Collapse and the interplay between essentiality and impact in socioecological systems

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We study conditions leading to collapse on a nonequilibrium toy model introduced here for the interaction dynamics between a social and an ecological system based on the concept of essentiality of services and goods. One key difference from previous models is the separation between purely environmental collapse and that caused by an imbalance in the population consumption of essential goods. By studying different regimes defined by phenomenological parameters, we identify sustainable and unsustainable phases as well as the likelihood of collapse. The behavior of the stochastic version of the model is analyzed with a combination of analytical and computational techniques introduced here and shown to be consistent with key features of such processes in real life.

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I. INTRODUCTION

After the publication of the landmark book by Meadows *et al.* [1], much attention has been paid to the environmental impact of human activities. In the following decades, the idea of sustainable development became more influential and later on originated what has been called sustainable development goals [2].

Limiting the usage of natural resources might however be in conflict with other national interests. In practice, nations will prioritize strategic needs, such as sustained economic growth, defense against internal and external threats, and infrastructure, over constraints on the use of natural resources, especially if the latter are too restrictive. This situation can be exemplified by the difficulties in enforcing worldwide agreements, such as the Paris Agreement [3], and the slowness and hesitancy in adopting sustainable technologies evidenced by the failure in avoiding the current environmental crisis despite decades of warnings. The necessary arrangements to meet tight environmental goals in order to avoid ecological crisis ultimately lead to discontentment of interest groups and even social unrest when enforced without consent, as in the extreme cases of food and water rationing during shortage times.

Conflicting interests between ecological impact and other dimensions of sustainability have been addressed before by mathematical models which, although different, bear elements similar to our approach. For instance, the challenge to devise viable strategies to regulate the use of decreasing fishing stocks in the face of illegal overfishing due to economical interests has been addressed by means of differential equations by Eisenack *et al.* [4].

When the target resources comprise basic needs, such as food and water, this situation is quite clear. It is important to realize though that the necessary conditions leading to social unrest vary with time, place, and culture. Repression of democratic rights in many westernlike cultures would be enough to cause social upheaval [5], even if this is not a need whose deprivation would directly result in death. An extended shortage of electrical power, which would not even exist two centuries ago, might lead, among another things, to a high social cost [6].

How the concept of essentiality, i.e., what exactly is considered to be an essential good in a certain context, is related to the emergence of social collapse has been already discussed in the social sciences literature, but it remains challenging to mathematically model in a meaningful way [7,8].

The view that a fulfillment of needs and aspirations has to be addressed to avoid social disruption has been expressed in the so-called Brutland Report [9] by the World Commission on Environment and Development and reiterated by Ostrom [10]. More recently, other authors have addressed similar questions by considering how economic value and economic growth influence social collapse [11–13].

A first attempt at a simplified model was made in [14], where the concept and word *essentiality* was first introduced in this subject. The approach aimed to highlight conceptual points rather than analyze the details of its dynamics. There it was considered a certain society inserted into a wider environment from which resources are extracted to supply the consumer needs of its individuals, a model that was named SUEland.

SUE stands for superfluous and essential and the name was intended to emphasize the idea that the agents comprising the social system of SUE land represent individuals that make consumer choices which not only have an environmental impact, but also contribute to the system's social well-being through the level of essentiality of the acquired products. In other words, the model includes the often neglected (although well-known) fact that people can choose to spend their money on superfluous rather than essential goods in order to fulfill aspirations not strictly required for biological survival.

Defining and measuring essentiality is a research line with several subtleties. We leave a more focused discussion of that subject to an upcoming work in a more appropriate outlet. Here we assume that such a definition is possible in principle. This allows us to attribute two dimensions to the sustainability of goods and products—essentiality and impact—which, for simplicity, we consider to assume only two discrete values each. Such an approach reduces to four the sustainable classes of goods comprising the possible buying choices of SUEland citizens: (i) high impact and high essentiality (for which we use the symbol $++$), (ii) high impact and low essentiality ($+-$), (iii) low impact and high essentiality ($-+$), and (iv) low impact and low essentiality ($--$). In [14] they were called the Knight's, Noble's, Peasant's, and Jester's baskets, respectively, for illustration purposes. They can be thought of as a coarse graining of the product space by clustering goods with close values of essentiality and impact.

It is worth stressing that selecting essentiality and impact as the main attributes for sustainability, and furthermore taking them as discrete, is a simplifying assumption used as an initial step to facilitate the analytical treatment of the model as we will see. A further step towards a more realistic modeling would clearly be to use a continuous distribution of values. Evidence of the shape of these distributions is being sought by the authors, for instance, by surveys about essentiality perception among the population [8]. One can understand the chosen discretization as a coarse graining by substituting the whole distribution by typical or average values. We are currently analyzing the effect of considering different distributions for these values based on the data we are gathering.

Real-life societies are clearly inserted into and extract resources from their surrounding environments in order to function properly. SUEland is modeled along these same lines. In the simplified model we introduce here, the amount of extracted resources goes only into the manufacturing of the products bought by citizens and already includes any possible pollution or degradation due to the associated activities rendering resources not usable.

The interaction between these two systems, the society and the environment, forms what is called a socioecological system (SES). We will follow the dynamics of measures of impact and essentiality of the whole SES. These dynamics are modeled by stochastic processes, reflecting the nondeterministic character of buying choices. By combining simulations with analytical approximations, we can identify sustainable and unsustainable phases (according to definitions to be given explicitly in the following) and associated features.

Environmental collapse is known to have been an important factor in the fall of a number of civilizations [15] and has been accordingly given a special status in the literature on sustainability. Mathematical models on how the use of natural resources affects the fate of societies, in particular based on differential equations [16], have been proposed before. The objective, rather than a perfect simulation of real scenarios (which might even not be feasible), is to understand the main reasons leading to collapse and the existence of some general behavior. Our model differs from those in the way we take into consideration the role of essentiality.

The main research question we try to answer is whether we can further contribute to an increasing corpus of mathematical theory on sustainability management [17] (see, in particular, Chaps. 2 and 3 therein) by designing a toy model where the trade-off between ecological impact and other dimensions of sustainability, of which essentiality is chosen as a proxy, can represent at least qualitatively the behavior of real systems and provide both quantitative and qualitative insights into the limits of such a kind of modeling. In the following, we show that the answer is positive, albeit with limitations, many of which we expect can be overcome by additional sophistication of future models.

In Sec. II an overview of our nonequilibrium model is given, its details being explained in the subsequent sections. Section III introduces and analyzes the deterministic version of the discrete map that is used to model the dynamics of the renewable resources consumed by the society. This is a minimal model designed to capture the essential properties of the kind of renewable environment considered here, to which noise will be added later on.

Section IV presents a detailed account of the SES dynamics and how it is simulated. How the consumption process is carried out in SUEland is explained here by defining precisely the citizens' buying choices and how they contribute to the measures of impact and essentiality of the whole system. How essentiality and social collapse are defined is mainly discussed in Sec. V.

The simplest case is that in which there is no direct interaction between essentiality and impact in a citizen's decision on what to buy. In other words, the citizen is free to choose any combination of impact and essentiality when buying goods, without the value of one influencing the probability of choosing the other. The mathematical analysis of this case is made in Sec. VI. In Sec. VII, simulations and numerical calculations are presented for that scenario, showing how the indirect interaction between the two dimensions of sustainability can push in opposite directions towards collapse. A discussion of the model, our main conclusions, and future directions are presented in Sec. VIII.

II. SOCIOECOLOGICAL SYSTEM

A SES can be analyzed by considering two interacting subsystems: A society $\mathcal{S}$ and an environment $\mathcal{E}$. The society is always inserted in the environment, which renders the difference between both functional rather than physical. This can be represented by three concentric spheres, which we call spheres of sustainability [14], representing three interacting subsystems. The outermost one is the environment, the next one is the society, and the innermost is its economy $\mathcal{M}$. For the sake of mathematical modeling, it is convenient to consider the environment and society as formally separated interacting systems. The economy will be represented by the processes of wealth distribution and product consumption, but any other consequences it might have in a real society will be here neglected. We will not be worried about how economic resources are generated in this first approximation, assuming that they are available as needed.

