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Editors' Suggestion

Small-scale dynamo with finite correlation times

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Fluctuation dynamos occur in most turbulent plasmas in astrophysics and are the prime candidates for amplifying and maintaining cosmic magnetic fields. A few analytical models exist to describe their behavior, but they are based on simplifying assumptions. For instance, the well-known Kazantsev model assumes an incompressible flow that is δ-correlated in time. However, these assumptions can break down in the interstellar medium as it is highly compressible and the velocity field has a finite correlation time. Using the renewing flow method developed by Bhat and Subramanian (2014), we aim to extend Kazantsev's results to a more general class of turbulent flows. The cumulative effect of both compressibility and finite correlation time over the Kazantsev spectrum is studied analytically. We derive an equation for the longitudinal two-point magnetic correlation function in real space to first order in the correlation time τ and for an arbitrary degree of compressibility (DOC). This generalized Kazantsev equation encapsulates the original Kazantsev equation. In the limit of small Strouhal numbers $\mathrm{St} \propto \tau$ we use the Wentzel-Kramers-Brillouin approximation to derive the growth rate and scaling of the magnetic power spectrum. We find the result that the Kazantsev spectrum is preserved, i.e., $M_k(k) \sim k^{3/2}$. The growth rate is also negligibly affected by the finite correlation time; however, it is reduced by the finite magnetic diffusivity and the DOC together.

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I. INTRODUCTION

The vast majority of the baryonic matter is in a plasma state, and therefore a complete description of the universe needs to include a proper treatment of the electromagnetic force [1]. From observations it is known that the universe is highly magnetized. Indeed, magnetic fields are observed in almost all astrophysical bodies as, for instance, in asteroids [2], planets [3], stars [4,5], galaxies [6,7], or the intergalactic medium [8–10]. Due to the broad range of objects, the typical strength and correlation length of these magnetic fields are distributed over several orders of magnitude. As an example in Milky Way-like galaxies, the observed magnetic fields are of a few tens μG in strength and correlated on kiloparsec scales [11].

The most popular mechanism to explain the observed magnetic fields is the dynamo process which converts the kinetic energy of the flow to magnetic energy. In the absence of large-scale motions, small-scale or fluctuation dynamos[1] amplify the initial magnetic field exponentially [12], a process which is most efficient on the smallest scales of the system. In the kinematic stage of the dynamo, the magnetic field lines are frozen into the plasma. Due to the turbulent motion of the flow, the action of the small-scale dynamo is to randomly twist, stretch, and fold these lines which makes the magnetic field strength grow. However, activating the dynamo requires an already existing seed field. Although unclear, it is generally assumed that these seed fields were generated in the early universe [13] or through astrophysical processes such as the Biermann battery [14]. Schober *et al.* [15] also highlighted that the small-scale dynamo can only amplify the magnetic field for magnetic Reynolds numbers $R_\mathrm{M} \sim UL/\eta$ (U and L are, respectively, the typical velocity and length scale of the system) larger than a few hundred. In the nonlinear regime after saturation on the smallest scales, the peak of the magnetic energy shifts from smaller to larger scales and the magnetic energy increases following a power law [16]. The exact behavior of the dynamo depends on the magnetic Prandtl number $\mathrm{Pr}_\mathrm{M} = \nu/\eta$ and on the type of turbulence [16–18].

The small-scale dynamo is a key process in astrophysics. Indeed, the strength of the magnetic fields predicted from the early universe is not consistent with the observed typical value of a few μG in the inter-cluster medium [[19], and references therein] or in high redshift galaxies [20]. Small-scale dynamos could then provide an explanation for the fast amplification of magnetic fields in the radiation-dominated phase of the early universe [21], in young galaxies [22], and galaxy clusters [12]

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[1]The terms "small-scale dynamo" and "fluctuation dynamo" are used interchangeably in the literature.

as they can act on timescales much shorter than the age of the system. In the context of supernova-driven turbulence, it is expected to give rise to the far-infrared-radio correlation in galaxies [23] and potentially even dwarf galaxies [24]. Small-scale dynamos might also be involved in the formation of the first stars [25–27] and black holes [28–30]; and thus could affect the epoch of reionization.

Different approaches have been published along the years to model the complex behavior of dynamos. For instance, Adzhemyan *et al.* [31] discuss the turbulent dynamo in the framework of a quantum-field formulation of stochastic magnetohydrodynamics, where it is described as a mechanism of spontaneous symmetry breaking. An early theoretical description of the small-scale dynamo is given by Kazantsev [32]. His equation describes the time evolution of the two-point magnetic correlation function under the assumption of a Gaussian incompressible flow that is δ-correlated in time. Its derivation indicates that the magnetic power spectrum scales as $M_k(k) \sim k^{3/2}$ for $q \ll k \ll k_\eta$, where q is the forcing scale and k_η is the wave number above which diffusion of the magnetic field dominates. Following Kazantsev's work many authors have tried to extend this model [see, e.g., Refs. [33–36]]. Although some astrophysical objects host plasma that is well described by an incompressible flow (as, for instance, neutron stars [37]); Kazantsev's assumptions strongly simplify the behavior of most astrophysical bodies. Indeed the majority of the plasma in the universe is highly compressible as indicated by observations of compressive interstellar turbulence [38]. Moreover, in realistic flows the correlation time τ should be of the order of the smallest eddy turnover time. Thus the assumptions involved in the Kazantsev [32] derivation do not allow for an accurate description of all types of fluctuation dynamos.

In this work we aim to study the small-scale dynamo for the general case of a flow that is compressible and with finite correlations in time. Zeldovich *et al.* [39] pointed out that the so-called renovating flows represent a solvable analytical model to study the impact of the correlation time on small-scale dynamos. In this context, Bhat and Subramanian [36] developed a method to study the dynamo of incompressible flows. They found that the Kazantsev spectrum was not strongly affected by a finite correlation time, i.e., $M_k(k) \sim k^{3/2}$. However, the growth rate of the dynamo is reduced. On the other hand, Schekochihin *et al.* [40] found that a compressible flow that is δ-correlated in time also preserves the Kazantsev spectrum where compressibility also reduces the growth rate of the dynamo. As far as we know, although there are clues that the Kazantsev spectrum should be preserved in the interstellar medium (compressible and correlated in time flow), there is no previous theoretical study that demonstrates formally that the combined actions have no effect on the $M_k(k) \sim k^{3/2}$ spectrum. Rogachevskii and Kleeorin [41] used a path integral method to solve the induction equation and show that a dynamo can be activated for compressible flows that are correlated in time. Their results admit solutions consistent with the Kazantsev spectrum.

The present work assumes a simplified random flow that is compressible and correlated in time. We present here a generalization of the previous work by Bhat and Subramanian [36] by including the effect of compressibility. The paper is organized as follows: in Sec. II we briefly review the original Kazantsev theory. In Sec. III we present the renewing flow method used by Bhat and Subramanian [36]. In Sec. IV we give the derivation of the original Kazantsev equation (incompressible and δ-correlated in time flow) with the use of the renovating flow method. In Sec. V we present our generalization of the Kazantsev equation for a compressible flow that is correlated in time and study the Wentzel-Kramers-Brillouin (WKB) solutions in Sec. VI. Finally, we insert our results in the current context and draw our conclusions in Sec. VII.

II. KAZANTSEV THEORY

Dynamos in the context of an isotropic flow have been hypothesized since the 1950s [see, e.g., Refs. [42,43]]; however, the first one to give a complete theoretical framework was Kazantsev [32]. In his work an isotropic and homogeneous flow that is δ-correlated in time was proposed. In this section we review the basics of the derivation of the Kazantsev equation and its results, in particular we follow Subramanian [34] for the formalism.

We rewrite the velocity field as

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \delta \boldsymbol{u}, \tag{1}$$

where $\langle \boldsymbol{u} \rangle$ is the mean and $\delta \boldsymbol{u}$ the fluctuations. If we assume the fluctuations to be isotropic, homogeneous, Gaussian random with zero mean and δ-correlated in time, then we can set the correlation function to be

$$T_{ij}(r)\delta(t_1 - t_2) = \langle \delta u_i(\boldsymbol{x}, t_1)\delta u_j(\boldsymbol{y}, t_2)\rangle, \tag{2}$$

with $r = |\boldsymbol{x} - \boldsymbol{y}|$. For a helical field, any two-point correlation function can be expressed through longitudinal and transverse components [44] as

$$T_{ij}(r) = \hat{r}_{ij} T_{\rm L}(r) + \hat{P}_{ij} T_{\rm N}(r), \tag{3}$$

with $\hat{r}_{ij} = r_i r_j / r^2$ and $\hat{P}_{ij} = \delta_{ij} - \hat{r}_{ij}$. For a divergence-free vector field (in the case of velocity: an incompressible flow $\nabla \cdot \boldsymbol{u} = 0$) we can even show that the two components are related by

$$T_{\rm N} = T_{\rm L} + \frac{r}{2}\frac{d}{dr} T_{\rm L}. \tag{4}$$

A similar decomposition can be performed for the magnetic field. Since $\boldsymbol{B}$ is divergence-free, the magnetic correlation function can be expressed as

$$\begin{aligned} M_{ij}(r) &= \langle \delta B_i(\boldsymbol{x})\delta B_j(\boldsymbol{y})\rangle, \\ &= (\hat{r}_{ij} + \hat{P}_{ij}) M_{\rm L} + \hat{P}_{ij}\frac{r}{2}\frac{d}{dr} M_{\rm L}. \end{aligned} \tag{5}$$

The time derivative of the two-point magnetic correlation function is thus given by

$$\frac{\partial M_{ij}}{\partial t} = \left\langle \frac{\partial B_i}{\partial t} B_j \right\rangle + \left\langle \frac{\partial B_j}{\partial t} B_i \right\rangle - \frac{\partial \langle B_i B_j \rangle}{\partial t}. \tag{6}$$

Inserting this expression in the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}), \tag{7}$$

and using the averaged induction equation

$$\frac{\partial\langle\boldsymbol{B}\rangle}{\partial t} = \nabla\times(\langle\boldsymbol{u}\rangle\times\langle\boldsymbol{B}\rangle - [\eta + T_{\rm L}(0)]\nabla\langle\boldsymbol{B}\rangle), \tag{8}$$

Subramanian [34] found an equation for the time evolution of the longitudinal two-point magnetic correlation function

$$\frac{\partial M_{\rm L}}{\partial t} = 2\kappa_{\rm diff}M_{\rm L}'' + 2\left(\frac{4\kappa_{\rm diff}}{r} + \kappa_{\rm diff}'\right)M_{\rm L}' + \frac{4}{r^2}\left(T_{\rm N} - T_{\rm L} - rT_{\rm N}' - rT_{\rm L}'\right)M_{\rm L}. \tag{9}$$

In this expression $\kappa_{\rm diff} \equiv \eta + T_{\rm L}(0) - T_{\rm L}(r)$ and a prime denotes a derivative with respect to r. If we further suppose that the time and spatial dependencies are independent, then we can use the ansatz

$$M_{\rm L}(r,t) = \frac{1}{r^2\sqrt{\kappa_{\rm diff}}}\psi(r)e^{2\Gamma t}. \tag{10}$$

This form is convenient as it highlights a formal similarity to quantum mechanics. We insert the ansatz into Eq. (9) and find

$$-\kappa_{\rm diff}\frac{d^2\psi}{dr^2} + U(r)\psi = -\Gamma\psi. \tag{11}$$

This equation has the form of a Schrödinger equation and is often referred to as the Kazantsev equation in the literature; however, in this work we will refer to Eq. (9) as the Kazantsev equation instead. The function $U(r)$ is equivalent to a potential and is given by

$$U(r) \equiv \frac{\kappa_{\rm diff}''}{2} - \frac{(\kappa_{\rm diff})'}{4\kappa_{\rm diff}} + \frac{2\kappa_{\rm diff}}{r^2} + \frac{2T_{\rm N}'}{r} + \frac{2(T_{\rm L} - T_{\rm N})}{r^2}. \tag{12}$$

Note that in the derivation of this equation we did not assume at any point that the flow is incompressible.

Schekochihin *et al.* [35] studied the Kazantsev equation in Fourier space in the sub-diffusion limit such that $k_f \ll k \ll k_\eta$, with k_f being the forcing scale for a single scaled flow and k_η the Fourier conjugate of the magnetic diffusion length scale (the scale at which the magnetic diffusion is important). If incompressibility is assumed, then the Kazantsev equation can be rewritten as [see, e.g., Refs. [33,45]]

$$\frac{\partial M_k}{\partial t} = \frac{\gamma}{5}\left(k^2\frac{\partial^2 M_k}{\partial k^2} - 2k\frac{\partial M_k}{\partial k} + 6M_k\right) - 2\eta k^2 M_k, \tag{13}$$

where γ is a constant that characterizes the flow and $M_k(k,t)$ represents the magnetic power spectrum. Compared to $M_{\rm L}(r)$ it characterizes the magnetic correlation function in Fourier space, formally we have the following relation:

$$\begin{aligned}\langle\hat{B}_i(\boldsymbol{k},t)\hat{B}_j^*(\boldsymbol{k}',t')\rangle &= (2\pi)^3\hat{M}_{ij}(k,t)\delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta(t-t')\\ &= (2\pi)^3\frac{M_k(k,t)}{4\pi k^2}\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta(t-t'),\end{aligned} \tag{14}$$

with $\hat{A}^*$ being the complex conjugate of the Fourier transform $\hat{A}$. The solution of the Fourier space Kazantsev equation is given by

$$M_k(k,t) = M_0e^{\gamma\lambda t}k^{3/2}K_{\rm Mc}(k/k_0), \quad {\rm Mc} = \sqrt{5\left(\lambda - \frac{3}{4}\right)}, \tag{15}$$

where $K_{\rm Mc}$ is the Macdonald function, λ the normalized growth rate and $k_0 = (\gamma/10)^{1/2}$. The magnetic power spectrum thus scales mostly as $M_k(k) \sim k^{3/2}$ in the subdiffuse limit, which we refer to as the Kazantsev spectrum. The magnetic spectrum grows exponentially in time, with a growth rate given by $3\gamma/4$ for an incompressible flow that is δ-correlated in time.

