Elastic constants of zero-temperature amorphous solids

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Elastic constants of zero-temperature amorphous solids are given as the difference between the Born term, which results from a hypothetical affine deformation of an amorphous solid, and a correction term, which originates from the fact that the deformation of an amorphous solid due to an applied stress is, at the microscopic level, nonaffine. Both terms are non-negative and thus it is *a priori* not obvious that the resulting elastic constants are non-negative. In particular, theories that approximate the correction term may spuriously predict negative elastic constants and thus an instability of an amorphous solid. Here we derive alternative expressions for elastic constants of zero-temperature amorphous solids that are explicitly non-negative. These expressions provide a useful blueprint for approximate theories for elastic constants and sound damping in zero-temperature amorphous solids.

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I. INTRODUCTION

Elastic constants of zero-temperature crystalline solids consisting of particles with pairwise additive interactions and with one atom per unit cell are given by the well-known Born formulas [1]. However, already for crystalline solids with more than one atom per unit cell the elastic constants are given as the difference between the Born term and a correction term originating from an internal relaxation within the unit cell upon deformation [2,3]. The correction term becomes more important for amorphous solids which are completely devoid of a crystalline lattice [4,5]. The correction term is rationalized in terms of the so-called *nonaffine displacement field*, which originates from the inherent disorder of amorphous solids [4–7]. In the following we will refer to the two terms in the standard expression for the elastic constants of zero-temperature amorphous solids [5,8,9] as the Born term and the correction or nonaffine term.

Both the Born term and the nonaffine term are positive definite. Thus, it is *a priori* not obvious whether their difference is non-negative, as elastic constants should be. This problem becomes important if an approximate theory is used to calculate the correction term. If such a theory overestimates the magnitude of the correction term, it may predict a spurious instability of the amorphous solid in question.

We recall that a similar situation occurred in the context of the description of the dynamics of stochastic systems, e.g., overdamped colloidal suspensions and kinetically constrained models. In this case, interaction-induced or constraint-induced change of a relaxation rate was initially described in terms of a correction term that was *subtracted* from the relaxation rate of the noninteracting system. Approximate theories for the correction term often resulted in a qualitatively incorrect description of the relaxation [10] or predicted spurious dynamic transitions [11]. This situation was resolved after the formal expression for the relaxation rate was rewritten in terms of a new quantity called the irreducible memory function [12,13], resulting in an expression that was explicitly non-negative. This new formulation led to useful approximate descriptions of strongly interacting colloidal suspensions [14].

Here we achieve a similar goal for the elastic constants of zero-temperature amorphous solids. We derive alternative expressions for elastic constants that are explicitly non-negative. The elastic constants are expressed as a ratio of the Born term squared and a sum of the Born term and an alternative nonaffine correction term, which is positive definite. We expect that the alternative expressions will allow for a reformulation of existing approximate descriptions of sound propagation in amorphous solids.

II. STATEMENT OF THE PROBLEM

We consider an amorphous zero-temperature solid consisting of N particles interacting via a spherically symmetric pairwise additive potential. We assume that particles' positions correspond to a local minimum of the potential energy, i.e., an inherent structure [15]. We are interested in small displacements of the particles from their inherent structure positions. The quantity that describes the response of the system to external perturbations is the Hessian,

$$\mathcal{H}_{il} = -\frac{\partial \mathbf{F}_i(\{\mathbf{R}_m\})}{\partial \mathbf{R}_l}, \tag{1}$$

where $\mathbf{R}_i$, $i = 1, \ldots, N$ denote the inherent structure positions of the particles and $\mathbf{F}_i(\{\mathbf{R}_m\})$ is the total force acting on particle i, which depends on the subset $\{\mathbf{R}_m\}$ of other particles,

$$\mathbf{F}_i(\{\mathbf{R}_m\}) = -\frac{\partial}{\partial \mathbf{R}_i} \sum_{j\neq i} V(R_{ij}), \tag{2}$$

with $R_{ij} = |\mathbf{R}_{ij}| \equiv |\mathbf{R}_i - \mathbf{R}_j|$ being the interparticle distance and $V(r)$ being the pair potential.

We note that each element $\mathcal{H}_{ij}$ is a 3×3 tensor; if needed, we will refer to the components of $\mathcal{H}_{ij}$ and other tensors using Greek indices, e.g., $\mathcal{H}_{i\alpha j\beta} = -\frac{\partial F_{i\alpha}(\{\mathbf{R}_m\})}{\partial R_{j\beta}}$.

Translational invariance implies that uniform displacements from the inherent structure positions do not induce a restoring force. In other words, the Hessian matrix has three linearly independent eigenvectors corresponding to zero eigenvalue. The components of these eigenvectors do not depend on the particles' positions and the eigenvectors can be chosen to point along directions of the coordinate system,

$$\boldsymbol{\mathcal{E}}_{0i}^{\alpha} = N^{-1/2}\hat{\boldsymbol{\alpha}}, \tag{3}$$

where $\hat{\boldsymbol{\alpha}}$ is a unit vector along the α axis.

In principle, there might be additional eigenvectors corresponding to zero eigenvalue, e.g., they would appear if our amorphous solid consisted of disconnected clusters. We assume that the three eigenvectors $\boldsymbol{\mathcal{E}}^{0\alpha}$ are the only eigenvectors of the Hessian corresponding to zero eigenvalue. In other words, we assume that the amorphous solid that we consider is stable. However, our analysis would also work for the rigid backbone of an amorphous solid which contains "rattlers."

