

Single active particle in a harmonic potential: Question about the existence of the Jarzynski relation

Grzegorz Szamel 

Department of Chemistry, Colorado State University, Fort Collins, Colorado 80523, USA



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The interest in active matter stimulates the need to generalize thermodynamic description and relations to active matter systems, which are intrinsically out of equilibrium. One important example is the Jarzynski relation, which links the exponential average of work done in an arbitrary process connecting two equilibrium states with the difference of the free energies of these states. Using a simple model system, a single thermal active Ornstein-Uhlenbeck particle in a harmonic potential, we show that if the standard stochastic thermodynamics definition of work is used, the Jarzynski relation is not generally valid for processes connecting stationary states of active matter systems.

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I. INTRODUCTION

Active matter systems [1–6] consist of particles that consume energy from their environment to propel themselves. These systems are intrinsically out of equilibrium and therefore, as a matter of principle, standard relations derived for equilibrium systems do not apply to them. Some of these relations can be generalized by introducing *effective* thermodynamic parameters, but, at least for now, there is no general framework for doing this and thus, the validity of such procedures has to be checked on a case by case basis.

The Jarzynski relation [7] applies to processes connecting two equilibrium states of a system in contact with a heat reservoir at temperature T (here and in the following we set Boltzmann constant $k_B = 1$). While the work done on such a system in an infinitely slow (quasistatic) process is unique and equal to the difference of the free energies of the two states, work done in an arbitrary process is generally different from the quasistatic work, it depends on the process realization, and for a given realization it may be lower than the free energy difference, violating the second law of thermodynamics. The Jarzynski relation introduces a constraint on the work fluctuations. It connects the exponential average of work w done in different realizations of an arbitrary process between two equilibrium states to the difference of the free energies of these states,

$$\langle \exp(-w/T) \rangle = \exp(-\Delta F/T). \quad (1)$$

The usefulness of relation (1) is in that it allows one to extract the free energy difference, i.e., equilibrium information, from an ensemble of nonequilibrium trajectories of the system [8].

It should be appreciated that in order to write down the Jarzynski relation one needs a characteristic energy scale. For transitions between equilibrium states there is a single, universal energy scale, set by the temperature of the reservoir, T .

If one tries to generalize the Jarzynski relation to processes involving active matter systems, in principle one needs to generalize the notion of the free energy. One can avoid this task by noticing that in the limit of infinitely slow processes

one expects the work to become a nonfluctuating quantity. This allows one to replace ΔF by work done in a quasistatic process, w^{qs} . The generalized Jarzynski relation would then connect the exponential average of the work done while driving an active matter system between two stationary states to the work done in a quasistatic process,

$$\langle \exp(-w/T) \rangle = \exp(-w^{qs}/T). \quad (2)$$

Here we investigate the existence of such a relation for a small active matter system using the standard stochastic thermodynamics definition of work.

The second problem with using the generalized Jarzynski relation to describe active matter systems is whether the thermal energy set by the temperature of the reservoir that enters into Eq. (2) is the appropriate characteristic energy scale for an active system. In the following, for brevity, we will often call this characteristic energy scale an effective temperature.

We recall that for active matter systems a number of different effective temperatures have been introduced [9–14]. It is *a priori* not clear whether one of these temperatures provides the characteristic energy scale that allows Eq. (2) to be generalized to active matter systems.

We study probably the simplest active matter system that can be externally manipulated, a single *thermal* active Ornstein-Uhlenbeck particle (AOUP) [12,15,16] in a harmonic potential. We consider two different classes of processes. In the first class we change the position of the minimum of the potential. In this case no work is done in the quasistatic process and the right-hand-side of the Jarzynski equality becomes equal to one. We show that in the limit of infinitely fast and slow but finite speed processes two different characteristic energy scales, i.e., two different effective temperatures, have to be used to keep the exponential average of the work equal to one.

In the second class of processes we change the force constant of the harmonic potential. In this case, the quasistatic work is nonzero. We show that for both infinitely fast and slow but finite speed processes there is no characteristic energy scale, i.e., no effective temperature, that makes the generalized Jarzynski relation valid.

Our results demonstrate that at least with the standard definition of work, relation (2) is not generally valid for processes connecting stationary states of active matter systems. We note that while Eq. (2) seems a natural extension of the original Jarzynski relation (1) to active matter systems, it was not derived by theoretical arguments [17]. Instead, we treat it as a working hypothesis whose validity we test for one specific system.

