Heat statistics in the relaxation process of the Edwards-Wilkinson elastic manifold

Yu-Xin Wu [1], Jin-Fu Chen [1], Ji-Hui Pei [1], Fan Zhang [1], and H. T. Quan [1,2,3,*]
[1] *School of Physics, Peking University, Beijing 100871, China*
[2] *Collaborative Innovation Center of Quantum Matter, Beijing 100871, China*
[3] *Frontiers Science Center for Nano-optoelectronics, Peking University, Beijing 100871, China*

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The stochastic thermodynamics of systems with a few degrees of freedom has been studied extensively so far. We would like to extend the study to systems with more degrees of freedom and even further-continuous fields with infinite degrees of freedom. The simplest case for a continuous stochastic field is the Edwards-Wilkinson elastic manifold. It is an exactly solvable model of which the heat statistics in the relaxation process can be calculated analytically. The cumulants require a cutoff spacing to avoid ultraviolet divergence. The scaling behavior of the heat cumulants with time and the system size as well as the large deviation rate function of the heat statistics in the large size limit is obtained.

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I. INTRODUCTION

Historically, people studied thermodynamics in macroscopic systems, such as ideal gas with up to 10^{23} molecules. Due to the huge number of degrees of freedom in the macroscopic scale, it is impossible to extract the trajectories of individual particles explicitly. Hence, it is not possible to study thermodynamics of macroscopic systems in arbitrary far from equilibrium processes. Nevertheless, for mesoscopic systems with only a few degrees of freedom, stochastic dynamics (Langevin equation, Fokker-Planck equation, master equation) provides detailed information about the system. Prominent examples of mesoscopic systems include colloidal particles, macromolecules, nanodevices, and so on [1,2]. In all these examples, researchers focus on the dynamics of a few degrees of freedom of the system whereas coarse graining all the degrees of freedom of the reservoir. Mesoscopic systems can be driven out of equilibrium by external driving, for instance, by varying the temperature or by controlling them with optical tweezers [3–18].

With the equation of motion, e.g., Langevin equation, Fokker-Planck equation, or master equation, researchers are able to establish a framework of thermodynamics for mesoscopic systems in arbitrarily far from equilibrium processes. This is stochastic thermodynamics in which thermodynamic quantities such as work, heat, and entropy production in nonequilibrium processes have been explored extensively in both classical and quantum realms [9,18–48]. In the study of work or heat distribution for extreme nonequilibrium processes, rare events with exponentially small probabilities have dominant contributions making finite sampling error particularly serious. Hence, previous studies, be it experimental or computer simulations, are predominantly for small systems, i.e., those with a few degrees of freedom [49]. Nevertheless, systems with a few degrees of freedom are too special. Therefore, it is desirable to extend the study of stochastic thermodynamics to more complicated systems. We, thus, would like to extend the studies to systems with more degrees of freedom, for example, stochastic fields. Hopefully, in some exactly solvable model, we can obtain analytical results about work and heat distribution. These rigorous results about work or heat distribution in systems with many degrees of freedom not only have pedagogical value but also may bring some insights to the understanding of thermodynamics in extreme nonequilibrium processes as P. W. Anderson once advocated, "More is different" [50]. Although many researchers are interested in the dynamic properties of stochastic fields [51–55], less research is carried out from the perspective of stochastic thermodynamics except [56–60] so far as we know.

In this article, we study the thermodynamics of an elastic manifold whose underlying dynamics is described by the Edwards-Wilkinson (EW) equation [61],

$$\partial_t h(\boldsymbol{x}, t) = \nu \nabla^2 h(\boldsymbol{x}, t) + \xi(\boldsymbol{x}, t), \tag{1}$$

where $h(\boldsymbol{x}, t)$ is the local height at spatial point $\boldsymbol{x}$ at time t, ν is the diffusive coefficient, and $\xi(\boldsymbol{x}, t)$ is the Gaussian white noise. The friction coefficient is set to unity.

The problem we analyze is the relaxation of an elastic manifold described by the EW equation. The elastic manifold is initially put in contact with a heat reservoir at the inverse temperature β'. After initial equilibration with the first heat reservoir at β', the system is detached from it, and is put in contact with a second heat reservoir at the inverse temperature β. The manifold, subsequently, tries to adapt to the working temperature [55]. The relaxation is characterized by the stochastic heat absorbed from/released into the surrounding reservoir during a period of time τ. We are interested in the average and fluctuation of the heat in such a process. We find several generic properties of the average and fluctuating heat in the relaxation process of the EW elastic manifold. By employing the Feynman-Kac method [45,62], we obtain

*Corresponding author: htquan@pku.edu.cn

analytical results of the characteristic function of heat for the EW model during an arbitrary relaxation period τ with an arbitrary diffusive coefficient ν and analyze the scaling behavior of the cumulants of heat with time. Analytical results of the heat statistics bring important insights into understanding the fluctuating property of heat in such a concrete and exactly solvable model. We also verify from the analytical results that the heat statistics satisfy the fluctuation theorem of heat exchange [63]. The large deviation rate function of heat statistics in the large size limit is also analyzed.

The rest of this article is organized as follows. In Sec. II, we introduce the EW model. In Sec. III, we define the stochastic heat and obtain analytical results of the characteristic function of heat using the Feynman-Kac approach. We also compute the cumulants of heat and discuss their scaling behavior with time and the system size. Conclusions are given in Sec. IV.

