Letter

Universal turbulent relaxation of fluids and plasmas by the principle of vanishing nonlinear transfers

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A 70-year-old problem of fluid and plasma relaxation has been revisited. A principal based on vanishing nonlinear transfer is proposed to develop a unified theory of the turbulent relaxation of neutral fluids and plasmas. Unlike previous studies, the proposed principle enables us to find the relaxed states unambiguously without going through any variational principle. The general relaxed states obtained herein are found to support naturally a pressure gradient which is consistent with several numerical studies. Relaxed states are reduced to Beltrami-type aligned states where the pressure gradient is negligibly small. According to the present theory, the relaxed states are attained in order to maximize a fluid entropy $\mathcal{S}$ calculated from the principles of statistical mechanics [Carnevale *et al.*, J. Phys. A: Math. Gen. **14**, 1701 (1981)]. This method can be extended to find the relaxed states for more complex flows.

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Self-organizing dynamic relaxation in neutral fluids and plasmas is an old but hardly understood subject. Although a considerable number of works have already been accomplished to explain the relaxed states in different flows, an unambiguous definition of such a state and a universal physical principle to achieve the same has yet to be agreed upon. Despite this fact, a relaxed state is often analytically obtained by extremizing (minimizing or maximizing) a target function (TF) subject to one or more constraints of the flow. Initially the observed alignment (also called the Beltrami-Taylor state, and hereafter referred to as the BT state) between the magnetic field [1] $\boldsymbol{b}$ and the current field $\boldsymbol{j}$ $(=\nabla \times \boldsymbol{b})$ in cosmic plasmas, i.e., $\boldsymbol{j} = \lambda \boldsymbol{b}$ (where λ is a scalar function of space), was analytically obtained by maximizing the total magnetic energy for a given mean-square current density [2]. Later, a similar state was obtained in a more convincing way by minimizing the magnetic energy for a constant magnetic helicity and λ was shown to be a global constant of the system [3,4]. One popular way to find the aligned states is based on the principle of selective decay where the relaxed states are obtained by varying the rapidly decaying quantity (chosen as the TF) subject to the invariance of the slowly decaying quantities (chosen as the constraints). For three-dimensional (3D) incompressible magnetohydrodynamics (MHD), the rate of decay of the total energy E $[=\int (u^2 + b^2)/2 \, d\tau]$ is found to be greater than that of the two helical invariants, namely the cross helicity H_C $(=\int \boldsymbol{u} \cdot \boldsymbol{b} \, d\tau)$ and the magnetic helicity H_M $(=\int \boldsymbol{a} \cdot \boldsymbol{b} \, d\tau)$, where $\boldsymbol{u}$ and $\boldsymbol{a}$ represent the fluid velocity and the magnetic vector potential, respectively, and the integration is done over the space. The self-organized states can therefore be obtained by varying

$$E - \lambda_1 H_M - \lambda_2 H_C \tag{1}$$

with respect to $\boldsymbol{a}$ and $\boldsymbol{u}$, respectively, where $\lambda_{1,2}$ denote the undetermined multipliers of Lagrange. Such a variation finally gives the relaxed configurations as

$$\nabla \times \boldsymbol{b} = 2\lambda_1 \boldsymbol{b} + \lambda_2 \boldsymbol{\omega} \tag{2}$$

and

$$\boldsymbol{u} = \lambda_2 \boldsymbol{b}, \tag{3}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Solving Eqs. (2) and (3), we get $\lambda_1 = 0$ and $\lambda_2 = \pm 1$, which, in turn, exactly correspond to the states

$$\boldsymbol{u} = \pm \boldsymbol{b}, \quad \text{and hence} \quad \boldsymbol{j} \times \boldsymbol{b} = \boldsymbol{\omega} \times \boldsymbol{u}, \tag{4}$$

previously obtained in Ref. [5]. Note that, for incompressible MHD, a false BT alignment condition may seem to be obtained if one substitutes Eq. (3) in Eq. (2) without solving for $\lambda_{1,2}$. Such a possibility is obviously eliminated as λ_1 vanishes. This clearly suggests that an alignment between $\boldsymbol{b}$ and $\boldsymbol{j}$ is only possible when $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are aligned. Using a similar formalism, relaxed states were also obtained for 3D Hall MHD (HMHD), where apart from E and H_M, the total generalized helicity H_G $[=\int (\boldsymbol{a} + d_i \boldsymbol{u}) \cdot (\boldsymbol{b} + d_i \boldsymbol{\omega}) \, d\tau]$ is also an inviscid invariant (d_i being the ion inertial length). The relaxed states can be obtained by varying

