Letter

Material surfaces in stochastic flows: Integrals of motion and intermittency

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We consider the line, surface, and volume elements of fluid in stationary isotropic incompressible stochastic flow in d-dimensional space and investigate the long-time evolution of their statistic properties. We report the discovery of a family of $d! - 1$ stochastical integrals of motion that are universal in the sense that their explicit form does not depend on the statistics of velocity. Only one of them has been discussed previously.

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I. INTRODUCTION

The evolution of material lines and surfaces in a turbulent flow is important for the theory of turbulence and turbulent transport [1–4]. It provides an inherently geometric view on turbulent mixing [5,6]. The study of material elements is of intrinsic interest and practical value for many applications; e.g., the evolution of infinitesimal material lines is identical to that of a frozen magnetic field in highly conducting media [7,8], and the material surfaces trace the constant-property surfaces of passive scalars in the limit of negligible molecular diffusivity [9–11], flamelet propagation for slow flame speeds compared with the Kolmogorov scale [12], or salinity waves in oceans [13]. Thus, good understanding of material element evolution is also necessary for problems of turbulent dynamo and combustion.

On the other hand, stochastical integrals of motion are some of the most important instruments to investigate systems far from equilibrium, as turbulent flow is. They help to reveal the basic mechanisms of turbulence [3]. In this letter, we find universal (i.e., independent of velocity statistics) integrals of motion for material elements.

The evolution of material line and area elements has been analyzed by many authors [1,2,6,10,14–16] theoretically, experimentally, and numerically, under different assumptions on the velocity field. Mathematically, infinitesimal material elements correspond to differential forms. Physically, infinitesimality corresponds to scales much less than the Kolmogorov viscous scale length (the so-called Batchelor regime). At these scales, separation of trajectories is exponential, the same as for the evolution of material line element length. The velocity field at such scales is linear; it is determined by the velocity gradient tensor $A_{ij}(t) = \partial_j v_i$. Thus, all statistical properties of the differential forms (and hence of lengths, squares, etc.) are completely determined by the statistics of $A_{ij}(t)$ along a liquid particle trajectory.

There exists an infinite set of time-invariant configurations; their explicit form generally depends on the statistics of the velocity gradient tensor along particle trajectories. However, in Refs. [17,18], a nontrivial integral of motion was found that is universal: Its expression does not depend on details of velocity statistics. It appears that there exists a family of such integrals; all of them are averages (or hypersurface integrals) of some powers of the absolute values of the differential forms. In this letter, we find them all; for d dimensional flow, there are $d! - 1$ integrals of motion.

The developed techniques also allow us to find various nontrivial time invariants expressed by ratios of different averages.

The existence of these universal integrals of motion is essentially nontrivial. It is a consequence of statistical isotropy of the flow, in combination with very particular properties of the evolution operator of material elements.

In the next section, we formulate the problem statement and the main results. In Secs. III and IV, we proceed to accurate analysis of the d-dimensional case. In Sec. V, we derive the stochastic integrals of motion. In the last section, we discuss briefly some properties and manifestations of the discovered integrals and some other possibilities to find integrals of motion.

II. PROBLEM STATEMENT AND RESULTS

Consider a d-dimensional space filled with fluid (continuous set of particles) that flows according to the equation:

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{u}(t, \mathbf{r}(t)), \tag{1}$$

where $\mathbf{u}$ is some random stochastically isotropic and homogenous (hereafter isotropic) stationary vector field with finite ($<\infty$) correlation time and length. Its statistics is assumed to be known. For instance, $\mathbf{u}$ may obey the Navier-Stokes equation with random forcing [19], or one can use the Gaussian δ-correlated velocity field (Kraichnan model [20]). We also assume the incompressibility condition $\nabla \cdot \mathbf{u} = 0$.

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We are interested in the evolution of material lines and ($k < d$-dimensional) hypersurfaces. Thus, we introduce a coordinate grid that is orthogonal at the initial moment:

$$\mathbf{r}(0, \mathbf{x}) = \sum \mathbf{e}_i x^i, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}.$$

Here, $\{\mathbf{e}_i\}$ is a set of orthonormal orts, and $\mathbf{x} = \{x^i\}$ is the Lagrangian marker of each particle. The grid is trapped in the stochastic flow; the position of every point of the grid changes according to Eq. (1), and the coordinate lines and planes become deformed and bent.