III. THE RENEWING FLOW METHOD

The renewing or renovating flow method was first proposed by Steenbeck and Krause [46]. Zeldovich *et al.* [39] highlighted that it provides an alternative to the unphysical assumption of velocities that are δ-correlated in time but remains analytically solvable. Several authors have used the method to obtain relevant results with finite correlation times [e.g. [47–51]]. In this work we employ the operator splitting method, used by Gilbert and Bayly [52] to recover the mean-field dynamo equations. Following the approach of Bhat and Subramanian [36], in a nonhelical flow, we impose a velocity field of the form

$$\boldsymbol{u} = \boldsymbol{a}\sin(\boldsymbol{q}\cdot\boldsymbol{x} + \psi). \tag{16}$$

We split the time into intervals of length τ which is the correlation time of the flow. In each of these τ-intervals we draw randomly $\boldsymbol{a}$, $\boldsymbol{q}$ and ψ such that the flow is overall isotropic, homogeneous, and with a zero mean [53]. Note that the flow is static only in intervals of the type $[(n-1)\tau, n\tau]$ (n being an integer) and renovates for each τ-interval.

To apply the operator splitting method we further divide the τ-intervals into two subintervals of duration $\tau/2$. In the first one we consider that the diffusion of the magnetic field is zero but the velocity is doubled, in the second one the velocity is now set to zero and diffusion acts as twice its original value. Using the induction equation [Eq. (7)] we need to solve the following problem:

$$\begin{aligned}\frac{\partial\boldsymbol{B}}{\partial t} &= \nabla\times 2\boldsymbol{u}\times\boldsymbol{B}, \quad t\in[(n-1)\tau, (n-1)\tau + \tau/2],\\ \frac{\partial\boldsymbol{B}}{\partial t} &= -2\eta\nabla\times\nabla\times\boldsymbol{B}, \quad t\in[(n-1)\tau + \tau/2, n\tau].\end{aligned} \tag{17}$$

The validity and convergence of the operator splitting method is beyond the scope of this work, we refer interested readers to Holden *et al.* [54].

A. First subinterval

We consider only the ideal induction equation. In this case, due to the magnetic flux freezing, the magnetic field is given by the standard Cauchy solution [see Sec. 3.3 of Ref. [55]]

$$B_i(\boldsymbol{x},t) = \frac{J_{ij}(\boldsymbol{x}_0)}{|J_{ij}|}B_j(\boldsymbol{x}_0,t_0), \tag{18}$$

where we define $\boldsymbol{x}(\boldsymbol{x}_0,t_0)$ to be the Lagrangian position at a time t of a fluid element with an initial position $\boldsymbol{x}_0$ at time

t_0. The matrix J_{ij} is given by the coordinate transformation, namely,

$$J_{ij} = \frac{\partial x_i}{\partial x_{0,j}}, \tag{19}$$

and $|\cdot|$ denotes the determinant of the matrix.

B. Second subinterval

We consider only the diffusion of the magnetic field. It is straightforward to solve the equation of diffusion in Fourier space where we denote the Fourier transform of A by $\hat{A}$. We find the solution

$$\hat{B}_i(\boldsymbol{k}, t) = e^{-\eta k^2 \tau} \hat{B}_j(\boldsymbol{k}, t_1), \tag{20}$$

with $t_1 = t_0 + \tau/2$.

C. Complete time-step

We express the total magnetic field evolution in Fourier space, from Eqs. (18) and (20), as

$$\hat{B}_i(\boldsymbol{k}, t) = e^{-\eta k^2 \tau} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{J_{ij}(\boldsymbol{x}_0)}{|J_{ij}|} B_j(\boldsymbol{x}_0, t_0)\, d^3\boldsymbol{x}, \tag{21}$$

which describes the successive evolution through the two subintervals.

We are now ready to give an expression for the two-point correlation function of the magnetic field in Fourier space

$$\langle \hat{B}_i(\boldsymbol{k}, t)\hat{B}^*_h(\boldsymbol{p}, t)\rangle = e^{-\eta\tau(\boldsymbol{k}^2+\boldsymbol{p}^2)} \left\langle \int \frac{J_{ij}(\boldsymbol{x}_0)}{|J_{ij}|} \frac{J_{hl}(\boldsymbol{y}_0)}{|J_{hl}|} B_j(\boldsymbol{x}_0, t_0) \times B_l(\boldsymbol{y}_0, t_0) e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{p}\cdot\boldsymbol{y})}\, d^3\boldsymbol{x} d^3\boldsymbol{y} \right\rangle, \tag{22}$$

where $\langle\cdot\rangle$ denotes an average over the parameter space of the velocity flow and A^* is the complex conjugate. We can change the integration variables $\{\boldsymbol{x}, \boldsymbol{y}\} \to \{\boldsymbol{x}_0, \boldsymbol{y}_0\}$ such that the determinants of the two Jacobian matrices cancel. We can also argue that the initial magnetic field is no longer correlated with the renewing flow in the next subinterval, which allows us to split the averages. The final expression is then given by

$$\langle \hat{B}_i(\boldsymbol{k}, t)\hat{B}^*_h(\boldsymbol{p}, t)\rangle = e^{-\eta\tau(\boldsymbol{k}^2+\boldsymbol{p}^2)} \int \langle B_j(\boldsymbol{x}_0, t_0) B_l(\boldsymbol{y}_0, t_0)\rangle \times \langle J_{ij}(\boldsymbol{x}_0) J_{hl}(\boldsymbol{y}_0) e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{p}\cdot\boldsymbol{y})}\rangle d^3\boldsymbol{x}_0 d^3\boldsymbol{y}_0. \tag{23}$$

Note that in this expression $\boldsymbol{x}$ and $\boldsymbol{y}$ are functions of the initial positions.

As the flow is overall isotropic and homogeneous we expect that for an initial state of the magnetic field, which is also isotropic and homogeneous, these properties are conserved. Under such assumptions the two-point magnetic correlation function takes the following form:

$$\langle B_i(\boldsymbol{x}, t) B_j(\boldsymbol{y}, t)\rangle = M_{ij}(r, t), \tag{24}$$

where $r = |\boldsymbol{x} - \boldsymbol{y}|$. We can further introduce a new set of integration variables $\{\boldsymbol{x}_0, \boldsymbol{y}_0\} \to \{\boldsymbol{r_0} \equiv \boldsymbol{x}_0 - \boldsymbol{y}_0, \boldsymbol{y}_0\}$. We rewrite the exponential part inside the integral as

$$-i[\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}_0) - \boldsymbol{p}\cdot(\boldsymbol{y} - \boldsymbol{y}_0) + \boldsymbol{k}\cdot\boldsymbol{r}_0 + (\boldsymbol{k} - \boldsymbol{p})\cdot\boldsymbol{y}_0]. \tag{25}$$

For now we assume that the evolution tensor, that is given by

$$R_{ijhl} \equiv \left\langle J_{ij}(\boldsymbol{x}_0) J_{hl}(\boldsymbol{y}_0) e^{-i[\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}_0) - \boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{y}_0)]}\right\rangle, \tag{26}$$

is independent of $\boldsymbol{y}_0$; which is convenient as we can rewrite Eq. (23) in the following form:

$$\langle \hat{B}_i(\boldsymbol{k}, t)\hat{B}^*_h(\boldsymbol{p}, t)\rangle = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{k}) e^{-2\eta\tau\boldsymbol{p}^2} \times \int e^{-i\boldsymbol{p}\cdot\boldsymbol{r}_0} R_{ijhl} M_{jl}(r_0, t_0)\, d^3\boldsymbol{r}_0, \tag{27}$$

once the integration over $d^3\boldsymbol{y}_0$ is performed. Note that the Dirac-δ function appears from the integration over $\boldsymbol{y}_0$ since the exponential is the only dependency on $\boldsymbol{y}_0$ and can be taken out of the flow parameters average. We assumed that R_{ijhl} only depends on $\boldsymbol{r}_0$ because this form of the equation is more compact; we will show in further sections (see Sec. IV C) that this assumption is valid, at least for the cases we consider.

IV. KAZANTSEV EQUATION FROM THE RENEWING FLOW METHOD

In his initial work, Kazantsev considered a flow that is δ-correlated in time and incompressible. This case is the easiest to treat with equations that are more or less tractable. We use this simplified treatment to present a detailed calculation in the framework of the renewing flow method. With the renewing flow method we consider the velocity field to be known, which constitutes the main difference with previous works on the topic.

A. Velocity flow parameters

The first step is to give a suitable parametrization of $\boldsymbol{a}$, $\boldsymbol{q}$ and ψ to ensure the statistical isotropy and homogeneity of the flow. We further impose an incompressible flow, which translates here to the requirement that $\boldsymbol{a}$ and $\boldsymbol{q}$ are orthogonal to each other.

Homogeneity. we draw ψ in each τ-interval from a uniform distribution in the range $[0, 2\pi]$.

Isotropy. we fix the value of q which is the norm of $\boldsymbol{q}$. The wave number $\boldsymbol{q}$ is randomly drawn from a sphere of radius q. The velocity orientation $\boldsymbol{a}$ is randomly drawn in the plane perpendicular to $\boldsymbol{q}$ such that $\langle \boldsymbol{u}\rangle = 0$.

To simplify the computations we change the average ensemble. Instead of averaging over the direction of $\boldsymbol{a}$ we prefer to use a new vector $\boldsymbol{A}$ which has a fixed norm and a direction drawn randomly. Then $\boldsymbol{A}$ and $\boldsymbol{q}$ define a plane where we can project the component of $\boldsymbol{A}$ that is orthogonal to $\boldsymbol{q}$. This is performed by

$$\tilde{P}_{ij} \equiv \delta_{ij} - \hat{q}_i\hat{q}_j, \quad a_i = \tilde{P}_{ij} A_j, \tag{28}$$

where $\hat{q}_i \equiv q_i/q$ is the normalized component of $\boldsymbol{q}$. Note also that we adopt the Einstein summation rule. Since $\boldsymbol{A}$ and $\boldsymbol{q}$ are two independent vectors this parametrization ensures $\langle \boldsymbol{u}\rangle = 0$. We directly see that a is not fixed in this context; however, we can evaluate it from A as

$$\begin{aligned} \langle a^2\rangle &= \langle a_i a_i\rangle = \langle \tilde{P}_{il} A_l \tilde{P}_{ih} A_h\rangle, \\ &\underset{\text{average of } \boldsymbol{A}}{=} \frac{A^2}{3} \langle \tilde{P}_{il}\tilde{P}_{ih}\delta_{lh}\rangle \underset{\text{average of } \boldsymbol{q}}{=} \frac{2A^2}{3}, \end{aligned} \tag{29}$$

where we used the fact that $\langle A_i A_j\rangle = \delta_{ij}/3$ for a random vector.

B. Two-point velocity correlation functions

To reconstruct the original Kanzatsev equation (9) we only need to compute the second-order velocity correlator. We use the definition of Bhat and Subramanian [36]

$$T_{ij} = \frac{\tau}{2}\langle u_i(\boldsymbol{x})u_j(\boldsymbol{y})\rangle \underset{\text{average of }\psi}{=} \frac{\tau}{4}\langle a_i a_j \cos(\boldsymbol{q}\cdot\boldsymbol{r})\rangle. \tag{30}$$

The factor $\tau/2$ is required here as the flow is correlated in time. It also ensures that in the limit $\tau \to 0$ we recover the Kazantsev equation. The initialization of $\boldsymbol{a}$ and $\boldsymbol{q}$ allows us to give an exact formula for this correlator. Using Eqs. (28) and (29) we average over the directions of $\boldsymbol{A}$ to eliminate it, the remaining average is thus only over the directions of $\boldsymbol{q}$ with

$$\begin{aligned} T_{ij} &= \frac{\tau}{4}\langle \tilde{P}_{il}A_l\tilde{P}_{jh}A_h\cos(\boldsymbol{q}\cdot\boldsymbol{r})\rangle = \frac{A^2\tau}{12}\langle \tilde{P}_{ij}\cos(\boldsymbol{q}\cdot\boldsymbol{r})\rangle \\ &= \frac{a^2\tau}{8}\left[\delta_{ij} + \frac{1}{q^2}\partial_i\partial_j\right]\langle\cos(\boldsymbol{q}\cdot\boldsymbol{r})\rangle, \end{aligned} \tag{31}$$

where we use the following notation: $\partial_i \equiv \partial/\partial r_i$. If we recall the proper definition of the average, then we can write

$$\begin{aligned} \langle\cos(\boldsymbol{q}\cdot\boldsymbol{r})\rangle &\equiv \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin(\theta)\cos(\boldsymbol{q}\cdot\boldsymbol{r})\,d\theta d\phi \\ &= \frac{1}{2}\int_0^{\pi}\sin(\theta)\cos(qr\cos(\theta))\,d\theta \\ &= j_0(qr), \end{aligned} \tag{32}$$

where $j_0(x)$ is the spherical Bessel function.