$$E - \lambda_1 H_M - \lambda_2 H_G \tag{5}$$

with respect to $\boldsymbol{a}$ and $\boldsymbol{u}$, thereby leading to

$$\nabla \times \boldsymbol{b} = 2(\lambda_1 + \lambda_2)\boldsymbol{b} + 2\lambda_2 d_i \boldsymbol{\omega} \tag{6}$$

and

$$\boldsymbol{u} = 2\lambda_2 d_i \boldsymbol{b} + 2\lambda_2 d_i^2 \boldsymbol{\omega}, \tag{7}$$

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respectively. Further simplification leads to the well-known double-curl Beltrami states for HMHD plasmas given by [6,7]

$$\nabla \times (\nabla \times \boldsymbol{b}) - \alpha(\nabla \times \boldsymbol{b}) + \beta \boldsymbol{b} = \boldsymbol{0}, \tag{8}$$

with $\alpha = (1 + 4\lambda_1\lambda_2 d_i^2)/2\lambda_2 d_i^2$ and $\beta = (\lambda_1 + \lambda_2)/\lambda_2 d_i^2$. However, the variational problem in Eq. (5) is mathematically ill posed as the decay rate of H_G may exceed that of E, and a mere permutation of E and H_G cannot solve this problem [8]. To get rid of this issue, the generalized enstrophy was chosen as the desired TF and through its variation a triple-curl Beltrami state in $\boldsymbol{b}$ was obtained. In contrast to the ordinary MHD, one can immediately see that the above relaxed state permits BT alignment as a natural solution. Without using Taylor's selective decay hypothesis, an interesting theory of BT relaxation was also proposed for resistive MHD using the Cauchy-Schwartz inequality [9].

Despite previous works, it has been observed that the relaxed state of an MHD plasma is rather given by a force-balanced minimum energy state supporting a finite pressure gradient as [10–12]

$$\boldsymbol{j} \times \boldsymbol{b} = \nabla p. \tag{9}$$

Such a state can trivially be obtained as the solution of a hydrostatic equilibrium. However, to explain this in general, a complementary approach was implemented by using the principle of minimum entropy production rate (MEPR) [13]. While for a low-β plasma, the BT state was approximately recovered using MEPR, a relaxed hydrodynamic state supporting the finite pressure gradient was analytically obtained later using the same principle [14–17]. In particular, using complex Chandrasekhar-Kendall functions, such a state was also justified from a triple-curl Beltrami alignment

$$\nabla \times \nabla \times (\nabla \times \boldsymbol{b}) = \lambda \boldsymbol{b} \tag{10}$$

in the absence of the mean plasma flow [18]. Although the principle of MEPR appears to be less ambiguous and more general than the method of selective decay, it is only able to describe the evolution of the states close to the states of relaxation. In addition, previous works only considered low-β plasmas, and hence a complete description of a plasma relaxation is still lacking [15,16].

Finding the relaxed states for a 3D hydrodynamic (HD) flow is tricky. Such a system permits two inviscid invariants, namely, the total kinetic energy E_K $(= \int u^2/2\, d\tau)$ and the total kinetic helicity H_K $(= \int \boldsymbol{u} \cdot \boldsymbol{\omega}\, d\tau)$. A Beltrami-type aligned state $\boldsymbol{u} = \lambda \boldsymbol{\omega}$ can simply be obtained by varying E_K for a constant H_K. However, as discussed previously, such a variation is mathematically ill posed as H_K may have a higher decay rate than E_K. One then needs a TF that decays faster than both E_K and H_K. The total enstrophy, Ω $(= \int \omega^2\, d\tau)$ indeed serves this purpose and varying this with E_K and H_K as constraints, we obtain

$$\nabla \times \boldsymbol{\omega} = \frac{\lambda_1}{2}\boldsymbol{u} + \lambda_2 \boldsymbol{\omega}, \tag{11}$$

