Bivariate moments of the two-point correlation function for embedded Gaussian unitary ensemble with k-body interactions

V. K. B. Kota*
Physical Research Laboratory, Ahmedabad 380 009, India

(Received 10 November 2022; accepted 6 May 2023; published 24 May 2023)

Embedded random matrix ensembles with k-body interactions are well established to be appropriate for many quantum systems. For these ensembles the two point correlation function is not yet derived, though these ensembles are introduced 50 years back. Two-point correlation function in eigenvalues of a random matrix ensemble is the ensemble average of the product of the density of eigenvalues at two eigenvalues, say E and E'. Fluctuation measures such as the number variance and Dyson-Mehta Δ_3 statistic are defined by the two-point function and so also the variance of the level motion in the ensemble. Recently, it is recognized that for the embedded ensembles with k-body interactions the one-point function (ensemble averaged density of eigenvalues) follows the so called q-normal distribution. With this, the eigenvalue density can be expanded by starting with the q-normal form and using the associated q-Hermite polynomials $He_\zeta(x|q)$. Covariances $\overline{S_\zeta S_{\zeta'}}$ (overline representing ensemble average) of the expansion coefficients S_ζ with $\zeta \geqslant 1$ here determine the two-point function as they are a linear combination of the bivariate moments Σ_{PQ} of the two-point function. Besides describing all these, in this paper formulas are derived for the bivariate moments Σ_{PQ} with $P+Q \leqslant 8$, of the two-point correlation function, for the embedded Gaussian unitary ensembles with k-body interactions [EGUE(k)] as appropriate for systems with m fermions in N single particle states. Used for obtaining the formulas is the $SU(N)$ Wigner-Racah algebra. These formulas with finite N corrections are used to derive formulas for the covariances $\overline{S_\zeta S_{\zeta'}}$ in the asymptotic limit. These show that the present work extends to all k values, the results known in the past in the two extreme limits with $k/m \to 0$ (same as $q \to 1$) and $k = m$ (same as $q = 0$).

DOI: 10.1103/PhysRevE.107.054128

I. INTRODUCTION

Classical random matrix ensembles, i.e., the Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, and GSE) are well known now in physics and need no introduction [1–3]. Hamiltonians (H) for atoms, atomic nuclei, molecules, mesoscopic systems such as quantum dots, etc., consist of a mean-field one-body part and a residual two-body interaction. With the two-body part sufficiently strong, energy levels of these systems in general exhibit quantum chaos and the appropriate random matrix ensembles for describing this, as recognized first in nuclear shell model studies [4–7], are the so-called embedded ensembles (EE) generated by k-body interactions [EE(k)] in many-particle (m particle with $m > k$) spaces (assumed is that the particles are in N number of single particle states with $N \gg m$). In particular, the embedded Gaussian orthogonal and unitary ensembles generated by k-body interactions [EGOE(k) and EGUE(k)], applicable to many fermion systems, have received considerable attention in the last two decades. Remarkably, for $m \gg k$ (with $N \gg m$), these ensembles generate Gaussian eigenvalue densities, i.e., the one point function in the eigenvalues (one, two, and higher point functions are defined by Dyson [8]). Here, it is important to note that for $m = k$, EE will reduce to the classical ensembles giving the well known Wigner semicircle form for the one-point function [6,7,9]. This important result is seen in a large number of numerical calculations and it is also proved analytically [6,9–11]. With E denoting eigenvalues and $\rho(E)$ the eigenvalue density for a given member of an ensemble of random matrices, the one point function is $\overline{\rho(E)}$ where the overline indicates ensemble average.

Turning to the two-point correlation function, though a large number of EGOE calculations showed that the spacing distribution, number variance, and the Dyson-Mehta Δ_3 statistic [12] and other measures of level fluctuations follow GOE, until today there was no success in deriving the two-point correlation function $\overline{\rho(x)\rho(y)}$ for EGOE(k) or EGUE(k), even in the limit of $k << m$. The earliest attempt is due to French [6,7,13] who has shown that EGOE(k) in the dilute limit (with k finite, $N \to \infty$, $m \to \infty$, and $m/N \to 0$) generates average-fluctuation separation that is absent in classical Gaussian ensembles. However, experimental confirmation of this feature is not yet available nor the formula for the two point function. The next attempt is due to Verbaarschot and Zirnbauer [14]. This is followed by an attempt due to Weidenmüller and collaborators [9,15]. However, as shown by Srednicki later [16], the results in [9] for the nature of level fluctuations generated by EGUE(k) are inconclusive. A significant result due to Weidenmüller *et al.* is that EE generate the so called cross correlations that are absent in classical

*vkbkota@prl.res.in

 ©2023 American Physical Society

ensembles; see [17,18] for results regarding cross correlations in EE. Here also, definitive experimental tests of these are not yet available.