C. Computation of the evolution tensor

To evaluate R_{ijhl} we first need to have an expression for J_{ij}. As we required that $\boldsymbol{a}$ and $\boldsymbol{q}$ are orthogonal we have

$$\frac{d(\boldsymbol{q}\cdot\boldsymbol{x}+\psi)}{dt} \equiv \frac{d\phi}{dt} = 2\boldsymbol{q}\cdot\boldsymbol{u} = 0, \tag{33}$$

such that ϕ is constant along the trajectory of a fluid element. So the equation $d\boldsymbol{x}/dt = 2\boldsymbol{u}$ can be easily integrated[2] and gives

$$x_i - x_{0,i} = a_i\tau\sin(\boldsymbol{q}\cdot\boldsymbol{x}_0+\psi) \tag{34}$$

for the Lagrangian positions. Using Eq. (19), it is straightforward to evaluate J_{ij}; from the last relation

$$J_{ij}(\boldsymbol{x}_0) = \delta_{ij} + \tau a_i q_j\cos(\boldsymbol{q}\cdot\boldsymbol{x}_0+\psi). \tag{35}$$

Bhat and Subramanian [36] motivated an expansion of the exponential of the evolution tensor [Eq. (26)] in the limit of small Strouhal numbers $\mathrm{St} = qa\tau \ll 1$. In the context of small-scale turbulent dynamos the magnetic spectrum in the kinematic regime peaks around the resistive scale [56–58] which can be evaluated to be $r_\eta \sim (l_0/R_\mathrm{M}^{1/2})$ with l_0 being the integral scale of the flow.[3] In the case considered here, the flow has only one typical scale $(1/q)$, thus $r_\eta \sim 1/(qR_\mathrm{M}^{1/2})$. We used $R_\mathrm{M} \sim a/(q\eta)$ for the magnetic Reynolds number which is usually very high in astrophysical objects [see Table 1 of Ref. [55]] such that qr_η is very small and hence $\sin(\boldsymbol{q}\cdot\boldsymbol{r_\eta}) \sim \boldsymbol{q}\cdot\boldsymbol{r_\eta}$. The phase of the exponential in Eq. (26) is then given by $aq\tau p_\eta r_\eta \sim qa\tau = \mathrm{St}$. Since the terms in the vicinity of the resistive scale will contribute more to the magnetic spectrum, the expansion of $\sin(\boldsymbol{q}\cdot\boldsymbol{x}_0+\psi) - \sin(\boldsymbol{q}\cdot\boldsymbol{y}_0+\psi) = \sin(\boldsymbol{q}\cdot\boldsymbol{r}_0/2)\cos(\boldsymbol{q}\cdot(\boldsymbol{x}_0+\boldsymbol{y}_0)+\psi)$ is reasonable. In this section, we only keep terms up to second order in τ and we will see that it leads to the original Kazantsev equation (9).

The equation (26) for R_{ijhl} can then be rewritten in the form

$$R_{ijhl} = \left\langle J_{ij}(\boldsymbol{x}_0)J_{hl}(\boldsymbol{y}_0)\left[1 - i\tau\beta\sigma - \frac{\tau^2\beta^2\sigma^2}{2!}\right]\right\rangle, \tag{36}$$

where $\beta = \sin(\boldsymbol{q}\cdot\boldsymbol{x}_0+\psi) - \sin(\boldsymbol{q}\cdot\boldsymbol{y}_0+\psi)$ and $\sigma = \boldsymbol{a}\cdot\boldsymbol{p}$. To continue further we make use of the average over ψ and we also introduce the notation $\phi_{x_0} = \boldsymbol{q}\cdot\boldsymbol{x}_0+\psi$. In fact, if we try to average a function of the type $\cos(n\phi_{x_0}+m\phi_{y_0})$ or $\sin(n\phi_{x_0}+m\phi_{y_0})$ with n and m being two integers, then we find that it always goes to zero except when $n=-m$. In particular, it highlights the fact, as we hypothesized in Sec. III, that R_{ijhl} is only dependent on $\boldsymbol{r}_0 = \boldsymbol{x}_0 - \boldsymbol{y}_0$.

Term by term evaluation of the average over ψ of Eq. (36), leads to the following expression:

$$\begin{aligned} R_{ijhl} = \Bigg\langle &\delta_{ij}\delta_{hl} + \frac{\tau^2 a_i q_j a_h q_l}{2}\cos(\boldsymbol{q}\cdot\boldsymbol{r_0}) \\ &- i\frac{\tau^2\sigma}{2}\sin(\boldsymbol{q}\cdot\boldsymbol{r_0})(\delta_{hl}a_i q_j + \delta_{ij}a_h q_l) \\ &- \frac{\tau^2\sigma^2}{2}(1-\cos(\boldsymbol{q}\cdot\boldsymbol{r_0}))\delta_{ij}\delta_{hl}\Bigg\rangle. \end{aligned} \tag{37}$$

Each term can then be matched with Eq. (30) to obtain

$$\begin{aligned} R_{ijhl} &= \delta_{ij}\delta_{hl} - 2\tau\partial_l\partial_j T_{ih} + 2i\tau p_m(\delta_{hl}\partial_j T_{im} + \delta_{ij}\partial_l T_{hm}) \\ &\quad -2\tau p_n p_m \delta_{ij}\delta_{hl}(T_{nm}(0) - T_{nm}), \end{aligned} \tag{38}$$

where we replaced q_i by suitable derivatives with respect to the components of $\boldsymbol{r}_0$ and σ by $a_m p_m$. This expression for R_{ijhl} cannot be simplified further and we need to go back to Eq. (27) and perform the integration.

D. Derivation of the Kazantsev equation

The original Kazantsev equation (9) describes the evolution of the two-point magnetic field correlation function in real space. Instead of evaluating $\hat{M}_{ih}(\boldsymbol{p},t)$ we take its inverse Fourier transform. Formally we get

$$M_{ih}(r,t) = \int e^{-2\eta\tau\boldsymbol{p}^2}e^{i\boldsymbol{p}(\boldsymbol{r}-\boldsymbol{r}_0)}R_{ijhl}M_{jl}(r_0,t_0)\,\frac{d^3\boldsymbol{r}_0 d^3\boldsymbol{p}}{(2\pi)^3}. \tag{39}$$

To further simplify this expression we assume that η is small, such that the exponential can also be expanded giving

[2]The factor 2 comes from the fact that in the first subinterval we have twice the initial velocity.

[3]This definition for the resistive scale is limited to the cases where the hydrodynamic Reynolds number R_e is close to unity. Some authors used a more general expression $k_\eta \propto R_\mathrm{M}^{1/2}R_e^{1/4}$ [40,77,87].

$\exp(-2\eta\tau \boldsymbol{p}^2) \sim 1 - 2\eta\tau \boldsymbol{p}^2$. This expansion is justified in the context of negligible η or large $R_{\rm M}$. Terms like $\eta\tau^2$ are also ignored, so the part $-2\eta\tau \boldsymbol{p}^2$ only contributes from the $\delta_{ij}\delta_{hl}$ term in the expression of R_{ijhl} as it is the only term that does not depend on τ^2. Once again we rewrite components of the wave vector (here $\boldsymbol{p}$) as derivatives with respect to the position (here $\boldsymbol{r}$) such that $p_j \rightarrow -i\partial_j$.

We adopt the notation $[\cdot]_{ij}$ for partial derivatives with respect to r_i and r_j. In the limit $\tau \rightarrow 0$ we can divide both sides by τ and replace $[M_{ih}(r,t) - M_{ih}(r,t_0)]/\tau \rightarrow \partial M_{ih}(r,t)/\partial t$ such that from Eq. (39) we arrive at [see Ref. [59], for a detailed calculation]

$$\frac{\partial M_{ih}(r,t)}{\partial t} = 2[M_{il}T_{jh}]_{jl} + 2[M_{jh}T_{il}]_{jl} - 2[M_{ih}T_{jl}]_{jl} - 2[M_{jl}T_{ih}]_{jl} + 2[M_{ih}(T_{\rm L}(0)+\eta)]_{jj}. \quad (40)$$

Note that $T_{\rm L}(0)$ appears from $T_{nm}(0) = \delta_{nm}T_{\rm L}(0)$. This result is very important for the formalism as we started from equation (37) that tracks the evolution of an initial state to the equation (40) for the two-point magnetic field correlation function that depends on other quantities evaluated at the same space-time positions.

We can even simplify the computation further by contracting Eq. (40) with $\hat{r}_{ih}$ on both sides to get an equation for $M_{\rm L}(\boldsymbol{r},t)$. We refer the reader to Table II for detailed expressions of different contractions that enter the computation. Using incompressibility we finally find

$$\frac{\partial M_{\rm L}(r,t)}{\partial t} = \frac{2}{r^4}\partial_r\{r^4[\eta + T_{\rm L}(0) - T_{\rm L}]\partial_r M_{\rm L}\} - \frac{2}{r}\big(r\partial_r^2 T_{\rm L} + 4\partial_r T_{\rm L}\big)M_{\rm L}, \quad (41)$$

which is exactly the incompressible Kazantsev equation (9) in the limit of a flow that is δ-correlated in time. In comparison to previous works [e.g., Refs. [33,41]], the input here is the velocity field that is used to solve directly the induction equation.

V. GENERALIZED KAZANTSEV EQUATION

In this section we will derive the equivalent of the Kazantsev equation in the context of the renewing flow method. Previous studies have analysed separately the effects of the finite correlation time [36] and the compressibility [40] of the flow. By generalized we mean that we relax the incompressibility assumption used in the previous work of Bhat and Subramanian [36]. Our new equations then include the contributions from the time correlation of the flow as well as its degree of compressibility.

A. Lagrangian positions

In the case of an incompressible flow $\xi \equiv \boldsymbol{a}\cdot\boldsymbol{q}$ was set to 0 (see Sec. IV). We can introduce a degree of compressibility by relaxing this condition, allowing ξ to be nonzero with $\xi \in [-aq; aq]$. This allows us to include the nontrivial contribution from the compressibility of the flow. We can no longer apply the same reasoning as before (see Sec. IV C) since this time we have $d\phi/dt = 2\xi\sin(\phi)$. If we integrate this expression over the first subinterval, then we find

$$|\tan(\phi/2)| = e^{\xi\tau}|\tan(\phi_0/2)|. \quad (42)$$

We defined $\phi = \boldsymbol{q}\cdot\boldsymbol{x} + \psi$ to be the phase of the velocity field at the final position (after a time $\tau/2$) and $\phi_0 = \boldsymbol{q}\cdot\boldsymbol{x}_0 + \psi$ to be the phase of the initial position.

Furthermore, by integrating the velocity field we get

$$x_i - x_{0,i} = \int \frac{dx_i}{dt}\, dt = \frac{a_i}{\xi}(\phi - \phi_0). \quad (43)$$

However, this formula cannot be inverted. The idea is thus to use Eq. (42) to isolate ϕ to plug it into Eq. (43) such that we get an expression of the Lagrangian positions $\boldsymbol{x}$ that depends only on the initial position $\boldsymbol{x}_0$.

We have imposed a peculiar velocity field that is periodic with respect to the variable ϕ with a period of 2π. It is then expected that the displacement $\boldsymbol{x} - \boldsymbol{x}_0$ also possesses this periodicity. Furthermore, the velocity field is static in a τ-interval which means that fluid elements are permanently pushed in the direction of $\boldsymbol{a}$ until they reach a zero of the velocity field and stop moving. As a result a fluid element with initial position $\phi_0 \in [n\pi, (n+1)\pi]$ will have a position after a time $\tau/2$ such that $\phi \in [n\pi, (n+1)\pi]$ where n is an integer. Equation (42) can thus be inverted, leading to

$$\phi/2 - \pi\lfloor\phi/(2\pi) + 1/2\rfloor = \arctan\left(e^{\xi\tau}\tan(\phi_0/2)\right). \quad (44)$$

Recall that in Sec. IV C we motivated an expansion with respect to a small Strouhal number St. We motivate the same idea here as $|\xi\tau| = |aq\tau\cos(\gamma)| < \mathrm{St}$ with γ being the angle between $\boldsymbol{a}$ and $\boldsymbol{q}$. With a similar argument we can show that any new term depends directly on St raised to some higher powers. To include effects due to finite correlation times we keep terms up to fourth order in τ. The expansion of the right-hand side of Eq. (44) also harbors a floor function that will cancel the one on the left-hand side.