Societies are complex systems, with a range of different processes running in parallel. Because we are mainly concerned with the sustainability of natural resources consumption, we will ignore them in the present study. Consumption is modeled by citizens' buying choices, which determine the extraction of resources from the environment. These choices result from a competition between the needs and aspirations of each individual and external pressures, such as environmental awareness or governmental policies. The only *impact* of the society on the environment in our analysis comes from such consumption and the role of the environment is only to provide a certain amount of available natural resources, which can be either renewable (within certain limits) or not, to manufacture products, satisfying society's demand. The agents we use in our agent-based simulation of SUEland's population represent individual citizens, but our results are valid for cases where they are groups of people, companies, or other institutions which contribute for the consumption of natural resources.

The nonequilibrium dynamics of our model is comprised of two different timescales. The macroscopic one determines the evolution of the environment and the income distribution among citizens, providing them with the necessary economic resources for buying products. Within this scale, there is a microscopic dynamics: Citizens buy products until their economic resources are exhausted. Each decision is taken stochastically, influenced by the factors we have already mentioned. By the end of each microscopic cycle, the product demand of the society is calculated and a related amount of resources necessary to produce the consumed goods is extracted from the environment. If the amount of resources required is higher than those available, we immediately stop the dynamics and declare an environmental collapse.

III. RENEWABLE RESOURCES: A MINIMAL MODEL

Let us call $R(t)$ the amount of available environmental resources at a certain time t in an arbitrary unit of measurement (e.g., carbon footprint, metric tons, and liters). In principle, like the society, the environment is a complex system with $R(t)$ being a result of the interactions of its many components. Because our main interest in this work is to study how the society's consumption affects the equilibrium between essentiality and impact, we will not model all details of the ecological system and concentrate only on this effect. We therefore use a minimal model for the dynamics of the resources given by a unidimensional discrete map

$$R(t+1) = f(R(t)), \tag{1}$$

where, at each macroscopic time step t, the environment provides a certain amount of extractable resources $R(t)$ to the society which is updated according to its consumption. This macroscopic timescale would be roughly equivalent to an actual month in the sense that this is the periodicity with which economic updates are carried out. The resources necessary to produce the goods consumed by the society in the previous microscopic cycle generate the environmental impact $I(t)$ and this quantity will be extracted from $R(t)$ at each update of $\mathcal{E}$.

A popular choice for a minimal model would be the logistic map. Its continuous version, the logistic equation, has been used before to model the evolution of resources for different systems [18,19] and is useful for the understanding of qualitative aspects of sustainability of natural resources. However, the usual logistic map does not have some properties we would desire in our case. In order to represent environment features that we think are important for our intended application, we will choose the simplest map satisfying the following criteria.

(i) There should be a maximum possible amount of resources available for consumption $R_{\max}$, the environmental limit, above which the resources cannot grow if they are renewable. We use it to define the relative amount of available resources $r(t) = R(t)/R_{\max}$.

(ii) If the resources are not renewable, the update equation is simply a linear discount of the society's amount of consumption, i.e., $r(t+1) = r(t) - \eta(t)$, where $\eta(t) = I(t)/R_{\max}$ is the relative impact.

(iii) If renewable, the resources should grow proportionally to the current amount $r(t) - \eta(t)$ whenever $0 < r(t) - \eta(t) \ll 1$. This is intended to model the fact that populations would grow exponentially when they have enough resources to reproduce freely.

(iv) The value $r(t) = 0$ is a fixed point of the dynamics, meaning that a depleted environment cannot recover.

(v) In particular, if $\eta(t) \geqslant r(t)$ at any time, this should imply $r(t+1) = 0$.

We now show that the following map obeys all of the above requirements:

$$r(t+1) = \Lambda(r(t) - \eta(t))\{1 + \alpha[1 - r(t) + \eta(t)]\}. \tag{2}$$

Here $0 \leqslant \alpha \leqslant 1$ is the growth rate of the environmental resources and $\Lambda(x)$ is the ramp function, which is 0 if $x < 0$ and x otherwise, automatically guaranteeing that conditions (iv) and (v) are satisfied.

Consider the case where $\eta(t) < r(t)$. If $\alpha = 0$, then $r(t+1) = r(t) - \eta(t)$, satisfying condition (ii). For $\alpha > 0$, the map can then be written as

$$r(t+1) = [r(t) - \eta(t)](1+\alpha) - \alpha[r(t) - \eta(t)]^2. \tag{3}$$

Because $0 \leqslant r(t) - \eta(t) \leqslant 1$, the leading term of the expression above satisfies (iii), while the quadratic term is a lesser-order perturbation.

It only remains to show that the model satisfies also condition (i). In fact, if $0 \leqslant r(t) \leqslant 1$ then the two inequalities $0 \leqslant r(t+1) \leqslant 1$ can be simultaneously solved for α, giving

$$r(t+1) \geqslant 0 \Rightarrow \alpha \geqslant \frac{1}{r(t) - \eta(t) - 1}, \tag{4}$$

$$r(t+1) \leqslant 1 \Rightarrow \alpha \leqslant \frac{1}{r(t) - \eta(t)}. \tag{5}$$

Because $0 < r(t) - \eta(t) \leqslant 1$, its maximum value is attained for $r(t) = 1$ and $\eta(t) = 0$, for which case $r(t) = 1$ is a fixed point for any α, as can be checked by plugging these values into Eq. (2). The first bound is therefore always negative and $\alpha \geqslant 0$ guarantees it is satisfied. The second bound is always greater than one and, as long as $\alpha \leqslant 1$, it is also obeyed.

Therefore, the map (2), with $\alpha \in [0, 1]$, satisfies all required conditions.

The nonequilibrium steady states of the system, the equivalent of its phases, are the fixed points of this map. We have already found that $r(t) = 0$ is always a fixed point and that $r(t) = 1$ is a fixed point when $\eta(t) = 0$. To obtain a more detailed characterization of the system, we will find its bifurcation diagram considering $\eta \in [0, 1]$ as a (constant) tuning parameter and drop its explicit dependence on t to simplify the notation (if $\eta > 1$, then the system is automatically in environmental collapse at the $r = 0$ fixed point). Later we will introduce a stochastic dynamics for η and analyze how it changes the current picture.

The stability analysis of one-dimensional maps is a standard procedure. We will give here only the results and direct the reader to usual textbooks for the technical aspects [20,21]. Our system has the two fixed points

$$r_{\pm}^{*} = \frac{1}{2} + \eta \pm \frac{1}{2}\sqrt{1 - \frac{4\eta}{\alpha}}. \tag{6}$$

For $\eta = 0$, we have $r_{+}^{*} = 1$ and $r_{-}^{*} = 0$, the latter coinciding with the collapse value, which is always fixed by design. When $0 < \eta < \alpha/4$, the above solution leads to two nonzero fixed points. The value of r_{+}^{*} is always greater than $\frac{1}{2} + \eta$. Notice also that the argument of the square root is always in the interval [0,1], which means that the last term is less than or equal to $\frac{1}{2}$, implying that r_{-}^{*} is also positive for $\eta > 0$. If $\eta = \alpha/4$, the two solutions collapse to the single value

$$r_{0}^{*} = \frac{1}{2}\left(1 + \frac{\alpha}{2}\right) \Rightarrow \frac{1}{2} \leqslant r_{0}^{*} \leqslant \frac{3}{4}. \tag{7}$$

It is easy to see that all solutions obey $r_{0}^{*} > \eta$, which justifies ignoring the step function.

The $r = 0$ fixed point exists for all values of η. It is easily seen to be attractive for any $\eta > 0$, being only repulsive in the particular case $\eta = 0$. For the other fixed points, we consider the following two cases separately: $\mathbf{0 \leqslant \boldsymbol{\eta} < \boldsymbol{\alpha}/4}$, in which the dynamics will always move towards r_{+}^{*} and away from r_{-}^{*}, rendering the former an attractive fixed point and the latter a repulsive one, and $\boldsymbol{\eta} = \boldsymbol{\alpha}/\mathbf{4}$, in which there is only one nonzero fixed point in this case, which is attractive for values above it and repulsive for values below it. In this latter case, the system evolves towards $r = 0$ over time.

