation values of work and coherence: The cumulants $\kappa_n(C)$ of C cancel in the sum due to the equality $\langle e^{-C}\rangle = 1$, and only work cumulants $\kappa_n(w)$ (e.g., the variance σ_w^2) and correlation terms (e.g., the covariance $\sigma_{w,C} = \langle wC\rangle - \langle w\rangle\langle C\rangle$) are present. For instance, if work and coherence are uncorrelated, we get $\kappa_n(s) = \beta^n\kappa_n(w) + \kappa_n(C)$ and so $\Delta F = \sum_{n=1}^{\infty}(-1)^{n+1}\beta^{n-1}\kappa_n(w)/n!$ and the coherence does not appear. If we consider a Gaussian probability distribution for the random variable s, we get

$$\Delta F = \langle w\rangle - \frac{\beta\sigma_w^2}{2} - \sigma_{w,C}. \tag{59}$$

For a given free energy change ΔF, from Eq. (59) we see that the average work extracted $W_{\text{ex}} = -\langle w\rangle$ in the process increases as the fluctuation of work becomes weak, i.e., the variance σ_w^2 decreases, and the work and coherence become strongly negative correlated, i.e., $\sigma_{w,C} < 0$, which clarifies the role of initial quantum coherence as useful resource. However, we note that Eq. (59) cannot be exactly satisfied for a global quench because we have to take into account also higher work cumulants and correlations which will contribute to the series in Eq. (58) due to large deviations. In particular, if we focus on the high temperature limit $\beta \to 0$, Eq. (58) reduces to

$$\langle w\rangle = \Delta F + \sum_{k=1}^{\infty}\frac{i^{k+1}}{k!}\partial_t^k\partial_u G(0,0), \tag{60}$$

where we have defined the function $G(u,t) = \ln\langle e^{iuw+itC}\rangle$. The derivatives are correlation terms, e.g., $\partial_t\partial_u G(0,0) = -\sigma_{w,C}$, $\partial_t^2\partial_u G(0,0) = 2i\langle C\rangle\sigma_{w,C} - i\sigma_{w,C^2}$ and $\partial_t^3\partial_u G(0,0) = 3(2\langle C\rangle^2 - \langle C^2\rangle)\sigma_{w,C} - 3\langle C\rangle\sigma_{w,C^2} + \sigma_{w,C^3}$. For the initial state $\rho_0 = \eta|\Psi_G(0)\rangle\langle\Psi_G(0)| + (1-\eta)\rho_G(0)$, we get the characteristic function of the coherence (see Appendix F)

$$\langle e^{itC}\rangle = D^{it}\left(\left(\eta + \frac{1-\eta}{D}\right)^{it+1} + (D-1)\left(\frac{1-\eta}{D}\right)^{it+1}\right), \tag{61}$$

where D is the dimension of the Hilbert space. Furthermore, by considering

$$\langle e^{iuw+itC}\rangle = \mathrm{Tr}\{\rho_0 e^{it\ln\rho_0}e^{-iuH/2-it\ln\Delta(\rho_0)/2}e^{iuH'} \times e^{-iuH/2-it\ln\Delta(\rho_0)/2}\}, \tag{62}$$

where for brevity we have defined $H = H(\lambda_0)$ and $H' = H^{(H)}(\lambda_\tau)$, we get

$$-i\partial_u G(0,t) = \frac{\left(\eta + \frac{1-\eta}{D}\right)^{it+1} w_1 + (D-1)\left(\frac{1-\eta}{D}\right)^{it+1} w_2}{\left(\eta + \frac{1-\eta}{D}\right)^{it+1} + (D-1)\left(\frac{1-\eta}{D}\right)^{it+1}}, \tag{63}$$

where $w_1 = \langle\Psi_G(0)|(H' - H)|\Psi_G(0)\rangle$ is the average work done starting from the coherent Gibbs state, which can be expressed as $w_1 = (\langle w\rangle - (1-\eta)\Delta F)/\eta$, and $w_2 = D\Delta F - w_1$, where $\Delta F = \mathrm{Tr}\{H(\lambda_\tau) - H(\lambda_0)\}/D$. Thus, the terms in Eq. (60) can be obtained by calculating the derivatives of Eq. (63) with respect to t. We note that for the Ising model we get $\Delta F = 0$, so in this limit the work extracted, i.e., Eq. (53) multiplied by η, completely comes from the correlations between work and coherence. Of course, the same situation occurs for a cyclic change of any Hamiltonian, i.e., such that $H(\lambda_\tau) = H(\lambda_0)$.

VII. CONCLUSIONS

We investigated the effects of the initial quantum coherence in the energy basis to the work done by quenching a transverse field of a one-dimensional Ising model. The work can be represented by considering a class of quasiprobability distributions. To study how the work statistics changes with the increasing of the system size, we calculated the exact formula of the characteristic function of work for an arbitrary size by imposing periodic boundary conditions. Then, we focused on the thermodynamic limit, and we showed that, for an initial coherent Gibbs state, by neglecting subdominant terms for the symmetric value $q = 1/2$ we get a Gaussian probability distribution of work, and so a noncontextual protocol. However, for $q \neq 1/2$, the quasiprobability of work can take negative values depending on the initial state. In contrast, for a local quench there are initial states such that any quasiprobability representation in the class is contextual as signaled by a negative fourth moment. We note that the quasiprobability distribution can be measured experimentally in different ways [13,14], also by using a qubit (see Appendix G). In the end, beyond the fundamental purposes of the paper, it is interesting to understand if the contextuality can be related to some advantages from a thermodynamic point of view, however, further investigations are needed to go in this direction. In particular, although the protocol tends to be noncontextual in the thermodynamic limit for a global quench, the initial quantum coherence can be still a useful resource for the work extraction in the protocol when it is correlated with the work.

ACKNOWLEDGMENTS

The authors acknowledge financial support from project BIRD 2021, "Correlations, dynamics and topology in long-range quantum systems" of the Department of Physics and Astronomy, University of Padova and from the EuropeanUnion-NextGenerationEU within the National Center for HPC, Big Data and Quantum Computing (Project No. CN00000013, CN1 Spoke 10 Quantum Computing).

APPENDIX A: WORK MOMENTS

Let us derive a closed formula for the work moments. We define $H = H(\lambda_0)$ and $H' = H^{(H)}(\lambda_\tau)$. The nth work moment can be calculated as

$$\langle w^n \rangle = (-i)^n \partial_u^n \chi_q(0) = \frac{(-i)^n \partial_u^n X_q(0)}{2} + \frac{(-i)^n \partial_u^n X_{1-q}(0)}{2}. \tag{A1}$$

To calculate $(-i)^n \partial_u^n X_q(0)$, we note that

$$X_q(u) = \mathrm{Tr}\{\rho_0(u) e^{iuH'}\}, \tag{A2}$$

where we have defined

$$\rho_0(u) = e^{-iuqH} \rho_0 e^{-iu(1-q)H}. \tag{A3}$$

Then,

$$(-i)^n \partial_u^n X_q(u) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Tr}\big\{((-i)^{n-k} \partial_u^{n-k} \rho_0(u)) H'^k e^{iuH'}\big\}, \tag{A4}$$

where we have noted that $(-i)^k \partial_u^k e^{iuH'} = H'^k e^{iuH'}$. It is easy to see that

$$(-i)^n \partial_u^n \rho_0(u) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (qH)^{n-k} \rho_0(u) ((1-q)H)^k, \tag{A5}$$

from which

$$(-i)^n \partial_u^n X_q(0) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} q^{n-k-l} (1-q)^l \times \mathrm{Tr}\{H^{n-k-l} \rho_0 H^l H'^k\}. \tag{A6}$$

APPENDIX B: QUASIPROBABILITY OF WORK

We consider two different initial states, a Gibbs state $\rho_G = e^{-\beta H(\lambda_0)}/Z$, and a coherent Gibbs state $|\Psi_G\rangle$. In particular, for $\phi_j = 0$, the state $|\Psi_G^s\rangle$ in Eq. (22) reads

$$\left|\Psi_G^s\right\rangle = \frac{1}{\sqrt{Z}} \otimes_{k \in K_s} \left(e^{\frac{\beta \epsilon_k}{4}} |\tilde{0}_k\rangle + e^{-\frac{\beta \epsilon_k}{4}} |\tilde{1}_k\rangle\right). \tag{B1}$$

