Detecting genuine tripartite entanglement via general symmetric informationally complete measurements

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We study multipartite entanglement and genuine tripartite entanglement based on general symmetric informationally complete positive operator valued measurements (GSIC-POVMs). By representing the bipartite density matrices in terms of GSIC-POVMs, we obtain the lower bound of the sum of squares of the corresponding probability. We then construct a special matrix with the correlation probability of GSIC-POVMs to derive useful and operational criteria to detect genuine tripartite entanglement. We also generalize the results to obtain a sufficient criterion to detect entanglement for multipartite quantum states in arbitrary dimensions. Detailed examples show that the new method can detect more entangled and genuine entangled states than previous criteria.

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I. INTRODUCTION

Quantum entanglement is one of the most fundamental resources in quantum information processing such as quantum cryptography [1,2], teleportation [3], and dense coding [4]. In particular, the genuine multipartite entanglement (GME) stands out with significant applications in quantum information processing [5,6]. The detection of entanglement and genuine multipartite entanglement are essential for multipartite quantum communication and quantum computing tasks.

There have been many results in detecting entanglement and GME. T. Gao *et al.* [7] proposed a criterion for k-nonseparability of multipartite quantum systems based on quantum Fisher information. The quantum Fisher information has been also used in the detection of entanglement [8]. In Ref. [9] the authors formulated a necessary and sufficient criteria to detect the entanglement of bipartite states by using positive linear map. GME criteria based on Heisenberg-Weyl representation [10], multiple sequential observers [11], and multisetting Bell-type inequalities [12] have been presented. The authors of Ref. [13] derived the criteria to detect entanglement and GME of three-qubit states. B. Chen *et al.* [14] provided a quantum entanglement criterion by using mutually unbiased measurements that can be experimentally implemented for arbitrary d-dimensional bipartite systems. The applications of mutually unbiased measurements in entanglement detection are examined in Ref. [15]. In Ref. [16] the entanglement criteria for continuous variable quantum states were derived based on mutually unbiased bases.

In addition to mutually unbiased measurements, the study on symmetric informationally complete positive operator valued measurements (SIC-POVMs) in quantum information tasks has received much attention. Most researches on SIC-POVMs are focused on rank-1 SIC-POVMs. By using SIC-POVMs the authors of Ref. [17] provided a stronger entanglement detection criterion than the computable cross-norm or realignment criterion based on local orthogonal observables. The existence of SIC-POVMs in arbitrary dimensions is still an open problem [18]. Analytical solutions have been only found in dimensions $d = 2 - 151, 168, 172, 195, 199, 228, 259, 323,$ and 844 [19]. Unlike the SIC-POVMs, the generalized SIC-POVMs (GSIC-POVMs) do exist for arbitrary dimensional systems. Gour and Kalev [20] constructed the set of GSIC-POVMs from generalized Gell-Mann matrices. The authors of Ref. [21] presented entanglement criteria for bipartite systems and multipartite systems of multilevel subsystems by using GSIC-POVMs. In [22] entanglement criteria based on GSIC-POVMs have been obtained and the validity and the power of entanglement detection have been formulated. The authors of Ref. [23] studied the quantum entanglement criteria by using GSIC-POVMs for both bipartite and multipartite systems. The authors of Ref. [24] presented an entanglement criterion for arbitrary high-dimensional bipartite systems based on GSIC-POVMs. In Ref. [25] the authors proposed entanglement criteria for tripartite systems via SIC-POVMs and GSIC-POVMs and extended the criteria to multipartite systems. The GME criterion has not been studied based on GSIC-POVMs yet. A fact is that the GSIC-POVMs approach would depend on some local measurements and can be relatively easy to be implemented experimentally. Moreover, many entanglement criteria become more complex when the dimension of the quantum system increases. In view of this, using GSIC-POVMs to study quantum entanglement will likely simplify the criteria, as the measurement operators satisfy the completeness and trace relations. We study not only arbitrary dimensional multipartite entanglement but also arbitrary dimensional tripartite GME by using GSIC-POVMs in this paper.

The paper is organized as follows. In Sec. II, we construct special matrices given by the probabilities from the local measurements of GSIC-POVMs and present the criterion for detecting genuine tripartite entanglement. By detailed examples, we show that our criterion is more efficient than the existing ones. In Sec. III, we present an approach to detect the entanglement of multipartite states in arbitrary dimensions, which can detect more entangled states than previous methods by a detailed example. Conclusions are given in Sec. IV.

II. GME FOR TRIPARTITE QUANTUM STATES BASED ON GSIC-POVM

We first review some basic knowledge of general symmetric informationally complete (GSIC) measurements. Let $H_f^{d_f}$ $(f=1,2,3)$ be d_f-dimensional Hilbert spaces. A set of d_f^2 positive operators $\{P_{\alpha_f}\}_{\alpha_f=1}^{d_f^2}$ in $\mathbb{C}^d$ is said to be GSIC measurements if

$$\sum_{\alpha_f=1}^{d_f^2} P_{\alpha_f}=I_f, \quad \mathrm{Tr}[(P_{\alpha_f})^2]=a_f, \tag{1}$$

$$\mathrm{Tr}(P_{\alpha_f}P_{\beta_f})=\frac{1-d_f a_f}{d_f\left(d_f^2-1\right)}, \tag{2}$$

where I_f is the identity operator, $\alpha_f,\beta_f\in\{1,2,\ldots,d_f^2\}$, $\alpha_f\neq\beta_f$, the parameter a_f satisfies $\frac{1}{d_f^3}<a_f\leqslant\frac{1}{d_f^2}$, $a_f=\frac{1}{d_f^2}$ if and only if all P_{α_f} are rank one, which gives rise to a SIC-POVMs. It can be shown that $\mathrm{Tr}(P_{\alpha_f})=\frac{1}{d_f}$ for all α_f.