The full bifurcation diagram for the particular value $\alpha = 1$, used for illustration of the qualitative features of the map, is presented in Fig. 1. For other nonzero values of α, it remains qualitatively the same. The attractive fixed points are depicted as solid lines, while the repulsive one is depicted as a dashed line. The labels sustainable, unsustainable, and instantaneous collapse indicate the asymptotic behavior if the map is initialized at points inside those regions. The sustainable region (green) comprises the area encircled by the nonzero fixed points and above them. It is characterized by an asymptotic nonzero value of the natural resources. The rest of the diagram (white and gray) is deemed unsustainable as the dynamics lead invariably to the zero fixed point. The gray area highlighted in the plot and labeled instantaneous collapse is the region where, by design, the system instantly is considered to have an environmental collapse.

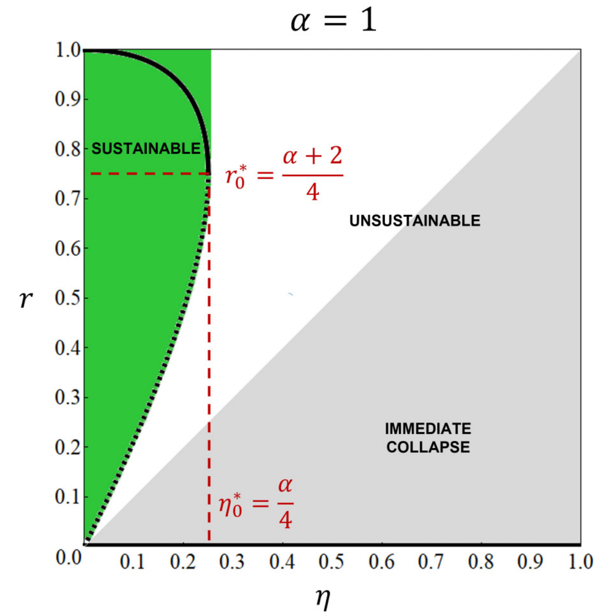

FIG. 1. Bifurcation diagram. Thick solid black lines indicate attractive (stable) fixed points for the natural resources r (r_{+}^{*} and 0) and the dashed black line the repulsive (unstable) one (r_{-}^{*}) for each value of the relative impact η. The regions on the plot show the asymptotic behavior if the dynamics is started in them. The value $\alpha = 1$ is used. The unsustainable region comprises both white and gray areas, the immediate collapse being a subregion of it. The dashed red lines, with the associated values for η and r, identify the point from where the sustainable region ceases to exist.

We can now see that the map describes a system with a threshold η_{0}^{*} for the extraction of resources which equals only one-fourth of the growth rate α. Above this, resources will be eventually depleted. At and below this value, the fate of the system depends on the initial amount of available resources. Even though the system can still collapse, as long as $\eta \leqslant \eta_{0}^{*}$, it is possible to have an attractive nonzero fixed point with constant resources in the long term.

It is important to look at what happens for perturbations $\delta\eta$ of the value of the impact η. In this case, the stability of the sustainable region against them will depend both on the size of the perturbation and on the current value of r. At the line r_{-}^{*} up to the point r_{0}^{*}, including it, any perturbation $\delta\eta > 0$ will throw the system into the unsustainable region, showing that the sustainability of the system at the threshold values for the impact is highly unstable to any increase of the latter. At the line r_{+}^{*}, the system is more robust. As long as $\delta\eta \leqslant \eta_{0}^{*} - \eta$, the system remains sustainable.

It is worth recalling that the value of the threshold comes from the simplified model above. The dynamics might change for more realistic situations, but due to the constraints used to build the equation, the existence of a threshold like this seems to be a sensible conjecture.

Being a minimal model, the above dynamical equation can be further modified to accommodate the influence of processes that might change consumption and renewal of resources, which should then result in a change of the stability diagram. For instance, higher efficiency of the resource can be translated, in a first approximation, as a multiplicative constant in the interval (0,1) in front of the extracted resources. This would appear both directly and as a factor on the ratio between consumption and growth that appears in the expression of the fixed points. Although we could generalize the model in such a way that all these possibilities could be included as additional parameters, we will keep the model simple in this initial analysis.

IV. DYNAMICS OF THE SOCIETY

We now describe in more detail the dynamics of the society which establishes the flow of money and extracted resources in the model. At the beginning of each macroscopic time step t, each citizen i of $\mathcal{S}$ receives a certain integer budget $b_i \geqslant 0$ according to a probability distribution representing $\mathcal{S}$'s income distribution and assumed to be known *a priori*. The total population size (number of citizens) is N.

Consumption proceeds at the microscopic timescale, whose discrete time steps we call τ. The citizens use their budgets to buy products from each of the four possible baskets according to predefined rules, which we leave unspecified for now. The microscopic cycle ends once every citizen has spent all their budget, at which point the next macroscopic step takes place.

Pareto analyzed data from different countries and found that the distribution of wealth could be approximated by a power-law distribution, which today is known as the Pareto distribution [22]. Due to the correlation between wealth and income, the latter has also been associated with income distributions. Modern analysis of income distribution data for several countries indicates that the population is separated in two main classes [23,24]: An upper class comprising about 3% of the population which is well fitted by a Pareto distribution and a lower-middle class whose income follows approximately a Boltzmann-Gibbs exponential distribution. Banerjee and Yakovenko [24] proposed a distribution capable of fitting both regimes

$$\mathcal{P}(x) = \frac{e^{-\lambda \arctan(x/x_0)}}{C[1+(x/x_0)^2]^{1+\gamma}}, \tag{8}$$

where x_0 is the crossover income between the two classes, λ is a temperature parameter regulating the spread of the curve, γ is a free adjustable parameter related to the exponent of the limiting Pareto distribution, and C is the normalization constant. This distribution is continuous with support on $[0, \infty]$.

Because we are working with integer values of the budget, it is convenient to convert the above distribution to discrete probabilities

$$W(b) = \frac{e^{-\lambda \arctan(b/b_0)}}{Z_W[1+(b/b_0)^2]^{1+\gamma}}, \tag{9}$$

with $b = b_{\min}, b_{\min}+1, \dots, b_{\max}$ and

$$Z_W = \sum_{b=b_{\min}}^{b_{\max}} \frac{e^{-\lambda \arctan(b/b_0)}}{[1+(b/b_0)^2]^{1+\gamma}}, \tag{10}$$

where $b_{\min}$ and $b_{\max}$ are the floor and ceiling of wages, respectively. Once budgets have been distributed, the buying choices of the citizens are updated. We assume that each product costs exactly one budget unit independently of its characteristics. The update order should depend on whether there is any influence between the choices of two different citizens. This will not be the case here and therefore the update order is irrelevant. The case of a network of influences is certainly close to real-life situations, but its analysis requires more careful consideration and will be left for future work. In the current model, there is also no possibility of saving any budget for the next round. It is probable that considering the possibility of savings might change the overall behavior, but like other possible variations of the model, it is beyond the scope of the present work.

We represent the buying choice of citizen i for the μth unit of its budget (i.e., $\mu = 1, 2, \dots, b_i$) by a two-dimensional binary variable $\sigma_i^\mu = (\iota_i^\mu, \varepsilon_i^\mu)$, $\iota_i^\mu, \varepsilon_i^\mu \in \{\pm 1\}$, where ι corresponds to the impact and ε to the essentiality. The plus sign means high and the minus sign means low for each of these variables. In principle, there is no reason for the values of impact and essentiality to have any direct dependence and can be considered as free parameters of the model. External policies and product availability can enforce some dependence, but this will not be analyzed here. Therefore, we assume that each item contributes with an environmental impact depending only on i_ι and an essentiality e_ε. While we allow the impact to assume any positive real value, for simplicity we take $e_+ = 1$ and $e_- = 0$, where 1 indicates an essential product and 0 a superfluous one.