It can be expressed as

$$|\Psi_G^+\rangle = \frac{1}{\sqrt{Z}} (\otimes_{k>0} |\Psi_k\rangle), \tag{B2}$$

$$|\Psi_G^-\rangle = \frac{1}{\sqrt{Z}} (\otimes_{k>0} |\Psi_k\rangle) \otimes |\Psi_0\rangle \otimes |\Psi_\pi\rangle, \tag{B3}$$

where $|\Psi_k\rangle = (|\tilde{0}_k\rangle + e^{-\frac{\beta \epsilon_k}{2}} |\tilde{1}_k\rangle) \otimes (e^{\frac{\beta \epsilon_k}{2}} |\tilde{0}_{-k}\rangle + |\tilde{1}_{-k}\rangle)$. Thus, by noting that $P_s = (I + s e^{i\pi N})/2$ and $e^{i\pi N} = \langle \tilde{0}_s | e^{i\pi N} | \tilde{0}_s \rangle e^{i\pi \sum_{k \in K_s} \alpha_k^\dagger \alpha_k}$, we get

$$X_q(u) = \frac{1}{2} \sum_s \mathrm{Tr}\big\{e^{-iuqH_s(\lambda_0)} \rho_0^s e^{-iu(1-q)H_s(\lambda_0)} e^{iuH_s^{(H)}(\lambda_\tau)}\big\} + \eta_s \mathrm{Tr}\big\{e^{-iuqH_s(\lambda_0)} e^{i\pi \sum_{k \in K_s} \alpha_k^\dagger \alpha_k} \rho_0^s e^{-iu(1-q)H_s(\lambda_0)} \times e^{iuH_s^{(H)}(\lambda_\tau)}\big\}, \tag{B4}$$

where we have defined $\eta_s = s\langle \tilde{0}_s | e^{i\pi N} | \tilde{0}_s \rangle$. Let us focus on the first term in the sum over s, which is

$$X_q^s(u) = \mathrm{Tr}\big\{e^{-iuqH_s(\lambda_0)} \rho_0^s e^{-iu(1-q)H_s(\lambda_0)} e^{iuH_s^{(H)}(\lambda_\tau)}\big\}. \tag{B5}$$

Then, e.g., for $s = -$, to evaluate the trace we can consider the basis formed by the vectors $|\{n_k\}\rangle = (\otimes_{k>0} |n_k n_{-k}\rangle) \otimes |n_0\rangle \otimes |n_\pi\rangle$, with $n_k = 0, 1$, where $|n_k n_{-k}\rangle = (a_k^\dagger)^{n_k} (a_{-k}^\dagger)^{n_{-k}} |0_k 0_{-k}\rangle$, where $|0_k\rangle$ is the vacuum state for the fermion a_k. Of course, $\{|n_k n_{-k}\rangle\}$ generates an invariant dynamically subspace, and in this subspace the Hamiltonian $H_s(\lambda)$ is the matrix $H_k(\lambda)$ such that

$$H_k(\lambda)|0_k0_{-k}\rangle = \quad 2(\lambda+\cos k)|0_k0_{-k}\rangle - 2i\sin k|1_k1_{-k}\rangle, \tag{B6}$$

$$H_k(\lambda)|1_k1_{-k}\rangle = \quad -2(\lambda+\cos k)|1_k1_{-k}\rangle + 2i\sin k|0_k0_{-k}\rangle, \tag{B7}$$

$$H_k(\lambda)|0_k1_{-k}\rangle = 0, \tag{B8}$$

$$H_k(\lambda)|1_k0_{-k}\rangle = 0. \tag{B9}$$

However, it is convenient to consider the initial eigenstates $|\tilde{n}_k\tilde{n}_{-k}\rangle$ such that

$$H_k(\lambda_0)|\tilde{n}_k\tilde{n}_{-k}\rangle = (\epsilon_k n_k + \epsilon_k(n_{-k}-1))|\tilde{n}_k\tilde{n}_{-k}\rangle. \tag{B10}$$

For our two initial states, it is equal to

$$X_q^s(u) = \frac{1}{Z}\prod_{k\in K_s:k\geqslant 0} X_q^{(k)}(u). \tag{B11}$$

For the Gibbs state, for $k>0$ and $k\neq\pi$, we have

$$X_q^{(k)}(u) = \sum_{n_k,n_{-k}} e^{(-iu-\beta)(\epsilon_k n_k+\epsilon_k(n_{-k}-1))} \times \langle\tilde{n}_k\tilde{n}_{-k}|U_{\tau,0}^\dagger e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\tilde{n}_k\tilde{n}_{-k}\rangle. \tag{B12}$$

To evaluate $X_q^{(k)}(u)$, we note that

$$e^{iuH_k(\lambda_\tau)} = e^{-iu\epsilon_k'\hat{d}_k'\cdot\vec{\tau}} = (\cos(u\epsilon_k')I - i\sin(u\epsilon_k')\hat{d}_k'\cdot\vec{\tau})\oplus I, \tag{B13}$$

where $\hat{d}_k' = \hat{d}_k(\lambda_\tau)$, $\epsilon_k' = \epsilon_k(\lambda_\tau)$, $\vec{\tau} = (\tau_1,\tau_2,\tau_3)^T$, where τ_i are the Pauli matrices, i.e., $\tau_3 = |0_k0_{-k}\rangle\langle 0_k0_{-k}| - |1_k1_{-k}\rangle\langle 1_k1_{-k}|$, and so on. We have to calculate

$$\langle\tilde{n}_k\tilde{n}_{-k}|U_{\tau,0}^\dagger e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\tilde{n}_k\tilde{n}_{-k}\rangle = \cos(u\epsilon_k') - i\sin(u\epsilon_k')\langle\tilde{n}_k\tilde{n}_{-k}|U_{\tau,0}^\dagger\hat{d}_k'\cdot\vec{\tau}U_{\tau,0}|\tilde{n}_k\tilde{n}_{-k}\rangle, \tag{B14}$$

with $(n_k,n_{-k}) = (0,0)$ and $(n_k,n_{-k}) = (1,1)$, while $\langle\tilde{0}_k\tilde{1}_{-k}|U_{\tau,0}^\dagger e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\tilde{0}_k\tilde{1}_{-k}\rangle = \langle\tilde{1}_k\tilde{0}_{-k}|U_{\tau,0}^\dagger e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\tilde{1}_k\tilde{0}_{-k}\rangle = 1$. In particular, since $\hat{d}_k'\cdot\vec{\tau}$ is traceless, we get $\langle\tilde{0}_k\tilde{0}_{-k}|U_{\tau,0}^\dagger\hat{d}_k'\cdot\vec{\tau}U_{\tau,0}|\tilde{0}_k\tilde{0}_{-k}\rangle + \langle\tilde{1}_k\tilde{1}_{-k}|U_{\tau,0}^\dagger\hat{d}_k'\cdot\vec{\tau}U_{\tau,0}|\tilde{1}_k\tilde{1}_{-k}\rangle = 0$, from which we get $X_q^{(k)}(u) = X_q^{(k),th}(u)$, with

$$X_q^{(k),th}(u) = 2(\cos((u-i\beta)\epsilon_k)\cos(u\epsilon_k') + \sin((u-i\beta)\epsilon_k) \times \sin(u\epsilon_k')\langle\tilde{0}_k\tilde{0}_{-k}|U_{\tau,0}^\dagger\hat{d}_k'\cdot\vec{\tau}U_{\tau,0}|\tilde{0}_k\tilde{0}_{-k}\rangle) + 1). \tag{B15}$$

In contrast, for the coherent Gibbs state, for $k>0$ and $k\neq\pi$ we get

$$X_q^{(k)}(u) = \langle\Psi_k(q-1)|U_{\tau,0}^\dagger e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\Psi_k(q)\rangle, \tag{B16}$$

where

$$|\Psi_k(q)\rangle = \left(|\tilde{0}_k\rangle + e^{-iuq\epsilon_k-\frac{\beta\epsilon_k}{2}}|\tilde{1}_k\rangle\right) \otimes \left(e^{iuq\epsilon_k+\frac{\beta\epsilon_k}{2}}|\tilde{0}_{-k}\rangle + |\tilde{1}_{-k}\rangle\right). \tag{B17}$$