We are now ready to plug the expression of ϕ from this expansion into Eq. (43)

$$x_i = x_{0,i} + a_i\tau\Bigg\{\sin(\phi_0) + \frac{\xi\tau}{4}\sin(2\phi_0) + \frac{\xi^2\tau^2}{12}[\sin(3\phi_0) - \sin(\phi_0)] + \frac{\xi^3\tau^3}{96}[3\sin(4\phi_0) - 4\sin(2\phi_0)]\Bigg\}, \quad (45)$$

which has the desired limit for $\xi \rightarrow 0$. It is straightforward to show that the Jacobian is then given by

$$J_{ij} = \delta_{ij} + a_i q_j\tau\Bigg\{\cos(\phi_0) + \frac{\xi\tau}{2}\cos(2\phi_0) + \frac{\xi^2\tau^2}{12}[3\cos(3\phi_0) - \cos(\phi_0)] + \frac{\xi^3\tau^3}{24}[3\cos(4\phi_0) - 2\cos(2\phi_0)]\Bigg\}. \quad (46)$$

B. Fourth-order velocity two-point function

To include finite correlation times we have to consider terms up to the fourth order in τ. The evolution tensor R_{ijhl} is then given by

$$R_{ijhl} = \left\langle J_{ij}J_{hl}\left[1 - i\tau\beta\sigma - \frac{\tau^2\beta^2\sigma^2}{2!} + i\frac{\tau^3\beta^3\sigma^3}{3!} + \frac{\tau^4\beta^4\sigma^4}{4!}\right]\right\rangle, \quad (47)$$

where the Jacobian matrices are given in Eq. (46) and

$$\beta = \sin(\phi_x) - \sin(\phi_y) + \frac{\tau\xi}{4}[\sin(2\phi_x) - \sin(2\phi_y)] + \frac{\tau^2\xi^2}{12}[\sin(3\phi_x) - \sin(3\phi_y) - \sin(\phi_x) + \sin(\phi_y)] + \frac{\tau^3\xi^3}{96}[3\sin(4\phi_x) - 3\sin(4\phi_y) - 4\sin(2\phi_x) + 4\sin(2\phi_y)]. \quad (48)$$

The evolution tensor now has dependencies on $a_i a_j a_h a_l$ due to the inclusion of τ^3 and τ^4 terms. It motivates the introduction of fourth-order two-point correlators that are defined in Bhat and Subramanian [59] by

$$\begin{aligned} T^{x^2y^2}_{ijhl} &= \tau^2\langle u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_h(\boldsymbol{y})u_l(\boldsymbol{y})\rangle, \\ T^{x^3y}_{ijhl} &= \tau^2\langle u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_h(\boldsymbol{x})u_l(\boldsymbol{y})\rangle, \\ T^{x^4}_{ijhl} &= \tau^2\langle u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_h(\boldsymbol{x})u_l(\boldsymbol{x})\rangle, \end{aligned} \quad (49)$$

where the factor τ^2 is included due to the time correlation of the flow. We can carry out the average over ψ such that the fourth-order correlators are given by

$$\begin{aligned} T^{x^2y^2}_{ijhl} &= \frac{\tau^2}{8}\langle a_i a_j a_h a_l[\cos(2\boldsymbol{q}\cdot\boldsymbol{r}) + 2]\rangle, \\ T^{x^3y}_{ijhl} &= \frac{3\tau^2}{8}\langle a_i a_j a_h a_l \cos(\boldsymbol{q}\cdot\boldsymbol{r})\rangle, \\ T^{x^4}_{ijhl} &= \frac{3\tau^2}{8}\langle a_i a_j a_h a_l\rangle. \end{aligned} \quad (50)$$

Note that $\boldsymbol{r}$ is still given by $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$.

Similar to the second-order velocity two-point correlation function we would like an expression for the fourth-order correlators in the case of an isotropic, homogeneous, and nonhelical velocity field. Following the ideas of De Karman and Howarth [44], Batchelor [60], Landau and Lifshitz [61], it can be shown that [62]

$$T_{ijhl}(r) = \hat{r}_{ijhl}\overline{T}_{\rm L}(r) + \hat{P}_{(ij}\hat{P}_{hl)}\overline{T}_{\rm N}(r) + \hat{P}_{(ij}\hat{r}_{hl)}\overline{T}_{\rm LN}(r), \quad (51)$$

where $\hat{r}_{ijhl} = r_i r_j r_h r_l/r^4$, $\hat{P}_{ij} = \delta_{ij} - r_i r_j/r^2$ and $\overline{T}_{\rm L/N/LN}$ are the longitudinal, transverse and mixed terms of the velocity correlation function. This formula has been derived and used by Bhat and Subramanian [36]. The bracket $(\cdot)$ operator denotes here the summation over all the different terms, formally $\hat{P}_{(ij}\hat{P}_{hl)} = \hat{P}_{ij}\hat{P}_{hl} + \hat{P}_{ih}\hat{P}_{jl} + \hat{P}_{il}\hat{P}_{jh}$ and $\hat{P}_{(ij}\hat{r}_{hl)} = \hat{P}_{ij}\hat{r}_{hl} + \hat{P}_{ih}\hat{r}_{jl} + \hat{P}_{il}\hat{r}_{jh} + \hat{P}_{hl}\hat{r}_{ij} + \hat{P}_{jl}\hat{r}_{ih} + \hat{P}_{jh}\hat{r}_{il}$. Knowing that it is straightforward to show that in the case of an incompressible flow the transverse, longitudinal, and mixed terms are related by

$$6\overline{T}_{\rm LN} = 2\overline{T}_{\rm L} + r\partial_r\overline{T}_{\rm L}, \quad 4\overline{T}_{\rm N} = 4\overline{T}_{\rm LN} + r\partial_r\overline{T}_{\rm LN}. \quad (52)$$

These two relations will be especially useful to check if our generalized Kazantsev equation has the right form when assuming incompressibility.

C. Generalized equation

The compressibility effects are characterized by the introduction of ξ and ξ^2 in the evolution tensor and the Jacobian matrices. These factors are not necessarily fixed between two τ intervals. To treat them we just need to recall that $\xi = a_i q_i$, such that the methodology explained in Secs. IV C and IV D can still be applied. Surprisingly we find that the compressibility only affects the fourth-order correlators, and Eq. (40) still holds for a velocity field that is δ-correlated in time in the compressible case. The resulting equation is given by

$$\begin{aligned} \frac{\partial M_{ih}}{\partial t} =\ & 2[M_{jh}T_{il}]_{jl} - 2[M_{ih}T_{jl}]_{jl} + 2[M_{il}T_{jh}]_{jl} - 2[M_{jl}T_{ih}]_{jl} + 2[M_{ih}T_{\rm L}(0)]_{jj}\} \ \xi^0\tau^2 \text{ terms} \\ & +2[M_{ih}\eta]_{jj}\} \text{ term due to resistive exponential expansion} \\ & +\tau\Big([M_{jl}\widetilde{T}_{ihmn}]_{mnjl} + \big[M_{ih}\big(\widetilde{T}_{mnst} + T^{x^4}_{mnst}/12\big)\big]_{mnst} - [M_{jh}\widetilde{T}_{imns}]_{mnsj} - [M_{il}\widetilde{T}_{hmns}]_{mnsl}\Big\} \ \xi^0\tau^4 \text{ terms} \\ & -[M_{jl}\partial_n\widetilde{T}_{ihmn}]_{mjl} - [M_{ih}\partial_t\widetilde{T}_{mnst}]_{mns} + [M_{jh}\partial_s\widetilde{T}_{imns}]_{mnj} + [M_{il}\partial_s\widetilde{T}_{hmns}]_{mnl}\} \ \xi^1\tau^4 \text{ terms} \\ & \left.\begin{aligned} & + \frac{2}{3}[M_{jl}\partial_m\partial_n\widetilde{T}_{ihmn}]_{jl} + \frac{2}{3}[M_{ih}\partial_s\partial_t\widetilde{T}_{mnst}]_{mn} - \frac{2}{3}[M_{jh}\partial_n\partial_s\widetilde{T}_{imns}]_{mj} \\ & - \frac{2}{3}[M_{il}\partial_n\partial_s\widetilde{T}_{hmns}]_{ml} - \frac{5}{48}\big[M_{jl}\partial_m\partial_n T^{x^2y^2}_{ihmn}\big]_{jl} - \frac{5}{48}\big[M_{ih}\partial_s\partial_t T^{x^2y^2}_{mnst}\big]_{mn} \\ & + \frac{5}{48}\big[M_{jh}\partial_n\partial_s T^{x^2y^2}_{imns}\big]_{mj} + \frac{5}{48}\big[M_{il}\partial_n\partial_s T^{x^2y^2}_{hmns}\big]_{ml} + \frac{5}{48}\big[M_{ih}\partial_s\partial_t T^{x^2y^2}_{mnst}\big|_0\big]_{mn}\Big), \end{aligned}\right\} \xi^2\tau^4 \text{ terms} \end{aligned} \quad (53)$$

where we introduced the tensor $\widetilde{T}_{ijhl} = T^{x^2y^2}_{ijhl}/4 - T^{x^3y}_{ijhl}/3$. The compressibility adds the divergence of the fourth-order velocity correlators, which is evaluated to be zero in the case of an incompressible flow; such that the incompressible limit ($\xi = 0$) gives back Eq. (16) in Bhat and Subramanian [36].

To derive the equation for $M_{\rm L}$ we contract Eq. (53) with $\hat{r}_{ih}$. By doing so we need to evaluate every term in the equation using the velocity correlation functions described in Eqs. (3) and (51). Even if this computation does not involve very complicated algebra we will not detail our derivation as it is very extensive. However, we give in Table II the main tools and properties to perform the complete calculation. Another very useful expression to simplify the computation is

$$\partial_i\partial_j[\hat{r}_{ij}f(r)+\hat{P}_{ij}g(r)]=\partial_r^2 f(r)+\frac{2}{r}\partial_r[2f(r)-g(r)]+\frac{2}{r^2}[f(r)-g(r)], \tag{54}$$

where f and g are two arbitrary functions of r. Using these properties, simplifications, and denoting $\overline{T}_{\mathrm{L/N/LN}}$ the respective components of $\widetilde{T}_{ijhl}$ we eventually arrive at the generalized Kazantsev equation

$$\begin{aligned}
\frac{\partial M_{\mathrm{L}}}{\partial t} &= \partial_r^4 M_{\mathrm{L}}\left(\tau\left\{\overline{T}_{\mathrm{L}}+\frac{T_{\mathrm{L}}^{x^3y}(0)}{12}\right\}\right)+\partial_r^3 M_{\mathrm{L}}\left(\tau\left\{2\partial_r\overline{T}_{\mathrm{L}}+\frac{8}{r}\overline{T}_{\mathrm{L}}+\frac{2T_{\mathrm{L}}^{x^3y}(0)}{3r}\right\}\right)+\partial_r^2 M_{\mathrm{L}}\Bigg(-2T_{\mathrm{L}}+2T_{\mathrm{L}}(0)+2\eta\\
&+\tau\left\{\frac{5}{3}\partial_r^2\overline{T}_{\mathrm{L}}+\frac{41}{3r}\partial_r\overline{T}_{\mathrm{L}}-\frac{8}{3r}\partial_r\overline{T}_{\mathrm{LN}}+\frac{40}{3r^2}\overline{T}_{\mathrm{L}}-\frac{80}{3r^2}\overline{T}_{\mathrm{LN}}+\frac{32}{3r^2}\overline{T}_{\mathrm{N}}-\frac{5}{48}\partial_r^2T_{\mathrm{L}}^{x^2y^2}-\frac{20}{48r}\partial_rT_{\mathrm{L}}^{x^2y^2}\right.\\
&\left.+\frac{50}{48r}\partial_rT_{\mathrm{LN}}^{x^2y^2}-\frac{10}{48r^2}T_{\mathrm{L}}^{x^2y^2}+\frac{110}{48r^2}T_{\mathrm{LN}}^{x^2y^2}-\frac{80}{48r^2}T_{\mathrm{N}}^{x^2y^2}+\frac{2T_{\mathrm{L}}^{x^3y}(0)}{3r^2}+K\right\}\Bigg)\\
&+\partial_r M_{\mathrm{L}}\Bigg(-2\partial_rT_{\mathrm{L}}-\frac{8}{r}T_{\mathrm{L}}+\frac{8}{r}T_{\mathrm{L}}(0)+\frac{8\eta}{r}+\tau\left\{\frac{2}{3}\partial_r^3\overline{T}_{\mathrm{L}}+\frac{25}{3r}\partial_r^2\overline{T}_{\mathrm{L}}-\frac{8}{3r}\partial_r^2\overline{T}_{\mathrm{LN}}+\frac{55}{3r^2}\partial_r\overline{T}_{\mathrm{L}}-\frac{104}{3r^2}\partial_r\overline{T}_{\mathrm{LN}}\right.\\
&+\frac{32}{3r^2}\partial_r\overline{T}_{\mathrm{N}}+\frac{8}{3r^3}\overline{T}_{\mathrm{L}}-\frac{160}{3r^3}\overline{T}_{\mathrm{LN}}+\frac{64}{3r^3}\overline{T}_{\mathrm{N}}-\frac{5}{48}\partial_r^3T_{\mathrm{L}}^{x^2y^2}-\frac{40}{48r}\partial_r^2T_{\mathrm{L}}^{x^2y^2}+\frac{50}{48r}\partial_r^2T_{\mathrm{LN}}^{x^2y^2}-\frac{70}{48r^2}\partial_rT_{\mathrm{L}}^{x^2y^2}\\
&\left.+\frac{260}{48r^2}\partial_rT_{\mathrm{LN}}^{x^2y^2}-\frac{80}{48r^2}\partial_rT_{\mathrm{N}}^{x^2y^2}-\frac{20}{48r^3}T_{\mathrm{L}}^{x^2y^2}+\frac{220}{48r^3}T_{\mathrm{LN}}^{x^2y^2}-\frac{160}{48r^3}T_{\mathrm{N}}^{x^2y^2}-\frac{2T_{\mathrm{L}}^{x^3y}(0)}{3r^3}+\frac{4K}{r}\right\}\Bigg)\\
&+M_{\mathrm{L}}\Bigg(-\frac{4}{r}\partial_rT_{\mathrm{L}}-\frac{4}{r}\partial_rT_N-\frac{4}{r^2}T_{\mathrm{L}}+\frac{4}{r^2}T_N+\tau\left\{\frac{4}{3r}\partial_r^3\overline{T}_{\mathrm{L}}+\frac{4}{3r}\partial_r^3\overline{T}_{\mathrm{LN}}+\frac{20}{3r^2}\partial_r^2\overline{T}_{\mathrm{L}}\right.\\
&-\frac{12}{3r^2}\partial_r^2\overline{T}_{\mathrm{LN}}-\frac{16}{3r^2}\partial_r^2\overline{T}_{\mathrm{N}}+\frac{8}{3r^3}\partial_r\overline{T}_{\mathrm{L}}-\frac{104}{3r^3}\partial_r\overline{T}_{\mathrm{LN}}+\frac{48}{3r^3}\partial_r\overline{T}_{\mathrm{N}}-\frac{8}{3r^4}\overline{T}_{\mathrm{L}}-\frac{56}{3r^4}\overline{T}_{\mathrm{LN}}\\
&+\frac{80}{3r^4}\overline{T}_{\mathrm{N}}-\frac{10}{48r}\partial_r^3T_{\mathrm{L}}^{x^2y^2}-\frac{10}{48r}\partial_r^3T_{\mathrm{LN}}^{x^2y^2}-\frac{50}{48r^2}\partial_r^2T_{\mathrm{L}}^{x^2y^2}+\frac{30}{48r^2}\partial_r^2T_{\mathrm{LN}}^{x^2y^2}+\frac{40}{48r^2}\partial_r^2T_{\mathrm{N}}^{x^2y^2}\\
&\left.-\frac{20}{48r^3}\partial_rT_{\mathrm{L}}^{x^2y^2}+\frac{260}{48r^3}\partial_rT_{\mathrm{LN}}^{x^2y^2}-\frac{120}{48r^3}\partial_rT_{\mathrm{N}}^{x^2y^2}+\frac{20}{48r^4}T_{\mathrm{L}}^{x^2y^2}+\frac{140}{48r^4}T_{\mathrm{LN}}^{x^2y^2}-\frac{200}{48r^4}T_{\mathrm{N}}^{x^2y^2}\right\}\Bigg),
\end{aligned} \tag{55}$$