At the end of the macroscopic time step, we calculate the total (absolute) impact $I(t)$ of the society as the sum of the individual impacts $I_i(t)$ of each citizen i, which are given by

$$\begin{aligned} I_i(t) &= \sum_{\mu=1}^{b_i} \left(\frac{1+\iota_i^\mu}{2} i_+ + \frac{1-\iota_i^\mu}{2} i_- \right) \\ &= \sum_{\mu=1}^{b_i} \left(\iota_i^\mu \Delta_- + \Delta_+ \right), \end{aligned} \tag{11}$$

with $\Delta_\pm = (i_+ \pm i_-)/2$. This value will then be used to update the environment's amount of resources according to the discrete map of Sec. III, which is used to evaluate whether the system will suffer an environmental collapse. In the model we are introducing, however, environmental collapse is not the only form of collapse. A serious imbalance in the amount of essential and superfluous consumption in the society will also contribute to its demise. In order to keep track of a possible social (as opposed to environmental) collapse, we also keep track of the system's essentiality $E(t)$ at each t by an expression analogous to the one used for the impact, but making the exchange $\iota \to \varepsilon$. The rules by which this essentiality-based social collapse can occur are discussed in the next section.

V. ESSENTIALITY, HUMAN NEEDS, AND SOCIAL COLLAPSE

The psychologist Maslow proposed in a seminal paper [25] a hierarchy of needs and aspirations that humans would strive to fulfill in life, which became known as Maslow's hierarchy of needs. The proposed classification has been challenged since then and showed to strongly depend on cultural aspects [26]. Still, that first approach can be considered as an initial model that reflects a fundamental observation: When resources are limited, humans seem to favor using them to first fulfill (biological) basic needs (hunger and safety) and only then spend resources on aspirations (careers, entertainment, and well-being).

This insight is what we use in our model, ignoring the fine details of a realistic classification and simply assuming that the probability of individuals buying either essential or a superfluous products favors the former whenever they still have basic needs to fulfill.

To be more specific, we assume that there exists a certain level of essentiality that we call $\hat{E}$ that corresponds to the border between needs (essential products) and aspirations (superfluous products). It is most probable that such a sharp border is an idealization, but more studies are needed to understand it better, which we leave for extensions of the current model.

The nature of the quantity $\hat{E}$ depends on the kind of society in which the individual is inserted. In a democracy, one would expect this to be a level which is perceived as consensual within the society. For instance, although a particular wealthy individual might assume that a car is an essential item, they can agree that having a car might be ultimately a luxury in a society with a high level of poverty. On the other hand, $\hat{E}$ might be enforced by some kind of policy, an artificial yet illustrative example being the fictional society of the novel *1984* [27], in which people are allowed to have only certain kinds of items in controlled quantities. Anything above that being considered not only superfluous, but illegal.

In fact, although we are using the words "needs" and "aspirations" above, our model does not depend strongly on these concepts, but rather on the characterization of essential and superfluous products. In addition to the observation above, what is superfluous also depends on culture, as mentioned in the introductory section to this paper. Therefore, one product can change from one category to another (e.g., electrical power). The nature of the mathematical border $\hat{E}$ is another issue, of which our binary separation is a simplification. A more appropriate representation of such separation should depend on a continuous essentiality index, which represents a challenge to define and measure and which is currently under study by the authors.

A simple choice for a quantity describing the cost for choosing to buy a product with $\varepsilon \in \{\pm 1\}$ is then

$$H_E = \frac{\varepsilon}{2}\left[E - \left(\hat{E} - \frac{1}{2}\right)\right], \tag{12}$$

where E is the citizen's current level of *satisfied* essentiality (or simply their level of essentiality, for short), the factor $\frac{1}{2}$ was introduced for convenience, and the $\frac{1}{2}$ subtracting $\hat{E}$ creates a barrier halfway between $\hat{E}$ and $\hat{E}-1$ due to the fact that individual essentiality levels change by integer units in the current model. When external influences affecting the choice of essentiality are not present, we take the probability of each buying choice for a citizen to be of the Boltzmann-Gibbs form

$$\mathcal{P}(\varepsilon) = \frac{1}{Z_E} e^{-\beta H_E}, \tag{13}$$

with the normalization constant $Z_E = 2\cosh[\beta(E - \hat{E} + \frac{1}{2})/2]$. The parameter β regulates the rigidity with which an individual makes their choices. If $\beta = 0$, the boundary defined by $\hat{E}$ is completely ignored by the individual when making a decision about consumption. The greater the value of β is, the more likely it is for the individual to favor a decision that will bring them closer to $\hat{E}$. Still, for any finite value of β, there is a probability of making a superfluous choice even when the minimum amount of essentiality is not satisfied; only in the limit $\beta \to \infty$ superfluous choices are strictly forbidden if $E < \hat{E}$. It should be clear that, in realistic situations, β is hardly in this limit as humans do not fully abandon superfluous choices even in the harshest conditions. Entertainment in all forms, including harmful kinds like addictive drugs, can even become dominant in certain scenarios. These considerations connect the aspiration part of the hierarchy to matters concerning mental health and similar topics, which are beyond the aims of our present analysis.

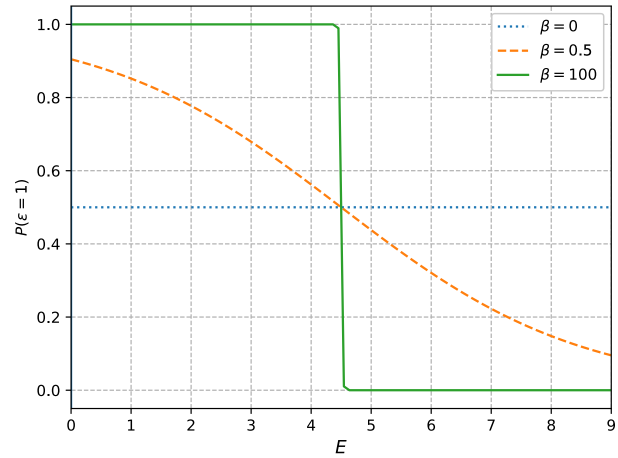

FIG. 2. Probability $\mathcal{P}(\varepsilon = 1)$ for an individual to buy an essential product given their current level E of essentiality. In the plot, $\hat{E} = 5$ for different values of β. Notice how the greater its value is, the sharper the barrier at $\hat{E} - 1/2$ becomes, meaning that in the infinite limit the individual will buy only essential products if $E < \hat{E}$ and only superfluous products if $E \geqslant \hat{E}$.

The above expression allows us to write the probability of buying an essential product ($\varepsilon = 1$) with the form

$$\mathcal{P}(\varepsilon = 1) = \frac{1}{1 + e^{\beta(E - \hat{E} + 1/2)}}, \tag{14}$$

known as a Fermi-Dirac distribution, where $\hat{E}$ plays the role of what is called a chemical potential [28], evidencing the role of β in smoothing the hard cutoff. Figure 2 illustrates the above probability for different values of β.

We are finally in a position of characterizing the conditions for an essentiality-based social collapse, which we will assume to be caused by a lack of balance between needs and aspirations of the society. Here we use a heuristic criterion for triggering social collapse based on previous work [14]. Social collapse is surely a complex subject, but we will constrain our study to essentiality-based social collapse as is appropriate for our model.

Consider the average value of the (fulfilled) essentiality over the whole population at the end of the tth macroscopic time step of the society's dynamics

$$\bar{E}(t) = \frac{E(t)}{N} = \frac{1}{N}\sum_{i=1}^{N} E_i(t). \quad (15)$$

If $\bar{E} \ll \hat{E}$, this means that the population either does not have enough resources to fulfill their basic needs or is spending all of its resources on superfluous products. Either way, biological survival is likely under stress. On the other hand, if $\bar{E} \gg \hat{E}$, then only basic needs are being satisfied within that society. As we have already mentioned, if we assume that superfluous products include items which are important for the psychological well-being of the population, we can argue that the lack of them might increase social stress.

Based on these assumptions, we previously introduced the essentiality balance B [14]. In its original version, it was comprised of a band around $\hat{E}$ within which it assumed the value $B = 1$, indicating the maximum allowed balance, and would decrease linearly outside of it. According to that model, a system would suffer from a social collapse whenever $B = 0$. In order to introduce a more realistic probabilistic element, we here define the essentiality balance as the probability of avoiding social collapse given a certain value of $\bar{E}$. Let us call $\omega_S \in \{0, 1\}$, with probability $B(\omega_S|\bar{E})$, a binary random variable that indicates social collapse when $\omega_S = 1$. Then $B(\omega_S = 0|\bar{E})$ indicates the probability of avoiding social collapse.