Furthermore, we will assume that the solid is, on average, isotropic. For this reason, it has only two independent elastic constants or, equivalently, two independent speeds of sound. To simplify the notation we formulate our approach as the theory for the speeds of sound. The actual elastic constants, the bulk modulus and the shear modulus, can be easily obtained from the speeds of sound.

To derive expressions for the speeds of sound, we consider the following question: if we apply an external periodic force on the particles, what will be the resulting displacement field in the limit of long wavelengths?

To be more specific, we assume that at initial time $t = 0$ we turn on a periodic force acting on the particles. Force on particle i is given by

$$\mathbf{f}_i = \hat{\mathbf{e}} e^{-i\mathbf{k}\cdot\mathbf{R}_i}, \tag{4}$$

where $\hat{\mathbf{e}}$ is a unit vector that is parallel, $\hat{\mathbf{e}}_L = \hat{\mathbf{k}}$, or perpendicular, $\hat{\mathbf{e}}_T$, to wave vector $\mathbf{k}$ for parallel (bulk) or transverse (shear) perturbations. Note that we assumed a unit amplitude of the force; since we are working in the harmonic approximation, the strength of the force does not play any role.

As a result of external force (4) the particles get displaced from their inherent structure positions. To mimic the procedure used in computer simulations (sample deformation followed by relaxation [16]), we assume that after the force is turned on, displacements $\mathbf{u}_i$ evolve according to overdamped dynamics with relaxation time τ,

$$\tau \partial_t \mathbf{u}_i(t) = -\sum_j \mathcal{H}_{ij} \cdot \mathbf{u}_j(t) + \mathbf{f}_i. \tag{5}$$

Here $\mathbf{u}_i$ is the displacement of particle i from its inherent structure position $\mathbf{R}_i$.

The dynamics described by Eqs. (5) is not the real dynamics of the system. However, it is a useful auxiliary process that allows us to reach the final state of the deformed system.

We are interested in the long-time limit of the displacements. We will analyze the evolution defined by Eqs. (5) in the Laplace space,

$$z\tau \mathbf{u}_i(z) = -\sum_j \mathcal{H}_{ij} \cdot \mathbf{u}_j(z) + \mathbf{f}_i/z. \tag{6}$$

Writing Eq. (6), we used the fact that before the force was turned on, the solid was un-deformed, i.e., $\mathbf{u}_i(t = 0) = 0$.

The formal solution of Eqs. (6) reads

$$\mathbf{u}_i(z) = \sum_j [z\tau + \mathcal{H}]_{ij}^{-1} \cdot \mathbf{f}_j/z. \tag{7}$$

In the small z limit, i.e., for $z\tau \ll 1$, the displacement field is given by

$$\mathbf{u}_i(z \to 0) = \sum_j [\mathcal{H}]_{ij}^{-1} \cdot \mathbf{f}_j/z. \tag{8}$$

Thus, the long-time limits of the displacements $\mathbf{u}_i(t \to \infty)$ satisfy the following equations:

$$\mathbf{u}_i(t \to \infty) = \sum_j [\mathcal{H}]_{ij}^{-1} \cdot \mathbf{f}_j. \tag{9}$$

Equations (9) express force balance after deformation and they may have been written directly. However, we find it convenient to use Eqs. (6) as the starting point of our analysis.

As discussed above, the real microscopic displacements are in general nonaffine. However, on the basis of the macroscopic theory of elasticity, we expect that after force (4) was applied, in the limit of small magnitude of the wave vector $k = |\mathbf{k}|$ there will be an affine component of the displacements that will be linearly related to the force, with the coefficient of proportionality that is proportional to the inverse of the product of the square of the speed of sound and the square of the wave vector $\mathbf{k}$. In the small z limit the affine component of the displacement field will be given by

$$\mathbf{u}_i^{\text{aff}}(z) = (k^2c^2)^{-1}\mathbf{f}_i/z. \tag{10}$$

To compare with relations derived from microscopic considerations we rewrite Eq. (10) as follows:

$$k^2c^2\mathbf{u}_i^{\text{aff}}(z) = \mathbf{f}_i/z. \tag{11}$$

Depending on whether the external force is parallel or perpendicular to the wave vector $\mathbf{k}$, one should use in Eqs. (10) and (11) the longitudinal c_L or the transverse c_T speed of sound.

Our goal is to derive, in the limit of $z\tau \to 0$, a microscopic version of Eq. (11). In this way we will obtain expressions for the speeds of sound. In the next section we re-derive the standard expression and in the following section we derive an alternative expression that is explicitly positive definite.

III. RE-DERIVATION OF THE STANDARD EXPRESSION FOR SPEEDS OF SOUND

We define projection operator $\mathcal{P}$ that selects the affine part of the displacement field and orthogonal projection $\mathcal{Q}$,

$$\mathbf{u}_i^{\text{aff}} = \mathcal{P}\mathbf{u}_i = e^{-i\mathbf{k}\cdot\mathbf{R}_i}\hat{\mathbf{e}}\frac{1}{N}\sum_j e^{i\mathbf{k}\cdot\mathbf{R}_j}\hat{\mathbf{e}} \cdot \mathbf{u}_j, \tag{12}$$

$$\mathcal{Q}\mathbf{u}_i = \mathbf{u}_i - \mathcal{P}\mathbf{u}_i. \tag{13}$$

Next, we apply $\mathcal{P}$ to both sides of Eq. (6) and we also insert $\mathcal{P}+\mathcal{Q}=1$ between the Hessian and the displacement field,

$$z\tau\mathcal{P}\mathbf{u}_i(z) = -\sum_j \mathcal{P}\mathcal{H}_{ij}\cdot(\mathcal{P}+\mathcal{Q})\mathbf{u}_j(z) + \mathcal{P}\mathbf{f}_i/z$$
$$\equiv -\sum_j \mathcal{P}\mathcal{H}_{ij}\cdot(\mathcal{P}+\mathcal{Q})\mathbf{u}_j(z) + \mathbf{f}_i/z. \tag{14}$$