Let ρ be a quantum state in $H_f^{d_f}$ and $\{P_{\alpha_f}\}_{\alpha_f=1}^{d_f^2}$ the GSIC measurements with parameter a_f satisfying $\mathrm{Tr}[(P^f_{\alpha_f})^2]=a_f$. We have the probability $p_{\alpha_f}=\langle P_{\alpha_f}\rangle=\mathrm{Tr}(\rho\cdot P^f_{\alpha_f})$ with respect to the measurement outcome α_f. The quantum state ρ can be expressed in terms of these probabilities [20],

$$\rho=\frac{d_f\left(d_f^2-1\right)}{a_f d_f^3-1}\sum_{\alpha_f=1}^{d_f^2}p_{\alpha_f}P^f_{\alpha_f}-\frac{d_f(1-a_f d_f)}{a_f d_f^3-1}\mathbb{I}_f.$$

The summation of the squared probabilities satisfies

$$\begin{aligned}\sum_{\alpha_f=1}^{d_f^2}p_{\alpha_f}^2&=\frac{\left(a_f d_f^3-1\right)\mathrm{Tr}(\rho^2)+d_f(1-a_f d_f)}{d_f(d_f^2-1)}\\&\leqslant\frac{a_f d_f^2+1}{d_f(d_f+1)},\end{aligned} \tag{3}$$

where the upper bound is saturated if and only if ρ is a pure state [24].

For bipartite states in $H_1^{d_1}\otimes H_2^{d_2}$ we define $P^{12}_{\alpha_1\alpha_2}=P^1_{\alpha_1}\otimes P^2_{\alpha_2}$, where $\alpha_1=1,2,\ldots,d_1^2$, $\alpha_2=1,2,\ldots,d_2^2$, $\{P^1_{\alpha_1}\}_{\alpha_1=1}^{d_1^2}$ with parameter a_1 and $\{P^2_{\alpha_2}\}_{\alpha_2=1}^{d_2^2}$ with parameter a_2 are associated with the two subsystems, respectively. We first show that the operators $P^{12}_{\alpha_1\alpha_2}$ are linearly independent. Assume that $\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}a_{\alpha_1\alpha_2}P^1_{\alpha_1}\otimes P^2_{\alpha_2}=0$ for some coefficients $a_{\alpha_1\alpha_2}$, which results in that

$$\begin{aligned}&\mathrm{Tr}\left[\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}a_{\alpha_1\alpha_2}\left(P^1_{\alpha_1}\otimes P^2_{\alpha_2}\right)\left(P^1_{\alpha_1'}\otimes P^2_{\alpha_2'}\right)\right]\\&\quad=\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}a_{\alpha_1\alpha_2}\mathrm{Tr}\left(P^1_{\alpha_1}\cdot P^1_{\alpha_1'}\right)\mathrm{Tr}\left(P^2_{\alpha_2}\cdot P^2_{\alpha_2'}\right)=0,\end{aligned}$$

where $\alpha_1'=1,2,\ldots,d_1^2,\alpha_2'=1,2,\ldots,d_2^2$. From (1) and (2) we obtain $d_1^2d_2^2$ equations of $a_{11},a_{12},\ldots,a_{d_1^2d_2^2}$. The coefficient matrix of the equations is $M=N_1\otimes N_2$, where

$$N_1=\begin{bmatrix}a_1&b_1&b_1&\cdots&b_1\\b_1&a_1&b_1&\cdots&b_1\\b_1&b_1&a_1&\cdots&b_1\\\vdots&\vdots&\vdots&\ddots&\vdots\\b_1&b_1&b_1&\cdots&a_1\end{bmatrix}_{d_1^2\times d_1^2},$$

$$N_2=\begin{bmatrix}a_2&b_2&b_2&\cdots&b_2\\b_2&a_2&b_2&\cdots&b_2\\b_2&b_2&a_2&\cdots&b_2\\\vdots&\vdots&\vdots&\ddots&\vdots\\b_2&b_2&b_2&\cdots&a_2\end{bmatrix}_{d_2^2\times d_2^2}$$

with $b_1=\frac{1-d_1a_1}{d_1(d_1^2-1)}$ and $b_2=\frac{1-d_2a_2}{d_2(d_2^2-1)}$. By a series of matrix transformations M can be transformed into an identity matrix, namely, $a_{11}=a_{12}=\cdots=a_{d_1^2d_2^2}=0$ and hence $P^1_{\alpha_1}\otimes P^2_{\alpha_2}$ are linear independent. Thus, a bipartite density matrix can be linearly expressed in terms of $P^1_{\alpha_1}\otimes P^2_{\alpha_2}$.