Thus, we get

$$X_q^{(k)}(u) = 2\bigg(\cos((u-i\beta)\epsilon_k)\cos(u\epsilon_k') - \frac{i}{2}\sin(u\epsilon_k') \times \langle\tilde{\Psi}_k(q-1)|U_{\tau,0}^\dagger\hat{d}_k'\cdot\vec{\tau}U_{\tau,0}|\tilde{\Psi}_k(q)\rangle + 1\bigg), \tag{B18}$$

where $|\tilde{\Psi}_k(q)\rangle = e^{iuq\epsilon_k+\beta\epsilon_k/2}|\tilde{0}_k\tilde{0}_{-k}\rangle + e^{-iuq\epsilon_k-\beta\epsilon_k/2}|\tilde{1}_k\tilde{1}_{-k}\rangle$. We get

$$X_q^{(k)}(u) = X_q^{(k),th}(u) + X_q^{(k),\mathrm{coh}}(u) \tag{B19}$$

where the coherent contribution is

$$X_q^{(k),\mathrm{coh}}(u) = -2i\sin(u\epsilon_k')\mathrm{Re}(e^{-iu(2q-1)\epsilon_k} \times \langle\tilde{0}_k\tilde{0}_{-k}|U_{\tau,0}^\dagger\hat{d}_k'\cdot\vec{\tau}U_{\tau,0}|\tilde{1}_k\tilde{1}_{-k}\rangle). \tag{B20}$$

To calculate the second term in the sum over s in Eq. (B4), we note that

$$e^{i\pi\sum_{k\in K_s}\alpha_k^\dagger\alpha_k} = (-1)^{\frac{L}{2}}e^{i\pi\sum_{k\in K_s}(\alpha_k^\dagger\alpha_k-\frac{1}{2})} \tag{B21}$$

$$= (-1)^{\frac{L}{2}}e^{iq\pi\sum_{k\in K_s}(\alpha_k^\dagger\alpha_k-\frac{1}{2})} \times e^{i(1-q)\pi\sum_{k\in K_s}(\alpha_k^\dagger\alpha_k-\frac{1}{2})}, \tag{B22}$$

then the second term is $\eta_s X_q'^s(u)$, where $X_q'^s(u)$ is obtained by multiplying $X_q^s(u)$ by $(-1)^{\frac{L}{2}}$ and by performing the substitution $u\epsilon_k\mapsto u\epsilon_k-\pi$, so

$$X_q'^s(u) = \frac{1}{Z}\prod_{k\in K_s:k\geqslant 0} X_q'^{(k)}(u), \tag{B23}$$

with

$$X_q'^{(k)}(u) = X_q^{(k)}(u) - 4 \tag{B24}$$

for $k>0$ and $k\neq\pi$. Then, we get

$$X_q(u) = \frac{1}{2}\sum_s X_q^s(u) + \eta_s X_q'^s(u). \tag{B25}$$

The partition function can be calculated as

$$Z = \mathrm{Tr}\{e^{-\beta H(\lambda_0)}\} = \sum_s \mathrm{Tr}\{P_s e^{-\beta H_s(\lambda_0)}\} \tag{B26}$$

$$= \frac{1}{2}\sum_s \mathrm{Tr}\{e^{-\beta H_s(\lambda_0)}\} + \eta_s\mathrm{Tr}\left\{e^{-\beta H_s(\lambda_0)+i\pi\sum_{k\in K_s}\alpha_k^\dagger\alpha_k}\right\} \tag{B27}$$

$$= \frac{1}{2}\sum_s\prod_{k\in K_s}2\cosh(\beta\epsilon_k/2) + \eta_s\prod_{k\in K_s}2\sinh(\beta\epsilon_k/2). \tag{B28}$$

Concerning the quasiprobability distribution of work $p_q(w)$, it can be calculated from the characteristic function as

$$p_q(w) = \int\frac{e^{-iuw}}{2\pi}\chi_q(u)du \tag{B29}$$

$$= \frac{1}{2}\int\frac{e^{-iuw}}{2\pi}(X_q(u) + X_{1-q}(u))du. \tag{B30}$$

Let us focus on the thermodynamic limit. For $|\lambda_0| > 1$, we get $Z \sim \prod_{k\in K_+} Z_k$ with $Z_k = 2\cosh(\beta\epsilon_k/2)$ and $X_q(u) \sim X_q^+(u)$. Then, $X_q(u)$ is the product of the characteristic functions having quasiprobability distributions

$$p_q^{(k)}(w) = \frac{1}{Z_k^2}\int \frac{e^{-iuw}}{2\pi} X_q^{(k)}(u)du, \tag{B31}$$

thus the quasiprobability distribution of work reads

$$p_q(w) = \frac{1}{2}\int\left(\prod_{k>0} p_q^{(k)}(w_k) + \prod_{k>0} p_{1-q}^{(k)}(w_k)\right) \times \delta\left(w - \sum_{k>0} w_k\right)\prod_{k>0} dw_k. \tag{B32}$$

We note that the average work can be calculated as

$$\langle w\rangle = -i\partial_u \chi_q(0) = -i\sum_{k>0}\frac{1}{Z_k^2}\partial_u X_q^{(k)}(0). \tag{B33}$$

On the other hand, for $|\lambda_0| < 1$, we get $Z \sim \prod_{k\in K_+} Z_k + \prod_{k\in K_+} Z'_k$ with $Z'_k = 2\sinh(\beta\epsilon_k/2)$, and $X_q(u) \sim X_q^+(u) + X_q'^+(u)$ from which

$$X_q(u) = \gamma\prod_{k>0}\frac{X_q^{(k)}(u)}{Z_k^2} + (1-\gamma)\prod_{k>0}\frac{X_q'^{(k)}(u)}{Z_k'^2}, \tag{B34}$$

where $\gamma = (\prod_{k>0} Z_k^2)/Z$. In the thermodynamic limit, we get

$$\gamma = \frac{e^{\frac{2L}{\pi}\int_0^\pi \cosh^2(\beta\epsilon_k/2)dk}}{e^{\frac{2L}{\pi}\int_0^\pi \cosh^2(\beta\epsilon_k/2)dk} + e^{\frac{2L}{\pi}\int_0^\pi \sinh^2(\beta\epsilon_k/2)dk}} \to 1 \tag{B35}$$

for a nonzero temperature, since $\int_0^\pi \cosh^2(\beta\epsilon_k/2)dk > \int_0^\pi \sinh^2(\beta\epsilon_k/2)dk$. In contrast, for $\beta\to\infty$, we get $Z'_k \sim Z_k$ so $\gamma = 1/2$, and $X_q'^{(k)}(u) \sim X_q^{(k)}(u)$. Then we get the same expression of the quasiprobability of work of Eq. (B32).

1. Sudden quench

Let us consider a sudden quench, i.e., the limit $\tau\to 0$, so $U_{\tau,0} = I$. For the Gibbs state, we get $X_q^{(k)}(u) = X_q^{(k),th}(u)$ with

$$X_q^{(k),th}(u) = 2(\cos((u-i\beta)\epsilon_k)\cos(u\epsilon'_k) + \sin((u-i\beta)\epsilon_k) \times \sin(u\epsilon'_k)\hat{d}_k\cdot\hat{d}'_k + 1) \tag{B36}$$

by noting that

$$\begin{aligned}\langle\tilde{0}_k\tilde{0}_{-k}|\hat{d}'_k\cdot\vec{\tau}|\tilde{0}_k\tilde{0}_{-k}\rangle &= \mathrm{Tr}\{\hat{d}'_k\cdot\vec{\tau}|\tilde{0}_k\tilde{0}_{-k}\rangle\langle\tilde{0}_k\tilde{0}_{-k}|\}\\ &= \tfrac{1}{2}\mathrm{Tr}\{\hat{d}'_k\cdot\vec{\tau}\hat{d}_k\cdot\vec{\tau}\}\\ &= \hat{d}_k\cdot\hat{d}'_k\end{aligned} \tag{B37}$$

since $\hat{d}_k\cdot\vec{\tau} = |\tilde{0}_k\tilde{0}_{-k}\rangle\langle\tilde{0}_k\tilde{0}_{-k}| - |\tilde{1}_k\tilde{1}_{-k}\rangle\langle\tilde{1}_k\tilde{1}_{-k}|$. On the other hand, for the coherent Gibbs state we get