II. THE MODEL

A d-dimensional elastic manifold, with finite size $2L$ in each direction, joggles under thermal noise. Its local height $h(\boldsymbol{x}, t)$ at spatial point $\boldsymbol{x}$ at time t evolves according to the EW equation Eq. (1), which takes the form of a multivariable overdamped Langevin equation [1]. The thermal noise $\xi(\boldsymbol{x}, t)$ is white in nature, i.e., $\langle\xi(\boldsymbol{x}, t)\rangle = 0$, $\langle\xi(\boldsymbol{x}, t)\xi(\boldsymbol{x}', t')\rangle = 2/\beta\delta^d(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')$. The EW energy is just that of a massless field with Hamiltonian $H_S = \nu\int d\boldsymbol{x}[\nabla h(\boldsymbol{x}, t)]/2$. Here, the subscript S refers to the system.

Initially, the system is prepared in an equilibrium state with the inverse temperature β' characterized by a Gibbs-Boltzmann distribution in the configuration space, i.e., the probability $\mathcal{P}(h, t)$ to find the system in the configuration $\{h(\boldsymbol{x}, t)\}$ is the Gibbs-Boltzmann distribution,

$$\mathcal{P}(h, 0) = \mathcal{N}'^{-1}\exp\left[-\beta'\frac{\nu}{2}\int d\boldsymbol{x}[\nabla h(\boldsymbol{x}, 0)]^2\right], \quad (2)$$

where $\mathcal{N}'$ is the normalization constant,

$$\mathcal{N}' = \int dh(\boldsymbol{x}, 0)\exp\left[-\beta'\frac{\nu}{2}\int d\boldsymbol{x}[\nabla h(\boldsymbol{x}, 0)]^2\right]. \quad (3)$$

Here, the integration in the normalization constant is taken over all possible initial configurations, whereas the one in the exponential factor is taken over all spatial points.

After initial equilibration, the system is detached from the first heat reservoir and is placed in contact with a second heat reservoir at the inverse temperature β, which is different from β'. The elastic manifold, subsequently, relaxes towards the equilibrium state at inverse temperature β since no external driving is involved. The heat absorbed/released is a fluctuating variable for the system undergoing stochastic motion. We are interested in the heat statistics in such a relaxation process.

For a finite-size manifold, we take periodic boundary conditions along each $\boldsymbol{x}$ direction. Following Refs. [1,52], we employ a Fourier representation of the height field,

$$h(\boldsymbol{x}, t) = \frac{1}{(2L)^d}\sum_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}h_{\boldsymbol{q}}(t), \quad (4)$$

$$h_{\boldsymbol{q}}(t) = \int d\boldsymbol{x}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}h(\boldsymbol{x}, t), \quad (5)$$

where $\boldsymbol{q}$ represents a wave vector with $q_j = n_j\pi/L$ [$j = x, y, z \ldots$, $n_j = \pm1, \pm2\cdots$, and $h_{\boldsymbol{q}=\boldsymbol{0}}(t) = 0$ for all time t] [55]. The evolution of the Fourier component is given by

$$\partial_t h_{\boldsymbol{q}}(t) = -\nu q^2 h_{\boldsymbol{q}}(t) + \xi_{\boldsymbol{q}}(t), \quad (6)$$

$$\langle\xi_{\boldsymbol{q}}(t)\rangle = 0, \quad (7)$$

$$\langle\xi_{\boldsymbol{q}}(t)\xi_{\boldsymbol{q}'}(t')\rangle = \frac{2(2L)^d}{\beta}\delta(t-t')\delta_{\boldsymbol{q},-\boldsymbol{q}'}. \quad (8)$$

The normalization constant in Eq. (3) can be computed as

$$\mathcal{N}' = \int d\{h_{\boldsymbol{q}}(0)\}\exp\left[-\frac{\beta'\nu}{(2L)^d}\sum_{\boldsymbol{q}(q_j\geqslant\pi/L)} q^2 h_{\boldsymbol{q}}(0)h_{-\boldsymbol{q}}(0)\right]$$
$$= \prod_{\boldsymbol{q}(q_j\geqslant\pi/L)}\frac{\pi(2L)^d}{\beta'\nu q^2}, \quad (9)$$

where q^2 stands for the modulus square of $\boldsymbol{q}$.

The probability density of system state $\mathcal{P}(h, t)$ evolves under the governing of the Fokker-Planck equation,

$$\frac{\partial\mathcal{P}(h, t)}{\partial t} = -\int d\boldsymbol{x}\frac{\delta}{\delta h}[\nu\nabla^2 h(\boldsymbol{x}, t)\mathcal{P}(h, t)]$$
$$+\frac{1}{\beta}\int d\boldsymbol{x}\frac{\delta^2}{\delta h^2}\mathcal{P}(h, t). \quad (10)$$