II. A THERMAL AOUP IN A HARMONIC POTENTIAL

We consider a single active particle moving in a harmonic potential. The particle is endowed with a self-propulsion force that evolves according to the Ornstein-Uhlenbeck stochastic process [18]. Following Ref. [19] we assume that the particle also experiences the standard thermal noise. The equations of motion read:

$$\gamma \dot{x} = -k(x - x_0) + f + \zeta, \quad \langle \zeta(t)\zeta(t') \rangle = 2\gamma T \delta(t - t'), \quad (3)$$

$$\tau_p \dot{f} = -f + \eta, \quad \langle \eta(t)\eta(t') \rangle = 2\gamma T_a \delta(t - t'). \quad (4)$$

In Eq. (3) γ is the friction coefficient, k is the force constant, x_0 is the location of the potential minimum, f is the self-propulsion, and ζ is the thermal noise characterized by temperature T . In Eq. (4) τ_p is the persistence time of the self-propulsion and η is the noise of the reservoir coupled to the self-propulsion, characterized by active temperature T_a . Equivalently, the particle can be described by the Fokker-Planck equation for a joint probability distribution of the position and the self-propulsion, $P(x, f; t)$,

$$\partial_t P(x, f; t) = \Omega P(x, f; t), \quad (5)$$

where the evolution operator Ω reads,

$$\Omega = -\gamma^{-1} \partial_x [-k(x - x_0) + f - T \partial_x] - \partial_f [-f/\tau_p - (\gamma T_a/\tau_p^2) \partial_f]. \quad (6)$$

We note that the so-called drift coefficients [18] in evolution operator (6) are linear in x and f and therefore, the stationary solution of Eq. (5) has a Gaussian form. In particular, the position distribution reads,

$$p^{ss}(x) \propto \exp \left[-\frac{(1/2)k(x - x_0)^2}{T + T_a/(k\tau_p/\gamma + 1)} \right]. \quad (7)$$

We follow standard stochastic thermodynamics [20,21] and define the work done while changing the parameter α of the potential $U(x) = (1/2)k(x - x_0)^2$ as

$$w = \int_0^\tau dt \dot{\alpha} \partial_\alpha U(x). \quad (8)$$

We note that the same definition of work was used in the analysis of active engines [22–24]. We consider two classes of processes, with $\alpha = x_0$ and $\alpha = k$.

III. WORK DONE BY SHIFTING THE POTENTIAL MINIMUM

In the case of moving the potential minimum, $x_0 \rightarrow x_0 + \Delta x_0$, the work done in an infinitely slow (quasistatic) process

vanishes, $w^{qs} = 0$, and, as stated earlier, the right-hand-side of generalized Jarzynski relation, Eq. (2), is equal to one.

The work done in an instantaneous process is equal to $w^{ins} = (1/2)k(-2\Delta x_0(x - x_0) + \Delta x_0^2)$ and its distribution reads,

$$p^{ins}(w) = \langle \delta(w - (1/2)k(-2\Delta x_0(x - x_0) + \Delta x_0^2)) \rangle^{ss}, \quad (9)$$

where here and in the following $\langle \dots \rangle^{ss}$ denotes averaging over the stationary distribution. We note that using, in Eq. (9), the distribution just before x_0 is changed can be justified by considering a general finite speed process and taking the limit of an infinitely fast switch $x_0 \rightarrow x_0 + \Delta x_0$ (see Appendix A).

Explicit calculation shows that in this case the generalized Jarzynski relation is satisfied with $T_{eff1} = T + T_a/(k\tau_p/\gamma + 1)$,

$$\langle \exp(-w/T_{eff1}) \rangle^{ins} = 1, \quad T_{eff1} = T + T_a/(k\tau_p/\gamma + 1), \quad (10)$$

where $\langle \dots \rangle^{ins}$ denotes averaging over distribution of work done in an instantaneous process. We note that effective temperature (10) is the temperature that is obtained if stationary state distribution of particle positions, Eq. (7), is interpreted as the Gibbs measure, $p^{ss}(x) \propto \exp(-U(x)/T_{eff1})$. Furthermore, Eq. (10) is consistent with the result of Paneru *et al.* [25] who considered work extracted from an active information engine.

For finite-speed processes it is convenient to follow Mazonka and Jarzynski [26] and write a Fokker-Planck equation for a joint probability distribution for the position, self-propulsion, and work, $p(x, f, w; t)$,

$$\partial_t p(x, f, w; t) = [\Omega + \dot{x}_0 k(x - x_0) \partial_w] p(x, f, w; t). \quad (11)$$

Assuming that at the start of driving the particle is in the stationary state, we get the following initial condition for Eq. (11), $p(x, f, w; t=0) = p^{ss}(x, f) \delta(w)$. Once again we note that drift coefficients in Eq. (11) are linear. This fact and a Gaussian (albeit singular) initial condition $p(x, f, w; t=0)$ imply that distribution $p(x, f, w; t)$ is a Gaussian distribution with time-dependent coefficients. It follows that work distribution $p(w; t) = \int dx df p(x, f, w; t)$ is also a Gaussian and therefore it is fully characterized by the first two cumulants of the work. To calculate these cumulants we use Eq. (11), and for slow but finite speed driving we get the following result (see Appendix A for details of the calculation):

$$\langle w \rangle^{sl} = \dot{x}_0 \gamma \Delta x_0, \quad (12)$$

$$\sigma_w^2 = \langle w^2 \rangle^{sl} - (\langle w \rangle^{sl})^2 = 2\dot{x}_0 \gamma \Delta x_0 (T + T_a), \quad (13)$$

where $\langle \dots \rangle^{sl}$ denotes averaging over the work distribution for slow but finite speed driving. In the limit of infinitely slow driving, i.e., in the quasistatic limit $\dot{x}_0 \rightarrow 0$, the variance of the work vanishes, i.e., the work does not fluctuate, and the average work vanishes as well.