Then time evolution of the tangent vectors:

$$\mathbf{l}_i(t, \mathbf{x}) = \frac{\partial \mathbf{r}(t, \mathbf{x})}{\partial x^i}, \tag{2}$$

is described by the equation:

$$\frac{d\mathbf{l}_i}{dt} = \frac{d}{dt}\frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial}{\partial x^i}\mathbf{u} = \mathbf{A}\mathbf{l}_i, \quad \mathbf{l}_i(0) = \mathbf{e}_i, \tag{3}$$

where $\mathbf{A}(t, \mathbf{r})$ is the velocity gradient tensor, $A_{ij} = \partial u_i / \partial r_j$; the time derivative is taken along the particle trajectory, i.e., at some constant $\mathbf{x}$.

The tangent vector field $\mathbf{l}_i(t, \mathbf{x})$ is also called the Cartan 1-form and describes the evolution of an infinitesimal length element: The length of a segment of the frozen Lagrangian coordinate line $x^{j \neq i} = \text{const.}, x^i \in L_0$ is

$$L = \int_{L_0} \|\mathbf{l}_i(t, \mathbf{x})\| dx^i \quad \text{(no summation)}.$$

The square of a segment of a frozen Lagrangian coordinate plane is

$$S = \int_{\sigma_0} \|\mathbf{l}_i \wedge \mathbf{l}_j\| dx^i dx^j \quad \text{(no summation)}.$$

Thus, the surface element is described by the Cartan 2-form $s_{ij}^{(2)}(t, \mathbf{x}) = \mathbf{l}_i \wedge \mathbf{l}_j$, where $\wedge$ denotes the outside (vector) product.

Generally, the evolution of a k-dimensional Lagrangian coordinate hypersurface is described by the Cartan k-form:

$$S^{(k)}_{i_1..i_k}(t, \mathbf{x}) = \mathbf{l}_{i_1} \wedge \cdots \wedge \mathbf{l}_{i_k}. \tag{4}$$

From isotropy, it follows that all Lagrangian coordinate planes are equivalent: Averages of all quantities do not depend on their orientation and position. Thus, one can restrict the consideration to the set:

$$\mathbf{l}_1,$$
$$\mathbf{l}_1 \wedge \mathbf{l}_2,$$
$$\ldots,$$
$$\mathbf{l}_1 \wedge \mathbf{l}_2 \wedge \cdots \wedge \mathbf{l}_d,$$

and investigate the time evolution of the norms

$$\begin{aligned} s_1 &= \|\mathbf{l}_1\|, \\ s_2 &= \|\mathbf{l}_1 \wedge \mathbf{l}_2\|, \\ &\ldots, \\ s_{d-1} &= \|\mathbf{l}_1 \wedge \mathbf{l}_2 \wedge \cdots \wedge \mathbf{l}_{d-1}\|. \end{aligned} \tag{5}$$

For incompressible flow, s_d is constant.

We require the velocity field to satisfy the following condition: The statistics of $\mathbf{A}(t, \mathbf{x}) = \mathbf{A}[t, \mathbf{r}(t, \mathbf{x})]$ taken along an $\mathbf{x}$-particle trajectory is stationary with finite correlation time and the same for all trajectories. For incompressible flow, this condition holds as a result of isotropy.

Now we can formulate the main result of this letter.

Let $i \to \pi(i)$, $i = 1..d$ be a permutation $[\forall 1 \leqslant i \leqslant d : 1 \leqslant \pi(i) \leqslant d,\ \pi(i) \neq \pi(j)$ if $i \neq j]$. Then in long-time asymptotics, there exists a stochastic integral of motion:

$$\left\langle s_1^{\pi(2)-\pi(1)-1} s_2^{\pi(3)-\pi(2)-1} \ldots s_{d-1}^{\pi(d)-\pi(d-1)-1} \right\rangle = \text{const.} \tag{6}$$

The average in Eq. (6) can be taken either over the ensemble of realizations of $\mathbf{u}(\mathbf{r})$ for some chosen point of the Lagrangian grid or over any Lagrangian coordinate hyperplane in a given realization.

There are $d! - 1$ nontrivial permutations; thus, we get the same number of universal stochastic conservation laws. We stress that the only essential restrictions we use are the ones listed below Eq. (1); details of statistics do not matter.