In the Fourier space, the probability of the height field configuration is the product of the real and the imaginary parts over all modes,

$$\mathcal{P}(\{h_{\boldsymbol{q}}\}, t) = \prod_{\boldsymbol{q}(q_j\geqslant\frac{\pi}{L})}\mathcal{P}(h_{\boldsymbol{q}}, t) = \prod_{\boldsymbol{q}(q_j\geqslant\frac{\pi}{L})}\mathcal{P}^R(h^R_{\boldsymbol{q}}, t)\mathcal{P}^I(h^I_{\boldsymbol{q}}, t), \quad (11)$$

where

$$h^R_{\boldsymbol{q}} = \mathrm{Re}(h_{\boldsymbol{q}}), \quad h^I_{\boldsymbol{q}} = \mathrm{Im}(h_{\boldsymbol{q}}). \quad (12)$$

The Fokker-Planck equation in the Fourier space can be then written into two independent parts: the real part and the imaginary part [64],

$$\frac{\partial\mathcal{P}^{R,I}(h^{R,I}_{\boldsymbol{q}}, t)}{\partial t} = \frac{\partial\left(\nu q^2 h^{R,I}_{\boldsymbol{q}}\mathcal{P}^{R,I}\right)}{\partial h^{R,I}_{\boldsymbol{q}}} + \frac{(2L)^d}{2\beta}\frac{\partial^2\mathcal{P}^{R,I}}{\partial\left(h^{R,I}_{\boldsymbol{q}}\right)^2}. \quad (13)$$

It is noteworthy that Eq. (6) represents an overdamped Brownian motion for each individual mode. Thus, we obtain the analytical solution to the probability distribution $\mathcal{P}^{R,I}(h^{R,I}_{\boldsymbol{q}}, t)$, which is also thermal for time-dependent effective inverse temperature $\beta_{\boldsymbol{q}}(t)$,

$$\mathcal{P}^{R,I}(h^{R,I}_{\boldsymbol{q}}, t) = \sqrt{\frac{\beta_{\boldsymbol{q}}(t)\nu q^2}{\pi(2L)^d}}\exp\left[-\frac{\beta_{\boldsymbol{q}}(t)\nu q^2}{(2L)^d}\left(h^{R,I}_{\boldsymbol{q}}\right)^2\right]. \quad (14)$$

Here, the time-dependent effective inverse temperature $\beta_{\boldsymbol{q}}(t)$ is

$$\beta_{\boldsymbol{q}}(t) = \frac{\beta\beta'}{(\beta-\beta')\exp(-2\nu q^2 t)+\beta'}. \quad (15)$$

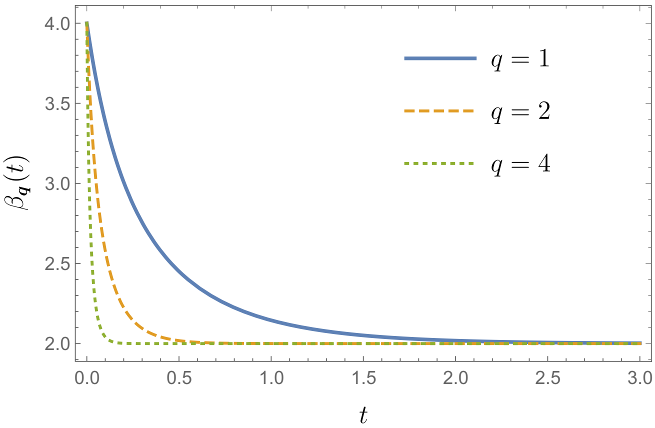

FIG. 1. The effective inverse temperature $\beta_{\boldsymbol{q}}(t)$ for different modes $q = 1, 2, 4$ (q stands for the modulus of mode $\boldsymbol{q}$) as a function of time. Parameters take values $d = 1,\ \nu = 1,\ \beta' = 4,$ and $\beta = 2$.

It transitions from the initial inverse temperature β' to the final β following Newton's law of cooling. As can be seen from Fig. 1, high-energy modes equilibrate more quickly than lower ones.

Having introduced the model, in the following, we will calculate the heat statistics in the relaxation process.

III. HEAT STATISTICS

In this section, we study heat statistics of the EW elastic manifold in the relaxation process. First, we obtain the analytical results of heat statistics and verify the fluctuation theorem of heat exchange. Second, we study the asymptotic behavior of the cumulants. Third, we calculate the large deviation function of heat statistics in the large size limit.

A. Characteristic function

Since no external driving is applied to the system, no work is performed during the relaxation process. The fluctuating heat Q absorbed from the heat reservoir equals the energy difference between the initial and the final states over a time period τ,

$$Q = H_S[h(\boldsymbol{x}, \tau)] - H_S[h(\boldsymbol{x}, 0)]. \tag{16}$$

The characteristic function of heat $\chi_\tau(u)$ is defined as the Fourier transform of the heat distribution,

$$\chi_\tau(u) = \int dQ \exp(iuQ)\mathcal{P}(Q, \tau). \tag{17}$$

Here, $\mathcal{P}(Q, \tau)$ stands for the probability of the heat Q transferred from the heat reservoir to the system during the period of time τ.