Recently, a new direction in exploration of EE has opened up with the analysis of quantum chaos in the Sachdev-Ye-Kitaev (SYK) model using random matrix theory by Verbaarschot and collaborators [19–24]. The most significant result in these papers, for the present purpose, is the recognition that the so-called q-normal distribution indeed gives the eigenvalue density in the SYK model. Bryc, Szablowski, Ismail, and others [25–29] earlier clearly showed that this q-normal distribution (see Sec. II for definition and other mathematical details) has a purely commutative and classical probabilistic meaning. With the q normal reducing to Gaussian form for $q = 1$ and semicircle form for $q = 0$, it immediately shows that EE(k) will generate q-normal form for the eigenvalue densities. Remarkably, it is seen that the lower order moments (up to the eighth order) of the eigenvalue density (one-point function) generated by EE(k) are essentially identical to the lower order moments given by q-normal distribution [30,31] with the fourth moment (this depends on k) determining the value of the q parameter. With this, there is a possibility that expansions for $\rho(E)$ starting from the q-normal form using the associated q-Hermite polynomials may allow us to understand the two-point function for EE(k) with k changing from $k = 2$ to m ($k = 1$ appears to be special [32,33]) just as it was done in the past for the classical Gaussian ensembles and also adopted for EGOE(k) with $k << m$ [6,7,34]. Interestingly, expansion involving q-Hermite polynomials is also employed in investigating level fluctuations in the SYK model [22]. Following this, we have revisited the problem of deriving the two-point correlation function for EE(k) and analytical formulas for the bivariate moments (to order eight) of the two-point function for EGUE(k) are presented in this paper. These will determine the covariances of the expansion coefficients appearing in the q-Hermite polynomial expansion of the eigenvalue density of the EGUE(k) ensemble members. It is expected that these results may yield the two-point function for EGUE(k) in the near future. Now we will give a preview.

In Sec. II, first for completeness, EGUE(k) is defined. Second, we introduce the two-point function and its integral version along with their relation to the number variance, Δ_3 statistic, and variance of the level motion in the ensemble. The q-normal form f_{qN}, along with q-Hermite polynomials, are defined and collected in addition to some of their properties. In Sec. III, using the expansion of the eigenvalue density in terms of q-normal f_{qN} and q-Hermite polynomials $He_\zeta(x|q)$, it is shown that the covariances of the expansion coefficients S_ζ ($\zeta = 1, 2, ..., \infty$) are related in a simple manner to the bivariate moments Σ_{PQ} of the two-point function. Following this, in Sec. IV formulas are derived for the bivariate moments Σ_{PQ} of the two point function for EGUE(k) for $P + Q \leqslant 8$ using the formulation in terms of $SU(N)$ Wigner-Racah algebra as described in [10]. Presented in Sec. V are asymptotic limit formulas for the covariances $\overline{S_\zeta S_{\zeta'}}$ for EGUE(k) ensemble. In addition, some general structures indicated by these formulas are also discussed and an expansion for the number variance is given. Finally, Sec. VI gives conclusions.

II. PRELIMINARIES : EGUE(k), TWO-POINT FUNCTION, q-NORMAL DISTRIBUTION, q-HERMITE POLYNOMIALS

A. EGUE(k) definition

Given a system of m spinless fermions distributed in N degenerate single particle (sp) states and interacting via k-body ($1 \leqslant k \leqslant m$) interactions, the EGUE(k) in m fermion spaces is generated by representing the k-particle H by GUE. For a more precise definition, first consider the sp states (denoted by ν_i) in increasing order, $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_N$. Now, a random k-body H in second quantized form is

$$H(k) = \sum_{\alpha,\,\beta} V_{\alpha,\beta}(k)\; \psi^\dagger(k;\alpha)\; \psi(k;\beta). \tag{1}$$