With respect to the operators $P^{12}_{\alpha_1 0}=P^1_{\alpha_1}\otimes\mathbb{I}_2$, $P^{12}_{0\alpha_2}=\mathbb{I}_1\otimes P^2_{\alpha_2}$ and $P^{12}_{\alpha_1\alpha_2}$, we have the probability $p^{12}_{\alpha_1 0}=\langle P^{12}_{\alpha_1 0}\rangle=\mathrm{Tr}(\rho P^{12}_{\alpha_1 0})$, $p^{12}_{0\alpha_2}=\langle P^{12}_{0\alpha_2}\rangle=\mathrm{Tr}(\rho P^{12}_{0\alpha_2})$ and $p^{12}_{\alpha_1\alpha_2}=\langle P^{12}_{\alpha_1\alpha_2}\rangle=\mathrm{Tr}(\rho P^{12}_{\alpha_1\alpha_2})$ associated with a bipartite state ρ. ρ can be expressed as

$$\begin{aligned}\rho=&\frac{d_1d_2\left(d_1^2-1\right)\left(d_2^2-1\right)}{\left(a_1d_1^3-1\right)\left(a_2d_2^3-1\right)}\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}p^{12}_{\alpha_1\alpha_2}P^{12}_{\alpha_1\alpha_2}-\frac{d_1d_2(1-a_2d_2)\left(d_1^2-1\right)}{\left(a_1d_1^3-1\right)\left(a_2d_2^3-1\right)}\sum_{\alpha_1=1}^{d_1^2}p^{12}_{\alpha_1 0}P^{12}_{\alpha_1 0}\\&-\frac{d_1d_2(1-a_1d_1)\left(d_2^2-1\right)}{\left(a_1d_1^3-1\right)\left(a_2d_2^3-1\right)}\sum_{\alpha_2=1}^{d_2^2}p^{12}_{0\alpha_2}P^{12}_{0\alpha_2}+\frac{d_1d_2(1-a_1d_1)(1-a_2d_2)}{\left(a_1d_1^3-1\right)\left(a_2d_2^3-1\right)}\mathbb{I}_{12},\end{aligned}$$

where $\mathbb{I}_{12}=\mathbb{I}_1\otimes\mathbb{I}_2$ is identity matrix. We have the following relation:

$$\mathrm{Tr}(\rho)=\frac{\left(d_1^2-1\right)\left(d_2^2-1\right)}{\left(a_1d_1^3-1\right)\left(a_2d_2^3-1\right)}\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}p^{12}_{\alpha_1\alpha_2}-\frac{d_2^2(1-a_2d_2)\left(d_1^2-1\right)}{\left(a_1d_1^3-1\right)\left(a_2d_2^3-1\right)}\sum_{\alpha_1=1}^{d_1^2}p^{12}_{\alpha_1 0}$$

$$-\frac{d_1^2(1-a_1d_1)\big(d_2^2-1\big)}{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}\sum_{\alpha_2=1}^{d_2^2}p^{12}_{0\alpha_2}+\frac{d_1^2d_2^2(1-a_1d_1)(1-a_2d_2)}{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}=1.$$

Lemma 1. For any bipartite quantum state ρ, the following inequality holds,

$$\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}\big(p^{12}_{\alpha_1\alpha_2}\big)^2\leqslant\frac{\big(a_1d_1^2+1\big)\big(a_2d_2^2+1\big)}{d_1d_2(d_1+1)(d_2+1)}. \tag{4}$$

Proof. For any bipartite pure state ρ, we have

$$\begin{aligned}\mathrm{Tr}(\rho^2)=&\frac{d_1d_2\big(d_1^2-1\big)\big(d_2^2-1\big)}{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}\big(p^{12}_{\alpha_1\alpha_2}\big)^2-\frac{d_1d_2(1-a_2d_2)\big(d_1^2-1\big)}{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}\sum_{\alpha_1=1}^{d_1^2}\big(p^{12}_{\alpha_10}\big)^2\\&-\frac{d_1d_2(1-a_1d_1)\big(d_2^2-1\big)}{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}\sum_{\alpha_2=1}^{d_2^2}\big(p^{12}_{0\alpha_2}\big)^2+\frac{d_1d_2(1-a_1d_1)(1-a_2d_2)}{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}=1.\end{aligned}$$

By using $\mathrm{Tr}(\rho_1^2)=\mathrm{Tr}(\rho_2^2)$, where ρ_1 and ρ_2 be the reduced density operators with respect to the subsystem $H_1^{d_1}$ and $H_2^{d_2}$, respectively, then we obtain

$$\begin{aligned}\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}\big(p^{12}_{\alpha_1\alpha_2}\big)^2=&\frac{\big(a_1d_1^3-1\big)\big(a_2d_2^3-1\big)}{d_1d_2\big(d_1^2-1\big)\big(d_2^2-1\big)}-\frac{d_2}{d_1}\frac{\big(a_1d_1^3-1\big)(a_2d_2-1)}{\big(d_1^2-1\big)\big(a_2d_2^3-1\big)}\sum_{\alpha_2=1}^{d_2^2}\big(p^{12}_{0\alpha_2}\big)^2\\&-\frac{d_1}{d_2}\frac{\big(a_2d_2^3-1\big)(a_1d_1-1)}{\big(d_2^2-1\big)\big(a_1d_1^3-1\big)}\sum_{\alpha_1=1}^{d_1^2}\big(p^{12}_{\alpha_10}\big)^2-\frac{d_2}{d_1}\frac{\big(a_1d_1^3-1\big)(a_2d_2-1)^2}{\big(d_1^2-1\big)\big(d_2^2-1\big)\big(a_2d_2^3-1\big)}\\&-\frac{d_1}{d_2}\frac{\big(a_2d_2^3-1\big)(a_1d_1-1)^2}{\big(d_1^2-1\big)\big(d_2^2-1\big)\big(a_1d_1^3-1\big)}+\frac{(a_1d_1-1)(a_2d_2-1)}{\big(d_1^2-1\big)\big(d_2^2-1\big)}\\\leqslant&\frac{\big(a_1d_1^2+1\big)\big(a_2d_2^2+1\big)}{d_1d_2(d_1+1)(d_2+1)}.\end{aligned}$$