According to Eq. (16), the characteristic function of heat of the relaxation process is encoded in the characteristic function of the initial and the final internal energy. This fact enables us to calculate the characteristic function of heat by utilizing the Fokker-Planck equation with a modified initial condition, i.e., the characteristic function of heat $\chi_\tau(u)$ can be calculated using the Feynman-Kac method [45,47,62],

$$\begin{aligned}\chi_\tau(u) &= \langle \exp(iuQ)\rangle \\ &= \int dh\, e^{iuH_S[h(\boldsymbol{x},\tau)]}\eta(h, \tau),\end{aligned} \tag{18}$$

where the probability-density-like function $\eta(h, \tau)$ satisfies Eqs. (10) and (13) with the initial condition,

$$\eta(h, 0) = e^{-iuH_S[h(\boldsymbol{x},0)]}\mathcal{P}(h, 0). \tag{19}$$

The probability-density-like function $\eta(h, \tau)$ is solved in the Fourier space (see the Appendix for a detailed derivation), and we obtain the characteristic function of heat for the relaxation process over a time period of τ,

$$\chi_\tau(u) = \beta\beta' \prod_{\boldsymbol{q}(q_j \geqslant \pi/L)} \frac{\exp(2\nu q^2\tau)}{-u(i\beta' - i\beta - u)[\exp(2\nu q^2\tau) - 1] + \beta\beta' \exp(2\nu q^2\tau)}. \tag{20}$$

The wave vector component in each direction only takes positive discrete values $q_j = n_j\pi/L,\ n_j = 1, 2\cdots$.

We do a self-consistent check of the analytic result Eq. (20) from three aspects:

(1) The distribution of heat satisfies the conservation of probability,

$$\chi_\tau(0) = 1. \tag{21}$$

(2) One can see the characteristic function of heat exhibits the following symmetry:

$$\chi_\tau(u) = \chi_\tau(i\beta' - i\beta - u), \tag{22}$$

indicating that the heat distribution satisfies the fluctuation theorem of heat exchange [23,47,63],

$$\langle e^{iuQ}\rangle = \langle e^{(-iu+\beta-\beta')Q}\rangle, \tag{23}$$

namely,

$$\frac{\mathcal{P}(Q, \tau)}{\mathcal{P}(-Q, \tau)} = e^{(\beta'-\beta)Q}. \tag{24}$$

By setting $u = 0$, we obtain the relation $\chi_\tau(i\beta' - i\beta) = 1$, which is exactly the fluctuation theorem of heat exchange in the integral form $\langle \exp[-(\beta' - \beta)Q]\rangle = 1$ [63].

Moreover, since every mode contributes independently, we can decompose the total heat exchange into the contribution from every mode $Q = \sum_{\boldsymbol{q}(q_j \geqslant \pi/L)} Q_{\boldsymbol{q}}$, where $Q_{\boldsymbol{q}} = \nu q^2(2L)^{-d}[h_{\boldsymbol{q}}(\tau)h_{-\boldsymbol{q}}(\tau) - h_{\boldsymbol{q}}(0)h_{-\boldsymbol{q}}(0)]$, and the heat distribution is in a product form $\mathcal{P}(Q, \tau) = \prod_{\boldsymbol{q}(q_j \geqslant \pi/L)} \mathcal{P}(Q_{\boldsymbol{q}}, \tau)$. The heat distribution of individual modes also satisfies a fluctuation theorem,

$$\frac{\mathcal{P}(Q_{\boldsymbol{q}}, \tau)}{\mathcal{P}(-Q_{\boldsymbol{q}}, \tau)} = e^{(\beta'-\beta)Q_{\boldsymbol{q}}}. \tag{25}$$

(3) In the long time limit $\tau \to \infty$, the characteristic function becomes

$$\lim_{\tau\to\infty} \chi_\tau(u) = \prod_{\boldsymbol{q}(q_j \geqslant \pi/L)} \frac{\beta\beta'}{(u+i\beta)(u-i\beta')}. \tag{26}$$

This result, independent of the relaxation dynamics, can be written in the form

$$\lim_{\tau\to\infty} \chi_\tau(u) = \langle e^{iuH_S(h(\boldsymbol{x},\tau))}\rangle_\beta \langle e^{-iuH_S[h(\boldsymbol{x},0)]}\rangle_{\beta'}, \tag{27}$$

where the initial distribution (thermal equilibrium with the inverse temperature β') and the final distribution (thermal equilibrium with the inverse temperature β) are sampled independently, reflecting the complete thermalization of the system [29]. This result agrees with our intuition.

B. Cumulants

The cumulants of heat can be derived by taking derivatives of the logarithm of the characteristic function $\chi_\tau(u)$ with respect to u at $u = 0$ with the first cumulant representing the average heat, and the second one standing for the variance.

The average heat is

$$\begin{aligned}\langle Q\rangle &= \frac{1}{i}\frac{d\ \ln\ \chi_\tau(u)}{du}\bigg|_{u=0}\\ &= \sum_{\boldsymbol{q}(\frac{\pi}{a}\geqslant q_j\geqslant\frac{\pi}{L})} \frac{[1-\exp(-2\nu q^2\tau)](\beta'-\beta)}{\beta\beta'}\\ &= \frac{\beta'-\beta}{\beta\beta'}\left(\frac{\pi}{L}\right)^{-d}\int_{\frac{\pi}{L}}^{\frac{\pi}{a}} d\boldsymbol{q}[1-\exp(-2\nu q^2\tau)].\end{aligned} \tag{28}$$

