Letter

Thermodynamic uncertainty relations for steady-state thermodynamics

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A system can be driven out of equilibrium by both time-dependent and nonconservative forces, which gives rise to a decomposition of the dissipation into two nonnegative components, called the excess and housekeeping entropy productions. We derive thermodynamic uncertainty relations for the excess and housekeeping entropy. These can be used as tools to estimate the individual components, which are in general difficult to measure directly. We introduce a decomposition of an arbitrary current into housekeeping and excess parts, which provide lower bounds on the respective entropy production. Furthermore, we also provide a geometric interpretation of the decomposition and show that the uncertainties of the two components are not independent, but rather have to obey a joint uncertainty relation, which also yields a tighter bound on the total entropy production. We apply our results to a paradigmatic example that illustrates the physical interpretation of the components of the current and how to estimate the entropy production.

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Introduction. There have been vast developments in experimental techniques for microscopic systems [1–3], which makes it possible to measure the thermodynamic quantities of microscopic systems such as heat and work. This has enabled direct applications of an emerging field of thermodynamics, called stochastic thermodynamics [1,4,5], where thermal fluctuation plays an important role in the nonequilibrium processes. A recently discovered nonequilibrium relation, the thermodynamic uncertainty relation (TUR) [6–26], states that, in the short-time limit [21,27],

$$\sigma^{\mathrm{tot}} \geqslant \frac{(j_d)^2}{\mathcal{D}_d}, \tag{1}$$

where σ^{tot} is the entropy production rate (EPR) of the total system and $j_d, \mathcal{D}_d$ are the average and variance of a generalized current, respectively (the precise definition will be given later). Since $(j_d)^2/\mathcal{D}_d$ reflects how reliably the fluctuating current takes its average value, inequality Eq. (1) can be viewed as a tradeoff relation between the dissipation and the current precision. Full statistics are needed to measure the EPR, and they are typically not readily available in experiments. Without requiring a specific model for the dynamics, the TUR allows one to estimate the EPR from a current's measurable average and variance.

The second law of thermodynamics dictates that the total EPR is always nonnegative at the level of ensemble average: $\sigma^{\mathrm{tot}} \geqslant 0$. In the presence of nonconservative driving, on the other hand, the ordinary second law does not provide a tight bound, as dissipation does not disappear in the steady state, which is out of equilibrium due to the driving. For such genuinely nonequilibrium situations, called steady-state thermodynamics, the second law can be refined by decomposing the total EPR into two nonnegative components: housekeeping (adiabatic) EPR σ^{hk} and excess (nonadiabatic) EPR σ^{ex}, that is, $\sigma^{\mathrm{tot}} = \sigma^{\mathrm{hk}} + \sigma^{\mathrm{ex}}$ [28–32]. Here, σ^{hk} quantifies the intrinsic dissipation due to the nonconservative force, whereas σ^{ex} quantifies the dissipation due to the timedependence of the system state. While these components offer detailed information about the nonequilibrium process, it is often challenging to measure them directly in experiments.

In this Letter, we derive a generalized TUR for the housekeeping and excess EPRs of overdamped Langevin dynamics and Markov jump processes by introducing two generalized currents: the housekeeping and excess currents. Just as information about the dissipation is contained in the usual current that vanishes in equilibrium, the introduced currents are nonequilibrium quantities possessing information about the corresponding EPRs.

For the Langevin dynamics, the generalized TUR has a geometrical representation connecting the EPRs and the currents, which we refer to as the projective TUR. As a corollary of the projective TUR, two separate TURs are derived, one for the housekeeping part and the other for the excess part (as also discussed in Ref. [33]), while our TUR indicates that the two TURs are not independent of each other. In addition, we derive TURs of this form for Markov jump processes and discuss its connection to the Langevin case.

The projective TUR further gives a tighter bound on the total EPR than the conventional TUR Eq. (1). It turns out that for a particular choice of the current coefficient, the generalized currents reduce to the usual current, implying that the housekeeping and excess EPRs can be estimated only from directly measurable quantities.