By employing the convexity property of the mixed state, we have $\sum_{\alpha_1=1}^{d_1^2}\sum_{\alpha_2=1}^{d_2^2}(p^{12}_{\alpha_1\alpha_2})^2\leqslant\frac{(a_1d_1^2+1)(a_2d_2^2+1)}{d_1d_2(d_1+1)(d_2+1)}$.

A tripartite state $\rho\in H=H_f^{d_f}\otimes H_{gh}^{d_{gh}}$ associated with subsystems f, g, and h is separable under bipartition $f|gh$ if $\rho=\sum_l z_l\rho_l^f\otimes\rho_l^{gh}$, $f\neq g\neq h\in\{1,2,3\}$, where the probabilities $z_l>0$, $\sum_l z_l=1$, $\rho_l^f=\frac{d_f(d_f^2-1)}{a_fd_f^3-1}\sum_{\alpha_f=1}^{d_f^2}p_{\alpha_f}P^f_{\alpha_f}-\frac{d_f(1-a_fd_f)}{a_fd_f^3-1}\mathbb{I}_f$, and $\rho_l^{gh}=\frac{d_gd_h(d_g^2-1)(d_h^2-1)}{(a_gd_g^3-1)(a_hd_h^3-1)}\sum_{\alpha_g=1}^{d_g^2}\sum_{\alpha_h=1}^{d_h^2}p^{gh}_{\alpha_g\alpha_h}P^{gh}_{\alpha_g\alpha_h}-\frac{d_gd_h(1-a_hd_h)(d_g^2-1)}{(a_gd_g^3-1)(a_hd_h^3-1)}\sum_{\alpha_g=1}^{d_g^2}p^{gh}_{\alpha_g0}P^{gh}_{\alpha_g0}-\frac{d_gd_h(1-a_gd_g)(d_h^2-1)}{(a_gd_g^3-1)(a_hd_h^3-1)}\sum_{\alpha_h=1}^{d_h^2}p^{gh}_{0\alpha_h}P^{gh}_{0\alpha_h}+\frac{d_gd_h(1-a_gd_g)(1-a_hd_h)}{(a_gd_g^3-1)(a_hd_h^3-1)}\mathbb{I}_{gh}$ are the density matrices associated with the subsystems f and gh, respectively. A quantum state is said to be genuine multipartite entangled if it cannot be written as a convex combination of biseparable states.

Denote $e_{f,\alpha_f}=\mathrm{Tr}(\rho P^f_{\alpha_f}\otimes\mathbb{I}_{gh})$, $e_{g0,\alpha_g}=\mathrm{Tr}(\rho\mathbb{I}_f\otimes P^{gh}_{\alpha_g0})$ and $e_{gh,\alpha_g\alpha_h}=\mathrm{Tr}(\rho\mathbb{I}_f\otimes P^{gh}_{\alpha_g\alpha_h})$ for $\alpha_f=1,2,\dots,d_f^2$; $\alpha_g=1,2,\dots,d_g^2$; and $\alpha_h=1,2,\dots,d_h^2$. We define $G^{f|gh}$ to be the correlation matrix constructed from e_{f,α_f}, e_{g0,α_g} and $e_{gh,\alpha_g\alpha_h}$.

$$G^{f|gh}=m|\beta\rangle\langle\gamma_{gh}|+F_f\otimes E_g, \tag{5}$$

where

$$\langle\gamma_{gh}|=(e_{gh,11}\cdots e_{gh,d_g^2d_h^2}0),\ |\beta\rangle=(e_{f,1}\ e_{f,2}\cdots e_{f,d_f^2})^T, \tag{6}$$

$$F_f=\begin{bmatrix}c\cdot e_{f,d_f^2}+b & c\cdot e_{f,d_f^2-1}+b & \cdots & c\cdot e_{f,1}+b\\ b & b & \cdots & b\end{bmatrix}^T, \tag{7}$$

$$E_g=\big(e_{g0,1}e_{g0,2}\cdots e_{g0,d_g^2}0\big), \tag{8}$$

T stands for the transpose; b, c and m are real numbers; $\langle\gamma_{gh}|$ and E_g are $(1)\times(2d_g^2d_h^2)$ and $(1)\times(d_g^2d_h^2)$ row vectors, respectively. Let $\|\cdot\|_{\mathrm{tr}}$ stand for the trace norm defined by $\|A\|_{\mathrm{tr}}=\sum_i\xi_i=\mathrm{Tr}\sqrt{AA^\dagger}$ with respect to a matrix $A\in\mathbb{R}^{m\times n}$, where ξ_i $(i=1,2,\dots,\min\{m,n\})$ are the singular values of the matrix A.