A cutoff π/a of the wave vector is needed to avoid ultraviolet divergence, i.e., we introduce a smallest spacing a in this elastic manifold [1,65,66]. Since we consider a continuous field, the cutoff spacing is always much smaller than the system size $a \ll L$. We will see that the choice of the value of a will influence the average heat [see Fig. 2(b) inset plot]. We rewrite the average heat $\langle Q\rangle$ with a change in the variable $\boldsymbol{s} = L\boldsymbol{q}$,

$$\langle Q\rangle = \frac{(\beta'-\beta)}{\beta\beta'\pi^d} f\left(\frac{\nu\tau}{L^2}\right), \tag{29}$$

where

$$\begin{aligned}f(r) &= \int_{\pi}^{\frac{L\pi}{a}} d\boldsymbol{s}[1-e^{-2rs^2}]\\ &= \left(\frac{L-a}{a}\pi\right)^d + \left(\frac{\pi}{8r}\right)^{\frac{d}{2}}\\ &\quad\times\left[\mathrm{Erf}(\pi\sqrt{2r}) - \mathrm{Erf}\left(\frac{\pi L\sqrt{2r}}{a}\right)\right]^d.\end{aligned} \tag{30}$$

$\mathrm{Erf}(r)$ is the error function.

In the following, we discuss the asymptotic behavior of the average heat as a function of time. For the one-dimensional case, the average heat as a function of time is illustrated in Fig. 2. At the initial stage for $\tau \ll a^2/\nu$,

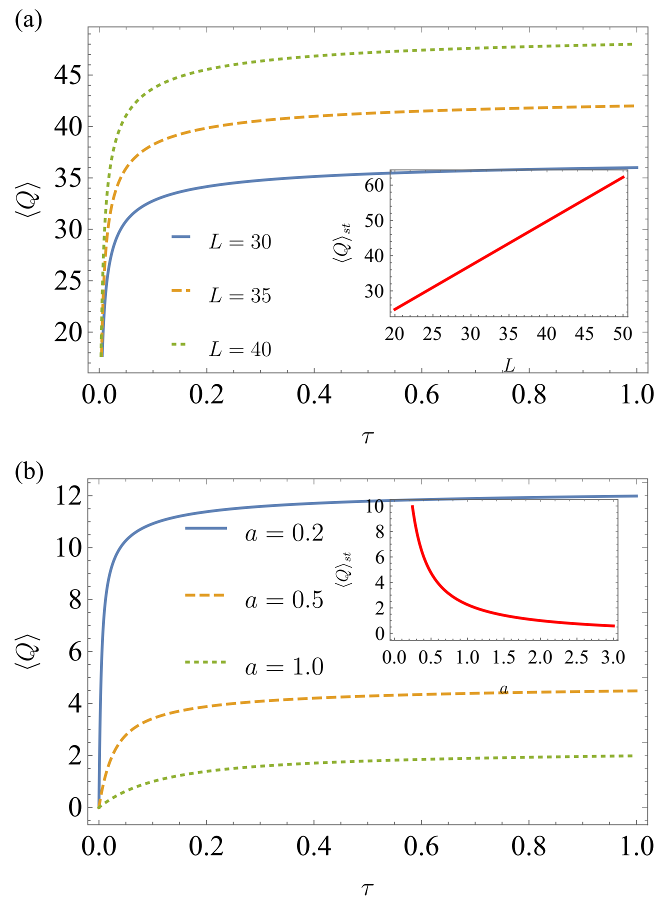

FIG. 2. Average heat as a function of time. Parameters for both panels: $d = 1$, $\nu = 1$, $\beta' = 4$, and $\beta = 2$. (a) $\langle Q\rangle$ as a function of τ for three system sizes $L = 30$, 35, and 40, fixing $a = 0.2$. The inset: the saturation value of average heat $\langle Q\rangle_{\mathrm{st}}$ as a function of system size L. (b) $\langle Q\rangle$ as a function of τ for three cutoff spacings $a = 0.2, 0.5,$ and 1.0, fixing $L = 10$. The inset: the saturation value of average heat $\langle Q\rangle_{\mathrm{st}}$ as a function of cutoff spacing a.

$$\langle Q\rangle \approx \frac{2\pi^2}{3a^2}\frac{(\beta'-\beta)}{\beta\beta'}\nu\tau\frac{L}{a}. \tag{31}$$

The average heat initially increases with time linearly. This is Newton's law of cooling.

For the intermediate time, $a^2/\nu \ll \tau \ll L^2/\nu$,

$$\langle Q\rangle \approx \frac{(\beta'-\beta)}{\beta\beta'}\frac{L}{a}\left(1-\frac{a}{\sqrt{8\nu}}\tau^{-1/2}\right). \tag{32}$$

It exhibits $\tau^{-1/2}$ scaling with time.

In the long time limit, for $\tau \gg L^2/\nu$,

$$\langle Q\rangle \to \frac{\beta'-\beta}{\beta\beta'}\frac{L}{a}, \tag{33}$$

the average heat saturates, which is a consequence of the equipartition theorem. The saturation value of heat is an extensive quantity, which scales linearly with the system size L. It will not diverge for a finite spacing a as a result of finite resolution.