$$X_q^{(k)}(u) = X_q^{(k),th}(u) + X_q^{(k),\mathrm{coh}}(u). \tag{B38}$$

To evaluate the coherent contribution $X_q^{(k),\mathrm{coh}}(u)$, we note that

$$\begin{aligned}\langle\tilde{0}_k\tilde{0}_{-k}|\hat{d}'_k\cdot\vec{\tau}|\tilde{1}_k\tilde{1}_{-k}\rangle &= \mathrm{Tr}\{\hat{d}_k\cdot\vec{\tau}\hat{d}'_k\cdot\vec{\tau}|\tilde{1}_k\tilde{1}_{-k}\rangle\langle\tilde{0}_k\tilde{0}_{-k}|\}\\ &= i\mathrm{Tr}\{(\hat{d}_k\times\hat{d}'_k)\cdot\vec{\tau}|\tilde{1}_k\tilde{1}_{-k}\rangle\langle\tilde{0}_k\tilde{0}_{-k}|\},\end{aligned} \tag{B39}$$

then

$$\langle\tilde{0}_k\tilde{0}_{-k}|\hat{d}'_k\cdot\vec{\tau}|\tilde{1}_k\tilde{1}_{-k}\rangle = i(\hat{d}_k\times\hat{d}'_k)_x\langle\tilde{0}_k\tilde{0}_{-k}|\tau_1|\tilde{1}_k\tilde{1}_{-k}\rangle, \tag{B40}$$

since $\hat{d}_k\times\hat{d}'_k$ has only an x component. Since $\langle\tilde{0}_k\tilde{0}_{-k}|\tau_1|\tilde{1}_k\tilde{1}_{-k}\rangle = 1$, we get

$$\langle\tilde{0}_k\tilde{0}_{-k}|\hat{d}'_k\cdot\vec{\tau}|\tilde{1}_k\tilde{1}_{-k}\rangle = i(\hat{d}_k\times\hat{d}'_k)_x. \tag{B41}$$

Thus, we get

$$X_q^{(k),\mathrm{coh}}(u) = -2i\sin(u\epsilon'_k)\sin(u(2q-1)\epsilon_k)(\hat{d}_k\times\hat{d}'_k)_x. \tag{B42}$$

For $\phi_j\neq 0$, we have $|\Psi_k\rangle = (|\tilde{0}_k\rangle + e^{i\phi_k - \frac{\beta\epsilon_k}{2}}|\tilde{1}_k\rangle)\otimes(e^{\frac{\beta\epsilon_k}{2}}|\tilde{0}_{-k}\rangle + e^{i\phi_{-k}}|\tilde{1}_{-k}\rangle)$, with $\phi_{-k} = \phi_k$. Thus, by considering the corresponding state $|\Psi_k(q)\rangle$, the only effect of phase ϕ_k is the shift $uq\epsilon_k \to uq\epsilon_k - \phi_k$, then we get

$$X_q^{(k),\mathrm{coh}}(u) = -2i\sin(u\epsilon'_k)\sin(u(2q-1)\epsilon_k - 2\phi_k)(\hat{d}_k\times\hat{d}'_k)_x. \tag{B43}$$

2. Arbitrary quench

In the end, let us consider an arbitrary quench. The time evolution acts as a rotation of the vector $\vec{d}_k(\lambda_\tau)$, so $U_{\tau,0}^\dagger\hat{d}_k(\lambda_\tau)\cdot\vec{\tau}U_{\tau,0} = \hat{d}'_k(\lambda_\tau)\cdot\vec{\tau}$, where for brevity $\hat{d}'_k = \hat{d}'_k(\lambda_\tau)$. Then, $X_q^{(k),th}(u)$ is still given by Eq. (B36) with the new vector $\hat{d}'_k$, and if $\hat{d}_k\times\hat{d}'_k$ also has a nonzero y component, then

$$\langle\tilde{0}_k\tilde{0}_{-k}|\hat{d}'_k\cdot\vec{\tau}|\tilde{1}_k\tilde{1}_{-k}\rangle = i(\hat{d}_k\times\hat{d}'_k)_x + (\hat{d}_k\times\hat{d}'_k)_y, \tag{B44}$$

from which the coherence contribution has a further term and reads

$$X_q^{(k),coh}(u) = -2i\sin(u\epsilon'_k)(\sin(u(2q-1)\epsilon_k - 2\phi_k)(\hat{d}_k\times\hat{d}'_k)_x + \cos(u(2q-1)\epsilon_k - 2\phi_k)(\hat{d}_k\times\hat{d}'_k)_y) \tag{B45}$$

3. Histogram

To determinate the quasiprobability distribution of work from the characteristic function $\chi_q(u)$, we consider the intervals $I_n = [w_n - \Delta w/2, w_n + \Delta w/2]$, where $w_n = n\Delta w$ with an n integer. Then, we can determinate the histogram by calculating the probability

$$p_n = \int_{I_n} p_q(w)dw = \frac{\Delta w}{2\pi}\int \chi_q(u)\mathrm{sinc}\left(\frac{u\Delta w}{2}\right)e^{-iuw_n}du, \tag{B46}$$

where $\mathrm{sinc}(x) = \sin(x)/x$. To calculate the integral, we can focus on the interval $[-2\pi K/\Delta w, 2\pi K/\Delta w]$ with K large enough. Of course, $p_q(w_n) \approx p_n/\Delta w$ for Δw small enough.

APPENDIX C: SUPERPOSITION OF TWO COHERENT GIBBS STATES

For simplicity, we consider the fermionic Hamiltonian H_+. We focus on the initial state

$$|\Psi\rangle = a|\Psi_{G,1}^+\rangle + b|\Psi_{G,2}^+\rangle, \tag{C1}$$

where $|\Psi^+_{G,i}\rangle$ is the coherent Gibbs state

$$|\Psi^+_{G,i}\rangle = \otimes_{k>0}\frac{|\Psi_{i,k}\rangle}{Z_{i,k}}, \tag{C2}$$

where

$$|\Psi_{i,k}\rangle = \left(|\tilde{0}_k\rangle + e^{i\phi_{i,k}-\frac{\beta_i\epsilon_k}{2}}|\tilde{1}_k\rangle\right)\otimes\left(e^{\frac{\beta_i\epsilon_k}{2}}|\tilde{0}_{-k}\rangle + e^{i\phi_{i,-k}}|\tilde{1}_{-k}\rangle\right). \tag{C3}$$

We will get

$$X^+_q(u) = |a|^2\prod_{k>0}\frac{X^{(k)}_{q,1}(u)}{Z^2_{1,k}} + |b|^2\prod_{k>0}\frac{X^{(k)}_{q,2}(u)}{Z^2_{2,k}} + ab^*\prod_{k>0}\frac{Y^{(k)}_{q,1}(u)}{Z_{1,k}Z_{2,k}} + a^*b\prod_{k>0}\frac{Y^{(k)}_{q,2}(u)}{Z_{1,k}Z_{2,k}}, \tag{C4}$$

where

$$X^{(k)}_{q,i}(u) = \langle\Psi_{i,k}(q-1)|U^\dagger_{\tau,0}e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\Psi_{i,k}(q)\rangle, \tag{C5}$$

$$Y^{(k)}_{q,1}(u) = \langle\Psi_{2,k}(q-1)|U^\dagger_{\tau,0}e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\Psi_{1,k}(q)\rangle, \tag{C6}$$

$$Y^{(k)}_{q,2}(u) = \langle\Psi_{1,k}(q-1)|U^\dagger_{\tau,0}e^{iuH_k(\lambda_\tau)}U_{\tau,0}|\Psi_{2,k}(q)\rangle, \tag{C7}$$

with

$$|\Psi_{i,k}\rangle = \left(|\tilde{0}_k\rangle + e^{-iuq\epsilon_k+i\phi_{i,k}-\frac{\beta_i\epsilon_k}{2}}|\tilde{1}_k\rangle\right)\otimes\left(e^{iuq\epsilon_k+\frac{\beta_i\epsilon_k}{2}}|\tilde{0}_{-k}\rangle + e^{i\phi_{i,-k}}|\tilde{1}_{-k}\rangle\right). \tag{C8}$$