which evidently permits $\boldsymbol{u}$-$\boldsymbol{\omega}$ alignment as a possible solution. Interestingly, for the HD case the above variational principle and the subsequent relaxed states in Eq. (11) can also be obtained using MEPR. Note that a $\boldsymbol{u}$-$\boldsymbol{\omega}$ aligned state was also obtained by varying Ω while keeping H_K as the only constraint [19,20]. However, their work used a heuristic $\boldsymbol{b}$-$\boldsymbol{\omega}$ analogy in the variational principle originally proposed by Ref. [4] and was inconclusive about the meaning of such relaxation. Similar to the 3D MHD case, the relaxed state of a 3D HD flow is also found to relax towards a state with a finite pressure gradient as [21,22]

$$\boldsymbol{u} \times \boldsymbol{\omega} = \nabla p, \tag{12}$$

and unfortunately such a state has not been theoretically obtained to date. As summarized, various competing theories of plasma relaxation have been being proposed for the last 70 years. Most of them predicted aligned relaxed states using the variational principle pivoted on different perspectives, thus leading to a nonunique choice of the TF and the constraints. Interestingly, a considerable drop of the nonlinear terms in the evolution equations was observed numerically [14,21,23–26] and it was realized that a dynamic relaxed state should be "as free from turbulence as possible" [8]. Nevertheless, a universal theory of turbulent relaxation in fluids and plasmas has yet to be developed to the best of our knowledge.

In this Letter, we concentrate on the dynamic relaxed states of a turbulent flow and propose a universal way to characterize such states in both neutral fluids and plasmas. By definition, turbulence is an out-of-equilibrium flow regime dominated by nonlinearity where the system conceives a large number of length scales and timescales. If $M = \int \boldsymbol{p} \cdot \boldsymbol{q}\, d\tau$ is an inviscid invariant of the flow, then $\partial_t \langle \boldsymbol{p} \cdot \boldsymbol{q} \rangle = \langle \mathcal{F}_M \rangle + \langle d_M \rangle + \langle f_M \rangle$, where $\mathcal{F}_M$ is the flux term, d_M the dissipative term, f_M the forcing term, and $\langle \cdot \rangle$ denotes the statistical average which becomes identical to the space average for homogeneous turbulence. $\mathcal{F}_M$ can be written as a pure divergence term which vanishes due to the Gauss divergence theorem, leading to a statistical stationary state given by $\langle d_M \rangle = -\langle f_M \rangle$. For scale-dependent transfers, one has to consider the evolution of $\mathcal{R}_M = \langle \boldsymbol{p} \cdot \boldsymbol{q}' + \boldsymbol{p}' \cdot \boldsymbol{q} \rangle/2$, which is the symmetric two-point correlator of M. Here, the unprimed and primed quantities represent the corresponding field properties at point $\boldsymbol{x}$ and $\boldsymbol{x}'$ $(\equiv \boldsymbol{x} + \boldsymbol{r})$, respectively, and are independent of each other. For homogeneous turbulence, any correlation function of primed and unprimed variables becomes scale dependent, i.e., a function of $\boldsymbol{r}$ only. The evolution equation of the correlator $\mathcal{R}_M$ can be written as

$$\partial_t \mathcal{R}_M = \langle \mathcal{F}_{\rm tr}^M \rangle + \langle f_c^M \rangle + \langle d_c^M \rangle, \tag{13}$$

where $\langle \mathcal{F}_{\rm tr}^M \rangle$, $\langle f_c^M \rangle$, and $\langle d_c^M \rangle$ represent the scale-dependent rates of nonlinear transfer, injection, and dissipation of M, respectively. Near the injection scale, Eq. (13) reduces to $\partial_t \mathcal{R}_M = \langle f_c^M \rangle$ and a stationary state can be achieved when $\langle f_c^M \rangle = 0$ (decaying turbulence). For the so-called inertial range, where $\langle f_c^M \rangle$ is taken as a constant input (as a uniform background) and $\langle d_c^M \rangle$ can be neglected, Eq. (13) reduces to $\partial_t \mathcal{R}_M = \langle \mathcal{F}_{\rm tr}^M \rangle + \langle f_c^M \rangle$. Now for $\langle f_c^M \rangle \neq 0$, a stationary state leads to $\langle \mathcal{F}_{\rm tr}^M \rangle = -\langle f_c^M \rangle$ and we obtain the exact relations in forced stationary turbulence [27–32]. If the energy input is removed, i.e., $\langle f_c^M \rangle = 0$, then for all scales inside the inertial range, a nonstationary transient state is achieved as $\partial_t \mathcal{R}_M = \langle \mathcal{F}_{\rm tr}^M \rangle \neq 0$. Since the inertial range length scales can neither inject nor dissipate but can only nonlinearly transfer invariants to the subsequent scales, it is reasonable to expect that a trivial