Specifically, we demonstrate the application of our TUR to a paradigmatic example of a rocking ratchet in which a time-periodic, nonconservative force drives persistent particle transport on a space-periodic potential. The excess EPR is estimated with high accuracy by using the interrelation with the housekeeping counterparts.

Main result. We consider the general overdamped Langevin equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), t) + \sqrt{2}G(t)\boldsymbol{\xi}(t), \tag{2}$$

where $\boldsymbol{f}$ is the force exerting on the Brownian particle and the mobility is taken to be unity. $\boldsymbol{\xi}$ is mutually independent white Gaussian noise, and its components satisfy $\langle \xi_i \rangle = 0$ ($i = 1, \dots, N$) and $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t - t')$. $G(t)$ represents the strength of the noise. The corresponding Fokker-Planck equation is written as

$$\partial_t p(\boldsymbol{x}, t) = -\nabla^T \boldsymbol{j}(\boldsymbol{x}, t), \tag{3}$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = (\boldsymbol{f}(\boldsymbol{x}, t) - D\nabla)p(\boldsymbol{x}, t), \tag{4}$$

where $D(t) := G(t)G(t)^T$ is the diffusion matrix (^{T} denotes the transpose of a vector or matrix). We assume that G has full rank and does not depend on $\boldsymbol{x}$. In addition, when the system is coupled to multiple reservoirs, it is assumed that D is diagonal, i.e., there is no direct interaction between the reservoirs.

The dissipation is locally well characterized by the mean local velocity [1], $\boldsymbol{\nu}(\boldsymbol{x}, t) := \boldsymbol{j}(\boldsymbol{x}, t)/p(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, t) - D\nabla \ln p(\boldsymbol{x}, t)$. The mean local velocity has the meaning of the average velocity of the Brownian particle under the condition located at that point $\boldsymbol{x}$, which amounts to the dynamical contribution $\boldsymbol{f}$ and the dissipative contribution $-D\nabla \ln p$. If $\boldsymbol{f}$ contains only the conservative force and the system is coupled to a single reservoir, the system will eventually relax to the equilibrium state, which satisfies the detailed balance condition $\boldsymbol{\nu} = 0$. On the other hand, if this condition is violated and the parameters of the dynamics are fixed at time t, the system will instead relax to the steady state determined by the parameters, called the instantaneous steady state. Let $p^{\mathrm{st}}(\boldsymbol{x}, t)$ denote the probability distribution of the instantaneous steady state whose time dependence represents the parameters at time t.

We introduce the housekeeping and excess currents as $j_d^{\mathrm{hk}} := \int d\boldsymbol{x}\boldsymbol{d}^T \boldsymbol{\nu}^{\mathrm{st}} p$, $j_d^{\mathrm{ex}} := \int d\boldsymbol{x}\boldsymbol{d}^T(\boldsymbol{\nu} - \boldsymbol{\nu}^{\mathrm{st}})p$, respectively. ν^{st} denotes the steady-state value of the mean local velocity and captures the flow of the particle driven by the nonconservative force. The current coefficient $\boldsymbol{d}(\boldsymbol{x}, t)$ quantifies how much weight is given to the particle displacement $d\boldsymbol{x}$. The housekeeping current incorporates the deviation of the (instantaneous) steady state from the equilibrium state, while the excess current incorporates the deviation of the nonequilibrium state from the steady state. $j_d^{\mathrm{hk}} = j_d$, $j_d^{\mathrm{ex}} = 0$ holds for the steady state, and $j_d^{\mathrm{ex}} = j_d$, $j_d^{\mathrm{hk}} = 0$ holds for the detailed balanced system. By definition, the usual current is decomposed into these currents as $j_d^{\mathrm{hk}} + j_d^{\mathrm{ex}} = j_d = \int d\boldsymbol{x}\boldsymbol{d}^T \boldsymbol{\nu} p$. The measurability of these currents is discussed later.