Then, we apply apply $\mathcal{Q}$ to both sides of Eq. (6) and again we insert the sum of $\mathcal{P}$ and $\mathcal{Q}$ between the Hessian and the displacement field,

$$z\tau\mathcal{Q}\mathbf{u}_i(z) = -\sum_j \mathcal{Q}\mathcal{H}_{ij}\cdot(\mathcal{P}+\mathcal{Q})\mathbf{u}_j(z). \tag{15}$$

Finally, we formally solve Eq. (15) for $\mathcal{Q}\mathbf{u}_i(z)$, substitute the result into Eq. (14), and in this way we obtain

$$\left\{z\tau + \mathcal{PHP} - \mathcal{PHQ}\frac{1}{z\tau+\mathcal{QHQ}}\mathcal{QHP}\right\}\mathcal{P}\mathbf{u} = \mathbf{f}/z, \tag{16}$$

where to simplify the notation we omitted summations over particles and indices of the Hessian and the displacement and force fields.

To recover standard expressions for the speeds of sound we need to investigate the small wave vector limit of the second and third term at the left-hand side of Eq. (16) and then take the small $z\tau$ limit. The sum of the second and third terms will give the product of the squares of the wave vector and the speed of sound, see Eq. (11).

Restoring summation over particles and the indices, we can analyze the second term at the left-hand side of (16) as follows:

$$\sum_j \mathcal{P}\mathcal{H}_{ij}\mathcal{P}\cdot\mathcal{P}\mathbf{u}_j$$
$$= \frac{1}{N^2}\hat{\mathbf{e}}e^{-i\mathbf{k}\cdot\mathbf{R}_i}\sum_{m,l}\hat{\mathbf{e}}\cdot\mathcal{H}_{lm}\cdot\hat{\mathbf{e}}e^{-i\mathbf{k}\cdot\mathbf{R}_{lm}}\sum_j e^{i\mathbf{k}\cdot\mathbf{R}_j}\hat{\mathbf{e}}\cdot\mathbf{u}_j$$
$$= \left[\frac{k^2}{2N}\sum_l\sum_{m\neq l}\hat{\mathbf{e}}\cdot\frac{\partial^2 V(R_{lm})}{\partial\mathbf{R}_l^2}\cdot\hat{\mathbf{e}}(\hat{\mathbf{k}}\cdot\mathbf{R}_{lm})^2 + o(k^2)\right]\mathcal{P}\mathbf{u}_i, \tag{17}$$

where we used the fact that the zeroth order in k term vanishes due to translational symmetry and the first-order term vanishes due to $i\leftrightarrow j$ symmetry.

For an isotropic system tensorial quantity

$$\mathcal{V}(\hat{\mathbf{k}}) = \frac{1}{2N}\sum_l\sum_{m\neq l}\frac{\partial^2 V(R_{lm})}{\partial\mathbf{R}_l^2}(\hat{\mathbf{k}}\cdot\mathbf{R}_{lm})^2 \tag{18}$$

consists of two independent components, longitudinal and transverse. These components are proportional to Born approximations for the longitudinal and transverse speed of sound squared,

$$\mathcal{V}(\hat{\mathbf{k}}) = c_{LB}^2\hat{\mathbf{k}}\hat{\mathbf{k}} + c_{TB}^2(1-\hat{\mathbf{k}}\hat{\mathbf{k}}), \tag{19}$$

where c_{LB} and c_{TB} are given by

$$c_{LB}^2 = \frac{1}{2N}\sum_i\sum_{j\neq i}\hat{\mathbf{k}}\cdot\frac{\partial^2 V(R_{ij})}{\partial\mathbf{R}_i^2}\cdot\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{R}_{ij})^2 \tag{20}$$

and

$$c_{TB}^2 = \frac{1}{4N}[\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}}]:\sum_i\sum_{j\neq i}\frac{\partial^2 V(R_{ij})}{\partial\mathbf{R}_i^2}(\hat{\mathbf{k}}\cdot\mathbf{R}_{ij})^2, \tag{21}$$

where the double dot product denotes contraction of the two tensors. Combining Eqs. (17)–(19), we get the following expression for the small wave-vector limit of the second term at the left-hand side of Eq. (16),

$$\sum_j \mathcal{P}\mathcal{H}_{ij}\mathcal{P}\cdot\mathcal{P}\mathbf{u}_j$$
$$= \left[c_{LB}^2k^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2 + c_{TB}^2k^2(1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2)\right]\mathcal{P}\mathbf{u}_i. \tag{22}$$