For a Gaussian distribution of work, generalized Jarzynski relation (2) is satisfied with T replaced by effective temperature T_{eff} if [26]

$$\langle w \rangle = w^{qs} + \sigma_w^2/(2T_{eff}). \quad (14)$$

Thus, results (12) and (13) imply that for slow but finite speed driving the generalized Jarzynski relation is satisfied with

$$T_{\text{eff } 2} = T + T_a,$$

$$\langle \exp(-w/T_{\text{eff } 2}) \rangle^{\text{sl}} = 1, \quad T_{\text{eff } 2} = T + T_a \quad (15)$$

(recall that $w^{\text{qs}} = 0$ for moving the potential minimum). We note that the effective temperature that enters into the generalized Jarzynski relation for slow but finite speed driving is the same as the effective temperature obtained from the fluctuation-dissipation ratio in the limit of small frequencies (see Appendix B for details).

We conclude that two different effective temperatures, i.e., two different characteristic energy scales, are required to make the generalized Jarzynski relation valid for this very simple class of processes.

IV. WORK DONE BY CHANGING THE FORCE CONSTANT

Next, we consider work done by increasing the force constant of the potential, $k \rightarrow k + \Delta k$. To simplify the notation in this section we set the potential minimum at $x_0 = 0$. For the increase of the force constant, the work done in an infinitely slow (quasistatic) process is nonzero. The quasistatic work can be calculated by rewriting Eq. (8) with $\alpha = k$ as an integration over k of $\partial_k U(x) = \frac{x^2}{2}$ averaged over stationary state distribution, Eq. (7),

$$w^{\text{qs}} = \int_k^{k+\Delta k} dk_1 \left\langle \frac{x^2}{2} \right\rangle^{\text{ss}}. \quad (16)$$

The result reads

$$w^{\text{qs}} = \frac{T + T_a}{2} \ln \frac{k + \Delta k}{k} - \frac{T_a}{2} \ln \frac{(k + \Delta k)\tau_p/\gamma + 1}{k\tau_p/\gamma + 1}. \quad (17)$$

The work done in an instantaneous process is equal to $w^{\text{ins}} = (1/2)\Delta k x^2$ and its distribution reads

$$\begin{aligned} p^{\text{ins}}(w) &= \langle \delta(w - (1/2)\Delta k x^2) \rangle^{\text{ss}} \\ &= \sqrt{\frac{k}{\pi w \Delta k T_{\text{eff } 1}}} e^{-\frac{k w}{\Delta k T_{\text{eff } 1}}}, \end{aligned} \quad (18)$$

where $T_{\text{eff } 1}$ is defined in Eq. (10). We remark that in Eq. (18), using the distribution just before k is changed follows by analogy with Eq. (9), where it can be justified by an explicit calculation.

Explicit calculation shows that for instantaneous changes of the force constant, $T_{\text{eff } 1}$ cannot be used in the generalized Jarzynski relation,

$$\langle \exp(-w/T_{\text{eff } 1}) \rangle^{\text{ins}} = \sqrt{\frac{k + \Delta k}{k}} \neq \exp(-w^{\text{qs}}/T_{\text{eff } 1}). \quad (19)$$

In fact, there is no effective temperature, i.e., no characteristic energy scale, that is independent of the change of the force constant and that leads to the generalized Jarzynski relation for work distribution (18).

To investigate the existence of the generalized Jarzynski relation for a slow but finite rate increase of the force constant, we derive an approximate distribution of work done in this process. To this end we follow Speck [27] who derived the analogous distribution for work done on a Brownian particle in a harmonic potential. The calculation is somewhat tedious but straightforward; it is presented in Appendix C. The approximate distribution of work for a slow but finite speed

change of the force constant is a Gaussian with cumulants that are given by the following, rather complicated, expressions:

$$\begin{aligned} \langle w \rangle^{\text{sl}} &= w^{\text{qs}} + \dot{k} \int_k^{k+\Delta k} dk_1 \left[\frac{\gamma T}{4k_1^3} \right. \\ &\quad \left. + \frac{\gamma T_a(4(k_1\tau_p/\gamma)^2 + 3k_1\tau_p/\gamma + 1)}{4k_1^3(k_1\tau_p/\gamma + 1)^3} \right], \end{aligned} \quad (20)$$

$$\begin{aligned} \sigma_w^2 &= 2\dot{k} \int_k^{k+\Delta k} dk_1 \left[\frac{\gamma T^2}{4k_1^3} + \frac{\gamma T T_a(2k_1\tau_p/\gamma + 1)}{2k_1^3(k_1\tau_p/\gamma + 1)^2} \right. \\ &\quad \left. + \frac{\gamma T_a^2((k_1\tau_p/\gamma)^2 + 3k_1\tau_p/\gamma + 1)}{4k_1^3(k_1\tau_p/\gamma + 1)^3} \right]. \end{aligned} \quad (21)$$

Cumulants (20) and (21) do not satisfy relation (14) and therefore, again, there is no effective temperature, i.e., no characteristic energy scale, that leads to the generalized Jarzynski relation for a slow but finite speed process in which the force constant is increased.