Our main result, the projective TUR, is now stated as

$$\frac{\left(j_d^{\mathrm{hk}}\right)^2}{\sigma^{\mathrm{hk}}} + \frac{\left(j_d^{\mathrm{ex}}\right)^2}{\sigma^{\mathrm{ex}}} \leqslant \mathcal{D}_d, \tag{5}$$

where $\mathcal{D}_d := \int d\boldsymbol{x}\boldsymbol{d}^T D\boldsymbol{d}$ is the (time-rescaled) variance of the current. As a corollary, we can deduce the housekeeping and excess TURs:

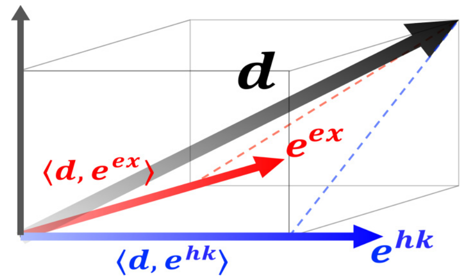

FIG. 1. Sketch of the projections of the current coefficient $\boldsymbol{d}$. Each axis represents the element of the basis of the vector field. The housekeeping (excess) current is represented by the blue (red) line segment, that is, the projected component to the housekeeping (excess) vector field. The length of the coefficient coincides with the square root of the current variance.

$$\sigma^{\mathrm{hk}} \geqslant \frac{\left(j_d^{\mathrm{hk}}\right)^2}{\mathcal{D}_d}, \quad \sigma^{\mathrm{ex}} \geqslant \frac{\left(j_d^{\mathrm{ex}}\right)^2}{\mathcal{D}_d}, \tag{6}$$

which have the same form as the conventional TUR Eq. (1) and have been discussed in Ref. [33]. Since a nonnegative term is removed from the left-hand side of Eq. (5), these TURs are looser than Eq. (5). Importantly, the projective TUR (5) indicates that the TUR bounds for the housekeeping and excess EPRs are not independent of each other, which is not evident in the looser version (6). In fact, if the uncertainty of the housekeeping part $\epsilon_{\mathrm{hk}} := \sigma^{\mathrm{hk}}\mathcal{D}_d/(j_d^{\mathrm{hk}})^2$ takes the value $s(\geqslant 1)$, the excess counterpart is bounded as $\epsilon_{\mathrm{ex}} := \sigma^{\mathrm{ex}}\mathcal{D}_d/(j_d^{\mathrm{ex}})^2 \geqslant s/(s-1) \geqslant 1$, which is tighter than the inequality (6). This tradeoff between the two uncertainties is a direct consequence of the projective TUR (5).

Furthermore, the projective TUR yields a TUR for the total dissipation as

$$\sigma^{\mathrm{tot}} \geqslant \frac{1}{\mathcal{D}_d} \max\left\{\left(j_d^{\mathrm{hk}} + j_d^{\mathrm{ex}}\right)^2, \left(j_d^{\mathrm{hk}} - j_d^{\mathrm{ex}}\right)^2\right\}. \tag{7}$$

When the product of the current components $j_d^{\mathrm{hk}} j_d^{\mathrm{ex}}$ is positive, this inequality reduces to the original TUR, Eq. (1). By contrast, when the product is negative, the TUR (7) offers a tighter lower bound than Eq. (1). Note that Eq. (7) is always tighter than the bound obtained by simply summing up Eq. (6).

For underdamped systems, the total EPR is not represented by the mean local velocity itself but by its irreversible part which is odd under the time reversal [34,35]. If the steady-state distribution $p^{\mathrm{st}}(x, v, t)$ meets the symmetry $p^{\mathrm{st}}(x, v, t) = p^{\mathrm{st}}(x, -v, t)$, the projective TUR (5) and its corollaries can be derived using the irreversible part in the same manner [36].