The analysis of the third term at the left-hand side of Eq. (16) is a bit more tedious; it is presented in Appendix A. The final result for the small wave vector and the subsequent small $z\tau$ limit of the third term is

$$-\sum_{j,l,m}\mathcal{P}\mathcal{H}_{il}\mathcal{Q}[z\tau+\mathcal{QHQ}]^{-1}_{lm}\mathcal{Q}\mathcal{H}_{mj}\mathcal{P}\cdot\mathcal{P}\mathbf{u}_j$$
$$= \left\{\Delta c_L^2k^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2 + \Delta c_T^2k^2[1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2]\right\}\mathcal{P}\mathbf{u}_i, \tag{23}$$

where the contributions to the speeds of sound due to nonaffine effects, Δc_L and Δc_T, can be written in terms of tensorial field $\mathcal{W}_j$,

$$\mathcal{W}_j(\hat{\mathbf{k}}) = \sum_{l\neq j}\frac{\partial^2 V(R_{jl})}{\partial\mathbf{R}_j^2}\hat{\mathbf{k}}\cdot\mathbf{R}_{jl}, \tag{24}$$

that quantifies the magnitude of the nonaffine response, see Appendix A. The expressions for Δc_L^2 and Δc_T^2 read

$$\Delta c_L^2 = -\hat{\mathbf{k}}\cdot\frac{1}{N}\sum_{l,m}\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}, \tag{25}$$

$$\Delta c_T^2 = -\frac{1}{2}[\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}}]:\frac{1}{N}\sum_{l,m}\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}}). \tag{26}$$

As discussed in Appendix A, terms $\sum_{l,m}\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}})$ in Eqs. (25) and (26) should be understood as $\sum_l\mathcal{W}_l(\hat{\mathbf{k}})\cdot\mathcal{U}_l(\hat{\mathbf{k}})$, where $\mathcal{W}_l(\hat{\mathbf{k}}) = \sum_m\mathcal{H}_{lm}\cdot\mathcal{U}_m(\hat{\mathbf{k}})$. The latter equation has a unique solution since $\mathcal{W}_m(\hat{\mathbf{k}})$ is orthogonal to all eigenvectors of the Hessian corresponding to zero eigenvalue. We note that since the Hessian is non-negative definite, contributions (25) and (26) are negative definite.

Combining Eq. (16) and Eqs. (22) and (23) and taking the $z\tau\to 0$ limit, we recover relation (11) between the affine component of the displacement field and the external force, with speeds of sound squared given by

$$c_L^2 = c_{LB}^2 - \frac{1}{N}\sum_{l,m}\hat{\mathbf{k}}\cdot\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}, \tag{27}$$

$$c_T^2 = c_{TB}^2 - \frac{1}{N}\sum_{l,m}\hat{\mathbf{e}}_T\cdot\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}})\cdot\hat{\mathbf{e}}_T. \tag{28}$$

Expressions (27) and (28) for the speeds of sound squared are equivalent to the standard expressions for elastic constants of zero-temperature amorphous solids [3,5,8]. As discussed earlier, expressions (27) and (28) are not explicitly non-negative and approximate theories for the second terms in these expressions may lead to spurious predictions of instabilities.

IV. ALTERNATIVE EXPRESSION FOR SPEEDS OF SOUND

We now derive alternative expressions for speeds of sound squared which make their non-negative property explicit.

We start by rewriting Eq. (14) as follows:

$$\mathcal{P}\mathbf{u}_i(z) = \sum_j [\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}_{ij}[-z\tau\mathcal{P}\mathbf{u}_j(z) + \mathbf{f}_j/z] - \sum_{j,l}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}_{ij}\mathcal{P}\mathcal{H}_{jl}\cdot\mathcal{Q}\mathbf{u}_l(z). \tag{29}$$

Then, we substitute the right-hand side of Eq. (29) into Eq. (15),

$$z\tau\mathcal{Q}\mathbf{u}_i(z) = -\sum_{j,l}\mathcal{Q}\mathcal{H}_{ij}\mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}_{jl}[-z\tau\mathcal{P}\mathbf{u}_l(z) + \mathbf{f}_l/z] - \sum_j\left[\mathcal{Q}\mathcal{H}_{ij} - \sum_{l,m}\mathcal{Q}\mathcal{H}_{il} \times \mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}_{lm}\mathcal{P}\mathcal{H}_{mj}\right]\cdot\mathcal{Q}\mathbf{u}_j(z). \tag{30}$$

Next, we solve Eq. (30) for $\mathcal{Q}\mathbf{u}_i$,

$$\mathcal{Q}\mathbf{u}_i(z) = -\sum_{l,m,j}\{z\tau + \mathcal{Q}\mathcal{H}\mathcal{Q} - \mathcal{Q}\mathcal{H}\mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}\mathcal{P}\mathcal{H}\mathcal{Q}\}^{-1}_{il} \times \mathcal{Q}\mathcal{H}_{lm}\mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}_{mj}\cdot[-z\tau\mathcal{P}\mathbf{u}_j(z) + \mathbf{f}_j/z]. \tag{31}$$

Finally, we substitute the right-hand side of Eq. (31) into Eq. (29) and solve for $\mathcal{P}\mathbf{u}$, and in this way we obtain the following relation:

$$\left\{z\tau + [\mathcal{P}\mathcal{H}\mathcal{P}]\left[\mathcal{P}\mathcal{H}\mathcal{P} + \mathcal{P}\mathcal{H}\mathcal{Q}\frac{1}{z\tau + \mathcal{Q}\mathcal{H}\mathcal{Q} - \mathcal{Q}\mathcal{H}\mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}\mathcal{P}\mathcal{H}\mathcal{Q}}\mathcal{Q}\mathcal{H}\mathcal{P}\right]^{-1}[\mathcal{P}\mathcal{H}\mathcal{P}]\right\}\mathcal{P}\mathbf{u} = \mathbf{f}/z, \tag{32}$$

where, once again, to simplify the notation we omitted summations over particles and indices of the Hessian, and the displacement and force fields. Equation (32) is the microscopic version of Eq. (11). It is the alternative to Eq. (16) that we were looking for.