V. DISCUSSION

Our results imply that if one uses the standard definition of work, Jarzynski relation (1) generally cannot be extended to active matter systems. For some classes of processes, depending on the speed of driving, different characteristic energy scales, i.e., different effective temperatures, have to be used to make generalized Jarzynski relation (2) valid. For other classes of processes there is no characteristic energy scale that would lead to the generalized Jarzynski relation.

We emphasize that this result follows if one uses the definition of work utilized in standard stochastic thermodynamics, Eq. (8). It is possible that other definitions of work, e.g., excess work defined by Hatano and Sasa [28], could lead to a generalized Jarzynski relation. This may seem plausible since, although in this work we showed that the Jarzynski relation generally cannot be extended to active matter systems, elsewhere [29] we showed that fluctuation theorems for different kinds of entropy are in general satisfied for active matter systems [30]. On the other hand, we recall that fluctuation theorems for entropy do not involve any characteristic energy scale and thus the issue that makes the generalized Jarzynski relation invalid does not occur. This subject is left for a future investigation.

Our finding is consistent with the fact that in out-of-equilibrium systems there is no unique characteristic energy scale that for equilibrium systems is set by the temperature. For this reason, if one uses relations that give thermodynamic temperature for equilibrium systems, one generally gets different effective temperatures. Simply speaking, there is no “effective temperature”. Only in certain cases, e.g., for glassy systems under shear [31,32] or athermal model foams [33], *a priori* different effective temperatures turn out to have the same value [34]. This result follows from a well understood theoretical argument involving the separation of time scales of different relaxation processes.

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APPENDIX A: WORK DONE WHILE SHIFTING THE POTENTIAL MINIMUM

Our calculation of the first two moments of the distribution of the work done while shifting the potential minimum follows a similar calculation presented in Ref. [26], which was concerned with a passive Brownian particle.

We consider a thermal AOUP in a harmonic potential. To simplify the notation in this section we set the initial potential minimum at $x_0(t=0) = 0$.

We assume that at time $t = 0$ the potential minimum starts moving with constant velocity $\dot{x}_0$. Using Eq. (8) of the main text we obtain the following equation of motion for the work done while shifting the potential minimum,

$$\partial_t w = -\dot{x}_0 k(x - \dot{x}_0 t). \quad (\text{A1})$$

Following Ref. [26] we switch to the reference frame of the moving potential minimum and introduce a new variable, $y = x - \dot{x}_0 t$. Next, we rewrite the Fokker-Planck equation, Eq. (11) of the main text, as an equation describing the joint probability distribution for the particle's position (in the reference frame of the moving potential minimum), self-propulsion, and work accumulated between the initial time and time t ,

$$\begin{aligned} \partial_t p(y, f, w; t) = & \left\{ -\gamma^{-1} \partial_y [-ky - \gamma \dot{x}_0 + f - T \partial_y] \right. \\ & - \partial_f [-f/\tau_p - (\gamma T_a/\tau_p^2) \partial_f] \\ & \left. + \dot{x}_0 k y \partial_w \right\} p(y, f, w; t). \end{aligned} \quad (\text{A2})$$

The initial condition for Eq. (A2) reads $p(y, f, w; t=0) = p^{ss}(y, f) \delta(w)$.

Starting from Eq. (A2) we can derive the following equations for the averages of w and y , and f ,

$$\partial_t \langle w \rangle = -\dot{x}_0 \langle y \rangle, \quad (\text{A3})$$

$$\partial_t \langle y \rangle = -\dot{x}_0 - (k/\gamma) \langle y \rangle + \gamma^{-1} \langle f \rangle, \quad (\text{A4})$$

$$\partial_t \langle f \rangle = -\tau_p^{-1} \langle f \rangle. \quad (\text{A5})$$

Initial conditions for these equation are $\langle w \rangle(t=0) = 0$, $\langle y \rangle(t=0) = 0$, and $\langle f \rangle(t=0) = 0$.

Solving Eqs. (A3)–(A5) we get $\langle w \rangle(t) = \dot{x}_0^2 \gamma t + (\dot{x}_0^2/k)(\exp(-kt/\gamma) - 1)$. Recalling that the change of the potential minimum over time τ is $\Delta x_0 \equiv \dot{x}_0 \tau$, we obtain

$$\langle w \rangle = \dot{x}_0 \gamma \Delta x_0 + (\dot{x}_0^2/k)(\exp(-k \Delta x / \gamma \dot{x}_0) - 1). \quad (\text{A6})$$

In the limit of slow but finite driving, Eq. (A6) leads to Eq. (12) of the main text. Incidentally, in the limit of infinitely fast process Eq. (A6) reproduces $\langle w \rangle^{\text{ins}} = k \Delta x_0^2 / 2$.