On the other hand, in Markov jump processes, the counterpart of Eq. (6) can be established by introducing the housekeeping and excess currents (see [36] for the derivation and illustration). However, inequalities (5) and (7) are not valid, because the mean local velocity does not satisfy an orthogonality condition mentioned below [Eq. (8)].

Derivation. We derive inequality (5) by projecting the current coefficient $\boldsymbol{d}$ into the housekeeping and excess vector fields. For vector fields $\boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{u}'(\boldsymbol{x})$, we define an inner product as $\langle \boldsymbol{u}, \boldsymbol{u}' \rangle := \int d\boldsymbol{x}\boldsymbol{u}^T D\boldsymbol{u}' p$ and the norm as $||\boldsymbol{u}|| := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}$.

At each time, the orthogonality condition

$$\langle D^{-1}\boldsymbol{\nu}^{\rm st}, D^{-1}(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st})\rangle = 0 \qquad (8)$$

holds [32], where D^{-1} denotes the inverse matrix of D. That is, the housekeeping and excess thermodynamic forces, $D^{-1}\boldsymbol{\nu}^{\rm st}$ and $D^{-1}(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st})$, are orthogonal to each other in terms of the inner product $\langle\cdot,\cdot\rangle$. Using this relation, each EPR can be written as $\sigma^{\rm tot} = ||D^{-1}\boldsymbol{\nu}||^2$, $\sigma^{\rm hk} = ||D^{-1}\boldsymbol{\nu}^{\rm st}||^2$, and $\sigma^{\rm ex} = ||D^{-1}(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st})||^2$. For a time-integrated generalized current $J_d(t) := \int^t ds\boldsymbol{d}(\boldsymbol{x}(s),s)\circ\dot{\boldsymbol{x}}$ with the Stratonovich product $\circ$, the average and variance of the (instantaneous) current are given by $j_d(t) := d\langle J_d(t)\rangle/dt = \langle \boldsymbol{d}, D^{-1}\boldsymbol{\nu}\rangle$ and $\mathcal{D}_d(t) := \lim_{\tau\to 0}{\rm Var}[J_d(t+\tau)-J_d(\tau)]/2\tau = \langle\boldsymbol{d},\boldsymbol{d}\rangle$, respectively [21].

Then, we define the housekeeping and excess vector fields, $\boldsymbol{e}^{\rm hk}$ and $\boldsymbol{e}^{\rm ex}$, as the normalized corresponding thermodynamic forces, that is, $\boldsymbol{e}^{\rm hk}(\boldsymbol{x},t) := D^{-1}\boldsymbol{\nu}^{\rm st}/||D^{-1}\boldsymbol{\nu}^{\rm st}||$, $\boldsymbol{e}^{\rm ex}(\boldsymbol{x},t) := D^{-1}(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st})/||D^{-1}(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st})||$. These vector fields satisfy $\langle\boldsymbol{e}^{\rm hk},\boldsymbol{e}^{\rm hk}\rangle = \langle\boldsymbol{e}^{\rm ex},\boldsymbol{e}^{\rm ex}\rangle = 1$ and $\langle\boldsymbol{e}^{\rm hk},\boldsymbol{e}^{\rm ex}\rangle = 0$ due to the normalization and the orthogonality condition [Eq. (8)].