Here, α (similarly β) are k-particle states (configurations) $|\nu_1^o, \nu_2^o, \dots, \nu_k^o\rangle$ in occupation number representation; ν_i^0 are occupied sp states. Distributing k fermions (following Pauli's exclusion principle) in N sp states will generate a complete set of these distinct configurations (α, β, ...) and the total number of these configurations is $\binom{N}{k}$. Operators $\psi^\dagger(k;\alpha)$ and $\psi(k;\beta)$, respectively, are k-particle creation and annihilation operators, i.e., $\psi^\dagger(k;\alpha) = \prod_{i=1}^k a^\dagger_{\nu_i^\alpha}$ and $\psi(k;\beta) = \prod_{j=1}^k a_{\nu_j^\beta}$; here, for example, ν_i^α is the ith occupied sp state for the k-particle configuration α. The one-particle creation ($a^\dagger_{\nu_i}$) and annihilation (a_{ν_j}) operators obey the usual anticommutation relations. In Eq. (1), $V_{\alpha,\beta}(k)$ matrix is chosen to be a $\binom{N}{k}$ dimensional GUE in k-particle spaces (V matrix is complex Hermitian). That means $V_{\alpha,\,\beta}(k)$ are antisymmetrized k-particle matrix elements chosen to be randomly distributed independent Gaussian variables with zero mean and variance,

$$\overline{V_{\alpha,\beta}(k)\; V_{\alpha',\beta'}(k)} = v^2\; \delta_{\alpha,\beta'}\; \delta_{\alpha',\beta}. \tag{2}$$

Here, the bar denotes ensemble averaging and we choose $v = 1$ without loss of generality. Distributing the m fermions in all possible ways in the N states generates the many-particle basis states (configurations) $|\nu_1^o, \nu_2^o, \dots, \nu_m^o\rangle$ in occupation number representation defining a $\binom{N}{m}$ dimensional Hilbert space. Action of the Hamiltonian operator $H(k)$ defined by Eq. (1) on the above many-particle basis states generates an H matrix ensemble in m-particle spaces with dimension $\binom{N}{m}$ and this is the EGUE(k) ensemble—it is a random matrix ensemble in m-particle spaces generated by k-body interactions. Note that EGUE(k) has three parameters (N, m, k). See [7,15,18,35] for further details regarding not only EGUE(k), but also for EGOE(k), EGSE(k), and many other extensions of embedded ensembles including those for interacting boson systems. In the present paper we restrict to EGUE(k).

B. Two-point function

Let us begin with the ensemble averaged eigenvalue density or the one-point function $\overline{\rho(E)}$ of EGUE(k) where $\rho(E)$ is the eigenvalue density (normalized to unity and usually it is called frequency function in statistics) for each member of EGUE(k); E denotes energy eigenvalues, and some times we will use x or y to denote eigenvalues. The integral version of $\rho(E)$ is the distribution function $F(x)$ (also called stair case function),

$$F(x) = d \int_{-\infty}^{x} \rho(E)\, dE. \tag{3}$$

Note that $F(x)$ gives the number of levels up to the eigenvalue x and $d = \binom{N}{m}$ is the total number of eigenvalues, i.e., dimension of the given EGUE(k). Now, the two-point correlation function $S^\rho(x, y)$ for the eigenvalues and its integral version $S^F(x, y)$ are (here and elsewhere in this paper mostly we employ the notations used in [7])

$$\begin{aligned} S^\rho(x, y) &= \overline{\rho(x)\, \rho(y)} - \overline{\rho(x)}\ \overline{\rho(y)}, \\ S^F(x, y) &= d^2 \int_{-\infty}^{x} \int_{-\infty}^{y} S^\rho(x', y') dx' dy' \\ &= \overline{F(x)\, F(y)} - \overline{F(x)}\ \overline{F(y)}. \end{aligned} \tag{4}$$

From Eq. (4), as the bar denotes ensemble average, it is clear that S^ρ (and S^F) gives measures for level fluctuations and the simplest two-point measure is the number variance $\Sigma^2(\overline{n})$. Say there are n number of levels between energies x and y. Then $n = F(x) - F(y)$, and similarly $\overline{n} = \overline{F(x)} - \overline{F(y)}$. With this, a measure for fluctuation in number of levels, with $\overline{n}$ the average number of levels, is the number variance $\Sigma^2(\overline{n}) = \overline{(n - \overline{n})^2}$ and this is simply related to $S^F(x, y)$,

$$\Sigma^2(\overline{n}) = S^F(x, x) + S^F(y, y) - 2S^F(x, y). \tag{5}$$

In addition, the Δ_3 statistic is simply related to $\Sigma^2(\overline{n})$ [7],

$$\Delta_3(\overline{n}) = \frac{2}{\overline{n}^4} \int_0^{\overline{n}} (\overline{n}^3 - 2\overline{n}^2 r + r^3)\, \Sigma^2(r)\, dr. \tag{6}$$