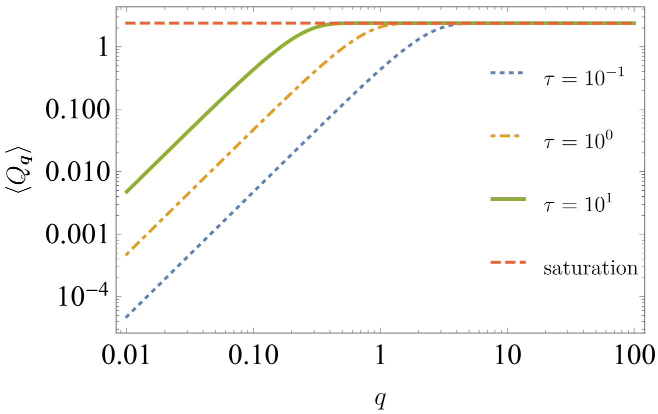

FIG. 3. Average heat of a mode $\boldsymbol{q}$ for different time durations. The parameters take values $L = 30,\ d = 1,\ \nu = 1,\ \beta' = 4,$ and $\beta = 2$ and the curves correspond to three values of time delay $\tau = 10^1,\ 10^0,$ and 10^{-1} from the bottom to the top. The dashed line stands for the saturation value.

From Eq. (28), one can see the average heat for every $\boldsymbol{q}$ mode is

$$\langle Q_{\boldsymbol{q}} \rangle = \frac{\beta' - \beta}{\beta\beta'} \left(\frac{\pi}{L}\right)^{-d} [1 - \exp(-2\nu q^2 \tau)]. \qquad (34)$$

As we can see from this equation and Fig. 3, heat transfer occurs mainly through high-energy modes, and thermalization occurs in high-energy modes more quickly than that in lower ones.

For fixed time duration τ, in the small wave vector limit, i.e., $2\nu q^2 \tau \ll 1$, it increases with time linearly

$$\langle Q_{\boldsymbol{q}} \rangle = 2\nu\tau \frac{\beta' - \beta}{\beta\beta'} \left(\frac{\pi}{L}\right)^{-d} q^2, \qquad (35)$$

which is the Newton's law of cooling.

On the other hand, if one takes the large wave vector limit, i.e., $2\nu q^2 \tau \gg 1$, the average heat reaches the asymptotic value,

$$\langle Q_{\boldsymbol{q}} \rangle = \frac{\beta' - \beta}{\beta\beta'} \left(\frac{\pi}{L}\right)^{-d}, \qquad (36)$$

which is the result of the equipartition theorem.

From the analytical result of heat statistics Eq. (20), we can also study the variance of heat. The variance of heat is defined as $\mathrm{var}(Q) = \langle Q^2 \rangle - \langle Q \rangle^2$ and can be calculated as

$$\begin{aligned} \mathrm{var}(Q) &= \frac{1}{i^2} \frac{d^2 \ln \chi_\tau(u)}{du^2} \bigg|_{u=0} \\ &= \left(\frac{\pi}{L}\right)^{-d} \frac{1}{\beta^2 \beta'^2} \int_{\frac{\pi}{L}}^{\frac{\pi}{a}} d\boldsymbol{q}\, e^{-4\nu q^2 \tau} (-1 + e^{2\nu q^2 \tau}) \\ &\quad \times [(-1 + e^{2\nu q^2 \tau})\beta^2 + 2\beta\beta' + (-1 + e^{2\nu q^2 \tau})\beta'^2] \\ &= \frac{1}{\beta^2 \beta'^2 \pi^d} g\left(\frac{\nu\tau}{L^2}\right), \end{aligned} \qquad (37)$$

where

$$\begin{aligned} g(r) = \int_{\pi}^{\frac{L\pi}{a}} d\boldsymbol{s} [&(\beta^2 + \beta'^2)(1 - 2e^{-2rs^2} + e^{-4rs^2}) \\ &+ 2\beta\beta'(-e^{-4rs^2} + e^{-2rs^2})]. \end{aligned}$$