If we choose $\{\boldsymbol{e}^\alpha\} = \{\boldsymbol{e}^{\rm hk},\boldsymbol{e}^{\rm ex},\cdots\}$ as an orthonormal basis of the space of the vector field, the current coefficient $\boldsymbol{d}$ can be expanded as $\boldsymbol{d} = \sum_\alpha\langle\boldsymbol{d},\boldsymbol{e}^\alpha\rangle\boldsymbol{e}^\alpha$. Then, the projective inequality

$$\langle\boldsymbol{d},\boldsymbol{d}\rangle = \sum_\alpha\langle\boldsymbol{d},\boldsymbol{e}^\alpha\rangle^2 \geqslant \langle\boldsymbol{d},\boldsymbol{e}^{\rm hk}\rangle^2 + \langle\boldsymbol{d},\boldsymbol{e}^{\rm ex}\rangle^2 \qquad (9)$$

leads to our main inequality Eq. (5) (see Fig. 1). As is evident from this derivation, additional basis components tighten the projective TUR (see Sec. I of the Supplemental Material [36]), but the physical meaning of the accompanying current component is not entirely obvious. The equality of Eq. (5) is achieved if and only if the current coefficient d contains no other orthonormal components e^α ($\alpha\neq$ hk, ex). In addition, if $\boldsymbol{d}$ has only housekeeping (excess) component, i.e., $\boldsymbol{d}\propto\boldsymbol{e}^{\rm hk}$ ($\boldsymbol{d}\propto\boldsymbol{e}^{\rm ex}$), the equality of the housekeeping (excess) TUR Eq. (5) is achieved, $\epsilon_{\rm hk} = 1$ ($\epsilon_{\rm ex} = 1$).

Meanwhile, Eq. (7) can be derived as

$$\begin{aligned}&\langle\boldsymbol{d}, D^{-1}(\boldsymbol{\nu}^{\rm st}\pm(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st}))\rangle^2\\ &\quad\leqslant \langle\boldsymbol{d},\boldsymbol{d}\rangle\langle D^{-1}(\boldsymbol{\nu}^{\rm st}\pm(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st})), D^{-1}(\boldsymbol{\nu}^{\rm st}\pm(\boldsymbol{\nu}-\boldsymbol{\nu}^{\rm st}))\rangle\\ &\quad= \langle\boldsymbol{d},\boldsymbol{d}\rangle\langle D^{-1}\boldsymbol{\nu}, D^{-1}\boldsymbol{\nu}\rangle,\end{aligned} \qquad (10)$$

where the Cauchy-Schwaltz inequality is applied in the second line. The equality of Eq. (7) holds if and only if the current coefficient d is proportional to the thermodynamic force $D^{-1}\boldsymbol{\nu}$ [6] or its dual $D^{-1}(\boldsymbol{\nu}-2\boldsymbol{\nu}^{\rm st})$, according to the sign of $j_d^{\rm hk}j_d^{\rm ex}$.

Application to rocking ratchet. As an illustration of Eq. (5), we consider a rocking ratchet [47,48], where the system is spatially periodic and a time-periodic force is driving a particle current [see Fig. 2(a)]. We estimate the excess EPR using the known expression of the instantaneous steady-state probability distribution.

The force term of the rocking ratchet consists of a conservative force $-\partial_x U(x)$ with a periodic potential $U(x) = U(x+L)$ and a time-periodic rocking (nonconservative) force $R(t) = R(t+\mathcal{T})$. After time, the system will relax to a periodic steady state eventually, which has the same spatial periodicity. Note that, even if the rocking force has no bias