For cyclic permutations, we get $d - 1$ integrals of motion:

$$\left\langle s_k^{-d} \right\rangle = \text{const.}, \quad k = 1 \ldots d-1. \tag{7}$$

The first of these integrals of motion is well known: For $k = 1$, $d = 3$, and the problem with discrete time, it was found in Ref. [18].

With account of isotropy, the ensemble averages can be written as integrals of some powers of k-dimensional hypersurface density $\sigma_k = s_k^{-1}$ over the k-hypersurface moving along with the flow:

$$\int \sigma_k^{d+1} dS = \text{const.} \tag{8}$$

A. Note on intermittency

As we will show below, in incompressible flow, the length of the material line increases exponentially on average as well as the square of the material surface, etc. To the contrary, $\langle s_1^{-d} \rangle = \text{const.}$, $\langle s_2^{-d} \rangle = \text{const.}$, ... For negative degrees $0 > a > -d$, $\langle s_k^a \rangle$ decreases exponentially as a function of time, while for $a < -d$, $\langle s_k^a \rangle$ increases. This is a manifestation of intermittency of material elements: Averages over positive degrees of s_k are mainly contributed by the regions where material elements stretch most intensively, while averages over negative degrees are dominated by even more rare regions where the material elements undergo exponential contraction. The integrals of motion correspond to a balance between high speed of contraction and low probability (frequency) of fast-contracting elements. Independently of intermittency or nonintermittency of isotropic stochastic flow, even if it is Gaussian, its transport properties (e.g., material elements) are intermittent.

III. CARTAN FORMS IN d DIMENSIONS

Consider the evolution matrix $\mathbf{Q}(t)$: $\mathbf{r}(t, \mathbf{x}) = \mathbf{Q}\mathbf{r}(0, \mathbf{x})$. Then from Eq. (1), it follows that

$$\frac{d\mathbf{Q}}{dt} = \mathbf{A}(t)\mathbf{Q}(t), \quad \mathbf{Q}(0) = \mathbf{I}.$$

The formal solution to this equation can be written by means of a T-exponent:

$$\mathbf{Q}(t) = \mathcal{T}\left\{\exp\left[\int_0^t \mathbf{A}(t')dt'\right]\right\}.$$

However, the matrices $\mathbf{A}(t')$ taken at different time moments do not commutate, which causes immense difficulties and makes the explicit expression for $\mathbf{Q}(t)$ impossible. To deal with this stochastic matrix equation, it is convenient to consider the Iwasawa decomposition of the matrix $\mathbf{Q}$:

$$\mathbf{Q}(t) = \mathbf{RDZ},$$

where $\mathbf{R}$ is an orthogonal matrix, $\mathbf{D}$ is a positive diagonal, and $\mathbf{Z}$ is an upper-triangular unipotent matrix:

$$\mathbf{RR}^T = \hat{\mathbf{I}}, \quad D_{ij} = \delta_{ij}D_i, \quad D_i > 0,$$
$$Z_{i>j} = 0, \quad Z_{jj} = 1.$$

From the multiplicative Oseledets theorem [21], it follows that almost surely there exist the limits:

$$\lim_{t\to\infty}\left(\frac{1}{t}\ln D_k\right) = \lambda_k, \quad \lambda_1 \geqslant \cdots \geqslant \lambda_d.$$

Now the Cartan forms in Eq. (4) can be written as

$$S^{(k)}_{1..k}(t,\mathbf{x}) = \mathbf{Qe}_1 \wedge \cdots \wedge \mathbf{Qe}_k = \mathbf{R}D_1\mathbf{e}_1 \wedge \cdots \wedge \mathbf{R}D_k\mathbf{e}_k.$$

Here, we make use of the fact that $\mathbf{Z}$ is upper-triangular. Thus, the norms in Eq. (5) take the form:

$$s_k = D_1D_2\dots D_k. \tag{9}$$

For our purposes, we are interested only in the evolution of the D_i components.