where $K=5C(0)/48$ is a constant with $C(r)=\partial_r^2(T_{\mathrm{L}}^{x^2y^2})+4\partial_r(T_{\mathrm{L}}^{x^2y^2})/r-10\partial_r(T_{\mathrm{LN}}^{x^2y^2})/r+2T_{\mathrm{LN}}^{x^2y^2}/r^2-22T_{\mathrm{L}}^{x^2y^2}/r^2+16T_{\mathrm{N}}^{x^2y^2}/r^2$. This equation has the most generic form if we assume only isotropy, homogeneity, and nonhelicity of the velocity flow in the vicinity of small St. To solve this equation we should define the boundary conditions. The magnetic field correlation function should go to zero for infinitely large space scales. Also we would require M_{L} to be finite in $r=0$ such that the autocorrelation of the magnetic field is a local maxima. These two conditions can be summarized by

$$\lim_{r\to 0}\partial_r M_{\mathrm{L}}(r,t)=0, \qquad \lim_{r\to\infty}M_{\mathrm{L}}(r,t)=0. \tag{56}$$

Note that if we assume incompressibility in Eq. (55) we retrieve Eq. (17) in Bhat and Subramanian [36]. Except for its length, the general aspect of the equation is unchanged for an arbitrary degree of compressibility (DOC). The most interesting difference arises in terms that depend on M_{L}. In the incompressible case, these terms cancel perfectly but not when the DOC is nonzero. We can already get the intuition that these terms will control the time growth rate of the magnetic correlation function.

D. Small-scale limit

In this section we discuss the limit of length scales much smaller than the turbulent forcing scale (i.e., $z\equiv qr\ll 1$) of Eq. (55). The Kazantsev spectrum $M_k(k)\sim k^{3/2}$ is predicted in the range $q\ll k\ll k_\eta$. Since we consider large R_{M}, it is sufficient to expand our generalized equation in the limit of small z. We introduce two different cases, which correspond to two initialization for $\boldsymbol{a}$ and $\boldsymbol{q}$. The first case is used to give a detailed derivation. However, the second case is more general and gives rise to a lengthy calculation so we will only present the results.

1. Two independent vectors

First consider the case where $\boldsymbol{a}$ and $\boldsymbol{q}$ are perfectly independent. It is straightforward to evaluate Eqs. (30) and (50) knowing that $\langle a_ia_ja_ha_l\rangle=\delta_{(ij}\delta_{hl)}/15$ for a random vector and we can directly plug the expansion of the correlators' components in Eq. (55). These considerations simplify strongly our generalized Kazantsev equation such that it reduces to the following expression in the limit $z \ll 1$:

$$\frac{\partial M_L}{\partial t} = q^2 T_L(0)\left[\left(\frac{2\eta}{T_L(0)} + \frac{z^2}{3}\right)\partial_z^2 M_L + \left(\frac{8\eta}{zT_L(0)} + 2z\right)\partial_z M_L + \frac{8}{3}M_L\right] + \frac{a^4q^4\tau^3}{160}\left[\frac{z^4}{10}\partial_z^4 M_L + \frac{8z^3}{5}\partial_z^3 M_L + \frac{958z^2}{135}\partial_z^2 M_L + \frac{404z}{45}\partial_z M_L + \frac{32}{27}M_L\right]. \quad (57)$$

We can further assume that $\tilde{M}_L$ is independent of space such that we use the ansatz $M_L(r,t) = \tilde{M}_L(z)e^{\gamma\tilde{t}}$ where $\tilde{t} = tT_L(0)q^2$ and γ is a normalized growth rate. We also set $\bar{\tau} = \tau T_L(0)q^2$ and rename $T_L(0) = \eta_t$ to stick to the conventions used in Bhat and Subramanian [36]. After some algebra we end up with

$$0 = \left(\frac{2\eta}{\eta_t} + \frac{z^2}{3}\right)\partial_z^2\tilde{M}_L + \left(\frac{8\eta}{z\eta_t} + 2z\right)\partial_z\tilde{M}_L + \left(\frac{8}{3} - \gamma\right)\tilde{M}_L + \frac{9\bar{\tau}}{10}\left[\frac{z^4}{10}\partial_z^4\tilde{M}_L + \frac{8z^3}{5}\partial_z^3\tilde{M}_L + \frac{958z^2}{135}\partial_z^2\tilde{M}_L + \frac{404z}{45}\partial_z\tilde{M}_L + \frac{32}{27}\tilde{M}_L\right]. \quad (58)$$

We now focus on the range $z_\eta = qr_\eta \ll z \ll 1$, where $\bar{\tau}$ terms cannot be neglected. Here, we use a Landau-Lifshitz approximation [63] and consider $\bar{\tau}$ to be a small parameter. To derive approximated expressions for the high order derivatives of $\tilde{M}_L$ as a function of the first and the second-order derivatives, we neglect $\bar{\tau}$ and $\sqrt{\eta/\eta_t}$ compared to z,

$$z^3\partial_z^3\tilde{M}_L = -8z^2\partial_z^2\tilde{M}_L + (3\gamma_0 - 8)\partial_z\tilde{M}_L,$$
$$z^4\partial_z^4\tilde{M}_L = (3\gamma_0 + 58)z^2\partial_z^2\tilde{M}_L - 10(3\gamma_0 - 14)\partial_z\tilde{M}_L, \quad (59)$$

where γ_0 is the growth rate for a δ-correlated in time flow. As a first approach (we will give in Sec. VI a more rigorous treatment) we neglect $\sqrt{\eta/\eta_t}$. The two expressions for high order derivatives can be plugged into Eq. (58) to obtain

$$0 = \frac{9\bar{\tau}}{10}\left[z^2\left(\frac{3\gamma_0}{10} + \frac{13}{135}\right)\partial_z^2\tilde{M}_L + z\left(\frac{9\gamma_0}{5} + \frac{78}{135}\right)\partial_z\tilde{M}_L + \frac{32}{27}\tilde{M}_L\right] + \frac{z^2}{3}\partial_z^2\tilde{M}_L + 2z\partial_z\tilde{M}_L + \left(\frac{8}{3} - \gamma\right)\tilde{M}_L. \quad (60)$$

It is obvious that this equation admits a power-law solution $\tilde{M}_L \sim z^{-\lambda}$. Solving for λ we find

$$\lambda = \frac{5}{2} \pm \frac{i}{2}\left[4\frac{8 + 16\bar{\tau}/5 - 3\gamma}{1 + 81\gamma_0\bar{\tau}/100 + 13\bar{\tau}/50} - 25\right]^{1/2}. \quad (61)$$

We find that the real part of λ is 5/2, which is exactly the same as Bhat and Subramanian [36] and is expected for a Kazantsev spectrum. Gruzinov *et al.* [64] have argued that the growth rate, in the limit of $R_M \to \infty$ is given by finding a value of λ such that $d\gamma/d\lambda = 0$. We can then plug that value into Eq. (61), γ is thus given by

$$\gamma = \tfrac{7}{12} - \tfrac{147}{320}\bar{\tau}, \quad (62)$$

where we used the self-consistent value for $\gamma_0 = 7/12$. Note that the complete expression for the growth rate of the dynamo is then $\gamma_{\text{tot}} = \gamma T_L(0)q^2$.

2. *Arbitrary degree of compressibility*

The main problem with the initialization that we just presented is that it does not include any parameter to control the DOC. We define the DOC by

$$\sigma_c \equiv \frac{\langle(\nabla\cdot\boldsymbol{u})^2\rangle}{\langle(\nabla\times\boldsymbol{u})^2\rangle} = \frac{\langle a_ia_jq_iq_j\rangle}{\langle\epsilon_{ijk}\epsilon_{ihl}a_ka_lq_jq_h\rangle}, \quad (63)$$

where ϵ_{ijk} is the Levi-Civita symbol; the third expression is obtained after averaging over ψ. The DOC is then zero for an incompressible flow and goes to infinity for a fully irrotational flow. To derive an equation for an arbitrary DOC we set $\boldsymbol{q}$ to be a random vector with norm q and $\boldsymbol{a}$ defined by

$$a_i = b[\tilde{P}_{ij}\hat{A}_j\sin(\theta) + \hat{q}_j\hat{A}_j\hat{q}_i\cos(\theta)], \quad (64)$$

where as before $\boldsymbol{A}$ is a random vector of norm A and $\hat{A}_j = A_j/A$. The two parameters b and θ (that are constant) allow to control, respectively, the norm of $\langle\boldsymbol{a}^2\rangle$ and the DOC. In such a parametrization the component of $\boldsymbol{A}$ along $\boldsymbol{q}$ is always rescaled by $\cos(\theta)$ whereas the component of $\boldsymbol{A}$ orthogonal to $\boldsymbol{q}$ in the plane described by $\boldsymbol{A} - \boldsymbol{q}$ is always recaled by $\sin(\theta)$. This parametrization is taken for convenience, and θ can be interpreted as the mean absolute angle between $\boldsymbol{a}$ and $\boldsymbol{q}$. Although this parametrization might seem arbitrary, we can show that the results we derive here are independent on the exact evaluation of a_i as long as σ_c is uniquely defined (see Appendix A). Under such considerations

$$\sigma_c = \frac{1}{2\tan(\theta)^2}, \quad \langle\boldsymbol{a}^2\rangle = b^2\left[\frac{2}{3}\sin(\theta)^2 + \frac{1}{3}\cos(\theta)^2\right]. \quad (65)$$

We directly see that the value $\theta = \pi/2$ represents the incompressible case and $\theta = 0$ the fully irrotational one. Note that due to the random behavior of $\boldsymbol{A}$ we have $\langle\xi\rangle = 0$.

We apply the exact same methodology as for the first initialization, which means we expand the two-point correlators, use the ansatz, and re-express the high order derivatives with the first and the second-order ones. The expressions for the velocity correlators with this parametrization can be found in the Appendix B [Eq. (B1)]. Furthermore, we define the two functions

$$\epsilon(\theta) = \frac{216}{5(\Omega+3)^2}\left[\frac{\Omega+3}{5-\Omega}\left(\frac{1}{24}\Omega_1 - \frac{3}{28}\Omega_2 + \frac{3}{40}\Omega_3\right) \times \left(5\gamma_0 - \frac{40}{\Omega+3}\right) + \frac{157}{420}\Omega_1 - \frac{599}{630}\Omega_2 + \frac{121}{180}\Omega_3\right],$$
$$\zeta(\theta) = \frac{216}{5(\Omega+3)^2}\left[\frac{2}{3}\Omega_1 - \frac{14}{9}\Omega_2 + \frac{8}{9}\Omega_3\right]. \quad (66)$$

The set of parameters Ω_i appears very naturally in the derivation of Eq. (B1) and depends only on θ; exact expressions are given in Eq. (B2). From the velocity correlators we have $\eta_t = \tau b^2(\Omega+3)/72$. However, the parameter b is still free and we can set it to $b^2 = 6a^2/(\Omega+3)$ such that $\langle \boldsymbol{a}^2\rangle = a^2$. As a result $\eta_t = \tau a^2/12$, and the normalized correlation time can be evaluated to $\bar{\tau} = \mathrm{St}^2/12$. The normalized correlation time is then fully controlled by St, independently on the choice of DOC.

The resulting equation is

$$\left(\frac{2\eta}{\eta_t} + z^2\frac{5-\Omega}{5(\Omega+3)}\right)\partial_z^2\tilde{M}_\mathrm{L} + \left(\frac{8\eta}{z\eta_t} + 6z\frac{5-\Omega}{5(\Omega+3)}\right)\partial_z\tilde{M}_\mathrm{L} + \left(\frac{8}{\Omega+3} - \gamma\right)\tilde{M}_\mathrm{L} + \bar{\tau}\left[\epsilon(\theta)z^2\partial_z^2\tilde{M}_\mathrm{L} + 6\epsilon(\theta)z\partial_z\tilde{M}_\mathrm{L} + \zeta(\theta)\tilde{M}_\mathrm{L}\right] = 0. \quad (67)$$

Similar to the first case we compute the growth rate and scale factor of the power-law solution in the limit of large R_M, we find that the real part of λ is still 5/2. The growth rate is given this time by

$$\gamma_0 = \frac{7+5\Omega}{4(\Omega+3)}, \qquad \gamma = \gamma_0 + \bar{\tau}\left[\zeta(\theta) - \frac{25}{4}\epsilon(\theta)\right]. \quad (68)$$

We already see from this quick evaluation that the Kazantsev spectrum seems to be preserved even with an arbitrary DOC and correlation time. However, to confirm this first approach and include the effects of finite magnetic Reynolds numbers, we need to study more carefully the solutions to Eq. (67).