Theorem 1. If a state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ is biseparable under the bipartition $f|gh$, then

$$\|G^{f|gh}\|_{\rm tr} \leqslant |m|\sqrt{\frac{a_f d_f^2+1}{d_f(d_f+1)}}\sqrt{\frac{(a_g d_g^2+1)(a_h d_h^2+1)}{d_g d_h(d_g+1)(d_h+1)}}+\left(|c|\sqrt{\frac{a_f d_f^2+1}{d_f(d_f+1)}}+\sqrt{2}|b|d_f\right)\sqrt{\frac{a_g d_g^2+1}{d_g(d_g+1)}}.$$

Proof. If a tripartite pure state ρ is separable under the bipartition $f|gh$, it can be expressed as $\rho = \rho^f \otimes \rho^{gh}$. From (5)–(8) we obtain

$$\begin{aligned}\|G^{f|gh}\|_{\rm tr} &\leqslant |m|\||\beta\rangle\langle\gamma_{gh}|\|_{\rm tr} + \|F_f \otimes E_g\|_{\rm tr}\\ &= |m|\||\beta\rangle\|\||\gamma_{gh}\rangle\| + \|F_f\|_{\rm tr}\|E_g\|_{\rm tr}\\ &\leqslant |m|\||\beta\rangle\|\||\gamma_{gh}\rangle\| + \left(|c|\sqrt{\sum_{\alpha_f=1}^{d_f^2} e_{f,\alpha_f}^2}+\sqrt{2}|b|d_f\right)\sqrt{\sum_{\alpha_g=1}^{d_g^2} e_{g0,\alpha_g}^2}\\ &= |m|\sqrt{\sum_{\alpha_f=1}^{d_f^2} e_{f,\alpha_f}^2}\sqrt{\sum_{\alpha_g=1}^{d_g^2}\sum_{\alpha_h=1}^{d_h^2} e_{gh,\alpha_g\alpha_h}^2} + \left(|c|\sqrt{\sum_{\alpha_f=1}^{d_f^2} e_{f,\alpha_f}^2}+\sqrt{2}|b|d_f\right)\sqrt{\sum_{\alpha_g=1}^{d_g^2} e_{g0,\alpha_g}^2}\\ &\leqslant |m|\sqrt{\frac{a_f d_f^2+1}{d_f(d_f+1)}}\sqrt{\frac{(a_g d_g^2+1)(a_h d_h^2+1)}{d_g d_h(d_g+1)(d_h+1)}}+\left(|c|\sqrt{\frac{a_f d_f^2+1}{d_f(d_f+1)}}+\sqrt{2}|b|d_f\right)\sqrt{\frac{a_g d_g^2+1}{d_g(d_g+1)}},\end{aligned} \tag{9}$$

where we have used $\|A+B\|_{\rm tr} \leqslant \|A\|_{\rm tr}+\|B\|_{\rm tr}$ in the first inequality and second inequality, $\||a\rangle\langle b|\|_{\rm tr} = \||a\rangle\|\||b\rangle\|$ for vectors $|a\rangle$ and $|b\rangle$ and $\|A\otimes B\|_{\rm tr} = \|A\|_{\rm tr}\cdot\|B\|_{\rm tr}$ for matrices A and B in the first equality and (3), (4) in the last inequality.

For tripartite mixed state $\rho = \sum_l z_l \rho_l^f \otimes \rho_l^{gh}$, from (9) and the convexity property we get

$$\|G^{f|gh}\|_{\rm tr} \leqslant |m|\sqrt{\frac{a_f d_f^2+1}{d_f(d_f+1)}}\sqrt{\frac{(a_g d_g^2+1)(a_h d_h^2+1)}{d_g d_h(d_g+1)(d_h+1)}}+\left(|c|\sqrt{\frac{a_f d_f^2+1}{d_f(d_f+1)}}+\sqrt{2}|b|d_f\right)\sqrt{\frac{a_g d_g^2+1}{d_g(d_g+1)}}. \qquad \blacksquare$$

Now we consider genuine tripartite entanglement. Let $\|G\|_{\rm tr} = \frac{1}{3}[\|G^{1|23}\|_{\rm tr}+\|G^{2|13}\|_{\rm tr}+\|G^{3|12}\|_{\rm tr}]$ and $Q = \max\{|m|\sqrt{\frac{(a_2d_2^2+1)(a_3d_3^2+1)}{d_2d_3(d_2+1)(d_3+1)}}\sqrt{\frac{a_1d_1^2+1}{d_1(d_1+1)}}+(|c|\sqrt{\frac{a_1d_1^2+1}{d_1(d_1+1)}}+\sqrt{2}|b|d_1)\sqrt{\frac{a_2d_2^2+1}{d_2(d_2+1)}}; |m|\sqrt{\frac{a_2d_2^2+1}{d_2(d_2+1)}}\sqrt{\frac{(a_1d_1^2+1)(a_3d_3^2+1)}{d_1d_3(d_1+1)(d_3+1)}}+(|c|\sqrt{\frac{a_2d_2^2+1}{d_2(d_2+1)}}+\sqrt{2}|b|d_2)\sqrt{\frac{a_1d_1^2+1}{d_1(d_1+1)}}; |m|\sqrt{\frac{a_3d_3^2+1}{d_3(d_3+1)}}\sqrt{\frac{(a_1d_1^2+1)(a_2d_2^2+1)}{d_1d_2(d_1+1)(d_2+1)}}+(|c|\sqrt{\frac{a_3d_3^2+1}{d_3(d_3+1)}}+\sqrt{2}|b|d_3)\sqrt{\frac{a_1d_1^2+1}{d_1(d_1+1)}}\}$, where b, c, and m are real numbers. We have the following theorem.