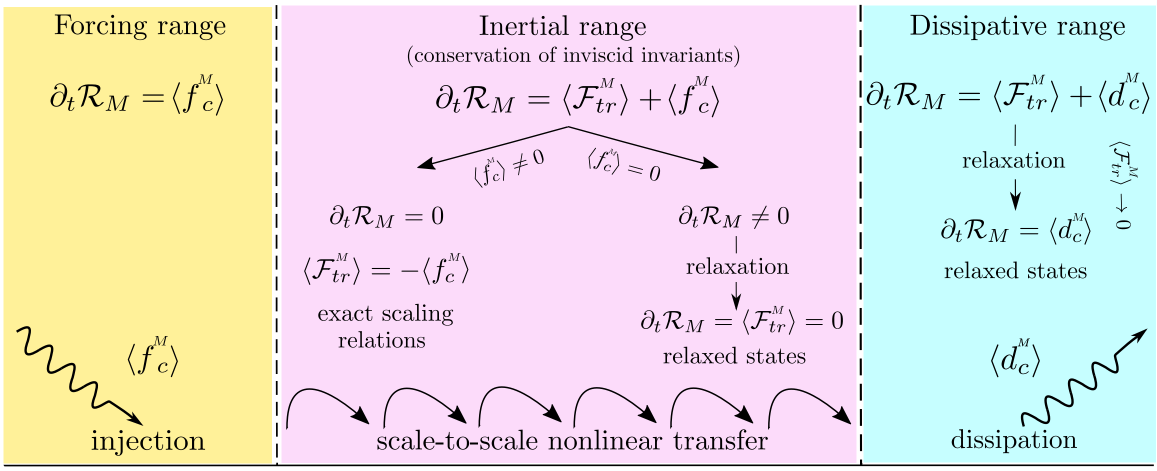

FIG. 1. Schematic diagram for principle of vanishing nonlinear transfer.

steady state is achieved at relaxation where $\langle \mathcal{F}_{\rm tr}^M \rangle$ vanishes. Such a state is called a "relaxed state" in the premise of our proposed principle which we call the principle of vanishing nonlinear transfer (PVNLT). For dissipative scales, $\langle \mathcal{F}_{\rm tr}^M \rangle$ can be considered as the input and hence one can write $\partial_t \mathcal{R}_M = \langle \mathcal{F}_{\rm tr}^M \rangle + \langle d_c^M \rangle$. A relaxed state ($\langle \mathcal{F}_{\rm tr}^M \rangle = 0$) therefore implies a nonstationary dissipative state for small scales. According to PVNLT, a turbulent system attains a nonstatic relaxed state in order to maintain the statistical stationarity for two-point correlators at all scales within the inertial range. A schematic diagram of the discussed principle is given in Fig. 1.