IV. GENERALIZED LYAPUNOV EXPONENTS

To calculate the correlators of s_k, we need the averages like $\langle D_1^{m_1}\dots D_d^{m_d}\rangle$. One can show [22] that each D_i satisfies the equation:

$$\frac{dD_i}{dt} = \xi_i D_i, \tag{10}$$

where $\boldsymbol{\xi}(t) = \{\xi_1,\dots,\xi_d\}$ is a set of stationary random processes that depend on $\mathbf{A}(t)$ in a rather complicated way [1]. For each component D_i, the solution of Eq. (10) is

$$D_i(t) = \exp\left[\int_0^t \xi_i(t')dt'\right] = \exp[t\bar{\xi}_i(t)],$$

where $\bar{\boldsymbol{\xi}}(t) = \frac{1}{t}\int_0^t \boldsymbol{\xi}(t')dt'$ is the time average of $\boldsymbol{\xi}$.

We assume that $\boldsymbol{\xi}(t)$ satisfies the large deviations principle [23], i.e., that the joint probability density of all $\bar{\xi}_i(t)$ at large $t\to\infty$ satisfies the relation:

$$\mathcal{P}_{\bar{\xi}}(a_1,\dots,a_d) \equiv \left\langle \prod_i \delta[\bar{\xi}_i(t) - a_i]\right\rangle \sim \exp[-tJ(a_1,\dots,a_d)].$$

[1]More precisely, $\boldsymbol{\xi}$ is the diagonal part of the statistically stationary random matrix $\mathbf{R}^{-1}\mathbf{AR}$.

Here, the angle brackets denote the ensemble average, and the sign $\sim$ means that

$$\lim_{t\to\infty}\frac{1}{t}\ln\mathcal{P}_{\bar{\xi}}(a_1,\dots,a_d) = -J(a_1,\dots,a_d).$$

The function J is called the Cramer function or effective action. It is concave and has the minimum $J_{\min} = 0$ at $\mathbf{a}_{\min} = \langle\boldsymbol{\xi}\rangle$. Then

$$\langle D_1^{m_1}\dots D_d^{m_d}\rangle = \left\langle\exp\left[\sum m_i\int_0^t \xi_i(t')dt'\right]\right\rangle \sim \int\exp\left\{t\left[\sum m_ia_i - J(\mathbf{a})\right]\right\}d\mathbf{a},$$

and there exists the limit:

$$\lim_{t\to\infty}\frac{1}{t}\ln\langle D_1^{m_1}\dots D_d^{m_d}\rangle = w_\xi(m_1,\dots,m_d), \quad m_k\in\mathbb{R}, \tag{11}$$

where w_ξ is the Legendre transform of J. The function w_ξ is called the generalized Lyapunov exponent (GLE) [22,24].

Generally, the statistics of $\boldsymbol{\xi}$ is not determined by the statistics of $\mathbf{A}$ at the same moment in time: it also depends on the prehistory. However, in Refs. [22,25], it was shown that, in the case of statistically isotropic $\mathbf{A}(t)$, there exists a simple relation between the statistics of $\boldsymbol{\xi}$ and $\mathbf{A}$, namely,

$$w_\xi(m_1,\dots,m_d) = w_A(m_1+\eta_1,\dots,m_d+\eta_d) - w_A(\eta_1,\dots,\eta_d), \tag{12}$$

where w_A is the cumulant-generating function corresponding to the diagonal elements of the matrix $\mathbf{A}$:

$$w_A(m_1,\dots m_d) = \lim_{t\to\infty}\frac{1}{t} \times \ln\left\langle\exp\left[\int(m_1A_{11}+\cdots+m_dA_{dd})dt\right]\right\rangle,$$

and η_k is a set of constants defined by

$$\eta_k = \frac{d+1}{2} - k. \tag{13}$$

This relation allows us to calculate the statistical characteristics of $\mathbf{D}$ for given statistics of $\mathbf{A}$, particularly to find all statistical moments.

V. STOCHASTIC INTEGRALS OF MOTION

From Eq. (11), we see that the average of some combination of powers of D_i remains constant if the corresponding w_ξ is equal to zero. Thus, we are interested in those sets $(m_1,\dots,m_d)$ that provide $w_\xi(m_1,\dots,m_d) = 0$.

Statistical isotropy and incompressibility of the flow require $\partial w_A/\partial A_{ii}(0) = \frac{1}{d}\langle\mathrm{tr}\mathbf{A}\rangle = 0$. Since $w_A(0) = 0$ and w_A is concave, this means that, for all nonzero arguments, w_A is positive.