We note that the structure of the matrix acting on $\mathcal{P}\mathbf{u}$ is very similar to that obtained by Kawasaki [13]. This matrix was obtained by simple manipulation of the same equations, Eqs. (14) and (15), that were used to obtain the standard relation between $\mathcal{P}\mathbf{u}$ and $\mathbf{f}/z$, Eq. (16). In particular, although a person familiar with Kawasaki's analysis can clearly see in Eq. (32) an object that could be called an "irreducible Hessian," we arrived at Eq. (32) without introducing such a concept.

We emphasize that the above formal construction makes physical sense only if operator $\mathcal{H} - \mathcal{H}\mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}\mathcal{P}\mathcal{H}$ is non-negative definite. We will return to this question at the end of this section.

Again, to recover the alternative expressions for the speeds of sound we need to investigate the small wave-vector limit of the second term at the left-hand side of Eq. (32) and then take the small $z\tau$ limit. The second term will become the product of the squares of the speed of sound and the wave vector, see Eq. (11).

We note that most objects involved in the small wave-vector limit of the second term at the left-hand side of Eq. (32) were already discussed in the context of the small wave-vector limit of Eq. (16). In Appendix B we discuss the additional term, $\mathcal{H}\mathcal{P}[\mathcal{P}\mathcal{H}\mathcal{P}]^{-1}\mathcal{P}\mathcal{H}$, and additional steps needed to derive the small limit of the second term at the left-hand side of Eq. (32).

Using the results of Appendix B we obtain the following expressions for the speeds of sound squared:

$$c_L^2 = \frac{c_{LB}^4}{c_{LB}^2 + \frac{1}{N}\sum_{l,m}\hat{\mathbf{k}}\cdot\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H} - \delta_L\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}}, \tag{33}$$

$$c_T^2 = \frac{c_{TB}^4}{c_{TB}^2 + \frac{1}{N}\sum_{l,m}\hat{\mathbf{e}}_T\cdot\mathcal{W}_l(\hat{\mathbf{k}})\cdot[\mathcal{H} - \delta_T\mathcal{H}]^{-1}_{lm}\cdot\mathcal{W}_m(\hat{\mathbf{k}})\cdot\hat{\mathbf{e}}_T}, \tag{34}$$

where

$$\delta_L\mathcal{H}_{lm} = \frac{1}{N}\mathcal{W}_l(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}c_{LB}^{-2}\hat{\mathbf{k}}\cdot\mathcal{W}_m(\hat{\mathbf{k}}), \tag{35}$$

and

$$\delta_T\mathcal{H}_{lm} = \frac{1}{N}\mathcal{W}_l(\hat{\mathbf{k}})\cdot\hat{\mathbf{e}}_T c_{TB}^{-2}\hat{\mathbf{e}}_T\cdot\mathcal{W}_m(\hat{\mathbf{k}}). \tag{36}$$

Equations (33) and (34) are the expressions for the speeds of sound squared that we propose as alternatives to standard expressions (27) and (28).

Finally, we need to examine the question of the non-negative definite character of the matrices that enter into Eqs. (33) and (34). We note that, as discussed in Appendix B, matrices $\mathcal{H} - \delta_L\mathcal{H}$ and $\mathcal{H} - \delta_T\mathcal{H}$ are obtained as $\mathbf{k} \to 0$ limits of the following matrix:

$$\mathcal{H}_{ij}(\mathbf{k}) - \sum_{m,n}\mathcal{H}_{in}(\mathbf{k})\cdot\hat{\mathbf{e}}\left[\sum_{k,l}\hat{\mathbf{e}}\cdot\mathcal{H}_{kl}(\mathbf{k})\cdot\hat{\mathbf{e}}\right]^{-1}\hat{\mathbf{e}}\cdot\mathcal{H}_{mj}(\mathbf{k}), \tag{37}$$

with $\hat{\mathbf{e}} = \hat{\mathbf{e}}_L$ and $\hat{\mathbf{e}} = \hat{\mathbf{e}}_T$. Next, we recall that the Hessian is non-negative definite. It follows that the wave vector–dependent Hessian $\mathcal{H}(\mathbf{k})$ is also non-negative definite and thus can be written in the following way in terms of its eigenvalues $\omega_a^2 \geqslant 0$ and corresponding eigenvectors $\boldsymbol{\mathcal{E}}_{ai}$, where a labels eigenvectors.

$$\mathcal{H}_{ij}(\mathbf{k}) = \sum_a \omega_a^2 \boldsymbol{\mathcal{E}}_{ai}\boldsymbol{\mathcal{E}}_{aj}. \tag{38}$$

Using (38) we can write a contraction of the matrix (37) with an arbitrary vector $\mathbf{a}_i$ as follows:

$$\sum_a \left[\sum_i \omega_\alpha \boldsymbol{\mathcal{E}}_{ai} \cdot \mathbf{a}_i\right]^2 - \frac{\left\{\sum_a \left[\sum_i \omega_a \boldsymbol{\mathcal{E}}_{ai} \cdot \mathbf{a}_i\right]\left[\sum_i \omega_a \boldsymbol{\mathcal{E}}_{ai} \cdot \hat{\mathbf{e}}\right]\right\}^2}{\sum_b \left[\sum_i \omega_b \boldsymbol{\mathcal{E}}_{bi} \cdot \hat{\mathbf{e}}\right]^2}. \tag{39}$$

Cauchy-Schwarz inequality implies that expression (39) is non-negative definite. Thus, matrices $\mathcal{H} - \delta_L\mathcal{H}$ and $\mathcal{H} - \delta_T\mathcal{H}$ are also non-negative definite, which makes our expressions for speeds of sound squared, Eqs. (33) and (34), well defined.