The above-mentioned macroscopic principle can indeed be explained using the principles of statistical mechanics. One can indeed formulate a Boltzmann H-theorem for ideal incompressible fluids and plasmas having a spectral cutoff [33,34]. Such systems always try to maximize a fluid entropy functional $\mathcal{S}$. For a turbulent system with inviscid invariant M, $\mathcal{S} \equiv \mathcal{S}[\widehat{\mathcal{R}}_M(\boldsymbol{k})]$, where $\widehat{\mathcal{R}}_M(\boldsymbol{k})$ is the Fourier transform of $\mathcal{R}_M$. Using second-order Markovian closure, it is shown that $d\mathcal{S}/dt \geqslant 0$. According to our theory, a relaxed state is obtained when $\mathcal{S}$ attains its maximum value. For a relaxed state, inside the inertial range, we therefore have $d\mathcal{S}/dt = 0 \Rightarrow \partial_t \widehat{\mathcal{R}}_M(\boldsymbol{k}) = 0 \Rightarrow \partial_t \mathcal{R}_M = 0 \Rightarrow \langle \mathcal{F}_{\rm tr}^M \rangle = 0$, thereby entailing PVNLT. As we shall see, this definition would help us in obtaining the aforementioned relaxed configurations in both neutral fluids and plasmas in a systematic manner. For 3D MHD flow, we define the symmetric two-point correlators for E, H_M, and H_C as $\mathcal{R}_E = \langle \boldsymbol{u} \cdot \boldsymbol{u}' + \boldsymbol{b} \cdot \boldsymbol{b}' \rangle/2$, $\mathcal{R}_{H_M} = \langle \boldsymbol{a} \cdot \boldsymbol{b}' + \boldsymbol{a}' \cdot \boldsymbol{b} \rangle/2$, and $\mathcal{R}_{H_C} = \langle \boldsymbol{u} \cdot \boldsymbol{b}' + \boldsymbol{u}' \cdot \boldsymbol{b} \rangle/2$, respectively, and the corresponding evolution equations are written as

$$\partial_t \mathcal{R}_E = \langle \mathcal{F}_{\rm tr}^E \rangle + \langle f_c^E \rangle + \langle d_c^E \rangle, \tag{14}$$

$$\partial_t \mathcal{R}_{H_M} = \langle \mathcal{F}_{\rm tr}^{H_M} \rangle + \langle f_c^{H_M} \rangle + \langle d_c^{H_M} \rangle, \tag{15}$$

$$\partial_t \mathcal{R}_{H_C} = \langle \mathcal{F}_{\rm tr}^{H_C} \rangle + \langle f_c^{H_C} \rangle + \langle d_c^{H_C} \rangle, \tag{16}$$

where

$$\langle \mathcal{F}_{\rm tr}^E \rangle = \tfrac{1}{2}\langle \boldsymbol{u}' \cdot (\boldsymbol{u} \times \boldsymbol{\omega} + \boldsymbol{j} \times \boldsymbol{b} - \nabla P_T) + \boldsymbol{u} \cdot (\boldsymbol{u}' \times \boldsymbol{\omega}' + \boldsymbol{j}' \times \boldsymbol{b}' - \nabla' P_T') + \boldsymbol{b}' \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) + \boldsymbol{b} \cdot \nabla' \times (\boldsymbol{u}' \times \boldsymbol{b}') \rangle, \tag{17}$$

$$\langle \mathcal{F}_{\rm tr}^{H_M} \rangle = \tfrac{1}{2}\langle \boldsymbol{a}' \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) + \boldsymbol{a} \cdot \nabla' \times (\boldsymbol{u}' \times \boldsymbol{b}') + \boldsymbol{b}' \cdot (\boldsymbol{u} \times \boldsymbol{b}) + \boldsymbol{b} \cdot (\boldsymbol{u}' \times \boldsymbol{b}') \rangle, \tag{18}$$

$$\langle \mathcal{F}_{\rm tr}^{H_C} \rangle = \tfrac{1}{2}\langle \boldsymbol{b}' \cdot (\boldsymbol{u} \times \boldsymbol{\omega} + \boldsymbol{j} \times \boldsymbol{b} - \nabla P_T) + \boldsymbol{b} \cdot (\boldsymbol{u}' \times \boldsymbol{\omega}' + \boldsymbol{j}' \times \boldsymbol{b}' - \nabla' P_T') + \boldsymbol{u}' \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) + \boldsymbol{u} \cdot \nabla' \times (\boldsymbol{u}' \times \boldsymbol{b}') \rangle, \tag{19}$$

with $P_T = p + u^2/2$ and omitting the gauge term in $\partial_t \boldsymbol{a}$. As per our definition above, for a relaxed state we have $\langle \mathcal{F}_{\rm tr}^E \rangle = \langle \mathcal{F}_{\rm tr}^{H_M} \rangle = \langle \mathcal{F}_{\rm tr}^{H_C} \rangle = 0$ at all scales within the inertial range. Furthermore, in homogeneous turbulence, for any solenoidal vector field $\boldsymbol{m}$ and scalar function θ, we have $\langle \boldsymbol{m}' \cdot (\nabla\theta) \rangle = -\langle \theta(\nabla' \cdot \boldsymbol{m}') \rangle = 0$. Using the aforementioned facts, for a nontrivial relaxed state (where none of the invariant vanishes identically), one should simultaneously have