Further, an approach to study $S^F(x, x)$ is to examine level motion in the ensemble. For example, variance of the fluctuation in an eigenvalue E, measured in units of the local level spacing $\overline{D(E)}$, is denoted by $\overline{\delta E^2}/\overline{D(E)}^2$. This is often called level motion variance. Then, it is easy to see that the variance of level motion is

$$\overline{\delta E^2}/\overline{D(E)}^2 = S^F(E, E). \tag{7}$$

Similarly, $S^F(x, y)$ and $S^\rho(x, y)$ can be probed or constructed using the bivariate moments $\widetilde{\Sigma}_{PQ}$ of $S^\rho(x, y)$,

$$\widetilde{\Sigma}_{PQ} = \int x^P\, y^Q\, S^\rho(x, y)\, dx dy = \overline{\langle H^P \rangle \langle H^Q \rangle} - \overline{\langle H^P \rangle}\ \ \overline{\langle H^Q \rangle} \tag{8}$$

with $|\alpha_i\rangle$, $i = 1, 2, \ldots, d$ denoting the m fermion basis states, the Pth moment of $\rho(E)$ is $\langle H^P \rangle = d^{-1}\, \mathrm{tr}(H^P)$, where $\mathrm{tr}(H^P)$ is the trace of H^P in m fermion space. Note that $\mathrm{tr}(H^P) = \sum_i \langle \alpha_i | H^P | \alpha_i \rangle = \sum_i (E_i)^P$ as traces are invariant under a unitary transformation [also, $\sum_i (E_i)^P = d \int E^P \rho(E)\, dE$]. It is easy to see from Eq. (8) that $\widetilde{\Sigma}_{PQ} = \widetilde{\Sigma}_{QP}$ and $\widetilde{\Sigma}_{P0} = 0$. Also, with

$$\Sigma_{PQ} = \overline{\langle H^P \rangle \langle H^Q \rangle}, \tag{9}$$

we have $\Sigma_{P,0} = \overline{\langle H^P \rangle^m}$, the Pth moment of $\overline{\rho(E)}$. Our purpose in this paper is to derive formulas for the bivariate moments Σ_{PQ} with $P + Q \leqslant 8$ (these are given in Sec. IV) as they will determine the lower order terms in an expansion of the two-point function and this is discussed in more detail in Sec. III. Before turning to these, in the next subsection we introduce the q-normal distribution and q-Hermite polynomials as the eigenvalue density for EE(k) (well demonstrated for EGOE(k) and EGUE(k) in [30]) is close to q normal and this reduces to Gaussian form for $k << m$ and semicircle for $k = m$. Thus, the q-normal form covers all k values.

C. q-normal distribution and q-Hermite polynomials

First, q numbers $[n]_q$ are defined by (with $[0]_q = 0$)

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{10}$$

Note that $[n]_{q \to 1} = n$. Similarly, q-factorial $[n]_q! = \Pi_{j=1}^{n} [j]_q$ with $[0]_q! = 1$. With this, the q binomials are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\, [k]_q!} \tag{11}$$

for $n \geqslant k \geqslant 0$ and 0 otherwise. Going further, the q-normal distribution $f_{qN}(x|q)$ [27,29], with x being a standardized variable (then x is zero centered with variance unity), is defined as

$$\begin{aligned} f_{qN}(x|q) = &\frac{\sqrt{1-q}\, \prod_{k'=0}^{\infty} (1 - q^{k'+1})}{2\pi\, \sqrt{4 - (1-q)x^2}} \\ &\times \prod_{k'=0}^{\infty} [(1 + q^{k'})^2 - (1-q) q^{k'} x^2]. \end{aligned} \tag{12}$$