Theorem 2. A quantum state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ is genuine tripartite entangled if $\|G\|_{\rm tr} > Q$.

Proof. Suppose ρ is a biseparable quantum state, $\rho = \sum_i o_i\rho_i^1 \otimes \rho_i^{23} + \sum_j r_j\rho_j^2 \otimes \rho_j^{13} + \sum_k s_k\rho_k^3 \otimes \rho_k^{12}$ with $0 \leqslant o_i, r_j, s_k \leqslant 1$ and $\sum_i o_i + \sum_j r_j + \sum_k s_k = 1$. By Theorem 1, we have that

$$\begin{aligned}\|G\|_{\rm tr} &= \frac{1}{3}\left[\sum_i o_i\|G^{1|23}\|_{\rm tr}+\sum_j r_j\|G^{2|13}\|_{\rm tr}+\sum_k s_k\|G^{3|12}\|_{\rm tr}\right]\\ &\leqslant \frac{1}{3}(Q+Q+Q) = Q.\end{aligned}$$

Consequently, if $\|G\|_{\rm tr} > Q$, then ρ is genuinely tripartite entangled. The above proof also applies to other bipartions. $\blacksquare$

To illustrate our results let us consider three-qubit systems. For $d = 2$ we give the following four matrices [24] for any nonzero $t \in [-\frac{1}{6\sqrt{6}}, \frac{1}{6\sqrt{6}}]$,

$$P_\alpha = \tfrac{1}{4}I + t(F_4 - 6F_\alpha), \quad \alpha = 1, 2, 3,$$
$$P_4 = \tfrac{1}{4}I + 3tF_4,$$

where

$$F_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad F_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix},$$
$$F_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad F_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}.$$

By calculation it is easy to show that $\sum_{\alpha=1}^4 P_\alpha = I$, $\mathrm{Tr}[(P_\alpha)^2] = \frac{1}{8}+27t^2$ and $\mathrm{Tr}(P_\alpha P_\beta) = \frac{1}{8}-9t^2$, $\alpha \neq \beta \in \{1,2,3,4\}$. And for any nonzero $t \in [-\frac{1}{6\sqrt{6}}, \frac{1}{6\sqrt{6}}]$, we have $\frac{1}{8} < \mathrm{Tr}[(P_\alpha)^2] \leqslant \frac{1}{4}$. Thus $\{P_\alpha\}_{\alpha=1}^4$ is a GSIC measurement.

Example 1. Consider the $2\times 2\times 2$ quantum state ρ_W,

$$\rho_W = \frac{1-x}{8}\mathbb{I}_8 + x|W\rangle\langle W|,\ 0 \leqslant x \leqslant 1,$$

where $|W\rangle = \frac{1}{\sqrt{3}}[|100\rangle + |010\rangle + |001\rangle]$. When $b = m = 0$ and $c = 1$, by Theorem 2 we have $\|G\|_{\rm tr} > \sqrt{\frac{4a_1+1}{6}}\sqrt{\frac{4a_2+1}{6}}$. The GME of ρ is shown in Table I for different values

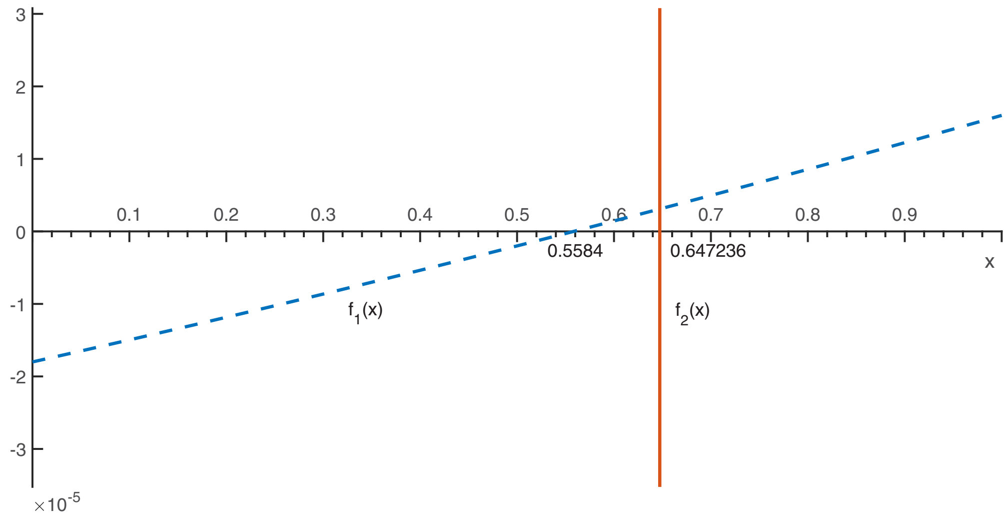

FIG. 1. $f_1(x)$ from our result (dotted blue line), $f_2(x)$ in Ref. [26] (solid red line).

of t. Moreover, when $t = 0.001$, we get $f_1(x) = \|G\|_{\rm tr} - \sqrt{\frac{4a_1+1}{6}}\sqrt{\frac{4a_2+1}{6}} > 0$, i.e., $0.5584 < x \leqslant 1$. In Ref. [26] ρ is shown to genuinely tripartite entangled if $f_2(x) = \frac{632x^2}{9(3x+1)} - 10 > 0$, i.e., $x > 0.647236$. Obviously our conclusion is better than the result in Ref. [26], see Fig. 1. Theorem 2 detects more GME states.