Then we can write

$$X^+_q(u) = |a|^2e^{Lg_{1,q}(u)} + |b|^2e^{Lg_{2,q}(u)} + ab^*e^{Ly_{1,q}(u)} + a^*be^{Ly_{2,q}(u)}, \tag{C9}$$

where

$$g_{i,q}(u) = \frac{1}{2\pi}\int_0^\pi \ln\left(\frac{X^{(k)}_{q,i}(u)}{Z^2_{i,k}}\right)dk, \tag{C10}$$

$$y_{i,q}(u) = \frac{1}{2\pi}\int_0^\pi \ln\left(\frac{Y^{(k)}_{q,i}(u)}{Z_{1,k}Z_{2,k}}\right)dk. \tag{C11}$$

As $L\to\infty$, the Fourier transform of $e^{Ly_{i,q}(u)}$ gives

$$p'_{i,q}(w) \sim \frac{e^{Ly_{i,q}(0)}}{\sqrt{2\pi x^{(2)}_{i,q}}}e^{-\frac{\left(w-x^{(1)}_{i,q}\right)^2}{2x^{(2)}_{i,q}}}, \tag{C12}$$

where $x^{(1)}_{i,q} = -i\partial_u y_{i,q}(0)L$ and $x^{(2)}_{i,q} = -\partial^2_u y_{i,q}(0)L$. For $q = 1/2$, we get $x^{(i)}_{2,1/2} = (x^{(i)}_{1,1/2})^*$, then the quasiprobability distribution of work reads

$$p_{1/2}(w) \sim |a|^2\frac{e^{-\frac{(w-\bar{w}_1)^2}{2v_{1,q}}}}{\sqrt{2\pi v_{1,q}}} + |b|^2\frac{e^{-\frac{(w-\bar{w}_2)^2}{2v_{2,q}}}}{\sqrt{2\pi v_{2,q}}} + 2\mathrm{Re}(ab^*p'_{1,1/2}(w)), \tag{C13}$$

where $\bar{w}_i = -i\partial_u g_{i,q}(0)L$ and $v_{i,q} = -\partial^2_u g_{i,q}(0)L$, so $p_{1/2}(w)$ can take negative values. However, since the real part of $y_{i,q}(0)$ is negative, $p'_{i,q}(w)$ tends exponentially to zero in the thermodynamic limit and $p_{1/2}(w)$ is the convex combination of two Gaussian probability distributions, which is positive.

1. Generalized coherent Gibbs state

For an arbitrary quench from the initial coherent Gibbs state, from Eq. (B45), we get

$$r_q = \frac{(2q-1)L}{2\pi}\int_0^\pi \frac{\epsilon_k\epsilon'_k}{\cosh^2(\beta\epsilon_k/2)}(\cos(2\phi_k)(\hat{d}_k\times\hat{d}'_k)_x + \sin(2\phi_k)(\hat{d}_k\times\hat{d}'_k)_y)dk, \tag{C14}$$

which is zero for $q = 1/2$. Let us focus on an initial state of the form

$$|\Psi^+\rangle = \otimes_{k>0}|\Psi_k\rangle, \tag{C15}$$

which generalizes the coherent Gibbs state $|\Psi^+_G\rangle$, where we have defined the states

$$|\Psi_k\rangle = \sum_{n_k,n_{-k}} c_{n_kn_{-k}}|\tilde{n}_k\tilde{n}_{-k}\rangle. \tag{C16}$$

This implies that $X_q(u)$ has the form in Eq. (33) with $Z_k = 1$, where $X^{(k)}_q(u)$ can be calculated from Eq. (B16) with

$$|\Psi_k(q)\rangle = c_{00}e^{iuq\epsilon_k}|\tilde{0}_k\tilde{0}_{-k}\rangle + c_{11}e^{-iuq\epsilon_k}|\tilde{1}_k\tilde{1}_{-k}\rangle + c_{01}|\tilde{0}_k\tilde{1}_{-k}\rangle + c_{10}|\tilde{1}_k\tilde{0}_{-k}\rangle. \tag{C17}$$

Then, the representation for $q = 1/2$ will be noncontextual. Let us show explicitly that $r_{1/2} = 0$. $X^{(k)}_q(u)$ reads

$$X^{(k)}_q(u) = X^{(k)}_{no}(u) + \delta X^{(k)}_q(u), \tag{C18}$$

where $X^{(k)}_{no}(u)$ does not depend on q, and

$$\delta X^{(k)}_q(u) = -2i\sin(u\epsilon'_k)\mathrm{Re}\left(c^*_{00}c_{11}e^{-iu(2q-1)\epsilon_k}\left(i(\hat{d}_k\times\hat{d}'_k)_x + (\hat{d}_k\times\hat{d}'_k)_y\right)\right). \tag{C19}$$

Then, $\partial^2_u\delta X^{(k)}_q(0)$ is imaginary and $\partial^2_u\delta X^{(k)}_q(0)\propto(1-2q)$. Similarly, it is easy to see that $\partial^2_u X^{(k)}_{no}(0)$ is real. Furthermore, $\partial_u X^{(k)}_q(0)$ is imaginary, then r_q is obtained by calculating an integral with respect to k of $\partial^2_u\delta X^{(k)}_q(0)$, so we get $r_q\propto(1-2q)$, which is zero for $q = 1/2$. In the end, we note that a linear combination of states of the form in Eq. (C15) will give for $q = 1/2$ a convex combination of Gaussian probability distributions, which is positive.

APPENDIX D: NEGATIVITY

To prove that $\mathcal{N} = 1$ implies, in general, that $p_q(w)\geqslant 0$, we can proceed ad absurdum. We write $p_q(w) = p(w) + \delta p(w)$, where $p(w)\geqslant 0$, $\int p(w)dw = 1$ and $\int\delta p(w)dw = 0$. If $p_q(w)\geqslant 0$ for $w\in I$ and $p_q(w) < 0$ for $w\in I'$, then $\delta p(w) < 0$ for $w\in I'$ and $I = I_+\cup I_-$ such that $\delta p(w)\geqslant 0$ for $w\in I_+$ and $\delta p(w) < 0$ for $w\in I_-$. Then, from $\mathcal{N} = 1$, we get the condition $p(I) - p(I') + \delta p(I_+) + \delta p(I_-) - \delta p(I') = 1$, where $p(I) = \int_I p(w)dw$ and so on, thus we get the system

$$\begin{aligned} p(I) + p(I') &= 1\\ p(I) &\geqslant 0\\ p(I') &\geqslant 0\\ \delta p(I_+) + \delta p(I-) + \delta p(I') &= 0\\ \delta p(I_+) &\geqslant 0\\ \delta p(I_-) &< 0\\ \delta p(I') &< 0\\ p(I) - p(I') + \delta p(I_+) + \delta p(I_-) - \delta p(I') &= 1, \end{aligned} \tag{D1}$$

which admits as solution $p(w)$ such that $0 \leqslant p(I) < 1$ and $p(I') = 1 - p(I)$ and $\delta p(w)$ such that $\delta p(I_+) > (1 - p(I) + p(I'))/2$, $\delta p(I_-) = (1 - 2\delta p(I_+) - p(I) + p(I'))/2$ and $\delta p(I') = -\delta p(I_-) - \delta p(I_+)$. Then $\delta p(I') = -p(I')$, so $p_q(I') = 0$, which implies that $p_q(w)$ is non-negative.