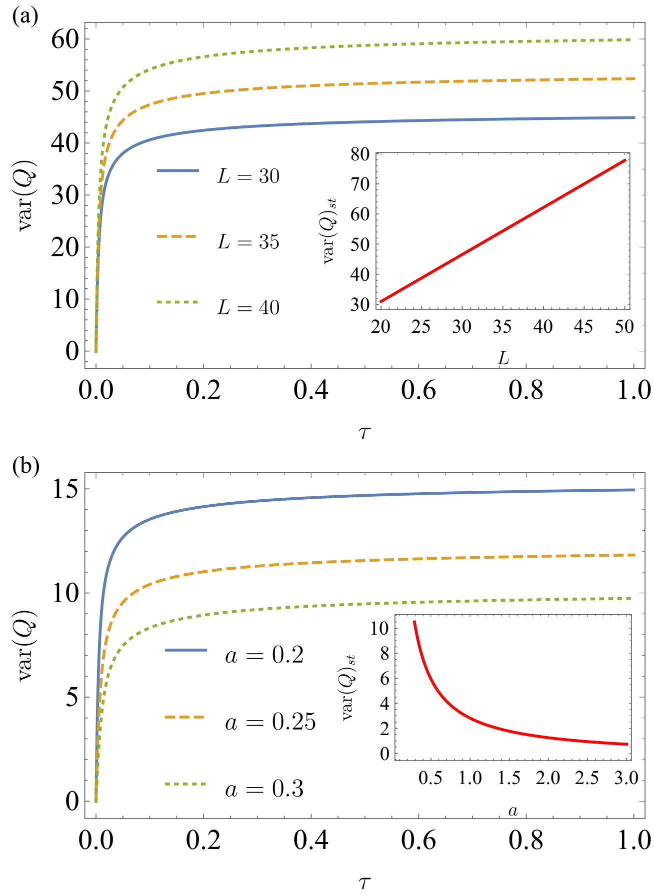

FIG. 4. Variance of heat as a function of time. Parameters for both panels: $d = 1,\ \nu = 1,\ \beta' = 4,$ and $\beta = 2$. (a) $\mathrm{var}(Q)$ as a function of τ for three system sizes $L = 30,\ 35,$ and 40, fixing $a = 0.2$. The inset: the saturation value of heat variance $\mathrm{var}(Q)_{\mathrm{st}}$ as a function of system size L. (b) $\mathrm{var}(Q)$ as a function of τ for three cutoff spacings $a = 0.2,\ 0.25,$ ans 0.3, fixing $L = 10$. The inset: the saturation value of heat variance $\mathrm{var}(Q)_{\mathrm{st}}$ as a function of cutoff spacing a.

In the one-dimensional case, for $\tau \ll a^2/\nu$, we have

$$\mathrm{var}(Q, \tau) \approx \frac{4\pi^2}{3a^2\beta\beta'} \nu\tau \frac{L}{a}. \qquad (38)$$

It grows with time linearly in the very beginning.

For $a^2/\nu \ll \tau \ll L^2/\nu$,

$$\mathrm{var}(Q, \tau) \approx \frac{4\pi^4 \nu^2 \tau^2}{5\beta^2 \beta'^2 a^4} (\beta^2 - 3\beta\beta' + \beta'^2) \frac{L}{a}. \qquad (39)$$

It scales as τ^2 as time elapses.

Finally, for $\tau \gg L^2/\nu$, it reaches the saturation value in the long time,

$$\mathrm{var}(Q, \tau) \approx \frac{\beta^2 + \beta'^2}{\beta^2 \beta'^2} \frac{L}{a}. \qquad (40)$$

As can be seen from Fig. 4, the variance of heat depends on the cutoff spacing a as well. Similar to the average heat, the saturation value of variance increases linearly with the system size L and will not diverge for finite spacing a. Higher order cumulants of heat can be analyzed in a similar way.

C. Large deviation rate function

We can also study the large deviation rate function of the heat statistics in the large size limit.

The scaled cumulant generating function (SCGF) $\phi(u,\tau)$ of heat per volume over time τ, which is defined through

$$\left\langle \exp\left[(2L)^d u \frac{Q}{(2L)^d}\right]\right\rangle \asymp_{L\to\infty} e^{(2L)^d\phi(u,\tau)}, \quad (41)$$

or

$$\begin{aligned}\phi(u,\tau) &= \lim_{L\to\infty}\frac{1}{(2L)^d}\ln\left\langle \exp\left[(2L)^d u\frac{Q}{(2L)^d}\right]\right\rangle \\ &= \lim_{L\to\infty}\frac{1}{(2L)^d}\ln\,\chi_\tau(-iu)\end{aligned} \quad (42)$$

can be computed by

$$\phi(u,\tau) = \lim_{L\to\infty} -\frac{1}{(2L)^d}\int_{\frac{\pi}{L}}^{\frac{\pi}{a}} d\boldsymbol{q}\,\ln\left(\frac{-u(\beta'-\beta+u)}{\beta\beta'}[1-\exp(-2\nu q^2\tau)]+1\right). \quad (43)$$

The large deviation rate function for heat per volume over time τ is just the Legendre-Fenchel transform of the SCGF [67],

$$\begin{aligned}I\left(\frac{Q}{(2L)^d},\tau\right) &= \lim_{L\to\infty} -\frac{1}{(2L)^d}\ln\mathcal{P}\left(\frac{Q}{(2L)^d},\tau\right)\\ &= \sup_{u\in\mathbb{R}}\left\{u\frac{Q}{(2L)^d}-\phi(u,\tau)\right\}.\end{aligned} \quad (44)$$

We emphasize that the large deviation rate function of work distribution in the large size limit has been studied in other models previously (see, e.g., Refs. [49,68]). Here we report the large deviation rate function of heat in the large size limit.

With the large deviation rate function Eq. (44), we can write down the probability distribution of heat per volume over time τ as

$$\mathcal{P}\left(\frac{Q}{(2L)^d},\tau\right) \asymp_{L\to\infty} \exp\left[-(2L)^d I\left(\frac{Q}{(2L)^d},\tau\right)\right], \quad (45)$$

which demonstrates the dependence of the heat distribution on the system size. And the fluctuation theorem of heat exchange Eq. (24) can also be formulated in terms of the large deviation rate function.

IV. CONCLUSION

Previously, the stochastic thermodynamics of systems with a few degrees of freedom have been studied extensively both in classical and quantum realms [9,18–48]. However, less is known in systems with many degrees of freedom. What new results the complexity of many degrees of freedom will bring to stochastic thermodynamics remains largely unexplored.