Example 2. Consider the quantum state $\rho_{\rm GHZ}$,

$$\rho_{\rm GHZ} = \frac{1-x}{d^3}\mathbb{I} + x|{\rm GHZ}\rangle\langle{\rm GHZ}|, \quad 0 \leqslant x \leqslant 1,$$

where $|{\rm GHZ}\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|iii\rangle$. When $d = 2$, $b = c = 0$ and $m = 1$, from Theorem 2 we obtain ρ is genuine tripartite entanglement if $\|G\|_{\rm tr} > \frac{1}{6}\sqrt{\frac{4a_1+1}{6}}\sqrt{(4a_2+1)(4a_3+1)}$. Table II shows the genuine tripartite entanglement of $\rho_{\rm GHZ}$ detected for different choices of t by using Theorem 2. Taking $t = 0.068$, we obtain $f(x) = \|G\|_{\rm tr} - \frac{1}{6}\sqrt{\frac{4a_1+1}{6}}\sqrt{(4a_2+1)(4a_3+1)} > 0$, i.e., $0.5169 < x \leqslant 1$. The result is better than the range of GME $0.733333 < x \leqslant 1$ given in Ref. [27]. By the corollary 1 in Ref. [26], ρ is genuinely entangled if $g(2,x) = \frac{120x^2}{2+6x} - 10 > 0$, i.e., $0.728714 < x \leqslant 1$. Our conclusion outperforms these existing results in detecting the genuine tripartite entanglement, see Fig. 2.

III. MULTIPARTITE ENTANGLEMENT CRITERION BASED ON GSIC-POVM

Now we study the entanglement for multipartite systems. For an n-partite state ρ in $H = H_1^{d_1}\otimes H_2^{d_2}\otimes\cdots\otimes H_n^{d_n}$, we consider GSIC-POVMs $\{P_{\alpha_i}^i\}_{\alpha_i=1}^{d_i^2}$ with parameter a_i, $i = 1, 2, \ldots, n$ for the subsystems. Any separable state is of the form,

$$\rho = \sum_l z_l\rho_l^1\otimes\rho_l^2\otimes\cdots\otimes\rho_l^n.$$

Denote $e_{i,\alpha_i} = {\rm Tr}(\rho\cdot\mathbb{I}_1\otimes\cdots\otimes\mathbb{I}_{i-1}\otimes P_{\alpha_i}^i\otimes\mathbb{I}_{i+1}\otimes\cdots\otimes\mathbb{I}_n)$, where $\alpha_i = 1, 2, \ldots, d_i^2$ and $i = 1, 2, \ldots, n$. We set

$$\begin{aligned} G^{1|2|\cdots|n} = {}& (|\beta_1\rangle\langle\gamma_1'| + F_1)\otimes(|\beta_2\rangle\langle\gamma_2'| + F_2)\otimes\cdots \\ & \otimes(|\beta_{n-1}\rangle\langle\gamma_{n-1}'| + F_{n-1})\otimes(|\gamma_n'\rangle\langle\beta_n| + F_n^T), \end{aligned} \tag{10}$$

where m_i, b_i and c_i $(i = 1, 2, \ldots, n)$ are real numbers,

$$|\gamma_i'\rangle = (m_i\ 0)^T, \quad |\beta_i\rangle = (e_{i,1}\ e_{i,2}\ \cdots\ e_{i,d_i^2})^T \tag{11}$$

and

$$F_i = \begin{bmatrix} c_i\cdot e_{i,d_i^2} + b_i & c_i\cdot e_{i,d_i^2-1} + b_i & \cdots & c_i\cdot e_{i,1} + b_i \\ b_i & b_i & \cdots & b_i \end{bmatrix}^T. \tag{12}$$

Theorem 3. For any n-partite separable state $\rho = \sum_l z_l\rho_l^1\otimes\rho_l^2\otimes\cdots\otimes\rho_l^n$, the inequality holds,

$$\|G^{1|2|\cdots|n}\|_{\rm tr} \leqslant \prod_i^n\left((|m_i| + |c_i|)\sqrt{\frac{a_id_i^2+1}{d_i(d_i+1)}} + \sqrt{2}|b_i|d_i\right). \tag{13}$$

TABLE I. The GME of ρ with respect to the coefficient t.

	Range of GME
$t = 0.068$	$0.5609 < x \leqslant 1$
$t = 0.041$	$0.5594 < x \leqslant 1$
$t = 0.001$	$0.5584 < x \leqslant 1$

TABLE II. The genuine entanglement ranges for $\rho_{\rm GHZ}$ with different t.