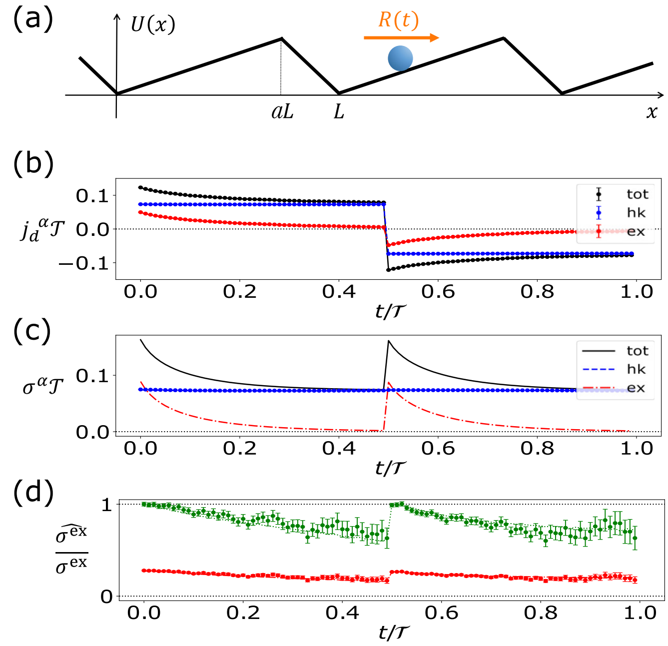

FIG. 2. (a) A sketch of the rocking ratchet. (b) The individual currents (α = tot, hk, ex) for $\boldsymbol{d}(x) = 1$. The points represent the calculation based on the Langevin equation, taking the sample averages. The lines represent the calculation based on solving the initial-value problem of the Fokker-Planck equation. They agree with each other quite well. (c) The calculation of the EPRs. Since we know the instantaneous steady state but not the periodic steady state, we can access the housekeeping EPR only through the Langevin sampling. (d) The estimation of the excess EPR based on Eq. (5) (green line) and Eq. (6) (red line) using the quantities obtained only by the Langevin sampling. The parameters used in calculation are $L = 1$, $\mathcal{T} = 1$, $D = 0.1$, $a = 0.8$, $U_0 = 0.2$, $R_0 = 0.1$. In the Langevin equation, we take the averages over 10^6 trajectories. In the Fokker-Planck equation, we use the resolution of 2^8 elements for the Fourier decomposition [49].

on average, a finite particle current can be obtained if the potential breaks the left-right symmetry [48]. Due to spacial periodicity, the infinite domain of the dynamics can be reduced to the unit domain $[0, L]$ [36,48].

We consider a saw-tooth potential such that $U(x) = U_0x/aL$ for $0 < x < aL$ $(0 < a < 1)$ and $U(x) = U_0(x-L)/(1-a)L$ for $aL < x < L$ Ref. [49], whereas the rocking force is simply chosen to be a square wave such that $R(t) = R_0$ $(t/\mathcal{T} < 0.5)$ and $R(t) = -R_0$ $(t/\mathcal{T} > 0.5)$. We numerically calculate the currents and the EPRs by sampling trajectories of the Langevin dynamics. We wait for a sufficiently long time that the system relaxes to its time-periodic state.

A crucial feature of this model is that we can easily calculate the instantaneous steady state [36,48]. Hence, we can obtain $j_d = \langle d\circ\dot{x}\rangle$, $j_d^{\rm hk} = \langle d\ \nu^{\rm st}\rangle$, $j_d^{\rm ex} = j_d - j_d^{\rm hk}$, and $\sigma^{\rm hk} = \langle\nu^{\rm st}\circ\dot{x}\rangle$ by taking the sample average, analogous to experimental measurements. Calculation convergence is sped up by converting the Stratonovich product $\circ\dot{x}$ to Itô product. It should be noted that $\boldsymbol{\nu}^{\rm st}$ is expected to be experimentally estimable in the other systems by optimizing the current coefficient [21,22].

On the other hand, σ^{tot} and σ^{ex} are inaccessible in this method because we do not have an explicit expression for the time-periodic state. Therefore, we evaluate them by applying our TUR, Eqs. (5) and (6), to this model. For this evaluation, we exactly calculate σ^{tot} and σ^{ex} by solving the Fokker-Planck equation, which is numerically much harder especially at low temperatures [49].

Plots of the currents and EPRs are shown in Figs. 2(b) and 2(c), respectively. Here, the coefficient $d(x) = 1$ is chosen so that the normal current has the meaning of a particle current. The currents change their signs around $t = 0, \mathcal{T}/2$ in accordance with the flip of the rocking force $R(t)$. Following this force switch, the housekeeping current and EPR become virtually stationary, and the excess parts display a relaxing behavior, clearly illustrating the physical meaning of the decomposition.