The $f_{qN}(x|q)$ is defined for x in the domain defined by $\mathbf{S}(q)$, where

$$\mathbf{S}(q) = \left(-\frac{2}{\sqrt{1-q}},\ +\frac{2}{\sqrt{1-q}} \right) \tag{13}$$

with q taking values 0 to 1 (in this paper). Note that $f_{qN}(x|q) = 0$ for x outside $\mathbf{S}(q)$ and the integral of $f_{qN}(x|q)$ is unity, i.e., $\int_{\mathbf{S}(q)} f_{qN}(x|q)\, dx = 1$. For $q = 1$, taking the limit properly will give $f_{qN}(x|1) = (1/\sqrt{2\pi}) \exp -x^2/2$, the Gaussian with $\mathbf{S}(q = 1) = (-\infty, \infty)$. Also, $f_{qN}(x|0) = (1/2\pi)\sqrt{4 - x^2}$, the semicircle with $\mathbf{S}(q = 0) = (-2, 2)$. If we put back the centroid ϵ and the width σ in f_{qN}, then $\mathbf{S}(q)$ changes to

$$\mathbf{S}(q : \epsilon, \sigma) = \left(\epsilon - \frac{2\sigma}{\sqrt{1-q}},\ \epsilon + \frac{2\sigma}{\sqrt{1-q}} \right).$$

All odd central moments of f_{qN} are zero and then the lowest shape parameter is excess or kurtosis γ_2 that is simply related to the reduced fourth central moment μ_4, $\gamma_2 = \mu_4 - 3$. For f_{qN} we have $\mu_4 = 2 + q$. Thus, μ_4 (or γ_2) determines the value of q [30]; see Eq. (39) ahead.

The q-Hermite polynomials $He_n(x|q)$, that are orthogonal with f_{qN} as the weight function, are defined by the recursion relation:

$$x\, He_n(x|q) = He_{n+1}(x|q) + [n]_q\, He_{n-1}(x|q) \tag{14}$$

with $He_0(x|q) = 1$ and $He_{-1}(x|q) = 0$. Note that for $q = 1$, the q-Hermite polynomials reduce to normal Hermite polynomials (related to Gaussian) and for $q = 0$ they will reduce to

Chebyshev polynomials (related to semicircle). The polynomials up to order four, for example, are

$$\begin{aligned} He_0(x|q) &= 1,\\ He_1(x|q) &= x,\\ He_2(x|q) &= x^2-1,\\ He_3(x|q) &= x^3-(2+q)x,\\ He_4(x|q) &= x^4-(3+2q+q^2)x^2+(1+q+q^2). \end{aligned} \tag{15}$$

Orthogonal property of $He_n(x|q)$'s that plays an important role in the discussion that follows, is

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} He_n(x|q)\, He_m(x|q)\, f_{qN}(x|q)\, dx = [n]_q!\, \delta_{mn}. \tag{16}$$

Using Eq. (16), it is easy to derive formulas for the lower order moments of f_{qN}.

With the ensemble averaged eigenvalue density $\overline{\rho(E)}$ for EGOE(k) or EGUE(k) being $f_{qN}(E)$, we can seek an expansion of the eigenvalue density $\rho(E)$ of the members of the ensemble in terms of the polynomial excitations of $f_{qN}(E)$ with the polynomials being obviously q-Hermite polynomials. This will allow us to study the two-point correlation function and we will turn to this in the following section. A similar study was made recently [22] for the two-point correlation function in the SYK model.

III. EIGENVALUE DENSITY IN TERMS OF q-HERMITE POLYNOMIALS AND THE COVARIANCES OF EXPANSION COEFFICIENTS DETERMINING TWO-POINT FUNCTION

A. Two-point function in terms of q-Hermite polynomials

Eigenvalue density $\rho(E)$ for various members of an embedded random matrix ensemble can be expanded in terms of q-Hermite polynomials starting with q normal, giving

$$\rho(E)\, dE = f_{qN}(\hat{E}|q)\left[1+\sum_{\zeta\geqslant 1}^{\infty} S_\zeta \frac{He_\zeta(\hat{E}|q)}{[\zeta]_q!}\right] d\hat{E};$$
$$\hat{E} = (E-E_c)/\sigma. \tag{17}$$