To evaluate the variance of the work we need to derive equations of motions for the second cumulants. We follow the notation of Ref. [26]:

$$\sigma_w^2 = \langle w^2 \rangle - \langle w \rangle^2, \quad (\text{A7})$$

$$c_{yw} = \langle yw \rangle - \langle y \rangle \langle w \rangle. \quad (\text{A8})$$

The derivation of equations of motion is straightforward but somewhat lengthy. The result is

$$\partial_t \sigma_w^2 = -2\dot{x}_0 k c_{yw}, \quad (\text{A9})$$

$$\partial_t c_{yw} = -(k/\gamma) c_{yw} + \gamma^{-1} c_{fw} - \dot{x}_0 k \sigma_y^2, \quad (\text{A10})$$

$$\partial_t \sigma_y^2 = -(2k/\gamma) \sigma_y^2 + (2/\gamma) c_{yf}, \quad (\text{A11})$$

$$\partial_t c_{yf} = -(k/\gamma + 1/\gamma) c_{yf} + \gamma^{-1} \sigma_f^2, \quad (\text{A12})$$

$$\partial_t \sigma_f^2 = -(2/\tau_p) \sigma_f^2 + (2\gamma T_a/\tau_p^2), \quad (\text{A13})$$

$$\partial_t c_{fw} = -\tau_p^{-1} c_{fw} - \dot{x}_0 k c_{yf}. \quad (\text{A14})$$

We note that Eqs. (A11)–(A13) do not couple to the other equations; since initial conditions for Eqs. (A11)–(A13) are stationary state averages, σ_y^2 , c_{yf} , and σ_f^2 will not change:

$$\sigma_y^2 = T/k + T_a/(k(k\tau_p/\gamma + 1)), \quad (\text{A15})$$

$$c_{yf} = T_a/(k\tau_p/\gamma + 1), \quad (\text{A16})$$

$$\sigma_f^2 = \gamma T_a/\tau_p. \quad (\text{A17})$$

The remaining equations can be integrated, starting from Eq. (A14), then moving to (A10) and finally (A9). Then we again recall that the change of the potential minimum over time τ is $\Delta x_0 \equiv \dot{x}_0 \tau$ and we get

$$\begin{aligned} \sigma_w^2 = & 2\dot{x}_0 k \Delta x_0 (\gamma \sigma_y^2 + \tau_p c_{yf}) \\ & + 2\dot{x}_0^2 \left(\gamma^2 \sigma_y^2 + \frac{\gamma \tau_p}{k\tau_p/\gamma - 1} c_{yf} \right) (\exp(-k \Delta x / \gamma \dot{x}_0) - 1) \\ & + 2\dot{x}_0^2 \frac{k^2 \tau_p^3}{\gamma(k\tau_p/\gamma - 1)} (\exp(-\Delta x / \tau_p \dot{x}_0) - 1). \end{aligned} \quad (\text{A18})$$

In the limit of slow but finite driving, Eq. (A18) gives Eq. (13) of the main text. Once again, it can be shown that in the limit of infinitely fast process Eq. (A18) reproduces the variance of distribution p^{ins} , Eq. (9) of the main text.

APPENDIX B: FLUCTUATION-DISSIPATION RATIO-BASED EFFECTIVE TEMPERATURE

The calculation outlined in this section generalizes that presented in Sec. IV of Ref. [12] for a single *athermal* AOUP in a harmonic potential. To simplify the notation, in this section we set the potential minimum at $x_0 = 0$.

Following Ref. [35] we define a frequency-dependent fluctuation-dissipation ratio-based effective temperature

$$T_{\text{eff}}^{\text{FDR}}(\omega) = \frac{\omega \text{Re}C(\omega)}{\chi''(\omega)}, \quad (\text{B1})$$

where $\text{Re}C(\omega)$ is the real part of the one-sided Fourier transform of the particle's position auto-correlation function, $\text{Re}C(\omega) = \text{Re} \int_0^\infty e^{i\omega t} \langle x(t)x(0) \rangle$, and $\chi''(\omega)$ is the imaginary part of the one-sided Fourier transform of the response function, $\chi''(\omega) = \text{Im} \int_0^\infty e^{i\omega t} R(t)$, where $R(t)$ describes the change of the particle's position due to an external force.

To calculate the position auto-correlation function we start from equations of motion, Eqs. (3) and (4) of the main text, and derive the following set of coupled equations for $\langle x(t)x(0) \rangle$ and $\langle f(t)x(0) \rangle$,

$$\gamma \partial_t \langle x(t)x(0) \rangle = -k \langle x(t)x(0) \rangle + \langle f(t)x(0) \rangle, \quad (\text{B2})$$

$$\tau_p \partial_t \langle f(t)x(0) \rangle = -\langle f(t)x(0) \rangle. \quad (\text{B3})$$

Since thermal noise is uncorrelated with the initial position of the particle, the above equations have the same form as those in Sec. IV B of Ref. [12].