APPENDIX E: GENERAL QUADRATIC FORM IN FERMI OPERATORS

We consider the initial Hamiltonian

$$H = \sum_{i,j} \left(a_i^\dagger A_{ij} a_j + \frac{1}{2}(a_i^\dagger B_{ij} a_j^\dagger + H.c.) \right) - \frac{1}{2} \sum_i A_{ii}, \tag{E1}$$

where A and B are real matrices such that $A^T = A$ and $B^T = -B$. The Hamiltonian can be diagonalized by performing the transformation

$$\alpha_k = \sum_i g_{ki} a_i + h_{ki} a_i^\dagger, \tag{E2}$$

so

$$H = \sum_k \epsilon_k \left(\alpha_k^\dagger \alpha_k - \frac{1}{2} \right). \tag{E3}$$

In detail, the matrices g and h are such that $\phi = g + h$ and $\psi = g - h$, where ϕ and ψ are orthogonal matrices such that $\psi^T \epsilon \phi = A + B$, where ϵ is the diagonal matrix with entries ϵ_k. The final time-evolved Hamiltonian is H' with matrices A' and B', and will be diagonalized by performing the transformation

$$\alpha'_k = \sum_i g'_{ki} a_i + h'_{ki} a_i^\dagger \tag{E4}$$

so

$$H' = \sum_k \epsilon'_k \left(\alpha_k'^\dagger \alpha'_k - \frac{1}{2} \right). \tag{E5}$$

Let us proceed with our investigation by considering the initial state

$$|\Psi_1\rangle = \frac{e^{\frac{\beta}{4}\sum_k \epsilon_k}}{\sqrt{Z_1}} \exp\left(\sum_k e^{-\frac{\beta \epsilon_k}{2}} \alpha_k^\dagger \right) |\tilde{0}\rangle. \tag{E6}$$

We note that for $\beta \to \infty$ we get $Z_1 \sim Z = \prod_k 2\cosh(\beta\epsilon_k/2)$ and $|\Psi_1\rangle \sim |\Psi_G\rangle$. We aim to calculate

$$X_q(u) = \langle \Psi_1 | e^{-iu(1-q)H} e^{iuH'} e^{-iuqH} | \Psi_1 \rangle. \tag{E7}$$

We consider the vacuum state $|\tilde{0}'\rangle$ of the fermions α'_k, we get the relation

$$|\tilde{0}\rangle = K e^{\frac{1}{2}\sum_{k,k'} G_{kk'} \alpha_k'^\dagger \alpha_{k'}'^\dagger} |\tilde{0}'\rangle, \tag{E8}$$

where G is solution of the equation $\tilde{g}G + \tilde{h} = 0$, where $\tilde{g} = gg'^T + hh'^T$ and $\tilde{h} = gh'^T + hg'^T$. In particular,

$$\alpha_k = \sum_{k'} \tilde{g}_{kk'} \alpha'_{k'} + \tilde{h}_{kk'} \alpha_{k'}'^\dagger. \tag{E9}$$

We get

$$X_q(u) = |K|^2 \frac{e^{(\beta+iu)\sum_k \epsilon_k/2 - iu\sum_k \epsilon'_k/2}}{Z_1} \langle \tilde{0}' | \exp\left(-\frac{1}{2} \sum_{k,k'} G_{kk'} \alpha'_k \alpha'_{k'} \right) \exp\left(\sum_k u_k \alpha'_k + v_k \alpha_k'^\dagger \right)$$
$$\times \exp\left(\sum_k u'_k \alpha_k'^\dagger + v'_k \alpha'_k \right) \exp\left(\frac{1}{2} \sum_{k,k'} \tilde{G}_{kk'} \alpha_k'^\dagger \alpha_{k'}'^\dagger \right) |\tilde{0}'\rangle, \tag{E10}$$

where $\tilde{G}_{kk'} = G_{kk'} e^{iu(\epsilon'_k + \epsilon'_{k'})}$ and

$$u_k = \sum_{k'} e^{-(\beta/2 + iu(1-q))\epsilon_{k'}} \tilde{g}_{k'k}, \tag{E11}$$

$$v_k = \sum_{k'} e^{-(\beta/2 + iu(1-q))\epsilon_{k'}} \tilde{h}_{k'k}, \tag{E12}$$

$$u'_k = \sum_{k'} e^{-(\beta/2 + iuq)\epsilon_{k'} + iu\epsilon'_k} \tilde{g}_{k'k}, \tag{E13}$$

$$v'_k = \sum_{k'} e^{-(\beta/2 + iuq)\epsilon_{k'} - iu\epsilon'_k} \tilde{h}_{k'k}. \tag{E14}$$

We note that

$$\exp\left(\sum_k u_k \alpha'_k + v_k \alpha_k'^\dagger \right) \exp\left(\sum_k u'_k \alpha_k'^\dagger + v'_k \alpha'_k \right) = 1$$
$$+ \sum_{k,k'} u_k u'_{k'} \alpha'_k \alpha_{k'}'^\dagger + u_k v'_{k'} \alpha'_k \alpha'_{k'} + v_k u'_{k'} \alpha_k'^\dagger \alpha_{k'}'^\dagger - v'_k v_{k'} \alpha'_k \alpha_{k'}'^\dagger + \sum_k v_k v'_k + \cdots, \tag{E15}$$

where we have omitted terms linear in the Fermi operators. Then, the overlap in Eq. (E10) can be easily calculated by using the coherent states $|\xi\rangle$ such that $\alpha'_k|\xi\rangle = \xi_k|\xi\rangle$. By using the identity $\int d\xi^* d\xi e^{-\sum_k \xi_k^* \xi_k} |\xi\rangle\langle\xi| = 1$, we get

$$X_q(u) \sim |K|^2 \frac{e^{(\beta+iu)\sum_k \epsilon_k/2 - iu\sum_k \epsilon'_k/2}}{Z_1} \left[\int d\xi^* d\xi \exp\left(-\frac{1}{2} \sum_{k,k'} G_{kk'} \xi_k \xi_{k'} + \sum_{k,k'} (u_k u'_{k'} \xi_k \xi_{k'}^* + u_k v'_{k'} \xi_k \xi_{k'} \right. \right.$$

$$+ v_k u'_{k'} \xi_k^* \xi_{k'}^* - v'_k v_{k'} \xi_k \xi_{k'}^*) - \sum_k \xi_k^* \xi_k + \frac{1}{2} \sum_{k,k'} \tilde{G}_{kk'} \xi_k^* \xi_{k'}^* \Big)$$

$$+ \sum_k v_k v'_k \int d\xi^* d\xi \exp\left(- \frac{1}{2} \sum_{k,k'} G_{kk'} - \sum_k \xi_k^* \xi_k + \frac{1}{2} \sum_{k,k'} \tilde{G}_{kk'} \xi_k^* \xi_{k'}^* \right) \Bigg]. \tag{E16}$$

By performing the integral, we get

$$X_q(u) \sim C e^{iu \sum_k (\epsilon_k - \epsilon'_k)/2} \left(\sqrt{\det(\Gamma(u))} + \sqrt{\det(\Gamma_0(u))} \sum_k v_k v'_k \right), \tag{E17}$$

where

$$\Gamma_0(u) = \begin{pmatrix} G & -I \\ I & -\tilde{G} \end{pmatrix} \tag{E18}$$

and

$$\Gamma(u) = \begin{pmatrix} G - M_1 & -I - M_2 \\ I + M_2^T & -\tilde{G} - M_3 \end{pmatrix} = \Gamma_0(u) + M(u), \tag{E19}$$

where $M_{1,kk'} = u_k v'_{k'} - u_{k'} v'_k$, $M_{2,kk'} = u_k u'_{k'} - v'_k v_{k'}$, and $M_{3,kk'} = v_k u'_{k'} - v_{k'} u'_k$. The constant C can be determined by requiring that $X_q(0) = 1$. The exact expression of $X_q(u)$ can be obtained by expanding $\sqrt{\det(\Gamma(u))}$ at the first order in $M(u)$, i.e.,

$$X_q(u) = C e^{iu \sum_k (\epsilon_k - \epsilon'_k)/2} \sqrt{\det(\Gamma_0(u))} \left(1 + \frac{1}{2} \mathrm{Tr}\{ \Gamma_0^{-1}(u) M(u) \} + \sum_k v_k v'_k \right). \tag{E20}$$

Concerning the coherent Gibbs state, for low temperatures $\beta \to \infty$ we get

$$|\Psi_G\rangle \sim \frac{e^{\frac{\beta}{4} \sum_k \epsilon_k}}{\sqrt{Z}} \left(1 + \sum_k e^{-\frac{\beta \epsilon_k}{2}} \alpha_k^\dagger + \sum_{k>k'} e^{-\frac{\beta(\epsilon_k + \epsilon_{k'})}{2}} \alpha_k^\dagger \alpha_{k'}^\dagger \right) |\tilde{0}\rangle. \tag{E21}$$