V. DISCUSSION

We derived exact formulas for elastic constants of zero-temperature elastic solids. In contrast to standard expressions, our formulas are explicitly non-negative.

In practical numerical calculations, elastic constants of zero-temperature elastic solids are determined by explicit deformations of these solids. This procedure requires some care since one needs to balance between two opposite goals: the need to impose small enough deformation to ensure linear response and the need to impose large enough deformation to generate a statistically meaningful response signal. However, it is simpler to implement than using standard exact formulas. We expect that our alternative formulas will also not be competitive with the explicit deformation procedure. However, we hope that these formulas will inspire future theoretical analyses of both elastic constants and speeds of sound and of sound damping in zero-temperature elastic solids.

There were several theoretical analyses of sound propagation in zero-temperature amorphous solids. In particular, the authors of Refs. [17–20] used different diagrammatical analyses to analyze sound propagation, i.e., speeds of sound, and sound attenuation in zero-temperature amorphous solids. Our present contribution suggests that resummations of classes of diagrams should be attempted that would reproduce the structure of our formulas for speeds of sound. We hope to develop such theories in the near future.

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APPENDIX A: ANALYSIS OF THE THIRD TERM IN EQ. (16)

The third term at the left-hand side of Eq. (16) involves "vertices" $\mathcal{P}\mathcal{H}_{il}\mathcal{Q}$ and $\mathcal{Q}\mathcal{H}_{mj}\mathcal{P}$ and the inverse of the projected Hessian $[z\tau + \mathcal{Q}\mathcal{H}\mathcal{Q}]^{-1}_{lm}$. In the small wave-vector limit the vertices are proportional to the magnitude of the wave vector and can be expressed in terms of tensorial field $\mathcal{W}_j$,

$$\mathcal{W}_j(\hat{\mathbf{k}}) = \sum_{l\neq j} \frac{\partial^2 V(R_{jl})}{\partial \mathbf{R}_j^2} \hat{\mathbf{k}} \cdot \mathbf{R}_{jl}. \tag{A1}$$

The action of the vertices on an arbitrary vector $\mathbf{a}_i$ defined at each inherent structure position reads

$$\sum_j \mathcal{Q}\mathcal{H}_{ij}\mathcal{P}\mathbf{a}_j = -i[k\mathcal{W}_i(\hat{\mathbf{k}}) + o(k)] \cdot \mathcal{P}\mathbf{a}_i, \tag{A2}$$

$$\sum_j \mathcal{P}\mathcal{H}_{ij}\mathcal{Q} \cdot \mathbf{a}_j = i[k\mathcal{P}\mathcal{W}_i(\hat{\mathbf{k}}) + o(k)] \cdot \mathbf{a}_i. \tag{A3}$$

Tensorial field $\mathcal{W}_j$ is closely related to vector field $\Xi_{j,\beta\delta}$ introduced by Lemaître and C. Maloney [5],

$$\Xi_{j,\beta\delta} = -\sum_{l\neq j} \frac{\partial^2 V(R_{jl})}{\partial R_{j\beta}\partial \mathbf{R}_j} R_{jl\delta}. \tag{A4}$$

Vector field $\Xi_{j,\beta\delta}$ describes forces due to an affine deformation. Specifically, $\Xi_{j,\beta\delta}$ is proportional to the force on particle j resulting from a deformation along the β direction that linearly depends on the δ coordinates. The $\alpha\beta$ component of $\mathcal{W}_j$ can be expressed in terms of contraction of α component of vector $\Xi_{j,\beta\gamma}$ with $\hat{\mathbf{k}}$,

$$\mathcal{W}_{j\alpha\beta}(\hat{\mathbf{k}}) = \sum_{l\neq j} \frac{\partial^2 V(R_{jl})}{\partial R_{j\alpha}\partial R_{j\beta}} R_{jl\delta}\hat{k}_\delta, \tag{A5}$$

where the Einstein summation convention over repeated Greek indices is adopted.

It follows that if we keep only the leading terms in the magnitude of the wave vector in the vertices, we can rewrite the third term at the left-hand side of Eq. (16) as

$$\begin{aligned}
&k^2\mathcal{P}\mathcal{W}_i(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{Q}\mathcal{H}\mathcal{Q}]^{-1}_{ij} \cdot \mathcal{W}_j(\hat{\mathbf{k}}) \cdot \mathcal{P}\mathbf{u}_j \\
&\quad = k^2 e^{-i\mathbf{k}\cdot\mathbf{R}_i}\hat{\mathbf{e}}\frac{1}{N}\sum_{lm} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{Q}_1\mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lm} \\
&\qquad \times \cdot\mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}\frac{1}{N}\sum_j e^{i\mathbf{k}\cdot\mathbf{R}_j}\hat{\mathbf{e}} \cdot \mathbf{u}_j \\
&\quad = k^2\frac{1}{N}\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{Q}_1\mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}\mathcal{P}\mathbf{u}_i,
\end{aligned} \tag{A6}$$

where we introduced wave vector–dependent Hessian matrix $\mathcal{H}(\mathbf{k})$,

$$\mathcal{H}_{il}(\mathbf{k}) = \mathcal{H}_{il}e^{i\mathbf{k}\cdot(\mathbf{R}_i - \mathbf{R}_l)}, \tag{A7}$$

and a (orthogonal) projection operator $\mathcal{Q}_1$ which acts on an arbitrary vector $\mathbf{a}_i$ as follows:

$$\mathcal{Q}_1 \mathbf{a}_i = \mathbf{a}_i - \hat{\mathbf{e}} N^{-1} \sum_j \hat{\mathbf{e}} \cdot \mathbf{a}_j. \tag{A8}$$