$$\boldsymbol{u} \times \boldsymbol{\omega} + \boldsymbol{j} \times \boldsymbol{b} = \nabla(P_T + \phi_0) \tag{20}$$

and

$$\boldsymbol{u} \times \boldsymbol{b} = \nabla \psi_0, \tag{21}$$

where ϕ_0 and ψ_0 are arbitrary scalar fields. The determination of ϕ_0 and ψ_0 is system specific. An alignment between $\boldsymbol{u}$ and $\boldsymbol{b}$ is usually observed in space plasmas, e.g., solar wind [35,36], leading to the choice $\nabla \psi_0 = \boldsymbol{0}$, which gives

$$\boldsymbol{u} \times \boldsymbol{\omega} + \boldsymbol{j} \times \boldsymbol{b} = \nabla(P_T + \phi_0) \tag{22}$$

and

$$\boldsymbol{u} \times \boldsymbol{b} = \boldsymbol{0}. \tag{23}$$

From Eq. (23), we have $\boldsymbol{u} = \lambda \boldsymbol{b}$ and using the fact that λ is a global constant, we have $\boldsymbol{\omega} = \lambda \boldsymbol{j}$ and

$$\boldsymbol{j} \times \boldsymbol{b} = \frac{\nabla(P_T + \phi_0)}{1 - \lambda^2}. \tag{24}$$

Furthermore, neglecting $\nabla\phi_0$ and using the identity $\boldsymbol{j}\times\boldsymbol{b}=(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}-\nabla(b^2/2)$, the above equation can be further reduced to

$$(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}=\frac{\nabla(p+u^2/2+(1-\lambda^2)b^2/2)}{1-\lambda^2}. \quad (25)$$

For an incompressible low-β plasma ($p\ll|\boldsymbol{b}|^2/2$) with negligible flow inertia ($|\boldsymbol{u}|\ll|\boldsymbol{b}|$), i.e., $1-\lambda^2\approx 1$, the above equation reduces to $(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\approx\nabla(b^2/2)$, thus resulting in a BT aligned state where $\boldsymbol{j}\times\boldsymbol{b}\approx\boldsymbol{0}$. Note that, in previous studies where higher-order multicurl Beltrami states were obtained as a result of an extremization principle [6–8,18,37], the aligned states could not be obtained as a natural limit of a relaxed state supporting the pressure gradient. However, in the current case, the relaxed states with pressure gradient emerge naturally and reduce to an aligned state in the appropriate limit. For the case of Alfvénic alignment ($\lambda=\pm1$), one obtains $\nabla P_T=\boldsymbol{0}$, thereby leading to $\boldsymbol{u}\times\boldsymbol{\omega}+\boldsymbol{j}\times\boldsymbol{b}=\boldsymbol{0}$. In the presence of the Hall term, similar as above, one can also construct two-point correlators $\mathcal{R}_E$, $\mathcal{R}_{H_M}$, and $\mathcal{R}_{H_G}$ corresponding to the inviscid invariants. For a relaxed state,

$$\boldsymbol{u}\times\boldsymbol{\omega}+\boldsymbol{j}\times\boldsymbol{b}=\nabla(P_T+\phi_1) \quad (26)$$

and

$$(\boldsymbol{u}-d_i\boldsymbol{j})\times\boldsymbol{b}=\nabla\psi_1. \quad (27)$$

For low-β plasma and assuming $\nabla\phi_1=\nabla\psi_1=\boldsymbol{0}$, the above two equations lead to

$$\boldsymbol{u}-d_i\boldsymbol{j}=\lambda_1\boldsymbol{b} \quad (28)$$

and

$$\boldsymbol{b}+d_i\boldsymbol{\omega}=\lambda_2\boldsymbol{u}, \quad (29)$$

which are identical to the states obtained in Eq. (10) of Ref. [6]. Further calculations leads to a double-curl Beltrami state similar to Eq. (8) and to Eq. (11) of Ref. [6]. Interestingly, our proposed relaxation principle can be shown to be consistent with numerically observed states obtained under certain initial conditions. It is well known that for a strongly helical system the final state is force free, whereas for high initial alignment the system ends up in a Alfvénic state [24]. The same results can be obtained through PVNLT as we explain below.