Like in the case of the natural resources map, we will use a simple model that obeys the following heuristic requirements.

(i) The probability of avoiding collapse is zero when no essential need is fulfilled, i.e., when $\bar{E} = 0$. In this case, either citizens do not have enough monetary resources to fulfill their basic needs or they are spending too much on superfluous products. In both cases, either by necessity or by choice, the system might come close to a collapse due to biological stress.

(ii) Starting from $\bar{E} = 0$, the probability of avoiding collapse increases with $\bar{E}$ as basic needs are increasingly fulfilled.

(iii) If too many resources are spent on essential needs, that might mean that aspirations are not being fulfilled. Whatever they are, that might lead to a breakdown of mental health, leading to a social collapse triggered by social and psychological stress. This means that the probability of avoiding collapse should decrease after reaching its maximum value.

(iv) The maximum value is 1 and within a band around $\bar{E} = \hat{E}$.

Straightforward calculations can show that the following piecewise function, an example of which is given in Fig. 3, satisfies all the above requirements:

$$B(\omega_S = 0|\bar{E}) = \begin{cases} q_- \bar{E} e^{-\beta_B(\bar{E}-\Theta_-)^2} & \text{for } \bar{E} \leqslant \hat{E} - h \\ q_+ \bar{E} e^{-\beta_B(\bar{E}-\Theta_+)^2} & \text{for } \bar{E} \geqslant \hat{E} + h \\ 1 & \text{otherwise.} \end{cases} \quad (16)$$

Here $\bar{E} \in [0, \infty)$, h is a real parameter which defines a symmetric band around $\hat{E}$ for which the probability is 1, β_B is a positive parameter that controls the decay of the probability outside that band, and the remaining parameters are given by

$$q_\pm = \frac{e^{1/4\beta_B(\hat{E}\pm h)^2}}{\hat{E} \pm h}, \quad (17)$$

$$\Theta_\pm = (\hat{E} \pm h) - \frac{1}{2\beta_B(\hat{E} \pm h)}. \quad (18)$$

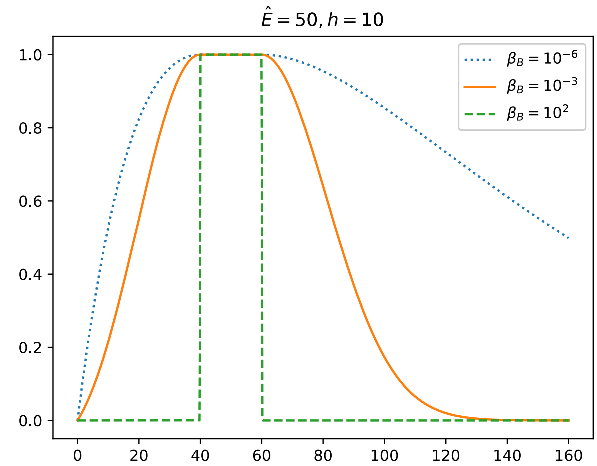

FIG. 3. Probability of the system avoiding social collapse as given in Eq. (16). In the plot, $\hat{E} = 50$ with a band of size $h = 10$ for different values of β_B.

In the same way as it is done for the environmental impact, at each new macroscopic time step, the system is tested for social collapse. While in the case of the impact there is a deterministic threshold (maximum amount of available resources) for the environmental collapse, social collapse will depend on a probabilistic test set by the distribution B.

The notion of collapse we try to capture with the model presented takes the real world as a key parameter. When the system is at one of two extremes, i.e., 100% essential or 100% superfluous, this should be seen as a sign of dysfunction, constant stress, or inability to satisfy human aspirations beyond the physiological needs. Maslow's [25] and other studies of human needs, such as Max-Neef's Human Scale Development [29], have inspired such an approach.

The reasoning is that, if a SES consumes only essential goods, considerations like the above suggest a higher probability that it is under constant stress. It indicates a lack of disposable income or resources to deal with crises or even just sustain a moment of indulgence. In practice, a more complete view of essentiality would consider each additional unit of consumption of the same essential good as having its marginal essentiality decreased when the system's needs have been partially fulfilled. For instance, the biological need for drinking water is around 2 L per day. Below such a threshold, the drinking water can be considered effectively essential. Still, each additional cup might decrease its essentiality. Above the threshold, units might be considered not essential anymore, being indulgent even if somewhat beneficial. In any case, they are not harmful or represent any dysfunction, although they may have essentiality and might be different or decreasing depending on when consumption happens. Above a certain limit though, consumption would address no actual need in the system. This is when it can be classified as superfluous and, even for drinking water, could be harmful.

VI. DECOUPLED DYNAMICS AND COLLAPSE

From here on, we assume that each citizen takes their consumption decisions without direct influence from the others. This will allow us to use the central-limit theorem (CLT) over the distributions of impact and essentiality to obtain a simplified effective dynamics. The case where peers can influence a citizen's decision is a common and important situation, but involves additional considerations and will be developed elsewhere. Indirect influence can come from governmental policies or cultural behaviors based on current levels of relevant variables, but this can be modeled as a constant local field and will not add any extra sophistication to the analysis.

How well this independence assumption reflects reality depends on the scenario being analyzed with the model. For instance, it can be used to simulate a temporary scenario in which information is prevented from flowing between agents or as a baseline to compare the effect of interactions. Interactions might be important in some situations and we are currently working on an extension of the model where people are considered as nodes on a social network and peer influence will affect choices of both impact and essentiality. That, however, adds additional levels of sophistication to the model that makes both the analytics and simulations more involved and will be left to be addressed in future works.

When such local fields are nonexistent, this means that citizens completely ignore the consequences of their choices on other citizens. This selfish behavior is well known in social sciences and leads to the famous tragedy of the commons [30], which describes the situation where the free sharing of a resource ends up exhausting it and causing its collapse due to individuals disregarding the needs of the others. For such a case, we can analyze separately the dynamics of the impact and of the essentiality. This is the scenario assumed in what follows.

The choice on the impact of the product to be bought can then be assumed to have a probability distribution

$$\mathcal{P}(\iota_i) = p\delta(\iota_i, +1) + (1-p)\delta(\iota_i, -1), \tag{19}$$

where $\delta(\cdot,\cdot)$ is a Kronecker delta. In other words, there is a probability p that a citizen will buy a high impact product and $1-p$ of a low impact one. Because impact choices and b_i are independent random variables, the mean and variance of the individual impacts $I_i(t)$ defined by Eq. (11) are calculated straightforwardly as

$$m \equiv \langle I_i(t)\rangle = [(2p-1)\Delta_- + \Delta_+]m_b, \tag{20}$$

$$\begin{aligned}\sigma^2 &\equiv \langle I_i^2(t)\rangle - \langle I_i(t)\rangle^2\\ &= [(2p-1)\Delta_- + \Delta_+]^2\sigma_b^2 + 4p(1-p)\Delta_-^2\, m_b,\end{aligned} \tag{21}$$

with m_b and σ_b^2 the mean and variance of the variables b_i, respectively. These quantities are the same for all citizens as the model considers that the I_i are independent and identically distributed random variables. Therefore, as the number of citizens N increases, the CLT guarantees that the relative impact $\eta = (1/R_{\max})\sum_i I_i$ can be approximated with increasing precision by a Gaussian-distributed random variable with mean $m_\eta = Nm/R_{\max}$ and variance $\sigma_\eta^2 = N\sigma^2/R_{\max}^2$. Each initial value of the resources will then result in a distribution of final values. This distribution can be obtained by simple population dynamics, where we define an initial population of values for r and update them for a number of time steps. Given a certain distribution of initial values, one can then obtain a probability distribution of final values after a certain number of time steps.