Finally, we show that in the small wave-vector limit projection operators $\mathcal{Q}_1$ in $\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}})[z\tau + \mathcal{Q}_1 \mathcal{H}(\mathbf{k}) \mathcal{Q}_1]^{-1}_{lm} \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}$ can be dropped. To this end we use the procedure described by Ernst and Dorfman [21]. We use the following operator identity:

$$\frac{1}{z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1} = \frac{1}{z\tau + \mathcal{H}(\mathbf{k})} + \frac{1}{z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1} \mathcal{H}(\mathbf{k}) \mathcal{P}_1 \frac{1}{z\tau + \mathcal{H}(\mathbf{k})}, \tag{A9}$$

where projection operator $\mathcal{P}_1$ acts on an arbitrary vector $\mathbf{a}_i$ as follows:

$$\mathcal{P}_1 \mathbf{a}_i = \hat{\mathbf{e}} N^{-1} \sum_j \hat{\mathbf{e}} \cdot \mathbf{a}_j. \tag{A10}$$

Using Eq. (A9) and recalling that $\mathcal{W}_l \mathcal{Q}_1 = \mathcal{W}_l$, we rewrite $\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{Q}_1 \mathcal{H}(\mathbf{k}) \mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}$ as follows:

$$\begin{aligned}
&\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&= \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&+ \sum_{l,m,k,n} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lk} \cdot \mathcal{H}_{kn}(\mathbf{k}) \cdot \\
&\times \mathcal{P}_1 [z\tau + \mathcal{H}(\mathbf{k})]^{-1}_{nm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}.
\end{aligned} \tag{A11}$$

In the second term at the right-hand side of Eq. (A11), we note that

$$\sum_n \mathcal{H}_{kn}(\mathbf{k}) \mathcal{P}_1 = [-ik\mathcal{W}_k(\hat{\mathbf{k}}) + o(k)] \mathcal{P}_1. \tag{A12}$$

Furthermore, we note that

$$\begin{aligned}
&z\tau N^{-1} \sum_{n,m} \hat{\mathbf{e}} \cdot [z\tau + \mathcal{H}(\mathbf{k})]^{-1}_{nm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&= -ikN^{-1} \sum_{n,m} \hat{\mathbf{e}} \cdot \mathcal{W}_n(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})]^{-1}_{nm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&+ O(k).
\end{aligned} \tag{A13}$$

Combining Eqs. (A11)–(A13) we obtain

$$\begin{aligned}
&\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&= \frac{z\tau \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}}{z\tau - k^2 \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}}.
\end{aligned} \tag{A14}$$

Equation (A14) implies that in the small wave-vector limit we can drop projections $\mathcal{Q}_1$ in the last line of Eq. (A6). Note that in the same limit we have $\mathcal{H}(\mathbf{k}) \to \mathcal{H}$.

The final issue concerns the small $z\tau$ limit of the expression $\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}$. We recall that the Hessian has zero eigenvalues. However, the small $z\tau$ limit of $\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}$ is well defined for the following reason. First, we note that

$$\begin{aligned}
&\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&= \sum_l \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot \mathcal{U}_l(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}},
\end{aligned} \tag{A15}$$

where $\mathcal{U}_l(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}$ satisfies the following equation:

$$\mathcal{W}_l(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} = \sum_m [z\tau + \mathcal{H}]_{lm} \cdot \mathcal{U}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}. \tag{A16}$$

We note that $\mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}$ is orthogonal to the space spanned by the eigenvectors of $\mathcal{H}$ and thus in the $z\tau \to 0$ limit the solution of Eq. (A16) is well defined.

We conclude that in the small wave vector and small $z\tau$ limit the last line of Eq. (A6) can be written as

$$k^2 \frac{1}{N} \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [\mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}. \tag{A17}$$

Finally, we note that for an isotropic system tensorial quantity

$$\sum_{l,m} \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [\mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \tag{A18}$$

consists of two independent components, longitudinal and transverse. These components are the contributions to the squares of the speeds of sound due to nonaffine effects. They are the differences between the actual speeds of sound squared and their Born values,

$$\Delta c_L^2 = -\hat{\mathbf{k}} \cdot \frac{1}{N} \sum_{l,m} \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [\mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}, \tag{A19}$$

$$\Delta c_T^2 = -\frac{1}{2} [\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}] : \frac{1}{N} \sum_{lm} \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [\mathcal{H}]^{-1}_{l,m} \cdot \mathcal{W}_m(\hat{\mathbf{k}}). \tag{A20}$$

APPENDIX B: SMALL WAVE-VECTOR LIMIT OF THE SECOND TERM IN EQ. (32)

We start with the analysis of term $\mathcal{PHQ}\{z\tau + \mathcal{QHQ} - \mathcal{QHP}[\mathcal{PHP}]^{-1}\mathcal{PHQ}\}^{-1}\mathcal{QHP}$. Following the transformations similar to those used in writing Eq. (A6), if we keep only the leading terms in the magnitude of the wave vector in "outside" vertices $\mathcal{PHQ}$ and $\mathcal{QHP}$, we can rewrite this term as

$$\begin{aligned}
&k^2 \frac{1}{N} \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{Q}_1 \mathcal{H}(\mathbf{k}) \mathcal{Q}_1 \\
&- \mathcal{Q}_1 \mathcal{H}(\mathbf{k}) \mathcal{P}_1 [\mathcal{P}_1 \mathcal{H}(\mathbf{k}) \mathcal{P}_1]^{-1} \mathcal{P}_1 \mathcal{H}(\mathbf{k}) \mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \mathcal{P},
\end{aligned} \tag{B1}$$

where wave vector–dependent Hessian is defined in Eq. (A7) and projection operators $\mathcal{P}_1$ and $\mathcal{Q}_1$ are defined in Eqs. (A10) and (A8), respectively.