The function $w_\xi(m_1,\dots,m_d)$ can be obtained from w_A in accordance with Eq. (12); thus, for all those points $(m_1^*,\dots,m_d^*)$ for which $w_A(m_1^*+\eta_1,\dots,m_d^*+\eta_d) = w_A(\eta_1,\dots,\eta_d)\equiv w_A^*$, we have (Fig. 1)

$$w_\xi(m_1^*,\dots,m_d^*) = 0. \tag{14}$$

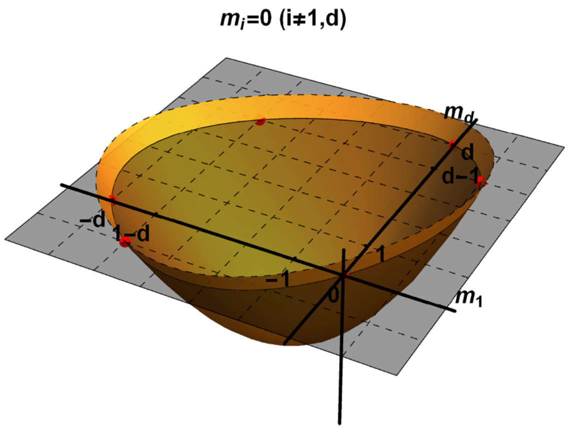

FIG. 1. Illustration of the surface $w_\xi(m_1, 0, \dots, 0, m_d)$ for $d \geqslant 3$ (yellow surface). The transection with the grey horizontal plane singles out the line $w_\xi = 0$. For all the points (m_1, m_d) in this line, $\langle D_1^{m_1} D_d^{m_d}\rangle$ conserve for given statistics of $\mathbf{A}$. The red points belong to the line $w_\xi = 0$ independently of the statistics; the corresponding sets $\{m_1^*, m_d^*\}$ produce universal integrals of motion.

Each of these points corresponds to some stochastic integral of motion:

$$\left\langle D_1^{m_1^*} \dots D_d^{m_d^*}\right\rangle = \text{const.}$$

In terms of the Cartan forms in Eq. (5), considering Eq. (9), this can be written as

$$\left\langle s_1^{m_1^*-m_2^*} \dots s_{d-1}^{m_{d-1}^*-m_d^*} s_d^{m_d^*}\right\rangle = \text{const.}$$

Thus, for any stationary Lagrangian statistics of velocity gradients, there exists a $d!-1$-parametric family of stochastic integrals of motion composed of s_i.

The solution of Eq. (14) depends on the statistics; the powers m_i^* for any flow are determined by the specific form of the function w_A. However, it appears that all possible surfaces $w_\xi(m_1^*, \dots, m_d^*) = 0$ corresponding to different statistics have several points where they all intersect. Indeed, since the process $\mathbf{A}(t)$ is isotropic, all its diagonal elements have identical statistical properties, and w_A is symmetric with respect to permutation of its arguments: If $\pi : 1 \to \pi(1), \dots, d \to \pi(d)$ is a permutation, then

$$w_A(m_1, \dots, m_d) = w_A(m_{\pi(1)}, \dots, m_{\pi(d)}). \tag{15}$$

This is valid for the set $\{\eta_1, \dots, \eta_d\}$. Hence, independently of the details of statistics, the set $m_i^* = \eta_{\pi(i)} - \eta_i$ is the solution of Eq. (14); making use of Eq. (13), we find $m_i^* = i - \pi(i)$, and

$$\left\langle s_1^{\pi(2)-\pi(1)-1} s_2^{\pi(3)-\pi(2)-1} \dots s_{d-1}^{\pi(d)-\pi(d-1)-1} s_d^{d-\pi(d)}\right\rangle = \text{const.} \tag{16}$$

Considering incompressibility, $s_d = \text{const.}$, and we arrive at Eq. (6).

There are $d!$ permutations of the set in Eq. (13); identical transform $\{\eta_i\} \to \{\eta_i\}$ corresponds to the trivial integral of motion, $\langle 1\rangle = 1$; for the rest of the permutations, we get the same number of integrals.

VI. DISCUSSION

In this letter, we find $d! - 1$ universal stochastic integrals of motion expressed in terms of Cartano forms or infinitesimal material lines and hypersurfaces in a stochastic flow. The universality means that the explicit forms of the integrals do not depend on the details of velocity statistics. The only requirement is that all correlators of the velocity gradient tensor along the trajectory of any particle are isotropic and stationary, with finite correlation time, and independent of the choice of a particle. This requirement holds for incompressible isotropic and homogenous flows with finite time and length correlation.