The presence of the thermal noise influences the initial conditions for Eqs. (B2) and (B3),

$$\langle x(0)x(0) \rangle \equiv \langle x^2 \rangle = \frac{T}{k} + \frac{T_a}{k(k\tau_p/\gamma + 1)}, \quad (\text{B4})$$

$$\langle f(0)x(0) \rangle \equiv \langle fx \rangle = k \langle x^2 \rangle - T = \frac{T_a}{k\tau_p/\gamma + 1}. \quad (\text{B5})$$

Equations of motion, (B2) and (B3), with initial conditions, (B4) and (B5), lead to the following expression for the position auto-correlation function,

$$\begin{aligned} \langle x(t)x(0) \rangle &= \frac{T_a/k}{(k\tau_p/\gamma)^2 - 1} ((k\tau_p/\gamma)e^{-t/\tau_p} - e^{-kt/\gamma}) \\ &+ (T/k)e^{-kt/\gamma}. \end{aligned} \quad (\text{B6})$$

To evaluate the response function, we add to Eq. (3) of the main text a weak, time-dependent external force $F^{\text{ext}}(t)$ and then derive coupled equations of motion for the resulting change of the average position of the particle and of the self-propulsion:

$$\gamma \partial_t \delta \langle x(t) \rangle = \delta \langle f(t) \rangle - k \delta \langle x(t) \rangle + F^{\text{ext}}(t) \quad (\text{B7})$$

$$\tau_p \partial_t \delta \langle f(t) \rangle = -\delta \langle f(t) \rangle. \quad (\text{B8})$$

The initial conditions for these equations are $\delta \langle x(t=0) \rangle = 0 = \delta \langle f(t=0) \rangle$.

Solving Eqs. (B7) and (B8) we get $\delta \langle f(t) \rangle \equiv 0$ and

$$\delta \langle x(t) \rangle = \frac{1}{\gamma} \int_0^t dt' e^{-k(t-t')/\gamma} F^{\text{ext}}(t'). \quad (\text{B9})$$

The response function thus is given by

$$R(t) = (1/\gamma)e^{-kt/\gamma}. \quad (\text{B10})$$

Response function (B10) is the same as that derived in Ref. [12] for an athermal AOUP.

Using Eqs. (B6) and (B10), from Eq. (B1) we get

$$T_{\text{eff}}^{\text{FDR}}(\omega) = \frac{T_a}{1 + \tau_p^2 \omega^2} + T. \quad (\text{B11})$$

In the small frequency limit, $T_{\text{eff}}^{\text{FDR}}(\omega)$ becomes $T_a + T$ and thus it coincides with the effective temperature that makes the Jarzynski relation valid for the work distribution in a slow but finite shift of the potential minimum. We note that $T_a + T$ is also the effective temperature that is obtained from the long-time diffusion coefficient of a free thermal AOUP.

APPENDIX C: WORK DISTRIBUTION FOR SLOW BUT FINITE INCREASE OF THE FORCE CONSTANT

Our calculation of the approximate distribution of the work done while increasing the force constant follows a similar calculation presented in Ref. [27], which was concerned with a passive Brownian particle.

We consider a thermal AOUP in a harmonic potential. To simplify the notation in this section we set the potential minimum at $x_0 = 0$.

We assume that at time $t = 0$ the force constant starts increasing with constant rate $\dot{k}$. Using Eq. (8) of the main text we obtain the following equation of motion for the work done while increasing the force constant,

$$\partial_t w = \frac{\dot{k}}{2} x^2. \quad (\text{C1})$$

Next, we write an evolution equation that is similar to Eq. (11) of the main text, which describes the time dependence of the joint probability distribution for the particle's position, self-propulsion, and work accumulated between the initial time and time t ,

$$\begin{aligned} \partial_t p(x, f, w; t) &= \left\{ -\gamma^{-1} \partial_x [-kx + f - T \partial_x] \right. \\ &\quad - \partial_f [-f/\tau_p - (\gamma T_a/\tau_p^2) \partial_f] \\ &\quad \left. - (\dot{k}x^2/2) \partial_w \right\} p(x, f, w; t). \end{aligned} \quad (\text{C2})$$

The initial condition for Eq. (C2) is $p(x, f, w; t=0) = p^{\text{ss}}(x, f) \delta(w)$.

We note that one of the drift coefficients in Eq. (C2) is quadratic and thus the time-dependent distribution $p(x, f, w; t)$ does not have Gaussian form. In fact, since for a process in which the force increases the time derivative of the work, Eq. (C1), is always positive, $p(x, f, w; t) = 0$ for $w < 0$ and thus distribution $p(x, f, w; t)$ cannot be a Gaussian.