Consider now the threshold relative impact η_0^*. Even if $m_\eta < \eta_0^*$, the probability that, at any time step, $\eta \leqslant \eta_0^*$ is given by

$$\mathcal{P}(\eta \leqslant \eta_0^*) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{m_\eta - \eta_0^*}{\sqrt{2\sigma_\eta^2}}\right), \tag{22}$$

which, for large N, behaves like

$$\mathcal{P}(\eta \leqslant \eta_0^*) \propto N^{-1/2} e^{-Nm/2\sigma^2}, \tag{23}$$

i.e., it decreases exponentially with the population size. Therefore, the longer the system remains in the unsustainable region (also, the farthest it is from η_0^*), the larger the chances are of an environmental collapse. A large enough variance can even take the system straight to the immediate collapse region.

Because essentiality is decoupled from impact under the above assumptions, we can analyze its dynamics separately. Each citizen i receives b_i units to spend at the beginning of the month and buy b_i different products according to the probability given by Eq. (13). As stated before, buying will proceed until each citizen has spent all its units. When the choices are independent (i.e., citizens do not influence one another), one can ignore the collective microscopic time and consider the evolution of the essentiality of each citizen in an effective "proper time" $\theta = 0, \ldots, b_i$, where each tick of the clock corresponds to a unit spent by the citizen. Therefore, at effective microscopic time step θ, the essentiality of the ith citizen is

$$E_i(\theta) = \begin{cases} E_0, & \theta = 0\\ \sum_{\mu=1}^{\theta} \frac{1+\varepsilon_i^\mu}{2}, & \theta \geqslant 1.\end{cases} \tag{24}$$

The dynamics of E_i is that of a first-order Markov chain. For the specific binary values of essentiality we are using, we have that $E_i(\theta) \leqslant b_i$. The transition probability from state $E_i = k$ to state $E_i = l$, $k, l = 0, \ldots, b_i$, is then

$$\begin{aligned}T_{lk}^{(i)} &\equiv \mathcal{P}(E_i(\theta) = l | E_i(\theta-1) = k)\\ &= \frac{\delta_{kl}}{1+e^{-\beta(k-\hat{E}+1/2)}} + \frac{\delta_{k+1,l}}{1+e^{\beta(k-\hat{E}+1/2)}},\end{aligned} \tag{25}$$

according to Eq. (13). This leads to the $(b_i+1)\times(b_i+1)$ transition matrix for the chain

$$T^{(i)} = \begin{pmatrix} p_0 & 0 & 0 & \cdots & 0 & 0\\ 1-p_0 & p_1 & 0 & \cdots & 0 & 0\\ 0 & 1-p_1 & p_2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & 1-p_{b_i-1} & p_{b_i}\end{pmatrix}, \tag{26}$$

with

$$p_n = \frac{1}{1+e^{-\beta(n-\hat{E}+1/2)}}, \quad n < b_i, \tag{27}$$

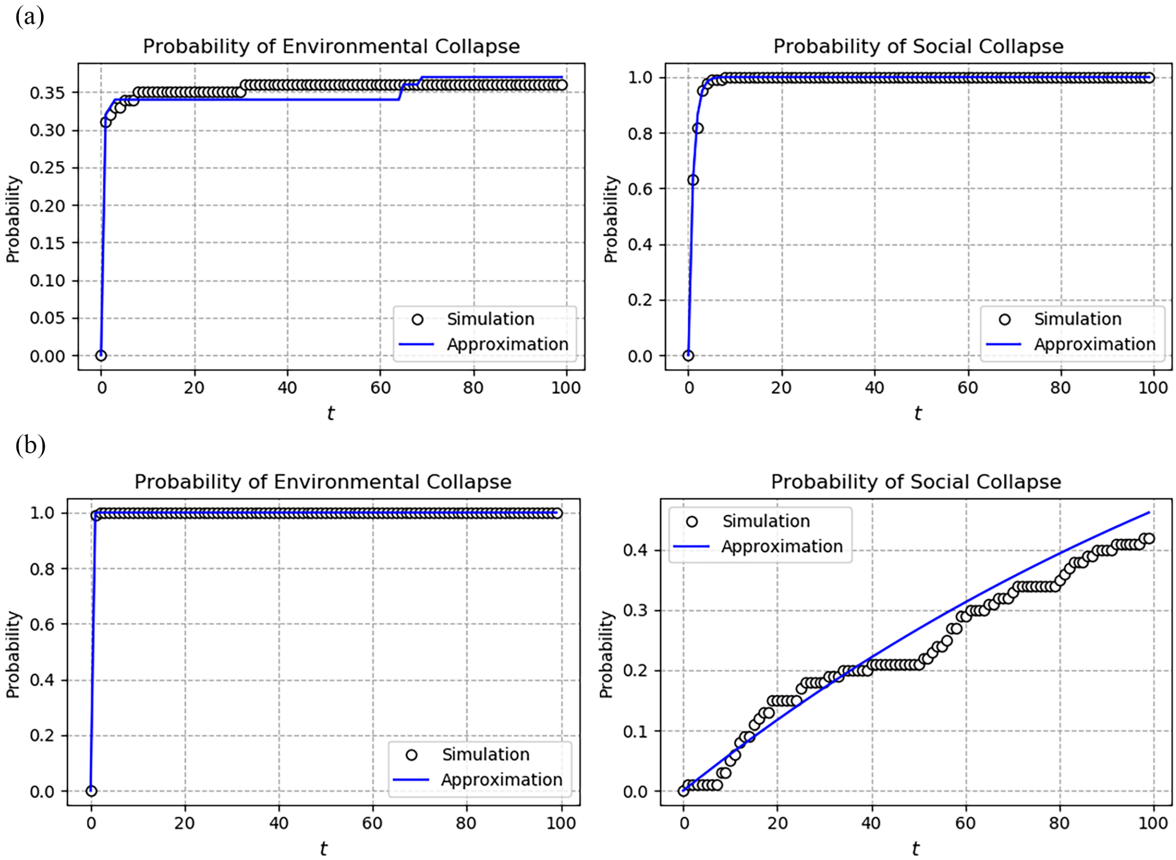

FIG. 4. Results of simulation (open circles) and the corresponding approximations (lines) of the SUE model with parameters given in Table I and (a) $b_{\max} = 4$ and (b) $b_{\max} = 15$.

and $p_{b_i} = 1$. Because $T^{(i)}$ is lower triangular, its eigenvalues are trivially $\lambda^\mu = p_\mu$, $\mu = 0, \ldots, b_i$.

The probability of collapse at that specific time step is then

$$\mathcal{P}(\omega_S = 1) = \int d\bar{E}\, B(\omega_S = 1|\bar{E})\mathcal{P}(\bar{E}) = 1 - \int d\bar{E}\, B(\omega_S = 0|\bar{E})\mathcal{P}(\bar{E}), \tag{28}$$

and we need to calculate the probability distribution for $\bar{E}$, which is the (scaled) convolution of the individual distributions of essentialities at the end of the microscopic cycle. Dropping the i indices for convenience, they are themselves

$$\mathcal{P}(E(\theta = b)) = \sum_b \mathcal{P}(E(\theta = b)|b)\mathcal{P}(b), \tag{29}$$

which means that they are all the same for every citizen. Therefore, $\mathcal{P}(\bar{E})$ can, in the limit of large N, be approximated by a Gaussian with the mean and variance equal to those of the individual essentialities. Let us define the vector $\pi(\theta)$ by its components as

$$\pi(\theta)_n = \mathcal{P}(E(\theta) = n) = [T^\theta \pi(0)]_n, \tag{30}$$

with $\pi(0)_n = \delta_{nE_0}$, where E_0 is an initial value of the essentiality which we assume to be the same for all citizens. For $\theta \geqslant 1$, we have

$$\mathcal{P}(E(\theta) = l) = \sum_k \mathcal{P}(E(\theta) = l|E(\theta - 1) = k)\mathcal{P}(E(\theta - 1) = k), \tag{31}$$

and we always start the process with $E(0) = E_0$. Any required averages, including the mean and variance, can then be obtained from the kth raw moments

$$\langle E_f^k \rangle = \sum_{b=b_{\min}}^{b_{\max}} \sum_{n=0}^{b} n^k T^b_{nE_0}\mathcal{P}(b), \tag{32}$$

where E_f means the final value of the essentiality at the end of the microscopic cycle. In particular, let us define $m_E = \langle E_f \rangle$ and $\sigma_E^2 = \langle E_f^2 \rangle - m_E^2$. These moments can be calculated numerically, with the most costly steps being the matrix powers T^b. This can be further developed analytically in terms of the left and right eigenvectors of T, which are explicitly given for the sake of completion in the Appendix. However, the entries of these eigenvectors will require recurrent expressions which will also need to be calculated numerically in practice, which does not seem to lead to significant improvements in terms of computational time.