From the isotropy of the flow, it follows that one can take an average along an arbitrary generic line or (hyper)surface instead of the ensemble average (the characteristic scale of the surface must be much more than the correlation length). Thus, in Eq. (6), it is possible to replace the average over an ensemble of liquid particles with an average taken over some material line or surface or even the whole space: In terms of the Lagrangian (frozen) coordinates $\mathbf{x}$,

$$\int s_1^{\pi(2)-\pi(1)-1} s_2^{\pi(3)-\pi(2)-1} \dots s_{d-1}^{\pi(d)-\pi(d-1)-1} d\mathbf{x} = \text{const.} \tag{17}$$

(The integral can be taken over any subset of the coordinates $\{x_1, \dots, x_d\}$.) Stochastic invariance implies that the integral does not change exponentially as a function of time; it may still have a power-law time dependence, which is a result of pre-exponential multipliers in $\mathbf{D}$.

For the particular case of cyclic permutations, we get

$$\left\langle s_k^{-d}\right\rangle = \text{const.}$$

For $k = 1$, this expression corresponds to the integral of motion found in Refs. [17,18] for random processes with discrete time. We note that, in these works, the average was found to conserve exactly, while in our investigation, we only prove the conservation to logarithmic accuracy because of the pre-exponential multiplier. However, the result in Ref. [18] gives reason to suppose that, at least for time much longer than the conservation time, all the integrals of motion found in this letter conserve exactly (or with accuracy higher than logarithmic). The proof of this is the subject of further research.

According to Eq. (17), the integrals of motion can also be written as integrals over a material element. Let a k-dimensional material hypersurface $\{x_1, \dots, x_k\}$ be marked by a passive scalar; let its initial hypersurface density be uniform, $\sigma_k(0, \mathbf{x}) = 1$. Then as time goes, the hypersurface density changes in accordance with the change of the hypersquare: $\sigma_k(t, \mathbf{x}) = 1/s_k(t, \mathbf{x})$. We choose a fragment of the hypersurface; let its initial hypersquare be S_0. Then making use of Eqs. (2) and (5), we pass from the integration over the Lagrangian coordinates to the integration over the invariant measure (i.e., over the square of the hypersurface): $\int dx_1 \dots dx_k = \int dS s_k^{-1}$. Now one can write the cyclic-permutation integrals of motion in the form:

$$\left\langle s_k^{-d}\right\rangle = \lim_{S_0\to\infty} \frac{1}{S_0} \int dx_1 \dots dx_k \sigma_k^d = \lim_{S_0\to\infty} \frac{1}{S_0} \int dS \sigma_k^{d+1}.$$

Thus, we arrive at Eq. (8).

This interpretation of the stochastic integrals allows a visualization that illustrates their relation to the intermittency. Let a k-dimensional hypersurface be initially marked uniformly by some scalar (paint). As time goes on, the hypersurface undergoes deformations, stretches, and bends. The average density of the paint decreases exponentially, inversely to the increase of the square. However, there are always some rare and small regions where the density increases. The balance between the small number of these regions and the very high density of paint in them results in the existence of the time invariants in Eq. (8). The higher-order statistical moments grow, while the lower-order moments decrease as a function of time.

Returning back to Eq. (6), we now present a recurrent procedure to obtain the complete set of these time invariants in $d+1$-dimensional space from the set of the time invariants in d dimensions.

First, we note that, being written in the form in Eq. (16), which includes the multiplier s_d, the d-dimensional integral of motion is at the same time the integral of motion for the $(d+1)$-dimensional case; it corresponds to the permutations $\{\pi(1), \dots, \pi(d), d+1\}$. We now write it in the form:

$$\left\langle s_1^{\alpha_1} \dots s_d^{\alpha_d} s_{d+1}^{\alpha_{d+1}} \right\rangle,$$

where

$$\begin{aligned} \alpha_k &= \pi(k+1) - \pi(k) - 1, \quad 1 \leqslant k \leqslant d, \\ \alpha_{d+1} &= 0, \\ \pi(d+1) &= d+1. \end{aligned}$$