We define

$$u_{kq} = e^{-(\beta/2 + iu(1-q))\epsilon_k} \tilde{g}_{kq}, \tag{E22}$$

$$v_{kq} = e^{-(\beta/2 + iu(1-q))\epsilon_k} \tilde{h}_{kq}, \tag{E23}$$

$$u'_{kq} = e^{-(\beta/2 + iuq)\epsilon_k + iu\epsilon'_q} \tilde{g}_{kq}, \tag{E24}$$

$$v'_{kq} = e^{-(\beta/2 + iuq)\epsilon_k - iu\epsilon'_q} \tilde{h}_{kq}, \tag{E25}$$

so $u_k = \sum_{k'} u_{k'k}$ and so on, then the matrices V_1, V_2, V_3, V'_1, V'_2, and V'_3 with elements $V_{1,qq'} = \sum_{k,k'} s_{k,k'} u_{kq} u_{k'q'}$, $V_{2,qq'} = \sum_{k,k'} s_{k,k'} u_{kq} v_{k'q'}$, $V_{3,qq'} = \sum_{k,k'} s_{k,k'} v_{kq} v_{k'q'}$, $V'_{1,qq'} = \sum_{k,k'} s_{k,k'} v'_{kq} v'_{k'q'}$, $V'_{2,qq'} = \sum_{k,k'} s_{k,k'} v'_{kq} u'_{k'q'}$, $V'_{3,qq'} = \sum_{k,k'} s_{k,k'} u'_{kq} u'_{k'q'}$, where $s_{k,k'} = 1$ if $k > k'$, $s_{k,k'} = -1$ if $k < k'$ and $s_{k,k} = 0$. Thus, by proceeding similarly, we get at second order

$$X_q(u) \sim C e^{iu \sum_k (\epsilon_k - \epsilon'_k)/2} \sqrt{\det(\Gamma_0(u))} \left(1 + \frac{1}{2} \mathrm{Tr}\{ \Gamma_0^{-1}(u) M(u) \} + \sum_k v_k v'_k + \frac{1}{2} \mathrm{Tr}\{ \Gamma_0^{-1}(u) (V(u) - V'(u)) \} + \frac{1}{2} \mathrm{Tr}\{ V_2 - V'_2 \} \right), \tag{E26}$$

where we have defined the matrices

$$V(u) = \begin{pmatrix} V_1 & V_2 \\ -V_2^T & V_3 \end{pmatrix}, \quad V'(u) = \begin{pmatrix} V'_1 & V'_2 \\ -V_2'^T & V'_3 \end{pmatrix}. \tag{E27}$$

We note that for an initial state that is the ground state of H, we get the characteristic function

$$\chi^{(0)}(u) = e^{iu \sum_k (\epsilon_k - \epsilon'_k)/2} \sqrt{\frac{\det(\Gamma_0(u))}{\det(\Gamma_0(0))}}, \tag{E28}$$

which is obtained from $X_q(u)$ in the limit $\beta \to \infty$. Alternatively, by considering $\theta^T = (\xi^T, \xi^{*T})$, Eq. (E20) can be derived with the help of the identity

$$\int d\theta \theta_i \theta_j e^{-\frac{1}{2} \theta^T \Gamma_0 \theta} = -\frac{1}{2} \mathrm{Tr}\{ \Gamma_0^{-1} X_{ij} \} \sqrt{\det(\Gamma_0)}, \tag{E29}$$

where $X_{ij} = |i\rangle\langle j| - |j\rangle\langle i|$, and $|i\rangle$ is the unit vector with only the ith component which is nonzero. Actually, $\sqrt{\det(\Gamma_0)}$ is the Pfaffian of Γ_0. To prove it, we note that

$$\int d\theta\theta_i\theta_j e^{-\frac{1}{2}\theta^T\Gamma_0\theta} = \frac{1}{\epsilon}\left(\int d\theta(1+\epsilon\theta_i\theta_j)e^{-\frac{1}{2}\theta^T\Gamma_0\theta} - \int d\theta e^{-\frac{1}{2}\theta^T\Gamma_0\theta}\right). \tag{E30}$$

The second integral is $\int d\theta e^{-\frac{1}{2}\theta^T\Gamma_0\theta} = \sqrt{\det(\Gamma_0)}$. By considering the limit $\epsilon \to 0$, we get

$$\int d\theta\theta_i\theta_j e^{-\frac{1}{2}\theta^T\Gamma_0\theta} \sim \frac{1}{\epsilon}\left(\int d\theta e^{\frac{\epsilon}{2}(\theta_i\theta_j-\theta_j\theta_i)}e^{-\frac{1}{2}\theta^T\Gamma_0\theta} - \sqrt{\det(\Gamma_0)}\right), \tag{E31}$$

then

$$\int d\theta\theta_i\theta_j e^{-\frac{1}{2}\theta^T\Gamma_0\theta} \sim \frac{1}{\epsilon}(\sqrt{\det(\Gamma_0-\epsilon X_{ij})} - \sqrt{\det(\Gamma_0)}); \tag{E32}$$

by evaluating the limit $\epsilon \to 0$, we get Eq. (E29). Similarly, we have the identity

$$\int d\theta\theta_i\theta_j\theta_k\theta_l e^{-\frac{1}{2}\theta^T\Gamma_0\theta} = -\frac{1}{2}\mathrm{Tr}\{\Gamma_0^{-1}X_{ij}\Gamma_0^{-1}X_{kl}\}\sqrt{\det(\Gamma_0)} + \frac{1}{4}\mathrm{Tr}\{\Gamma_0^{-1}X_{ij}\}\mathrm{Tr}\{\Gamma_0^{-1}X_{kl}\}\sqrt{\det(\Gamma_0)}. \tag{E33}$$

To prove it, we consider that

$$\int d\theta\theta_i\theta_j\theta_k\theta_l e^{-\frac{1}{2}\theta^T\Gamma_0\theta} = \frac{1}{\epsilon}\left(\int d\theta\theta_i\theta_j(1+\epsilon\theta_k\theta_l)e^{-\frac{1}{2}\theta^T\Gamma_0\theta} - \int d\theta\theta_i\theta_j e^{-\frac{1}{2}\theta^T\Gamma_0\theta}\right), \tag{E34}$$

which, in the limit $\epsilon \to 0$ can be evaluated with the help of the identity in Eq. (E29). We get

$$\int d\theta\theta_i\theta_j\theta_k\theta_l e^{-\frac{1}{2}\theta^T\Gamma_0\theta} \sim \frac{1}{2\epsilon}\left(\mathrm{Tr}\{\Gamma_0^{-1}X_{ij}\}\sqrt{\det(\Gamma_0)} - \mathrm{Tr}\{(\Gamma_0-\epsilon X_{kl})^{-1}X_{ij}\}\sqrt{\det(\Gamma_0-\epsilon X_{kl})}\right); \tag{E35}$$

by evaluating the limit $\epsilon \to 0$, we get Eq. (E33). In the end, we consider the initial state in Eq. (E21), which is

$$|\Psi_2\rangle = \frac{e^{\frac{\beta}{4}\sum_k\epsilon_k}}{\sqrt{Z_2}}\left(1+\sum_k e^{-\frac{\beta\epsilon_k}{2}}\alpha_k^\dagger + \frac{1}{2}\sum_{k,k'} s_{k,k'}e^{-\frac{\beta(\epsilon_k+\epsilon_{k'})}{2}}\alpha_k^\dagger\alpha_{k'}^\dagger\right)|\tilde{0}\rangle. \tag{E36}$$

By using the identities in Eqs. (E29) and (E33), we get

$$\begin{aligned}
X_q(u) = Ce^{iu\sum_k(\epsilon_k-\epsilon_k')/2}\sqrt{\det(\Gamma_0(u))}\Big(&1+\frac{1}{2}\mathrm{Tr}\{\Gamma_0^{-1}(u)M(u)\}\\
&+\sum_k v_k v_k' + \frac{1}{2}\mathrm{Tr}\{\Gamma_0^{-1}(u)(V(u)-V'(u))\}\\
&+\frac{1}{2}\mathrm{Tr}\{V_2-V_2'\} - \frac{1}{4}\mathrm{Tr}\{V_2\}\mathrm{Tr}\{V_2'\}\\
&-\frac{1}{4}\mathrm{Tr}\{V_2\}\mathrm{Tr}\{\Gamma_0^{-1}(u)V'(u)\}\\
&-\frac{1}{4}\mathrm{Tr}\{V_2'\}\mathrm{Tr}\{\Gamma_0^{-1}(u)V(u)\} - \frac{1}{2}\mathrm{Tr}\{V_3V_1'\}\\
&+\frac{1}{2}\mathrm{Tr}\{\Gamma_0^{-1}(u)V''(u)\} + \frac{1}{2}\mathrm{Tr}\{\Gamma_0^{-1}(u)V(u)\Gamma_0^{-1}(u)V'(u)\}\\
&-\frac{1}{4}\mathrm{Tr}\{\Gamma_0^{-1}(u)V(u)\}\mathrm{Tr}\{\Gamma_0^{-1}(u)V'(u)\}\Big),
\end{aligned} \tag{E37}$$

where we have defined

$$V''(u) = \begin{pmatrix} V_2V_1' + V_1'V_2^T & V_2V_2' - V_1'V_3 \\ V_3V_1' - V_2'^TV_2^T & V_3V_2' + V_2'^TV_3 \end{pmatrix}. \tag{E38}$$

If we introduce a relative phase ϕ_k, we have to multiply u_{kq} and v_{kq} by $e^{-i\phi_k}$ and u_{kq}' and v_{kq}' by $e^{i\phi_k}$.