VI. FINITE MAGNETIC RESISTIVITY SOLUTIONS

The scaling solution that has been derived in the previous section only works if the term $\sqrt{\eta/\eta_t}$ is neglected. However, to include effects due to a finite magnetic resistivity we should not systematically neglect it. A WKB approximation can be used to evaluate the solution of Eq. (67) including the finite resistivity. An explicit derivation of the WKB solutions can be found in Appendix C. We only review the main results obtained for the magnetic power spectrum and the growth rate of the dynamo including a finite magnetic Reynolds number.

A. Growth rate

The normalized growth rate of the dynamo that includes contributions from the magnetic resistivity (through R_M), compressibility (through Ω and θ), and finite correlation time (through $\bar{\tau}$) is found to be

$$\gamma = -\left(\frac{\pi}{\ln(R_\mathrm{M})}\right)^2\left[\frac{5-\Omega}{5(\Omega+3)} + \bar{\tau}\epsilon(\theta)\right] + \frac{7+5\Omega}{4(\Omega+3)} + \bar{\tau}\left[\zeta(\theta) - \frac{25}{4}\epsilon(\theta)\right],$$
$$\equiv -\left(\frac{\pi}{\ln(R_\mathrm{M})}\right)^2\gamma_{R_\mathrm{M}} + \gamma_0 + \bar{\tau}\gamma_1, \quad (69)$$

where the functions $\epsilon(\theta)$ and $\zeta(\theta)$ are given in Eq. (66) and we have introduced the different components of the growth rate γ_{R_M}, γ_0, and γ_1. In Table I we list the different parameters of the flow and the magnetic spectrum for three regimes, namely incompressible ($\nabla\cdot\boldsymbol{u} = 0$), irrotational ($\nabla\times\boldsymbol{u} = 0$), and the intermediate case treated in Sec. V D 1. To get a better intuition on the results presented here, we display in Fig. 1 the DOC dependency of the two main contributions to the growth rate. From the evolution of γ_0 it is very clear that the compressibility tends to decrease the growth rate of the magnetic energy spectrum of the dynamo. Moreover, γ_0 is

TABLE I. Presentation of the velocity field and magnetic spectrum parameters for three types of flow: incompressible, irrotational, and intermediate (see Sec. V D).

Parameters	Incompressible	Intermediate	Irrotational
θ	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0
σ_c	0	$\frac{1}{2}$	∞
λ_k	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
γ_0	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{1}{4}$
γ_1	$-\frac{135}{224}$	$-\frac{147}{320}$	$-\frac{1017}{2240}$
γ_{R_M}	$\frac{1}{5}+\bar{\tau}\frac{27}{280}$	$\frac{1}{3}+\bar{\tau}\frac{293}{1200}$	$\frac{3}{5}+\bar{\tau}\frac{3429}{2800}$
$R_\mathrm{M,thresh}$	$\sim 3\times 10^5$	$\sim 1.5\times 10^5$	$\sim 7\times 10^4$

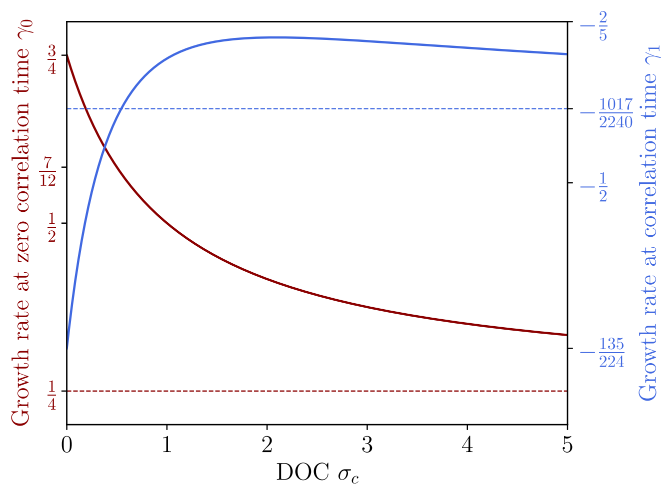

FIG. 1. Evolution of the two main contributions to the magnetic spectrum growth rate with respect to the DOC for large magnetic Reynolds number. The left axis represents the main contribution γ_0 and the right axis the contribution to the growth rate related to the correlation time γ_1 of Eq. (69). Dashed lines correspond to the value for a fully irrotational flow ($\sigma_c \to \infty$).

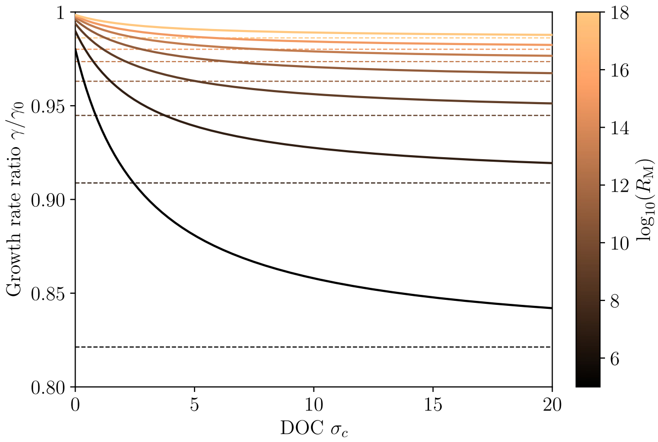

FIG. 2. Ratio of the growth rate and its main contribution for a few values of R_M and $\mathrm{St} = 10^{-2}$. Dashed lines correspond to the value for a fully irrotational flow ($\sigma_c \to \infty$).

comprised between 0.75 and 0.25 which indicates that the dynamo action always exists. It is interesting to note that γ_0 is a monotonously decreasing function of the DOC, whereas γ_1 has a maximum around $\sigma_c \sim 2$.

In Fig. 2 we study more precisely the dependence of the growth rate on R_M. As expected from Eq. (69), γ increases as R_M increases. In practice, due to the WKB approximation, there is a limiting value on the magnetic Reynolds number ($R_{M,\mathrm{thresh}}$) for which Eq. (69) is valid such that we need to keep $R_M > R_{M,\mathrm{thresh}}$. The impact on the total growth rate of the DOC is stronger when R_M is small. In this work, we only considered first-order corrections; and discrepancies can already represent ~85% for the lowest values of R_M presented. However, the correlation time has a negligible impact on the total growth rate in the limit $\mathrm{St} \ll 1$. Indeed, the correlation time enters the computation through St which itself contributes through $\bar{\tau} \propto \mathrm{St}^2 \ll 1$.

B. Magnetic power spectrum

In the range of interest, $z_\eta \ll z \ll 1$, we find the solution for the longitudinal two-point magnetic correlation function

$$M_L(z,t) = e^{\gamma\tilde{t}} z^{-5/2} M_0 \cos\left[\frac{\pi}{\ln(R_M)} \ln\left(\frac{z}{z_0}\right)\right], \quad (70)$$

where γ is given by Eq. (69). This result is already very interesting as we can identify a power law $z^{-5/2}$ independent of the correlation time that dominates the spectrum compared to the slowly varying $\cos[\ln(z)]$ function. However, the Kazantsev spectrum we are interested in predicts that the magnetic power spectrum scales as $M_k(k) \sim k^{-3/2}$ in the range $q \ll k \ll k_\eta$. We can show (see Appendix D) that the magnetic power spectrum and the longitudinal two-point correlation function are related by

$$M_k(k,t) = \frac{1}{\pi} \int (kr)^3 M_L(r,t) j_1(kr)\, dr. \quad (71)$$

The Bessel function $j_1(x)$ is very peaked around $x \sim 2$, so the dominant part of the integral is around $k \sim 1/r$. Thus, for $M_L(r) \sim r^\lambda$ we have $M_k(k) \sim k^{\lambda_k}$ with $\lambda_k = -(1+\lambda)$. Plugging in $\lambda = -5/2$ gives the well-known Kazantsev spectrum $M_k(k) \sim k^{3/2}$ even for the compressible and time-correlated flow considered here. Note that the main contribution to the power spectrum derived does not depend on any of the parameters of the flow. The last row of Table I corresponds to the minimal value of R_M for which the WKB approximation holds (see Appendix C 3). We see that for most astrophysical application of the small-scale dynamo our derived results remain valid.

VII. DISCUSSION AND CONCLUSIONS

Several authors have previously modeled the kinematic phase of the small-scale dynamo with the Kazantsev theory [see, e.g., Refs. [33,35,40,65]]. They found that the Kazantsev spectrum is preserved, even for a compressible flow. However, they often assumed Gaussian statistics of the velocity field, such that the flow is δ-correlated in time. There are several examples of analytic treatments that include a finite correlation time: Kolekar *et al.* [50] and Lamburt and Sokoloff [66] used a similar approach to our work but in the context of the mean-field dynamos, Schekochihin and Kulsrud [67] and Kleeorin *et al.* [68] considered a general case of fluctuation dynamo, and Bhat and Subramanian [59] solved the incompressible case. Most of the theoretical studies, if not all, have found that the Kazantsev spectrum is preserved even if compressibility, finite correlation time, or finite resistivity are considered. Our work shows that the combined effect of the three on the Kazantsev spectrum is negligible, as Eq. (70) scales mostly with the power law $z^{-5/2}$. However, our results are only derived for first-order corrections from the correlation time, as higher-order corrections are usually hard to treat.

Besides the shape of the magnetic energy spectrum, the dynamo growth rate γ is of particular interest. Our results for γ are similar to the ones obtained by Kulsrud and Anderson [33] and Schekochihin *et al.* [35] for the limit of incompressibility. Schekochihin *et al.* [40] also derived a formula for the growth rate for an arbitrary DOC for a δ-correlated in time flow. They found that γ ranges between $3/4$ for an incompressible flow and $1/8$ for a fully irrotational flow. It does not match our results by a factor of two in the limit of the fully irrotational flow. Similarly, the growth rate related to the finite resistivity γ_{R_M} matches their result in the incompressible case but is overestimated by a factor of 2 in the fully compressible limit. The discrepancy can be solved when we consider instead of γ alone the complete growth rate, namely $\gamma\eta_t q^2$. In their paper Schekochihin *et al.* [40] defined the initial growth rate from the velocity correlators in Fourier space while we define it from real space. If we transfer a factor $(\Omega+3)/4$ from η_t to γ, then our growth rate matches theirs in both limits. It is worth mentioning that Illarionov and Sokoloff [69] derived a growth rate that depends on the Strouhal number but does not match ours exactly. However, they also found that $\gamma_{\mathrm{tot}} \propto aq\mathrm{St}$ and that $\bar{\tau}$ reduces the growth rate. Although, we derived an expression for the growth rate that includes the DOC, a finite resistivity and time correlations our treatment of turbulence is very simple due to the imposed velocity field. A more rigorous treatment [15] highlights that the growth rate might also be a power law of the Reynolds number. Rogachevskii and Kleeorin [41] used a different approach to the problem as they impose directly a velocity correlation function instead of the velocity field itself. Their results match ours for the magnetic power spectrum as long as $T_L(r) \sim r^2$; but the growth rate differs as their velocity spectrum is different from ours. This highlights that the Kazantsev spectrum should be preserved for a large class of flows. Using a similar approach, some authors [70–72] studied more deeply the influence of the Prandtl number on the growth rate or magnetic spectrum while we fix the Reynolds number in our work. These papers suggest that the growth rate we derived should also depend on the Prandtl number if an even more general case is considered; this relation is beyond the scope of this paper as we want to focus on the effect of compressibility.

From a numerical point of view it seems indeed that a slope close to $3/2$ in the magnetic energy spectrum can be observed in both incompressible and compressible magnetohydrodynamic simulations at large length scales [see, e.g., Refs. [73–77]]. Although the slope measured in simulations is often close to the theoretical prediction, small discrepancies can still arise. One of the possible explanations can directly appear from the size of the simulation box. Indeed k has to be small but simulations are limited in resolution. This often can lead to an insufficient separation of spatial scales. A related problem is the assumption of very large hydrodynamic and/or magnetic Reynolds numbers in theoretical models. The required large values of these two numbers make a comparison between numerical simulations and theory hard. Kopyev *et al.* [78] also found that time irreversible flows can generate a nontrivial deviation to the Kazantsev spectrum. Regarding the growth rate of the dynamo, its reduction by the correlation time has also been observed in numerical studies [79]. Further discussions of the current state of dynamo numerical simulations can be found in Brandenburg *et al.* [80].

In conclusion, we have given an example of an analytical treatment for the fluctuation dynamo in the most generic case of a compressible flow with a finite correlation time. To this end, we proposed a framework to study the cumulative effects of a finite correlation time and an arbitrary degree of compressibility by generalizing the former work of Bhat and Subramanian [36]. We used the renovating flow method which assumes a very crude flow that does not allow for a very complex modeling of turbulence but keep the analytical treatment tractable. We derived a generalization to the Kazantsev equation in real space [Eq. (55)] that is valid at any scale. We note however that if we assume an incompressible flow that is δ-correlated in time at this point we retrieve the original Kazantsev equation. This equation describes the time evolution of the two-point magnetic correlation function M_L from the velocity correlators and the spatial derivatives of M_L up to the fourth order. We then studied solutions for length scales much smaller than turbulent forcing scale (i.e., $qr \ll 1$).