Second, we make a cyclic permutation of the set $\{\pi(1), \dots, \pi(d+1)\}$, shifting it by i:

$$\tilde{\pi}(k) = \begin{cases} \pi(k-i+d+1), & k < i, \\ \pi(k-i), & k > i. \end{cases}$$

In accordance with Eq. (6) for $d+1$ dimensions, this new $\tilde{\pi}(k)$ corresponds to $\langle s_1^{\beta_1} \dots s_d^{\beta_d} \rangle$, where

$$\beta_k = \tilde{\pi}(k+1) - \tilde{\pi}(k) - 1 = \begin{cases} \alpha_{k-i+d+1}, & k < i, \\ \alpha_{k-i}, & i < k \leqslant d, \end{cases}$$

$$\beta_i = \pi(1) - (d+1) - 1 = -\sum_1^d \alpha_i - (d+1).$$

This set of β_k determines the new $(d+1)$ invariant. Since any permutation of $\{1, \dots, d+1\}$ is a cyclic permutation of some $\{\pi(1), \dots, \pi(d), d+1\}$, this procedure allows us to find all the $(d+1)$ invariants from known d invariants.

If one wants to continue this recursion, one has to restore the form in Eq. (16), i.e., to find the power β_{d+1} of s_{d+1}. To this purpose, it is helpful to use the [evident from Eq. (16)] property $\beta_1 + 2\beta_2 + \dots + (d+1)\beta_{d+1} = 0$ for $(d+1)$-dimensional time invariants.

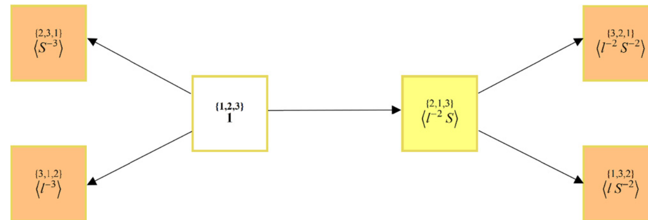

FIG. 2. Integrals of motion for incompressible flow in three dimensions and the illustration of the recurrent procedure. Each square would generate 3 more squares in the case of four dimensions.

For example, in two dimensions, we have one nontrivial permutation: (2,1), with corresponding integral of motion $\langle s_1^{-2} s_2^1 \rangle$; in three dimensions, we rewrite it as $\langle s_1^{-2} s_2^1 s_3^0 \rangle$, with corresponding permutation 213. The further cyclic permutations lead to $(321, i=1)$ $\langle s_1^{-2} s_2^{-2} \rangle = \langle s_1^{-2} s_2^{-2} s_3^2 \rangle$ and $(132, i=2)$ $\langle s_1^1 s_2^{-2} \rangle = \langle s_1^1 s_2^{-2} s_3^1 \rangle$. The other integrals of motion come from the cyclic permutations of the ordered set $(1, 2, 3)$. Figure 2 presents this procedure with a list of corresponding stochastic integrals of motion for the three-dimensional case.

The developed mechanism also allows us to construct time invariants composed of two and more averages. The permutation properties of w_A are not restricted to the sets $\{m_i^*\}$. From Eqs. (12) and (15), it follows that, for any set $\{m_i\}$, the set $\{m_i' = m_{\pi(i)} + \eta_{\pi(i)} - \eta_i\}$ provides the same w_ξ, $w_\xi(m_1, \dots, m_d) = w_\xi(m_1', \dots, m_d')$, and the corresponding moments grow with the same rate. Hence, the ratio $\langle D_1^{m_1'} \dots D_d^{m_d'} \rangle / \langle D_1^{m_1} \dots D_d^{m_d} \rangle$ neither increases nor decreases exponentially, and we get one more set of stochastic time invariants:

$$\frac{\left\langle s_1^{m_1'-m_2'} \dots s_{d-1}^{m_{d-1}'-m_d'} \right\rangle}{\left\langle s_1^{m_1-m_2} \dots s_{d-1}^{m_{d-1}-m_d} \right\rangle} = \text{const.},$$

$$m_i' = m_{\pi(i)} + i - \pi(i).$$

The intermittent nature of a stochastic flow provides a wide range of stochastic integrals of motion; some of them are universal and independent of statistic properties of the flow. The formalism of GLEs appears to be a useful tool to find them. We believe that they can be observed in direct numerical simulations.

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