If A and B are complex matrices, we get g and h complex. In this case, we have the same formulas, with $\tilde{g} = gg'^\dagger + hh'^T$ and $\tilde{h} = gh'^T + hg'^\dagger$, and in Γ_0 in Eq. (E18), we have G^* instead of G, and in u_k, v_k, u_{kq}, and v_{kq} we have $\tilde{g}^*$ and $\tilde{h}^*$ instead of $\tilde{g}$ and $\tilde{h}$.

APPENDIX F: INITIAL QUANTUM COHERENCE

We consider the initial state $\rho_0 = \eta|\Psi_G(0)\rangle\langle\Psi_G(0)| + (1-\eta)\rho_G(0)$, and we get

$$\langle e^{itC}\rangle = \mathrm{Tr}\{\rho_0 e^{i\ln\rho_0} e^{i\ln\rho_G(0)t}\} = \mathrm{Tr}\{\rho_0^{1+it}\}D^{it} \tag{F1}$$

since $\rho_G(0) = I/D$ is the completely mixed state. Then, since the eigenvalues of ρ_0 are $\eta + (1-\eta)/D$ and $(1-\eta)/D$ which is $D-1$-fold degenerate, by evaluating the trace we get

$$\langle e^{itC}\rangle = D^{it}((\eta + (1-\eta)/D)^{1+it} + (D-1)((1-\eta)/D)^{1+it}), \tag{F2}$$

which is Eq. (61). Concerning $\langle e^{iuw+itC}\rangle$, it can be easily derived from the joint quasiprobability distribution of the work and coherence given in Ref. [13]. By doing a symmetric choice of the parameters $q = q' = 1/2$ we get Eq. (62), from which

$$-i\partial_u G(0,t) = -i\partial_u \ln\langle e^{itC+iuw}\rangle|_{u=0} = \frac{\mathrm{Tr}\{\rho_0 e^{i\ln\rho_0 t}(H'-H)\}}{\mathrm{Tr}\{\rho_0 e^{i\ln\rho_0 t}\}}, \tag{F3}$$

and by proceeding similarly we get Eq. (63).

APPENDIX G: MEASURING THE CHARACTERISTIC FUNCTION

The characteristic function can be measured as observed in Ref. [13]. Here we note the detector can be a qubit in the initial state $\rho_D(t_i)$ with Hamiltonian $H_D = \omega|e\rangle\langle e|$. We consider the interactions with the system described by $H_I = -\delta_e|e\rangle\langle e| - \delta_g|g\rangle\langle g|$ and $H_I' = -\delta_e'|e\rangle\langle e| - \delta_g'|g\rangle\langle g|$, where $|g\rangle$ is the ground state of the qubit and $|e\rangle$ is the excited state. The total system is in the initial state $\rho_D(t_i)\otimes\rho_0$ at the initial time $t_i = -t_D$. In the time interval $(-t_D, 0)$, the time evolution is generated by the total Hamiltonian $H_{\mathrm{tot}} = H(\lambda_0) + H_D + H_I$. Then, in the time interval $(0,\tau)$, the qubit and the system do not interact and the quench is performed. Finally, in the time interval $(\tau, \tau + t_D')$, the time evolution is generated by the total Hamiltonian $H_{\mathrm{tot}}' = H(\lambda_\tau) + H_D + H_I'$. The coherence of the qubit at the final time $t_f = \tau + t_D'$ reads

$$\langle e|\rho_D(t_f)|g\rangle = \langle e|\rho_D(t_i)|g\rangle e^{-i\omega(t_f-t_i)}\mathrm{Tr}\{e^{-i(1-\delta_e)t_D H(\lambda_0)}\rho_0 \times e^{i(1-\delta_g)t_D H(\lambda_0)}U_{\tau,0}^\dagger e^{i(\delta_e'-\delta_g')t_D' H(\lambda_\tau)}U_{\tau,0}\}, \tag{G1}$$

from which we can determine $X_q(u)$.

[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[2] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[3] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. **83**, 1405 (2011).
[4] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. **65**, 239 (2016).
[5] J. Dziarmaga, Adv. Phys. **59**, 1063 (2010).
[6] J. Goold, F. Plastina, A. Gambassi, and A. Silva, in *Thermodynamics in the Quantum Regime: Fundamental Aspects and New Directions*, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, Fundamental Theories of Physics Vol. 195 (Springer, Cham, 2019), pp. 317–336.
[7] P. Talkner, E. Lutz, and P. Hänggi, Phys. Rev. E **75**, 050102(R) (2007).
[8] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyan, M. Huber, and A. Acin, Phys. Rev. Lett. **118**, 070601 (2017).
[9] M. Lostaglio, Phys. Rev. Lett. **120**, 040602 (2018).
[10] E. Wigner, Phys. Rev. **40**, 749 (1932).
[11] P. Solinas and S. Gasparinetti, Phys. Rev. E **92**, 042150 (2015).
[12] A. E. Allahverdyan, Phys. Rev. E **90**, 032137 (2014).
[13] G. Francica, Phys. Rev. E **105**, 014101 (2022).
[14] G. Francica, Phys. Rev. E **106**, 054129 (2022).
[15] B.-M. Xu, J. Zou, L.-S. Guo, and X.-M. Kong, Phys. Rev. A **97**, 052122 (2018).
[16] María Díaz, G. Guarnieri, and M. Paternostro, Entropy **22**, 1223 (2020).
[17] A. Gambassi and A. Silva, Phys. Rev. Lett. **109**, 250602 (2012).
[18] S. Sotiriadis, A. Gambassi, and A. Silva, Phys. Rev. E **87**, 052129 (2013).
[19] G. Perfetto, L. Piroli, and A. Gambassi, Phys. Rev. E **100**, 032114 (2019).
[20] A. Silva, Phys. Rev. Lett. **101**, 120603 (2008).
[21] R. Dorner, J. Goold, C. Cormick, M. Paternostro, and V. Vedral, Phys. Rev. Lett. **109**, 160601 (2012).
[22] L. Fusco, S. Pigeon, T. J. G. Apollaro, A. Xuereb, L. Mazzola, M. Campisi, A. Ferraro, M. Paternostro, and G. De Chiara, Phys. Rev. X **4**, 031029 (2014).
[23] A. Russomanno, S. Sharma, A. Dutta, and G. E. Santoro, J. Stat. Mech. (2015) P08030.
[24] N. O. Abeling and S. Kehrein, Phys. Rev. B **93**, 104302 (2016).
[25] K. Zawadzki, A. Kiely, G. T. Landi, and S. Campbell, Phys. Rev. A **107**, 012209 (2023).
[26] Z. Fei and H. T. Quan, Phys. Rev. Res. **1**, 033175 (2019).
[27] Q. Wang, D. Cao, and H. T. Quan, Phys. Rev. E **98**, 022107 (2018).
[28] Z. Fei, N. Freitas, V. Cavina, H. T. Quan, and M. Esposito, Phys. Rev. Lett. **124**, 170603 (2020).
[29] R. W. Spekkens, Phys. Rev. Lett. **101**, 020401 (2008).
[30] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997).