By the use of the WKB approximation, we derived formulas for the growth rate and slope of the magnetic power spectrum $M_k(k)$ for large magnetic Reynolds number $R_M \gg 1$ and small Strouhal number $\mathrm{St} \ll 1$. In particular, it allowed to capture the effect of finite magnetic diffusivity. Furthermore, we could define a lower bound on R_M for which our results should hold, $R_{M,\mathrm{thresh}} \sim 10^5$, which is smaller than most of the typical values in astrophysical objects. Although the growth rate showed dependencies on both the degree of compressibility and the correlation time, the Kazantsev spectrum seemed to be preserved, i.e., $M_k(k) \sim k^{3/2}$, independently of τ or σ_c. Our results are derived in a very special context, namely for a renovating flow. But our predictions regarding the magnetic field spectrum seem robust in the sense that both numerical and theoretical studies agree with the conservation of the Kazantsev spectrum for compressible and time-correlated flows.

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APPENDIX A: GENERAL INITIALIZATION FOR δ-CORRELATED IN TIME FLOW

We will review an even more general initialization than the one presented in Sec. V D 2. We only consider a δ-correlated in time flow but this discussion could in principle be generalized to a finite correlation time. A very general expression for $\boldsymbol{a}$ that preserves isotropy is

$$a_i = b(\tilde{P}_{ij}\hat{A}_j f_1 + \hat{q}_j\hat{A}_j\hat{q}_i f_2), \tag{A1}$$

where f_1 and f_2 are two constants. To control the norm a we should impose that $f_{1/2}$ are between minus one and one. It is then straightforward to show that $\sigma_c = f_2^2/(2f_1^2)$. Once again we compute the velocity correlators and plug in Eq. (55). To simplify this derivation we also neglect the resistivity η. We find the equation

$$\frac{z^2}{5}\frac{2f_1^2+3f_2^2}{2f_1^2+f_2^2}\partial_z^2\tilde{M}_L + \frac{6z}{5}\frac{2f_1^2+3f_2^2}{2f_1^2+f_2^2}\partial_z\tilde{M}_L + \left(4\frac{f_1^2+f_2^2}{2f_1^2+f_2^2}-\gamma\right)\tilde{M}_L = 0, \tag{A2}$$

that again allows some power-law solution. If we follow the same approach than in Sec. V D 2, then we find

$$\lambda = \frac{5}{2} \pm ig(f_1, f_2, \gamma), \quad \gamma = \frac{6f_1^2+f_2^2}{4(2f_1^2+f_2^2)}, \tag{A3}$$

with $g(f_1, f_2, \gamma)$ a function that characterizes the growth rate. Once again the power spectrum slope is constant and $\gamma = 3/4$ for an incompressible flow, $\gamma = 1/4$ for a fully irrotational one and $\gamma = 7/12$ if $f_1 = f_2$. If we define

$$f_1 = \sin\theta, \quad f_2 = \cos\theta, \tag{A4}$$

then we retrieve the initialization presented. This is convenient as we reduced the number of parameters to only θ to completely and uniquely define σ_c. Also it presents the option to work with another more natural parameter as in this case f_1^2 and f_2^2 are related by $f_1^2 + f_2^2 = 1$. In fact, we just showed that the exact initialization does not matter as long as σ_c is uniquely defined and that we can choose the most convenient one. Note also that $f_2 = \sqrt{2}\cos(\theta)$ is also an option that keeps the norm of $\boldsymbol{a}$ independent of the DOC.

TABLE II. Summary of the most basic tools to contract Eq. (53) with $\hat{r}_{ih}$.

Expression	Reduced form
$\hat{r}_{ii}$	1
$\hat{P}_{ii}$	2
$\hat{P}_{ij}\hat{r}_{il}$	0
$\hat{P}_{ij}\hat{P}_{il}$	$\hat{P}_{jl}$
∂_i	$\hat{r}_i\partial_r$
$\partial_i\partial_j$	$\hat{r}_{ij}\partial_r^2 + \hat{P}_{ij}\frac{\partial_r}{r}$
$\partial_i\hat{r}_j$	$\frac{1}{r}\hat{P}_{ij}$
$\partial_i\hat{r}_{jl}$	$\frac{1}{r}(\hat{r}_l\hat{P}_{ij} + \hat{r}_j\hat{P}_{il})$
$\partial_i\hat{P}_{jl}$	$-\frac{1}{r}(\hat{r}_l\hat{P}_{ij} + \hat{r}_j\hat{P}_{il})$
T_{ii}	$2T_{\rm N} + T_{\rm L}$
$\hat{r}_iT_{ij}$	$\hat{r}_jT_{\rm L}$
$\hat{r}_{ij}T_{ij}$	$T_{\rm L}$
$\hat{P}_{ij}T_{ij}$	$2T_{\rm N}$
$M_{ij}T_{il}$	$\hat{P}_{jl}M_{\rm N}T_{\rm N} + \hat{r}_{jl}M_{\rm L}T_{\rm L}$
T_{iijj}	$\overline{T}_{\rm L} + 8\overline{T}_{\rm N} + 4\overline{T}_{\rm LN}$
T_{iijl}	$\hat{r}_{jl}\overline{T}_{\rm L} + 4\hat{P}_{jl}\overline{T}_{\rm N} + (\hat{P}_{jl} + 2\hat{r}_{jl})\overline{T}_{\rm LN}$
$\hat{r}_iT_{ijhl}$	$\hat{r}_{jhl}\overline{T}_{\rm L} + (\hat{P}_{jl}\hat{r}_h + \hat{P}_{jh}\hat{r}_l + \hat{P}_{hl}\hat{r}_j)\overline{T}_{\rm LN}$
$\hat{r}_{ij}T_{ijhl}$	$\hat{r}_{hl}\overline{T}_{\rm L} + \hat{P}_{hl}\overline{T}_{\rm LN}$
$\hat{P}_{ij}T_{ijhl}$	$4\hat{P}_{hl}\overline{T}_{\rm N} + 2\hat{r}_{hl}\overline{T}_{\rm LN}$
∂_iT_{ijhl}	$\hat{r}_{jhl}\big(\partial_r\overline{T}_{\rm L} + \frac{2}{r}\overline{T}_{\rm L} - \frac{6}{r}\overline{T}_{\rm LN}\big) + (\hat{r}_j\hat{P}_{hl} + \hat{r}_h\hat{P}_{jl} + \hat{r}_l\hat{P}_{jh})\big(\partial_r\overline{T}_{\rm LN} + \frac{4}{r}\overline{T}_{\rm LN} - \frac{4}{r}\overline{T}_{\rm N}\big)$
$\partial_i\partial_jT_{ijhl}$	$\hat{r}_{hl}\big(\partial_r^2\overline{T}_{\rm L} + \frac{4}{r}\partial_r\overline{T}_{\rm L} - \frac{10}{r}\partial_r\overline{T}_{\rm LN} + \frac{2}{r^2}\overline{T}_{\rm L} - \frac{22}{r^2}\overline{T}_{\rm LN} + \frac{16}{r^2}\overline{T}_{\rm N}\big)$ $+\hat{P}_{hl}\big(\partial_r^2\overline{T}_{\rm LN} + \frac{8}{r}\partial_r\overline{T}_{\rm LN} - \frac{4}{r}\partial_r\overline{T}_{\rm N} + \frac{12}{r^2}\overline{T}_{\rm LN} - \frac{12}{r^2}\overline{T}_{\rm N}\big)$

APPENDIX B: COMPLEMENTARY EXPRESSIONS

We display in Table II the main tools used to contract Eq. (53) with $\hat{r}_{ij}$.

If we carry out all the algebra of Sec. V D 2, then we get the following expressions for the two-point velocity correlators:

$$T_{ij} = \frac{\tau b^2}{12}\left\{\hat{r}_{ij}\left(\frac{\Omega+1}{2} + \Omega\partial_z^2\right) + \hat{P}_{ij}\left(\frac{\Omega+1}{2} + \Omega\frac{\partial_z}{z}\right)\right\}j_0(z),$$

$$T_{ijhl}^{x^2y^2} = \frac{\tau^2b^4}{120}\left\{\hat{r}_{ijhl}\left[\frac{3}{16}\Omega_1\partial_z^4 + \frac{3}{2}\Omega_2\partial_z^2 + 3\Omega_3 + 6\frac{\Omega_{\rm tot}}{j_0(2z)}\right] + \hat{r}_{(ij}\hat{r}_{hl)}\left[\frac{3}{16}\Omega_1\left(\frac{\partial_z^2}{z^2} - \frac{\partial_z}{z^3}\right) + \frac{1}{2}\Omega_2\frac{\partial_z}{z} + \Omega_3 + 2\frac{\Omega_{\rm tot}}{j_0(2z)}\right]\right.$$
$$\left. + \hat{P}_{(ij}\hat{r}_{hl)}\left[\frac{3}{16}\Omega_1\left(\frac{\partial_z^3}{z} - 2\frac{\partial_z^2}{z^2} + 2\frac{\partial_z}{z^3}\right) + \frac{1}{4}\Omega_2\left(\partial_z^2 + \frac{\partial_z}{z}\right) + \Omega_3 + 2\frac{\Omega_{\rm tot}}{j_0(2z)}\right]\right\}j_0(2z),$$

$$T_{ijhl}^{x^3y} = \frac{\tau^2b^4}{40}\left\{\hat{r}_{ijhl}\left[3\Omega_1\partial_z^4 + 6\Omega_2\partial_z^2 + 3\Omega_3\right] + \hat{r}_{(ij}\hat{r}_{hl)}\left[3\Omega_1\left(\frac{\partial_z^2}{z^2} - \frac{\partial_z}{z^3}\right) + 2\Omega_2\frac{\partial_z}{z} + \Omega_3\right]\right.$$
$$\left. + \hat{P}_{(ij}\hat{r}_{hl)}\left[3\Omega_1\left(\frac{\partial_z^3}{z} - 2\frac{\partial_z^2}{z^2} + 2\frac{\partial_z}{z^3}\right) + \Omega_2\left(\partial_z^2 + \frac{\partial_z}{z}\right) + \Omega_3\right]\right\}j_0(z), \tag{B1}$$

where the set of five parameters is defined as follows:

$$\Omega = 2\sin(\theta)^2 - 1, \quad \Omega_1 = \Omega^2, \quad \Omega_2 = \Omega\frac{\Omega+1}{2},$$
$$\Omega_3 = \left(\frac{\Omega+1}{2}\right)^2, \quad \Omega_{\rm tot} = \frac{\Omega_1}{5} - \frac{2\Omega_2}{3} + \Omega_3. \tag{B2}$$

These five parameters appear very naturally in the derivation of the velocity correlators that is why we decided not to reduce the expressions to a single dependency on Ω. Note that $\Omega = 1$ for an incompressible flow and $\Omega = -1$ for a fully irrotational one.

APPENDIX C: WENTZEL-KRAMERS-BRILLOUIN SOLUTIONS DERIVATION

The scaling solution that has been derived in the previous section only works if the term $\sqrt{\eta/\eta_t}$ is neglected. However, to include effects due to a finite magnetic resistivity we should not systematically neglect it. A WKB approximation can be used to evaluate the solution of Eq. (67) including the finite resistivity.

1. WKB approximation

The WKB approximation is first introduced in 1926 [81,82]. In particular, this approximation method has been extensively used in quantum mechanics to solve the Schrödinger equation [83,84]. Formally the method can be used to solve equations of the type

$$\frac{d^2\Theta}{dx^2}+p(x)\Theta=0, \tag{C1}$$

where the WKB solutions to this equation are linear combinations of

$$\Theta=\frac{1}{p^{1/4}}\exp\left(\pm i\int p^{1/2}\,dx\right). \tag{C2}$$

We call turning points the value of x where $p(x)$ is zero. In a given interval if $p(x)<0$ the solution is in the form of growing and decaying exponential however if $p(x)>0$ we have an oscillatory regime. Moreover, the solutions need to satisfy boundary conditions; it is especially common to impose $\Theta(x)\to 0$ for $x\to\pm\infty$.