For a strongly helical system, $\boldsymbol{a}$ and $\boldsymbol{b}$ are highly aligned, thus one can take $|\boldsymbol{a}\times\boldsymbol{b}|\sim 0$ which implies $|\boldsymbol{j}\times\boldsymbol{b}|\sim 0$. Hence, we can drop all the terms containing $\boldsymbol{j}\times\boldsymbol{b}$ from Eqs. (17)–(19). The relaxed states are obtained as

$$\boldsymbol{u}\times\boldsymbol{\omega}=\nabla(P_T+\phi_0) \quad (30)$$

and

$$\boldsymbol{u}\times\boldsymbol{b}=\nabla\psi_0. \quad (31)$$

Assuming $\nabla\phi_0=\nabla\psi_0=\boldsymbol{0}$ and combining above two equations we get $\boldsymbol{j}\times\boldsymbol{b}=\nabla P_T/\lambda^2$, where $\lambda(\neq 0)$ is a constant. In the limit of low-β plasma, the given state further reduces to a BT aligned state $\boldsymbol{j}\times\boldsymbol{b}=\boldsymbol{0}$. Similarly, for a large initial alignment of $\boldsymbol{u}$ and $\boldsymbol{b}$, $|\boldsymbol{u}\times\boldsymbol{b}|\sim 0$. The relaxed state obtained in this case is given by $\boldsymbol{u}\times\boldsymbol{\omega}+\boldsymbol{j}\times\boldsymbol{b}=\nabla(P_T+\phi_0)$. Again, assuming $\nabla\phi_0=\boldsymbol{0}$ and low-β plasma, the relaxed state reduces to $\boldsymbol{u}\times\boldsymbol{\omega}+\boldsymbol{j}\times\boldsymbol{b}=\boldsymbol{0}$ which is the most general solution. Now, high alignment between $\boldsymbol{u}$ and $\boldsymbol{b}$ implies $\boldsymbol{u}=\lambda\boldsymbol{b}$ and the general state reduces to $(1-\lambda^2)\boldsymbol{j}\times\boldsymbol{b}=\boldsymbol{0}$. If initially one chooses H_M to be low enough, then $\boldsymbol{j}\times\boldsymbol{b}$ cannot be neglected and we get $\lambda=\pm1$, leading to $\boldsymbol{u}=\pm\boldsymbol{b}$ (Alfvénic state).

For ordinary hydrodynamics, the correlators for E_K and H_K are written as $\mathcal{R}_{E_K}=\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle/2$ and $\mathcal{R}_{H_K}=\langle\boldsymbol{u}\cdot\boldsymbol{\omega}'+\boldsymbol{u}'\cdot\boldsymbol{\omega}\rangle/2$, respectively. In the relaxed state, the vanishing nonlinear transfer leads to

$$\boldsymbol{u}\times\boldsymbol{\omega}=\nabla(P_T+\phi_2). \quad (32)$$

Again one can assume $\nabla\phi_2=\boldsymbol{0}$ and the above state reduces to $\boldsymbol{u}\times\boldsymbol{\omega}=\nabla P_T$. As mentioned previously, such a relaxed state has been observed numerically in Refs. [21,22]. Unlike the MHD case, here a Beltrami alignment between $\boldsymbol{u}$ and $\boldsymbol{\omega}$ is not easily found.

In contrast to three-dimensional flows, the relaxed states in two dimensions are occasionally investigated [38]. In the case of 2D hydrodynamics, $\boldsymbol{\omega}$ is perpendicular to the plane of $\boldsymbol{u}$ and therefore H_K vanishes identically at every point. The enstrophy Ω is a new inviscid invariant along with E_K. A relaxed state was obtained through the variational principle by varying $\Omega-\lambda_1E_K$, with respect to $\boldsymbol{u}$, thereby leading to a double-curl Beltrami state in $\boldsymbol{u}$, given by

$$\nabla\times\boldsymbol{\omega}=\lambda_1\boldsymbol{u}. \quad (33)$$

It is easy to see that the above state also supports a $\boldsymbol{u}$-$\boldsymbol{\omega}$ alignment as a possible solution [39]. Similar to H_K, magnetic helicity H_M also vanishes trivially in 2D MHD. Instead, the mean-square vector potential $A\,(=\int a^2d\tau)$ is conserved along with E and H_C. The relaxed states are obtained by varying $E-\lambda_1A-\lambda_2H_C$, with respect to $\boldsymbol{a}$ and $\boldsymbol{u}$, respectively, thereby leading to