Figure 2(d) illustrates the estimation of the excess EPR. The green and red lines correspond to $\widehat{\sigma^{\mathrm{ex}}} = (j_d^{\mathrm{ex}})^2/(\mathcal{D}_d - (j_d^{\mathrm{hk}})^2/\sigma^{\mathrm{hk}})$ and $\widehat{\sigma^{\mathrm{ex}}} = (j_d^{\mathrm{ex}})^2/\mathcal{D}_d$, respectively. The projective TUR (5) allows for around 80% estimate using measurable quantities, as shown by the green line, making it far more accurate than the excess TUR (6). Moreover, the excess EPR is estimated almost exactly immediately after each force flip, which indicates that the current coefficient $d(x) = 1$ is expanded only by e^{hk} and e^{ex}.

Discussion. We have extended the thermodynamic uncertainty relation to the framework of steady-state thermodynamics, which is the projective TUR (5). We show that the housekeeping/excess decomposition can also be applied to currents as well as the entropy production, and that the respective components satisfy a TUR both separately and together. The newly introduced currents, j_d^{hk}, j_d^{ex}, contain information on the housekeeping and excess EPRs. Our TUR yields various corollaries that can be used according to the experimental restriction.

In the Supplemental Material [36], we clarify the measurable condition of the housekeeping and excess currents. This situation is depicted with a model of two-dimensional Brownian motion, where the total work current is divided into the housekeeping and excess parts and they coincide with the usual work currents of the nonconservative and conservative force. On top of them, using the corollary (7), the total EPR is estimated with higher accuracy than the conventional TUR Eq. (1).

We illustrated our TUR in a paradigmatic example of a rocking ratchet, where the excess EPR is estimated by using measurable quantities with a remarkably high degree of accuracy. This rocking ratchet can be experimentally realized by nanofluidic circuitry [50]. The relative magnitude of the housekeeping and excess currents can tell us whether the particle transport is primarily attributed to the nonconservative force or the timedependence of the system state. Likewise, detailed information about the currents and EPRs can help us to understand the dynamics of nonequilibrium processes and to optimize thermodynamic machines such as ratchets and heat engines.

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[1] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[2] J. M. Parrondo, J. M. Horowitz, and T. Sagawa, Nat. Phys. **11**, 131 (2015).
[3] S. Ciliberto, Phys. Rev. X **7**, 021051 (2017).
[4] U. Seifert, Eur. Phys. J. B **64**, 423 (2008).
[5] K. Sekimoto, *Stochastic Energetics*, Lecture Notes in Physics, Vol. 799 (Springer, New York, 2010).
[6] A. C. Barato and U. Seifert, Phys. Rev. Lett. **114**, 158101 (2015).
[7] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Phys. Rev. Lett. **116**, 120601 (2016).
[8] T. R. Gingrich, G. M. Rotskoff, and J. M. Horowitz, J. Phys. A: Math. Theor. **50**, 184004 (2017).
[9] K. Proesmans and C. Van den Broeck, Europhys. Lett. **119**, 20001 (2017).
[10] J. P. Garrahan, Phys. Rev. E **95**, 032134 (2017).
[11] T. R. Gingrich and J. M. Horowitz, Phys. Rev. Lett. **119**, 170601 (2017).
[12] A. Dechant, J. Phys. A: Math. Theor. **52**, 035001 (2018).
[13] K. Brandner, T. Hanazato, and K. Saito, Phys. Rev. Lett. **120**, 090601 (2018).
[14] J. Li, J. M. Horowitz, T. R. Gingrich, and N. Fakhri, Nat. Commun. **10**, 1666 (2019).
[15] Y. Hasegawa and T. Van Vu, Phys. Rev. E **99**, 062126 (2019).
[16] T. Van Vu and Y. Hasegawa, Phys. Rev. E **100**, 032130 (2019).
[17] T. Koyuk and U. Seifert, Phys. Rev. Lett. **122**, 230601 (2019).
[18] T. Koyuk and U. Seifert, Phys. Rev. Lett. **125**, 260604 (2020).
[19] K. Liu, Z. Gong, and M. Ueda, Phys. Rev. Lett. **125**, 140602 (2020).
[20] A. Dechant and S.-i. Sasa, Proc. Natl. Acad. Sci. USA **117**, 6430 (2020).
[21] S. Otsubo, S. Ito, A. Dechant, and T. Sagawa, Phys. Rev. E **101**, 062106 (2020).
[22] S. Otsubo, S. K. Manikandan, T. Sagawa, and S. Krishnamurthy, Commun. Phys. **5**, 11 (2022).
[23] N. Shiraishi, J. Stat. Phys. **185**, 19 (2021).
[24] A. Dechant and S.-i. Sasa, Phys. Rev. Res. **3**, L042012 (2021).
[25] J. S. Lee, J.-M. Park, and H. Park, Phys. Rev. E **104**, L052102 (2021).
[26] A. Dechant, arXiv:2202.10696.
[27] N. Shiraishi, K. Saito, and H. Tasaki, Phys. Rev. Lett. **117**, 190601 (2016).