2. Magnetic spectrum and growth rate at finite magnetic Reynolds number

In the context of dynamos the WKB approximation is commonly used to derive the growth rate of the two-point correlation function of magnetic fluctuations. Reconsider Eq. (67), which is valid in the limit $z\ll 1$. To apply the WKB approximation we define a new coordinate, which is more convenient to use [34], as $e^x=\bar{z}\equiv\sqrt{\eta_t/\eta}z$. With this new coordinate Eq. (67) becomes

$$\begin{aligned}&\left(\frac{d^2\tilde{M}_{\mathrm{L}}}{dx^2}-\frac{d\tilde{M}_{\mathrm{L}}}{dx}\right)\left(\bar{\tau}\epsilon(\theta)+\frac{5-\Omega}{5(\Omega+3)}+\frac{2}{\bar{z}^2}\right)\\&+\frac{d\tilde{M}_{\mathrm{L}}}{dx}\left(6\bar{\tau}\epsilon(\theta)+6\frac{5-\Omega}{5(\Omega+3)}+\frac{8}{\bar{z}^2}\right)\\&+\tilde{M}_{\mathrm{L}}\left(\bar{\tau}\zeta(\theta)+\frac{8}{\Omega+3}-\gamma\right)=0.\end{aligned} \tag{C3}$$

To simplify notations we rewrite

$$\left(\frac{d^2\tilde{M}_{\mathrm{L}}}{dx^2}-\frac{d\tilde{M}_{\mathrm{L}}}{dx}\right)A(x,\theta)+\frac{d\tilde{M}_{\mathrm{L}}}{dx}B(x,\theta)+\tilde{M}_{\mathrm{L}}C(x,\theta)=0, \tag{C4}$$

where the three functions are simply

$$\begin{aligned}A(x,\theta)&=\bar{\tau}\epsilon(\theta)+\frac{5-\Omega}{5(\Omega+3)}+\frac{2}{\bar{z}^2},\\B(x,\theta)&=6\bar{\tau}\epsilon(\theta)+6\frac{5-\Omega}{5(\Omega+3)}+\frac{8}{\bar{z}^2},\\C(x,\theta)&=\bar{\tau}\zeta(\theta)+\frac{8}{\Omega+3}-\gamma.\end{aligned} \tag{C5}$$

We further assume that $\tilde{M}_{\mathrm{L}}$ can be expressed as a product of two functions $\tilde{M}_{\mathrm{L}}=g(x)W(x)$. The idea is to impose certain relations on $g(x)$ such that all first-order derivatives of $W(x)$ are canceled, leading us to an equation that has the desired form. If we take

$$\frac{dg}{dx}=g\frac{A(x,\theta)-B(x,\theta)}{2A(x,\theta)}, \tag{C6}$$

then we find the desired equation Eq. (C1) for $W(x)$ with

$$p(x)=\frac{1}{A^2}\left[AC-\frac{1}{2}(B'A-A'B)-\frac{1}{4}(A-B)^2\right], \tag{C7}$$

where primes denote derivative with respect to x and the three functions are given by Eq. (C5). After some computation we can even show that

$$p(x)=\frac{A_0\bar{z}^4-B_0\bar{z}^2-9}{(2+F\bar{z}^2)^2}, \tag{C8}$$

where, for convenience, we set the following three functions of the DOC:

$$\begin{aligned}A_0&=\left(\bar{\tau}\epsilon(\theta)+\frac{5-\Omega}{5(\Omega+3)}\right)\\&\quad\times\left\{\frac{7+5\Omega}{4(\Omega+3)}+\bar{\tau}\left[\zeta(\theta)-\frac{25}{4}\epsilon(\theta)\right]-\gamma\right\},\\B_0&=2\gamma+19\bar{\tau}\epsilon(\theta)-2\bar{\tau}\zeta(\theta)+\frac{15-19\Omega}{5(\Omega+3)},\\F&=\bar{\tau}\epsilon(\theta)+\frac{5-\Omega}{5(\Omega+3)}.\end{aligned} \tag{C9}$$

Recall that we are interested in the solution for the range $z_\eta\ll z\ll 1$ which implies roughly that $1\ll\bar{z}\ll R_{\mathrm{M}}^{1/2}$. If we take the limit of very small $\bar{z}$, $x\to-\infty$, then we see that $p\to-9/4$. As $\bar{z}$ increases $p(x)$ increases too, let us call the first turning point $\bar{z}_0$. We can guess from the evaluation of γ in Sec. V D 2 that A_0 is very small compared to B_0. Indeed when plugging in the value for γ we found previously, we obtain that A_0 goes to zero while B_0 has a part independent on $\bar{\tau}$. In particular, it implies that $\bar{z}_0$ is large enough to neglect the constant terms in the equation of $p(x)$ (i.e., $\bar{z}_0\gg 1$). The opposite limit of very large $\bar{z}$, $x\to\infty$, is not described by Eq. (C3) as it is valid only in the small z limit. We need to go back to Eq. (55) and use that in the limit of very large z the velocity correlators and their derivatives should go to zero. After some computation we obtain for the highest contribution

$$p(x)\sim-2e^{2x}\frac{(1+\eta_t/\eta)\gamma_0}{V(\theta,\eta_t,\eta,\bar{\tau})^2}, \tag{C10}$$

such that $p(x)<0$ in this limit. Note that we do not need to specify the exact form of $V(\theta,\eta_t,\eta,\bar{\tau})$ as the denominator is always positive. In this formula we also neglected terms that depend on $\bar{\tau}$ in the numerator as they should always be smaller than η_t/η or γ_0 which are both positive. Such a form means that $p(x)$ must have gone through another zero at some point that we call $\bar{z}_1$. To simplify the treatment we will say that Eq. (C8) is valid for $z < 1$ and Eq. (C10) is valid for $z > 1$. The boundary between the two can be taken to be z_1 such that $\bar{z}_1 \sim R_{\mathrm{M}}^{1/2}$. In fact we will find that the final results have a small dependence on the exact value of z_1 such that we can approximate it without changing the conclusions [40,55,59]. To summarize we consider that we have damped solutions for $\bar{z} \ll \bar{z}_0$ and $\bar{z}_1 \ll \bar{z}$ and an oscillatory one for $\bar{z}_0 \ll \bar{z} \ll \bar{z}_1$. The exponentially growing solutions are discarded as $M_{\mathrm{L}}(z)$ must remain finite at both $z = 0$ and $z = \infty$.

For the oscillatory solution to match the two damped regime we have to require [85,86]

$$\int_{x_0}^{x_1} p(x)^{1/2}\, dx = \frac{(2n+1)\pi}{2}, \tag{C11}$$

where n is an integer. This condition is key to determine the growth rate γ of the two-point correlation of the magnetic field. In the context of this work we only consider the fastest eigen-mode given by $n = 0$. As we already mentioned the constant terms in Eq. (C8) can be neglected which makes the solution to Eq. (C11) exact. Evaluating the integral gives

$$\begin{aligned}\int_{x_0}^{x_1} p(x)^{1/2}\, dx &= \int_{\bar{z}_0}^{\bar{z}_1} \frac{p(z)^{1/2}}{z}\, dz,\\ &\simeq \int_{\bar{z}_0}^{\bar{z}_1} \frac{\sqrt{A_0 z^2 - B_0}}{F z^2}\, dz,\\ &= \frac{\sqrt{A_0}}{F}\left\{\ln\left(\frac{z_1}{z_0} + \sqrt{\frac{z_1^2}{z_0^2} - 1}\right) - \sqrt{1 - \frac{z_0^2}{z_1^2}}\right\},\end{aligned} \tag{C12}$$

where to go from the first to the second line we used that $\bar{z}_0 \sim \sqrt{B_0/A_0} > 0$. We can thus use the condition of Eq. (C11), square both sides, and isolate the growth rate. The growth rate is finally given by

$$\begin{aligned}\gamma = -\left(\frac{\pi}{\ln(R_{\mathrm{M}})}\right)^2\left[\frac{5-\Omega}{5(\Omega+3)} + \bar{\tau}\epsilon(\theta)\right]\\ + \frac{7+5\Omega}{4(\Omega+3)} + \bar{\tau}\left[\zeta(\theta) - \frac{25}{4}\epsilon(\theta)\right],\end{aligned} \tag{C13}$$

where again we plugged the self-consistent value for γ_0. Note that in this equation we also used the self-consistent evaluations $\bar{z}_0 \sim \ln(R_{\mathrm{M}})$ and $\bar{z}_1 \sim R_{\mathrm{M}}^{1/2}$, such that we neglected $\bar{z}_0$ compared to $\bar{z}_1$.

In the oscillatory range $1 \ll \bar{z}_0 \ll \bar{z} \ll \bar{z}_1$ the WKB solution is thus given by

$$W(x) \sim \left(\frac{\ln(R_{\mathrm{M}})}{\pi}\right)^{1/2} \cos\left[\frac{\pi}{\ln(R_{\mathrm{M}})} \ln\left(\frac{z}{z_0}\right)\right]. \tag{C14}$$

In this limit we see that Eq. (C6) can be simplified such that $g'(x) \to -5g(x)/2$ which gives $g(x) \sim e^{-5x/2}$. The two-point magnetic correlation function is then also scaling as $z^{-5/2}$. So finally we find the equation for the longitudinal two-point magnetic correlation function in the region $z_\eta \ll z \ll 1$,

$$M_{\mathrm{L}}(z,t) = e^{\gamma \bar{t}} z^{-5/2} M_0 \cos\left[\frac{\pi}{\ln(R_{\mathrm{M}})} \ln\left(\frac{z}{z_0}\right)\right], \tag{C15}$$

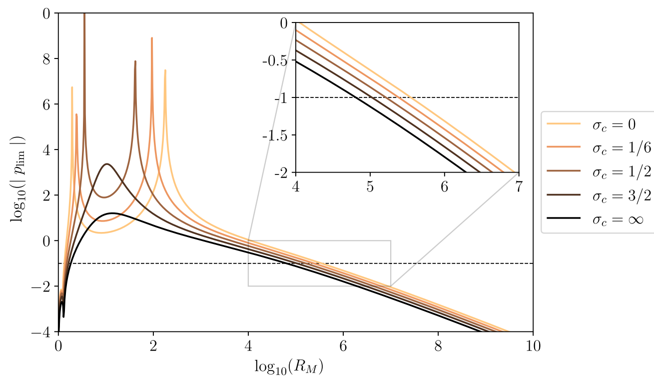

FIG. 3. p_{lim} as a function of the magnetic Reynolds number R_{M} for $\mathrm{St} = 10^{-2}$ and a couple of DOC σ_c. The dashed line corresponds to the threshold over p_{lim} such that the intersections with the curves give the threshold $R_{\mathrm{M,thresh}}$ for which the WKB approximation holds.

where γ is given by Eq. (C13).

3. Validity of the WKB approximation

It can be showed that if we plug the solutions of the WKB approximation into Eq. (C1) we arrive at the following equation:

$$\frac{d^2\Theta}{dx^2} + \left(1 + \frac{1}{4p(x)^2}\frac{d^2p}{dx^2} - \frac{3}{16p(x)^3}\left(\frac{dp}{dx}\right)^2\right)p(x)\Theta = 0, \tag{C16}$$

such that we retrieve the initial problem to solve only if

$$\begin{aligned}p_{\mathrm{lim}} &\equiv \frac{1}{4p(x)^2}\frac{d^2p}{dx^2} - \frac{3}{16p(x)^3}\left(\frac{dp}{dx}\right)^2\\ &= \frac{z^2p''(z) + zp'(z)}{4p(z)^2} - \frac{3z^2p'(z)^2}{16p(z)^3}\end{aligned} \tag{C17}$$

is very small compared to 1. Here primes denote derivatives with respect to the z variable. Furthermore, in a similar way to Schober *et al.* [15], we consider that the criterion of validity for our WKB approximation is $|p_{\mathrm{lim}}| < 0.1$. We find that p_{lim} depends not only on the magnetic Reynolds number but also on St, σ_c, and z_c. We define here z_c to be the scale at which we evaluate $p(z)$ and it derivatives. As the WKB approximation is valid between the two zeros of $p(z)$ we must impose $z_0 \ll z_c \ll z_1$. Until now, we only ask R_{M} to be very large, but the latter criterion gives us a way to quantify it. In particular, we use the expressions derived earlier for $p(z)$ and $\bar{\tau}$ to define a threshold on R_{M} for which we consider that the derived results are valid[4]. To respect the conditions imposed on z_c, we take $z_c = (z_0 + z_1)/2$. Although the scale can seem arbitrary, we find only a slight dependency on it as long as z_c is not too close to z_0 or z_1. In Fig. 3 we present p_{lim} for a fixed $\mathrm{St} = 10^{-2}$ and a few DOC. Again, St being tiny its exact value does not highly impact $R_{\mathrm{M,thresh}}$. It appears that the threshold of this work, regarding R_{M} is around 5×10^5. More precisely, the $R_{\mathrm{M,thresh}}$

[4]Note that this threshold on R_{M} is completely unrelated to the critical value of magnetic Reynolds number for which the dynamo can exist.

threshold decreases until it reaches $\sim 7 \times 10^4$ when the DOC goes to infinity. The results derived in this work concerning the magnetic field are thus valid for most astrophysical objects where the fluctuation dynamo plays a major role. Note that from Fig. 3 we also have a valid WKB approximation for very small $R_{\rm M}$. We can exclude this range of validity as we derived our generalized Kanzantsev equation (i.e., the expansion with respect to St) with the condition that $R_{\rm M}$ was a large number.

APPENDIX D: PROOF OF EQ. (71)

Let us start by expressing the magnetic power spectrum as the Fourier transform of the magnetic two-point correlation and take the Fourier transform of this expression

$$M_k(k) = 2\pi k^2 \hat{M}_{ii}(k) = \frac{k^2}{(2\pi)^2} \int M_{ii}(r) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r}. \tag{D1}$$

Now use the properties of $M(r)$ to derive the following:

$$\begin{aligned} M_k(k) &= \frac{k^2}{2\pi} \int r^2 \sin(\theta) M_{ii}(r) e^{ikr\cos(\theta)}\, dr d\theta, \\ &= \frac{ik}{2\pi} \int r M_{ii}(r)(e^{-ikr} - e^{ikr})\, dr, \\ &= \frac{1}{\pi} \int kr(3M_{\rm L}(r) + r\partial_r M_{\rm L}(r)) \sin(kr)\, dr, \\ &= \frac{1}{\pi} \int kr M_{\rm L}(r)(\sin(kr) - \cos(kr))\, dr, \\ &= \frac{1}{\pi} \int (kr)^3 M_{\rm L}(r) j_1(kr)\, dr, \end{aligned} \tag{D2}$$

where to go from the third to the fourth line we integrated by parts. It is pretty obvious from the definition of $rM_{\rm L}(r)$ that the boundary terms just go to zero.

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