$$\nabla\times\boldsymbol{b}=2\lambda_1\boldsymbol{a}+\lambda_2\boldsymbol{\omega} \quad (34)$$

and

$$\boldsymbol{u}=\lambda_2\boldsymbol{b}. \quad (35)$$

Combining Eqs. (34) and (35) one obtains

$$\nabla\times(\nabla\times\boldsymbol{a})=\lambda\boldsymbol{a}, \quad (36)$$

where $\lambda=2\lambda_1/(1-\lambda_2^2)$. Our proposed principle can be extended, without any problem, for two-dimensional flows as well. The symmetric two-point correlator for Ω is defined as $\mathcal{R}_\Omega=\langle\boldsymbol{\omega}\cdot\boldsymbol{\omega}'\rangle$. From $\partial_t\mathcal{R}_{E_K}$ and $\partial_t\mathcal{R}_\Omega$, at a relaxed state, one obtains $\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})=\boldsymbol{0}$ (similar to the 3D case). For a two-dimensional flow, further we have

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})=-(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}=\boldsymbol{u}\times(\nabla\times\boldsymbol{\omega})=\boldsymbol{0}, \quad (37)$$

leading to a double-curl Beltrami state in $\boldsymbol{u}$. This is in agreement with the relaxed state obtained due to minimization of Ω for a given E_K [38]. The correlator for A is written as $\mathcal{R}_A=\langle\boldsymbol{a}\cdot\boldsymbol{a}'\rangle$. From $\partial_t\mathcal{R}_E$, $\partial_t\mathcal{R}_A$, and $\partial_t\mathcal{R}_{H_C}$, for the relaxed state one obtains similar conditions as given in Eqs. (22) and (23). For a 2D flow, $\boldsymbol{a}$ is perpendicular to the plane of the flow containing $\boldsymbol{u}$ and $\boldsymbol{b}$. One can therefore say $\boldsymbol{a}\times(\nabla\times\boldsymbol{u})=\boldsymbol{0}$. Since from Eq. (23), $\boldsymbol{u}=\lambda\boldsymbol{b}$, the relaxed condition is given by $\boldsymbol{a}\times(\nabla\times\boldsymbol{b})=\boldsymbol{0}$, leading to a double-curl Beltrami state in $\boldsymbol{a}$, similar to Eq. (36). Note that the study of HMHD flow strictly in two dimensions leads to an inconsistency in the evolution equation of the vector potential ($\partial_t \boldsymbol{a}$). To get out of this issue, numerical studies have been done for 2.5D HMHD [40–42] where the velocity and the magnetic fields have three components without any functional dependence on z. The relaxed states for such a system are exactly similar to those obtained for a 3D HMHD flow.

The present Letter proposes a simple and fundamental solution to the long-standing problem of dynamic relaxation of fluids and plasmas in terms of PVNLT. The proposed principle is universal for incompressible fluids and plasmas consistent with a high Reynolds number turbulence regime. The BT aligned states are obtained in the limit of insignificant pressure gradient. Unlike previous approaches, our theory does not use the principle of selective decay and explains the dynamic relaxation as a state of maximum fluid entropy functional S and naturally connects the relaxed states with and without the pressure gradient. Note that for obtaining the relaxed states using PVNLT, one needs to have prior knowledge of all the inviscid invariants. However, unlike the method of selective decay, here, we do not require to compare the decay rates of those quantities in the presence of dissipation. Furthermore, our methodology is not affected by the direction of the cascades. Unlike the principle of MEPR, our analysis does not depend on the perturbation of states close to equilibrium. Interestingly, the alternative form of exact relations in turbulence directly shows that the turbulent flux vanishes in the relaxed states obtained by PVNLT [29,30]. Finally, our principle can also be extended to study the turbulent relaxation of other nontrivial systems, e.g., compressible fluids and plasmas, ferrofluids, and binary fluid systems, where, unlike $\boldsymbol{u}$ and $\boldsymbol{b}$, the field variables are not necessarily solenoidal.

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