In general, unless the distribution of E_f is pathological in some way, the CLT can be once again invoked to claim that the distribution of $\bar{E}$ can be well approximated by a Gaussian

TABLE I. Set of values used in the simulation.

Variable	Value	Variable	Value
$R_{\max}$	500	λ	1.0
N	100	b_0	50
α	1.0	γ	0
i_+	1.0	$\hat{E}$	4
i_-	0.5	β	50
p	0.6	β_B	0.5
E_0	0	h	0.7
$b_{\min}$	0		

with mean m_E and variance σ_E^2/N for large N. This allow us to find an analytical, albeit complicated, expression for the distribution for social collapse from Eq. (28), which we will not show here as it is straightforward to obtain.

We can combine both probabilities of collapse and, when essentiality and impact do not have any direct coupling, write the probability distribution for avoiding either collapse until time step T as

$$\mathcal{P}(\omega_E = 0, \omega_S = 0|T) = \mathcal{P}(\omega_S = 0|T) \times \int dr_0 \mathcal{P}(\omega_E = 0|T, r_0)\mathcal{P}(r_0), \tag{33}$$

where r_0 represents the initial value of the available resources. The first is given by $\mathcal{P}(\omega_S = 0|T) = \mathcal{P}(\omega_S = 0)^T$ as the probability of collapse does not depend on the specific time step. The second term can be calculated by population dynamics by measuring the fraction of collapsed systems on a certain population.

VII. INTERPLAY

Even without a direct interaction term between the decisions of consuming items according to essentiality and impact, there is an indirect interplay which is controlled by the average value of the wealth distribution. In the scenario presented here, citizens spend their whole budget at every macroscopic time step. Therefore, the larger the average wealth, the greater the environmental impact which, depending on the combination of the other parameters, can lead to an environmental collapse. With all other related parameters remaining the same, the solution would then be to keep the average wealth within levels that guarantee the environment enough room to renew at each iteration. On the other hand, by decreasing too much the average wealth, one can prevent citizens from fulfilling their basic needs, which might lead to a social collapse.

The situation described above is illustrated in Fig. 4, where the parameters of the model are all set to the same values given in Table I except for $b_{\max}$. By changing $b_{\max}$, we can change the value of the average wealth.

Figure 4(a) shows the results for $b_{\max} = 4$ and Fig. 4(b) for $b_{\max} = 15$, which lead to average wealth values of approximately $m_b = 1.957$ and 6.968, respectively. In the first case, the system reaches quickly an environmental steady state below collapse, but social collapse eventually takes place. The second case shows the opposite behavior; we observe a quick environmental collapse, but the probability of social collapse, although increasing, does that at a very slow pace.

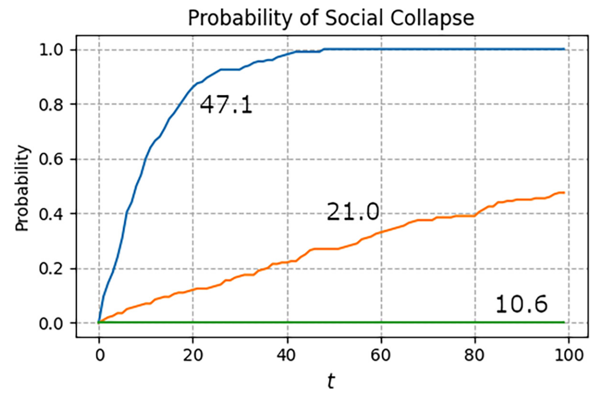

FIG. 5. Results of simulation of social collapse for approximately the same average b ($m_b \approx 7$) for different variances given next to the corresponding curve.

Figure 5 shows simulations for the probability of social collapse for values of $m_b \approx 7$ and different variances which are shown next to the corresponding curve. Most parameter values used for each case are the same as in Table I, except for the four parameters $b_{\max}$, b_0, γ, and λ. The values used are 11, 7735, 1, and -1000 for $\sigma_b^2 \approx 10.6$; 15, 50, 0, and 1 for $\sigma_b^2 \approx 21.0$; and 30, 11.4, 0, and 1 for $\sigma_b^2 \approx 47.1$.

The increasing variance might be understood as an increasing amount of inequality in the wealth distribution. The value of m_b is greater than the minimum acceptable value of essentiality, but as the inequality increases, there is a greater chance that many people will not be able the stay in the allowed essentiality band and that will increase the chances of social collapse.

Another interesting question is what happens if one allows some tolerance to higher values of the essentiality against social collapse. Figure 6 shows the results of simulations where we modify the probability of buying essential products by making $\beta = 0$, meaning that an individual will choose randomly between an essential and a superfluous product to buy. We then use an asymmetric interval around $\hat{E}$ in the probability of avoiding collapse and observe the effect of increasing the right-hand size of it.

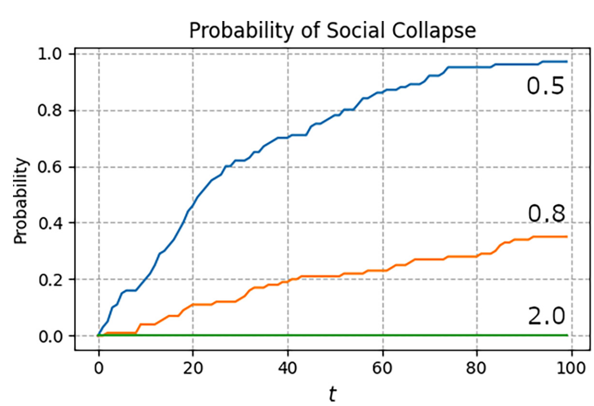

FIG. 6. Results of simulation of social collapse for different tolerances against collapse for high essentiality. The numbers close to the curve are the lengths of the right-hand-side intervals.

The plot shows how the probability of social collapse quickly decreases as the tolerance interval becomes wider. This is the case for the combination of parameters we used because the choice of parameter for the probability of buying an essential product guarantees that citizens will still by essential products even if their essentiality is fulfilled. That effect would not be seen in our previous simulations as we are assuming that, whenever the essentiality is fulfilled, the citizen starts to buy only superfluous products.

The simulations show that the model can capture well the intuitive behavior expected from the model. We also can see from the plots that the approximations we have derived are in good agreement with the actual results, even for population values which are not very large.

What combination of possible parameters reflects better real situations is an important aspect which has implications on the stability of the model dynamics. The model developed here needs to be compared to data collected from real situations and analyzed in order to evaluate the plausibility of its parameter space. The task involves research and data from different areas. Without a thorough consideration of the actual meaning of these numerical values in real situations, one might indeed risk carrying out an analysis that might miss relevant points and focus on potentially irrelevant parameter subspaces and studies in this direction are currently being carried out by the authors.

VIII. DISCUSSION

The model, as it was introduced and analyzed in this work, is clearly a toy model even though, by formalizing mathematically the concepts introduced by it, it allows an initial insight into the dynamics of the existing opposing forces when trying to prevent a serious loss of environmental resources under the constraint of fulfilling social needs of a population. Our intention was to lay down the basic blocks on a framework to describe and analyze mathematically the interplay between environmental impact and social health, here measured as a function of the proposed essentiality measure, of a socioecological system. It is true that real-life agents can have consumption behaviors which will in general be much more complex than the one considered in this work. The consequences of such behaviors, which might change the results of this paper, indeed cannot be captured by our analysis. However, as the model variables have clear interpretations, they not only provide insights into the behavior of such systems, but also clarify avenues for improvement of the model.

There are, in addition, several possibilities that were not considered here.