In this article, we extend previous studies about the stochastic thermodynamics of systems with a few degrees of freedom to a continuous field. We compute the heat statistics in the relaxation process of an exactly solvable model—an elastic manifold whose underlying dynamics can be described by the Edwards-Wilkinson equation. By employing the Feynman-Kac approach, we calculate analytically the characteristic function of heat for any relaxation time. The analytical results of heat statistics have pedagogical value and may bring important insights to the understanding of thermodynamics in the relaxation process of continuous fields. For example, the cumulants of heat in such a system with many degrees of freedom require a spatial cutoff to avoid the ultraviolet divergence, which is a consequence of finite resolution. We also analyze the scaling behavior of the cumulants with time and the system size. In addition, the large deviation rate function of heat in the large size limit is analyzed.

This paper can be regarded as an early step in the stochastic thermodynamics of continuous fields. More interesting problems remain to be explored, such as the definitions for the thermodynamic quantities in every space-time point, the extension to nonlinear models, the work statistics in the presence of external driving and so on. Studies about these issues will be given in our future work.

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APPENDIX: DERIVATION OF EQ. (20)

Similar to the probability density distribution, the modified function $\eta(h,t)$ can be written as the product of the imaginary part and the real part over all modes in the Fourier space,

$$\eta(\{h_{\boldsymbol{q}}\},t) = \prod_{\boldsymbol{q}(q_j\geqslant\pi/L)}\eta^R\left(h^R_{\boldsymbol{q}},t\right)\eta^I\left(h^I_{\boldsymbol{q}},t\right). \quad (A1)$$

The probability-density-like function $\eta^{R,I}(h^{R,I}_{\boldsymbol{q}},t)$ follows the same time evolution as $\mathcal{P}^{R,I}(h^{R,I}_{\boldsymbol{q}},t)$ in Eq. (13),

$$\frac{\partial\eta^{R,I}\left(h^{R,I}_{\boldsymbol{q}},t\right)}{\partial t} = \frac{\partial\left(\nu q^2 h^{R,I}_{\boldsymbol{q}}\eta^{R,I}\right)}{\partial h^{R,I}_{\boldsymbol{q}}} + \frac{(2L)^d}{2\beta}\frac{\partial^2\eta^{R,I}}{\partial\left(h^{R,I}_{\boldsymbol{q}}\right)^2}, \quad (A2)$$

with the initial condition,

$$\eta(h,0) = e^{-iuH_S(0)}\mathcal{P}(h,0). \quad (A3)$$

Due to the quadratic nature of the EW equation, we assume the time-dependent solution $\eta(h,t)$ takes a Gaussian form at any time,

$$\eta^{R,I}\left(h^{R,I}_{\boldsymbol{q}},t\right) = \sqrt{\frac{\beta'\nu q^2}{\pi(2L)^d}}\exp\left[-A(t)\left(h^{R,I}_{\boldsymbol{q}}\right)^2+B(t)\right]. \quad (A4)$$

The coefficients are governed by the following ordinary differential equations:

$$\dot{A}(t) = -\frac{2(2L)^d}{\beta}A^2(t) + 2A(t)\nu q^2, \quad \text{(A5)}$$

$$\dot{B}(t) = -\frac{(2L)^d}{\beta}A(t) + \nu q^2. \quad \text{(A6)}$$

The initial condition Eq. (A3) gives way to the initial values of the coefficients,

$$A(0) = \frac{(\beta' + iu)\nu}{(2L)^d}q^2, \quad \text{(A7)}$$

$$B(0) = 0. \quad \text{(A8)}$$

By solving the above equations we obtain

$$A(t) = \frac{1}{(2L)^d}\frac{e^{2\nu q^2 t}\beta(u - i\beta')\nu q^2}{(e^{2\nu q^2 t} - 1)u - i[\beta + (e^{2\nu q^2 t} - 1)\beta']}, \quad \text{(A9)}$$

$$B(t) = \nu q^2 t + \frac{1}{2}\ln\left[\frac{i\beta}{u - i\beta' + i\beta + (i\beta' - u)e^{2\nu q^2 t}}\right]. \quad \text{(A10)}$$

Substituting Eqs. (A9) and (A10) into Eq. (A4), we arrive at

$$\begin{aligned}
\eta(\{h_{\boldsymbol{q}}\}, t) &= \prod_{q_i \geqslant \pi/L} \eta^R\big(h^R_{\boldsymbol{q}}, t\big)\eta^I\big(h^I_{\boldsymbol{q}}, t\big) \\
&= \prod_{\boldsymbol{q}(q_i \geqslant \pi/L)} \frac{\beta'\nu q^2}{\pi(2L)^d}\frac{i\beta\, \exp(2\nu q^2 t)}{u - i\beta' + i\beta + (i\beta' - u)\exp(2\nu q^2 t)} \\
&\quad \times \exp\left\{-\frac{1}{(2L)^d}\frac{\exp(2\nu q^2 t)\beta(u - i\beta')\nu q^2}{[-1 + \exp(2\nu q^2 t)]u - i[\beta - \beta' + \beta'\exp(2\nu q^2 t)]}\left[\big(h^R_{\boldsymbol{q}}\big)^2 + \big(h^I_{\boldsymbol{q}}\big)^2\right]\right\}. \quad \text{(A11)}
\end{aligned}$$

Substituting it into Eq. (18), we obtain the characteristic function of heat Eq. (20) of the EW elastic manifold in the relaxation process.

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