Next, we use an identity analogous to Eq. (A9) to write the following relation:

$$\begin{aligned}
&\sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1 - \mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&\quad = \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k}) - \mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \\
&\qquad + \sum_{l,m,k,n} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H}(\mathbf{k})\mathcal{Q}_1 - \mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{Q}_1]^{-1}_{lk} \\
&\qquad \times \cdot [\mathcal{H}(\mathbf{k}) - \mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})]_{kn} \\
&\qquad \times \cdot \mathcal{P}_1[z\tau + \mathcal{H}(\mathbf{k}) - \mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})]^{-1}_{nm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}.
\end{aligned} \tag{B2}$$

Then we note that

$$\sum_n [\mathcal{H}(\mathbf{k}) - \mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})]_{kn}\mathcal{P}_1 = 0, \tag{B3}$$

which implies that the second term at the right-hand side of Eq. (B2) vanishes. It follows that projections $\mathcal{Q}_1$ in Eq. (B1) do not contribute and can be dropped.

To complete the analysis of term $\mathcal{PHQ}\{z\tau + \mathcal{QHQ} - \mathcal{QHP}[\mathcal{PHP}]^{-1}\mathcal{PHQ}\}^{-1}\mathcal{QHP}$ we need to consider the small wave-vector limit of $\mathcal{H}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}\mathcal{P}_1\mathcal{H}(\mathbf{k})$. We have

$$\begin{aligned}
\sum_{i,j} \mathcal{H}_{ki}(\mathbf{k})\mathcal{P}_1[\mathcal{P}_1\mathcal{H}(\mathbf{k})\mathcal{P}_1]^{-1}_{jl}\mathcal{P}_1\mathcal{H}_{jn}(\mathbf{k}) &= \sum_i \mathcal{H}_{ki} \cdot \hat{\mathbf{e}} e^{i\mathbf{k}\cdot\mathbf{R}_{ki}} \left\{c_{LB}^2 k^2 (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2 + c_{TB}^2 k^2 [1 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2]\right\}^{-1} \frac{1}{N} \sum_j e^{i\mathbf{k}\cdot\mathbf{R}_{jn}} \hat{\mathbf{e}} \cdot \mathcal{H}_{jn} \\
&= \mathcal{W}_k(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \left\{c_{LB}^2 (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2 + c_{TB}^2 [1 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2]\right\}^{-1} \frac{1}{N} \hat{\mathbf{e}} \cdot \mathcal{W}_n(\hat{\mathbf{k}}) + O(k).
\end{aligned} \tag{B4}$$

Thus, the small wave-vector limit of term $\mathcal{PHQ}\{z\tau + \mathcal{QHQ} - \mathcal{QHP}[\mathcal{PHP}]^{-1}\mathcal{PHQ}\}^{-1}\mathcal{QHP}$ reads

$$k^2 \frac{1}{N} \sum_{l,m} \hat{\mathbf{e}} \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot \left[z\tau + \mathcal{H} - \frac{1}{N}\mathcal{W}(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}} \times \left\{c_{LB}^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2 + c_{TB}^2[1 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}})^2]\right\}^{-1} \hat{\mathbf{e}} \cdot \mathcal{W}(\hat{\mathbf{k}})\right]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}\mathcal{P} \tag{B5}$$

To write down the expression for the second term at the left-hand side of Eq. (32) we need to include $\mathcal{PHP}$ terms. The resulting formula is rather long. To simplify it a bit, we will write it for the case of the transverse deformation, i.e., for $\hat{\mathbf{e}} = \hat{\mathbf{e}}_T$,

$$\begin{aligned}
&[\mathcal{PHP}]\left\{\mathcal{PHP} + \mathcal{PHQ}\frac{1}{z\tau + \mathcal{QHQ} - \mathcal{QHP}[\mathcal{PHP}]^{-1}\mathcal{PHQ}}\mathcal{QHP}\right\}^{-1}[\mathcal{PHP}]\Bigg|_T \\
&\quad \to k^2 c_{TB}^2 \left\{k^2 c_{TB}^2 + k^2 \frac{1}{N} \sum_{lm} \hat{\mathbf{e}}_T \cdot \mathcal{W}_l(\hat{\mathbf{k}}) \cdot [z\tau + \mathcal{H} - \delta_T\mathcal{H}]^{-1}_{lm} \cdot \mathcal{W}_m(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}_T\right\}^{-1} c_{TB}^2 k^2 \mathcal{P}
\end{aligned} \tag{B6}$$

where $\delta_T\mathcal{H}$ reads

$$\delta_T\mathcal{H} = \frac{1}{N}\mathcal{W}(\hat{\mathbf{k}}) \cdot \hat{\mathbf{e}}_T c_{TB}^{-2} \hat{\mathbf{e}}_T \cdot \mathcal{W}(\hat{\mathbf{k}}). \tag{B7}$$

Finally, we substitute expression (B6) into Eq. (32), take the small $z\tau$ limit, and obtain the microscopic version of Eq. (11), with the square of the transverse speed of sound given in Sec. IV, Eq. (34). The equation for the longitudinal speed of sound can be obtained by substituting $\hat{\mathbf{e}}_T \to \hat{\mathbf{e}}_L \equiv \hat{\mathbf{k}}$.

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