	Range of GME
$t = 0.055$	$0.5648 < x \leqslant 1$
$t = 0.060$	$0.5448 < x \leqslant 1$
$t = 0.068$	$0.5169 < x \leqslant 1$

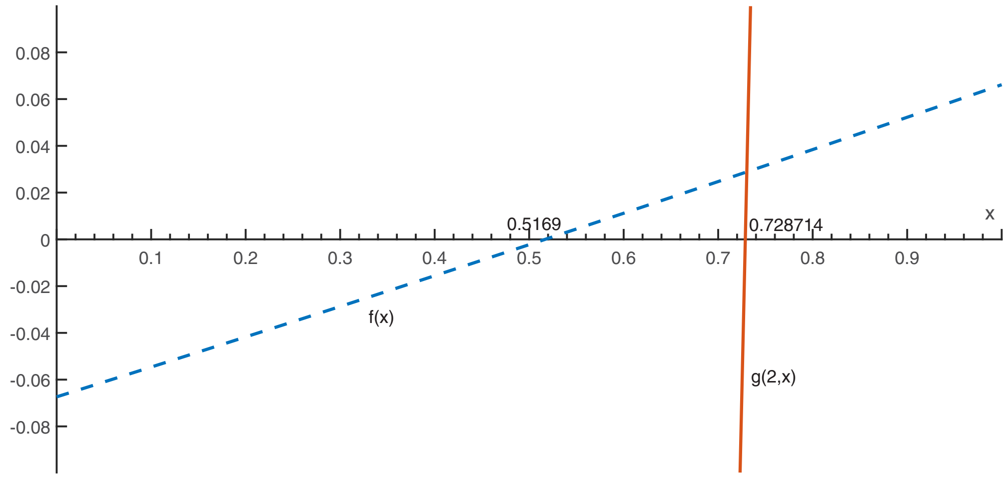

FIG. 2. $f(x)$ from our result (dotted blue line), $g(2,x)$ from the Corollary 1 in Ref. [26] (solid red line).

Proof. For an n-partite separable state, from (10)–(12) we obtain

$$\begin{aligned}
\|G^{1|2|\cdots|n}\|_{\rm tr} &= \sum_l z_l \prod_i^{n-1} \|(|\beta_i\rangle\langle\gamma_i^{'}| + F_i)\|_{\rm tr}\|(|\gamma_n^{'}\rangle\langle\beta_n| + F_n)\|_{\rm tr} \\
&\leqslant \sum_l z_l \prod_i^{n-1}(\||\beta_i\rangle\|\||\gamma_i^{'}\rangle\| + \|F_i\|_{\rm tr})(\||\gamma_n^{'}\rangle\|\||\beta_n\rangle\| + \|F_n\|_{\rm tr}) \\
&\leqslant \sum_l z_l \prod_i^{n}((|m_i| + |c_i|)\sqrt{\sum_{\alpha_i=1}^{d_i^2} e_{i,\alpha_i}^2} + \sqrt{2}|b_i|d_i) \\
&\leqslant \sum_l z_l \prod_i^{n}((|m_i| + |c_i|)\sqrt{\frac{a_i d_i^2+1}{d_i(d_i+1)}} + \sqrt{2}|b_i|d_i) \\
&= \prod_i^{n}((|m_i| + |c_i|)\sqrt{\frac{a_i d_i^2+1}{d_i(d_i+1)}} + \sqrt{2}|b_i|d_i),
\end{aligned}$$

where we have used $\|A\otimes B\|_{\rm tr} = \|A\|_{\rm tr}\cdot\|B\|_{\rm tr}$ for matrices A and B in the first equality, $\||a\rangle\langle b|\|_{\rm tr} = \||a\rangle\|\cdot\||b\rangle\|$ for vectors $|a\rangle$ and $|b\rangle$ in the first inequality and $\|A+B\|_{\rm tr} \leqslant \|A\|_{\rm tr} + \|B\|_{\rm tr}$ in the first and second inequality, and (3) in the third inequality. ■

Violation of the inequality (13) implies that the multipartite state is entangled.

Example 3. Consider the quantum state,

$$\rho = \frac{x}{8}\mathbb{I} + (1-x)|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|,\ 0 \leqslant x \leqslant 1,$$

where $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. When $b_i = c_i = 0$ and $m_i = 1$, $i = 1, 2, 3$, from Theorem 3, we have that ρ is entangled (not fully separable) when $\|G^{1|2|3}\|_{\rm tr} > \sqrt{\frac{4a_1+1}{6}}\sqrt{\frac{4a_2+1}{6}}\sqrt{\frac{4a_3+1}{6}}$. The range of entanglement of ρ is shown in Table III for different values of t. Taking $t = 0.068$, we get $f(x) = \|G\|_{\rm tr} - \sqrt{\frac{4a_1+1}{6}}\sqrt{\frac{4a_2+1}{6}}\sqrt{\frac{4a_3+1}{6}} > 0$, i.e., $0 \leqslant x < 0.4831$. Therefore, results are better than $0 \leqslant x < 0.2$ given in [28].

TABLE III. The entanglement ranges for ρ with different t.

	Range of entanglement
$t = 0.050$	$0 \leqslant x < 0.4130$
$t = 0.058$	$0 \leqslant x < 0.4474$
$t = 0.068$	$0 \leqslant x < 0.4831$

IV. CONCLUSION

We have studied multipartite entanglement and genuine tripartite entanglement by using GSIC-POVMs. Based on GSIC-POVMs we have obtained the criterion for detecting genuine tripartite entanglement. The criteria has been shown to be more efficient in detecting GME of some quantum states than some existing criteria. We have also studied the entanglement in multipartite quantum systems via GSIC-POVMs. Our separability criteria are also shown to be stronger than the existing one. Our approach may highlight further investigations on detection of other quantum correlations.

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