However, if we introduce the characteristic function [18],

$$\rho(x, f, \lambda; t) = \int dw e^{i\lambda w} p(x, f, w; t), \quad (\text{C3})$$

we note that the equation of motion for $\rho(x, f, \lambda; t)$,

$$\begin{aligned} \partial_t \rho(x, f, \lambda; t) &= \left\{ -\gamma^{-1} \partial_x [-kx + f - T \partial_x] \right. \\ &\quad - \partial_f [-f/\tau_p - (\gamma T_a/\tau_p^2) \partial_f] \\ &\quad \left. + i\lambda \dot{k}x^2/2 \right\} \rho(x, f, \lambda; t), \end{aligned} \quad (\text{C4})$$

allows for a solution that has a Gaussian form [27]. This fact allows us to derive a closed set of equations describing the time dependence of the characteristic function of the work distribution,

$$\psi(\lambda; t) = \int dx df \rho(x, f, \lambda; t). \quad (\text{C5})$$

We start with the equation that expresses the time derivative of $\psi(\lambda; t)$ in terms of the generalized second moment,

$$\partial_t \psi(\lambda; t) = i\lambda \dot{k} \phi_{x^2}(\lambda; t)/2, \quad (\text{C6})$$

where the generalized second moment $\phi_{x^2}(\lambda; t)$ reads

$$\phi_{x^2}(\lambda; t) = \int dx df x^2 \rho(x, f, \lambda; t). \quad (\text{C7})$$

The generalized second moment satisfies the following equation:

$$\partial_t \phi_{x^2}(\lambda; t) = \frac{2}{\gamma} (\phi_{xf}(\lambda; t) - k \phi_{x^2}(\lambda; t)) + \frac{2T}{\gamma} \psi(\lambda; t) + \frac{i\lambda \dot{k}}{2} \int dx df x^4 \rho(x, f, \lambda; t), \quad (\text{C8})$$

where $\phi_{xf}(\lambda; t)$ denotes the mixed generalized second moment,

$$\phi_{xf}(\lambda; t) = \int dx df x f \rho(x, f, \lambda; t). \quad (\text{C9})$$

To close the equations of motion we will also need $\phi_{f^2}(\lambda; t)$:

$$\phi_{f^2}(\lambda; t) = \int dx df f^2 \rho(x, f, \lambda; t). \quad (\text{C10})$$

Equations of motion for $\phi_{xf}(\lambda; t)$ and $\phi_{f^2}(\lambda; t)$ read:

$$\begin{aligned} \partial_t \phi_{xf}(\lambda; t) &= \frac{1}{\gamma} \phi_{f^2}(\lambda; t) - \left(\frac{k}{\gamma} + \frac{1}{\tau_p} \right) \phi_{xf}(\lambda; t) \\ &\quad + \frac{i\lambda \dot{k}}{2} \int dx df x^3 f \rho(x, f, \lambda; t), \end{aligned} \quad (\text{C11})$$

$$\begin{aligned} \partial_t \phi_{f^2}(\lambda; t) &= -\frac{2}{\tau_p} \phi_{f^2}(\lambda; t) + \frac{2\gamma T_a}{\tau_p^2} \psi(\lambda; t) \\ &\quad + \frac{i\lambda \dot{k}}{2} \int dx df x^2 f^2 \rho(x, f, \lambda; t). \end{aligned} \quad (\text{C12})$$

As noted above, equation of motion for $\rho(x, f, \lambda; t)$ allows for a solution that has a Gaussian form. However, due to the last term at the right-hand-side of Eq. (C4), this Gaussian distribution is not normalized. The un-normalized Gaussian form of $\rho(x, f, \lambda; t)$ allows us to express higher-order moments in terms of generalized second moments,

$$\int dx df x^4 \rho(x, f, \lambda; t) = \frac{3\phi_{x^2}^2(\lambda; t)}{\psi(\lambda; t)}, \quad (\text{C13})$$

$$\int dx df x^3 f \rho(x, f, \lambda; t) = \frac{3\phi_{x^2}(\lambda; t) \phi_{xf}(\lambda; t)}{\psi(\lambda; t)}, \quad (\text{C14})$$

$$\int dx df x^2 f^2 \rho(x, f, \lambda; t) = \frac{\phi_{x^2}(\lambda; t) \phi_{f^2}(\lambda; t)}{\psi(\lambda; t)} + \frac{2\phi_{xf}^2(\lambda; t)}{\psi(\lambda; t)}. \quad (\text{C15})$$

Using closures (C13)–(C15) in the equations of motion for the generalized moments we get the following closed set of equations:

$$\begin{aligned} \partial_t \phi_{x^2}(\lambda; t) &= \frac{2}{\gamma} (\phi_{xf}(\lambda; t) - k \phi_{x^2}(\lambda; t)) + \frac{2T}{\gamma} \psi(\lambda; t) \\ &\quad + \frac{3i\lambda \dot{k}}{2} \frac{\phi_{x^2}^2(\lambda; t)}{\psi(\lambda; t)}, \end{aligned} \quad (\text{C16})$$