(i) Credit. If citizens are allowed to borrow money or buy goods for paying at a later time, this would change the limits on the amount of goods that can be bought and one would expect that to drive the system deeper into the unsustainable region. Such a situation would depend on rules of credit, such as the total amount that can be borrowed, conditions of repayment, and deadlines.

(ii) Savings. The possibility of saving money at a certain time to spend that amount later can introduce an extra source of unpredictability in the model. If such actions are random, their effect might average out in the long term, but if they are coordinated (or driven by external factors), they might create sudden bursts on a system that is otherwise believed to be in a sustainable situation.

(iii) Price distributions. Instead of considering all products to have the same price, a more realistic situation would be to include a price distribution. For large populations, one would expect that the behavior of the system would then be dependent on general statistical features of the distribution, like whether it is uni- or multimodal, has a large tail, or has nonconverging moments.

(iv) Interactions. Considering citizens as agents of social networks with possibly complex structures and taking into consideration influences of peers in their consumption choices can result in a more diverse behavior of the model. Some similarity of our model to the Ashkin-Teller model [31] hints at a complex phase diagram with possible topological transitions. Such influences have been recently suggested and analyzed in models of opinion dynamics [32] and are a possible avenue for future exploration in our model.

Although all these modifications could bring the model closer to real situations, our objective was to explore the most fundamental aspects and construct a first framework to which these modifications can be added later on and analyzed properly.

The simulations presented show that the model can capture expected aspects of the system's behavior and that the techniques developed here for analysis, especially the derived approximations, are efficient tools of study. Some parameters and variables we chose for our model are clearly difficult to measure in real settings, for instance, the essentiality of products.

Several elements of the model presented here are currently being used by the authors in the formulation and implementation of environmental strategies. Running different scenarios with this model has proven, in practice, to improve reflection on the impact of strategic decisions on sustainability performance and reduced risk of greenwashing, among other benefits. Moreover, its explanatory power contributes to the understanding of the behavior of SESs going through transitions and/or perturbations, providing insights about future model improvements. For instance, the model provides a stronger viewpoint to explain the ban on production and imports of ozone-depleting refrigerants rather than the fridges themselves. China's increase in export tariffs when there is a shortage of energy to produce essential goods for the domestic market also gains a clearer understanding through the lenses of our model.

Earth is a nonequilibrium dynamical system, of which the societies are an inseparable part. In order to survive, they need to constantly reassess priorities and adapt their strategies in key economic sectors taking into consideration the value of their natural resources. At moments of crisis, such assessments imbue a sense of urgency and/or contribute to quicker changes. One of the main challenges in implementing urgent policies, especially when they constrain both economic activities or the normal pace of life, is to decide which activities are essential or not. We have seen first hand (e.g., during the COVID-19 pandemic) that such restrictive measures lead to the discontentment of macroscopic fractions of the society, and policy needs to balance on a fine line between avoiding greater losses and attaining the minimum necessary level of compliance. In any given crisis, be it political, economic, social, or ecological, SESs need to address what their essential activities are [33,34]. Even nowadays, only a small number of companies, for instance, assess the essentiality of their products, a scenario that our model could help change as it connects essentiality and impact to the possibility of devising more robust responses to crisis for any SES, keeping the system's resilience in both ecological and socioeconomical dimensions.

One question that might arise from our analysis is whether collapse is unavoidable given the results of our simulations. Whether or not collapse happens depends on how the dynamics of the system develops on the stability diagram of Fig. 1, which will be defined by the attraction basins of the fixed points. Here there is a subtle point. Once the system enters the unsustainable phase, collapse cannot be avoided unless the parameters defining the stability diagram change. If they do, then the phases can be readjusted and the system might be driven back to stability. This is good news, as collapse could be avoided by changes in consumption patterns or in the renewable rate of the system. Such a control scenario is not addressed here, but the authors have been considering such strategies. In our model, it is defined *ad hoc* that once all resources are used, the system immediately collapses, but in reality the collapse, even if initially unavoidable, will take a finite time to happen and such a delay might be used to divert resources in an attempt to reverse the situation. Once more, this would be an additional modification that we leave to explore in extensions of the current work.

There is a connection between our ideas, in particular the concept of collapse, and those developed in viability theory [35]. Roughly speaking, a viable system has the ability to persist and maintain its function over time in a certain environment. If one considers that a nonviable system is one that is unable to sustain itself and is at risk of collapsing, indeed the simplified model presented by us would imply a general nonviability of such systems. In our model, a lack of exact balance between essential and superfluous consumption will lead to that, as excessive consumption leads to resource depletion and environmental degradation, which immediately implies the system's inability to sustain itself. However, the analysis presented here also points to what features of the model lead to that, providing indications of how it can be modified to allow the viability of the system.

ACKNOWLEDGMENTS

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APPENDIX: EIGENVECTORS OF THE ESSENTIALITY MARKOV CHAIN

In order to find the first two moments of the Markov chain from Sec. VI, we need to calculate the bth power of the transition matrix, which can be done by using the expression

$$T^b = \sum_{\mu=0}^{b} (c^\mu)^{-1} (\lambda^\mu)^b A^\mu, \tag{A1}$$

where the matrices A^μ form the spectral set of T and are given by

$$A^\mu = u^\mu v^\mu, \tag{A2}$$

where u^μ and v^μ are the right and left eigenvectors of T for the eigenvalue λ^μ, respectively. The equation for the right eigenvalues reads $Tu^\mu = \lambda^\mu u^\mu$ and from this we have

$$p_0 u_0^\mu = \lambda^\mu u_0^\mu, \quad (1-p_n)u_n^\mu + p_{n+1}u_{n+1}^\mu = \lambda^\mu u_{n+1}^\mu, \quad n \geqslant 0. \tag{A3}$$

The second relation implies

$$u_n^\mu = \left(\frac{\lambda^\mu - p_{n+1}}{1-p_n}\right) u_{n+1}^\mu. \tag{A4}$$

Because the eigenvalues are $\lambda^\mu = p_\mu$, we have

$$u_n^\mu = 0, \quad n < \mu, \tag{A5}$$

$$u_n^\mu = \left(\prod_{m=n+1}^{b} \frac{\lambda^\mu - p_m}{1-p_{m-1}}\right) u_b^\mu \tag{A6}$$

and we are free to adjust the scale. There is a similar equation for the left eigenvalues $vT = \lambda v$ which gives

$$p_b v_b^\mu = \lambda^\mu v_b^\mu, \quad p_n v_n^\mu + (1-p_n)v_{n+1}^\mu = \lambda^\mu v_n^\mu, \quad n < b, \tag{A7}$$

leading to

$$v_n^\mu = \left(\frac{1-p_n}{\lambda^\mu - p_n}\right) v_{n+1}^\mu \tag{A8}$$

and

$$v_n^\mu = 0, \quad n > \mu, \tag{A9}$$

$$v_n^\mu = \left(\prod_{m=n}^{\mu-1} \frac{1-p_m}{\lambda^\mu - p_m}\right) v_\mu^\mu, \tag{A10}$$

and we are free to adjust the scale again.

It becomes convenient to normalize the eigenvectors in such a way that $c^\mu = v^\mu u^\mu = 1$. Given the constraints in the definitions of their coordinates, it is clear that $v^\mu u^\mu = v_\mu^\mu u_\mu^\mu$. We can then choose the normalizations of the eigenvectors such that $v_\mu^\mu = u_\mu^\mu = 1$. The last equality can be achieved by choosing

$$u_b^\mu = \prod_{m=\mu+1}^{b} \frac{1-p_{m-1}}{p_\mu - p_m}. \tag{A11}$$

Let us define the indicator function

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false.} \end{cases} \tag{A12}$$

Then we can write the eigenvectors as

$$u_n^\mu = \chi(n > \mu)\left(\prod_{m=\mu+1}^{n} \frac{1-p_{m-1}}{p_\mu - p_m}\right), \tag{A13}$$

$$v_n^\mu = \chi(n < \mu)\left(\prod_{m=n}^{\mu-1} \frac{1-p_m}{p_\mu - p_m}\right), \tag{A14}$$

$$u_\mu^\mu = v_\mu^\mu = 1. \tag{A15}$$

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