[28] T. Hatano and S.-i. Sasa, Phys. Rev. Lett. **86**, 3463 (2001).
[29] T. Speck and U. Seifert, J. Phys. A: Math. Gen. **38**, L581 (2005).
[30] M. Esposito and C. Van den Broeck, Phys. Rev. Lett. **104**, 090601 (2010).
[31] M. Esposito and C. Van den Broeck, Phys. Rev. E **82**, 011143 (2010).
[32] C. Van den Broeck and M. Esposito, Phys. Rev. E **82**, 011144 (2010).
[33] A. Dechant, S.-i. Sasa, and S. Ito, Phys. Rev. E **106**, 024125 (2022).
[34] R. E. Spinney and I. J. Ford, Phys. Rev. E **85**, 051113 (2012).
[35] J. Yeo, C. Kwon, H. K. Lee, and H. Park, J. Stat. Mech.: Theory Exp. (2016) 093205.
[36] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.107.L052101 for detailed information and further generalization, which includes Refs. [37–46].
[37] M. Nakazato and S. Ito, Phys. Rev. Res. **3**, 043093 (2021).
[38] A. Dechant, S.-i. Sasa, and S. Ito, Phys. Rev. Res. **4**, L012034 (2022).
[39] N. Shiraishi, K. Funo, and K. Saito, Phys. Rev. Lett. **121**, 070601 (2018).
[40] G. Benenti, G. Casati, K. Saito, and R. Whitney, Phys. Rep. **694**, 1 (2017).
[41] M. Esposito, K. Lindenberg, and C. van den Broeck, Europhys. Lett. **85**, 60010 (2009).
[42] P. Wu, R. Huang, C. Tischer, A. Jonas, and E.-L. Florin, Phys. Rev. Lett. **103**, 108101 (2009).
[43] K. Tomita and H. Tomita, Prog. Theor. Phys. **51**, 1731 (1974).
[44] T. Nemoto and S.-i. Sasa, Phys. Rev. E **84**, 061113 (2011).
[45] T. Kamijima, S. Otsubo, Y. Ashida, and T. Sagawa, Phys. Rev. E **104**, 044115 (2021).
[46] A. M. Lyapunov, Int. J. Control **55**, 531 (1992).
[47] R. Bartussek, P. Hänggi, and J. G. Kissner, Europhys. Lett. **28**, 459 (1994).
[48] P. Reimann, Phys. Rep. **361**, 57 (2002).
[49] E.-j. Kim and R. Hollerbach, Symmetry **14**, 314 (2022).
[50] M. J. Skaug, C. Schwemmer, S. Fringes, C. D. Rawlings, and A. W. Knoll, Science **359**, 1505 (2018).