$$\begin{aligned} \partial_t \phi_{xf}(\lambda; t) &= \frac{1}{\gamma} \phi_{f^2}(\lambda; t) - \left(\frac{k}{\gamma} + \frac{1}{\tau_p} \right) \phi_{xf}(\lambda; t) \\ &\quad + \frac{3i\lambda \dot{k}}{2} \frac{\phi_{xf}(\lambda; t) \phi_{x^2}(\lambda; t)}{\psi(\lambda; t)}, \end{aligned} \quad (\text{C17})$$

$$\begin{aligned} \partial_t \phi_{f^2}(\lambda; t) &= -\frac{2}{\tau_p} \phi_{f^2}(\lambda; t) + \frac{2\gamma T_a}{\tau_p^2} \psi(\lambda; t) \\ &\quad + \frac{i\lambda \dot{k}}{2} \frac{\phi_{x^2}(\lambda; t) \phi_{f^2}(\lambda; t)}{\psi(\lambda; t)} + i\lambda \dot{k} \frac{\phi_{xf}^2(\lambda; t)}{\psi(\lambda; t)}. \end{aligned} \quad (\text{C18})$$

Next, following Ref. [27] we assume constant rate of change of the force constant, $\dot{k} = \text{const}$ and expand generalized second moments in powers of $\dot{k}$,

$$\phi_{x^2}(\lambda; t) = \phi_{x^2}^{(0)}(\lambda; t) + \dot{k} \phi_{x^2}^{(1)}(\lambda; t) + \dots, \quad (\text{C19})$$

$$\phi_{xf}(\lambda; t) = \phi_{xf}^{(0)}(\lambda; t) + \dot{k} \phi_{xf}^{(1)}(\lambda; t) + \dots, \quad (\text{C20})$$

$$\phi_{f^2}(\lambda; t) = \phi_{f^2}^{(0)}(\lambda; t) + \dot{k} \phi_{f^2}^{(1)}(\lambda; t) + \dots \quad (\text{C21})$$

We substitute expansions (C19)–(C21) into the equations of motion. We also assume that the time derivatives are of order $\dot{k}$. In this way we get the following set of equations for the zeroth order terms:

$$0 = \frac{2}{\gamma} (\phi_{xf}^{(0)}(\lambda; t) - k \phi_{x^2}^{(0)}(\lambda; t)) + \frac{2T}{\gamma} \psi(\lambda; t), \quad (\text{C22})$$

$$0 = \frac{1}{\gamma} \phi_{f^2}^{(0)}(\lambda; t) - \left(\frac{k}{\gamma} + \frac{1}{\tau_p} \right) \phi_{xf}^{(0)}(\lambda; t), \quad (\text{C23})$$

$$0 = -\frac{2}{\tau_p} \phi_{f^2}^{(0)}(\lambda; t) + \frac{2\gamma T_a}{\tau_p^2} \psi(\lambda; t). \quad (\text{C24})$$

Solving these equations we get the following result for $\phi_{x^2}^{(0)}(\lambda; t)$:

$$\phi_{x^2}^{(0)}(\lambda; t) = \frac{T}{k} \psi(\lambda; t) + \frac{T_a}{k(k\tau_p/\gamma + 1)} \psi(\lambda; t). \quad (\text{C25})$$

We use result (C25) in Eq. (C6) and get the following result for $\psi(\lambda; t)$:

$$\begin{aligned} \psi(\lambda; t) &= \exp \left\{ \frac{i\lambda}{2} \left[(T + T_a) \ln \frac{k(t)}{k(0)} \right. \right. \\ &\quad \left. \left. - T_a \ln \frac{k(t)\tau_p/\gamma + 1}{k(0)\tau_p/\gamma + 1} \right] \right\}. \end{aligned} \quad (\text{C26})$$

Equation (C26) means that the work distribution is a delta function centered at the quasistatic work given by Eq. (17) of the main text.

Next, we consider terms of order $\dot{k}$ in the equations of motion. After some manipulations we get

$$\begin{aligned} \phi_{x^2}^{(1)}(\lambda; t) &= \left[\frac{\gamma T}{2k^3} + \frac{\gamma T_a (2(k\tau_p/\gamma)^2 + 3k\tau_p/\gamma + 1)}{2k^3(k\tau_p/\gamma + 1)^3} \right] \psi(\lambda; t) \\ &\quad + i\lambda \left[\frac{\gamma T^2}{2k^3} + \frac{\gamma T T_a (2k\tau_p/\gamma + 1)}{k^3(k\tau_p/\gamma + 1)^2} \right. \\ &\quad \left. + \frac{\gamma T_a^2 ((k\tau_p/\gamma)^2 + 3k\tau_p/\gamma + 1)}{2k^3(k\tau_p/\gamma + 1)^3} \right] \psi(\lambda; t). \end{aligned} \quad (\text{C27})$$

Using the right-hand-side of Eq. (C27) in Eq. (C6) we see that at the first order in $\dot{k}$ the work distribution is a Gaussian with the first cumulants given by expressions (20) and (21) of the main text.

In closing we note that the Gaussian form of the work distribution is only an approximation, since the true distribution vanishes for $w < 0$ and therefore cannot have Gaussian form.

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