Here, S_ζ are the expansion coefficients and the S_ζ should not be confused with $S^\rho(x,y)$ used for the two-point function. It is important to recall, as mentioned at the end of Sec. II B, the ensemble averaged eigenvalue density $\overline{\rho(E)}$ for EGUE(k) is f_{qN}, i.e.,

$$\overline{\rho(E)}\, dE = \sigma^{-1} f_{qN}(\hat{E})\, d\hat{E}. \tag{18}$$

Therefore in Eq. (17), E_c is the centroid and σ is the width of $\overline{\rho(E)}$. Now, using the expansion given by Eq. (17) the distribution function is

$$F(E) = F_{qN}(E) + d \sum_{\zeta\geqslant 1} \frac{S_\zeta}{[\zeta]_q!} \int_{-2/\sqrt{1-q}}^{\hat{E}} f_{qN}(\hat{E}'|q) \times He_\zeta(\hat{E}'|q)\, d\hat{E}', \tag{19}$$

and Eqs. (3) and(18) give

$$\overline{F(E)} = F_{qN}(E) = d\int_{-2/\sqrt{1-q}}^{\hat{E}} f_{qN}(\hat{E}'|q)\, d\hat{E}'. \tag{20}$$

In the limits $q=1$ (i.e., for Gaussians or in the limit $k << m$) and $q=0$ (semicircle or $k=m$ limit), the integrals in Eqs. (19) and (20) are easy to obtain. More importantly, the S_ζ in Eqs. (17) and (19) are for a given member of the EE(k) ensemble and it is easy to see that the ensemble average of S_ζ is zero, i.e., $\overline{S_\zeta}=0$. However, $\overline{S_\zeta S_{\zeta'}} \neq 0$ [22] and these determine the two-point function as discussed ahead. Before turning to this, let us add that in the past, using Eq. (17) with additional approximations, some aspects of the variance of level motion in embedded ensembles has been studied by many groups [6,7,36–39].

Equation (17) generates an expansion of the two-point function $S^\rho(x,y)$ in terms of q-Hermite polynomials (in the remainder of this paper, the symbols x and y are standardized variables, i.e., they denote $\hat{E}$),

$$S^\rho(x,y) = f_{qN}(x|q)\, f_{qN}(y|q) \times \sum_{\zeta,\zeta'=1}^{\infty} \overline{S_\zeta S_{\zeta'}}\, \frac{He_\zeta(x|q)}{[\zeta]_q!}\, \frac{He_{\zeta'}(y|q)}{[\zeta']_q!}. \tag{21}$$

Here, it is significant to note that the covariances $\overline{S_\zeta S_{\zeta'}}$ of the S_ζ's are related to the bivariate moments $\widetilde{\Sigma}_{PQ}$ of the two-point function and this is seen as follows. First,

$$\langle H^p\rangle = \overline{\langle H^p\rangle} + \sum_{\zeta\geqslant 1} S_\zeta \frac{\sigma^p}{[\zeta]_q!} \int_{S(q)} x^p\, f_{qN}(x|q)\, He_\zeta(x|q)\, dx. \tag{22}$$

Note that $\sigma^2 = \Sigma_{2,0} = \Sigma_{0,2}$. Now, writing x^p in terms of q-Hermite polynomials using "Proposition 1" in [28] and then applying Eq. (16) will simplify Eq. (22), giving

$$\langle H^p\rangle = \overline{\langle H^p\rangle} + \sum_{\zeta\geqslant 1} S_\zeta\, \sigma^p\, C_{\frac{p-\zeta}{2},p}(q);$$
$$C_{r,n}(q) = (1-q)^{-r} \sum_{j=0}^{r} (-1)^j q^{j(j+1)/2} \times \left\{\binom{n}{r-j} - \binom{n}{r-j-1}\right\} \begin{bmatrix} n-2r+j \\ j \end{bmatrix}_q. \tag{23}$$

This combined with Eq. (21) generates formulas for the reduced bivariate moments $\hat{\Sigma}_{PQ}$ in terms of the covariances $\overline{S_\zeta S_{\zeta'}}$,

$$\hat{\Sigma}_{PQ} = \frac{\widetilde{\Sigma}_{PQ}}{[\Sigma_{2,0}]^{(P+Q)/2}} = \sum_{\zeta,\zeta'=1}^{\infty} \overline{S_\zeta S_{\zeta'}}\, C_{\frac{P-\zeta}{2},P}(q) C_{\frac{Q-\zeta'}{2},Q}(q). \tag{24}$$

Note that $\widetilde{\Sigma}_{Pq}$ is defined by Eq. (8) and Σ_{PQ} by Eq. (9). Let us add that $\hat{\Sigma}_{PQ}=0$ for $P+Q$ is odd, and similarly $\overline{S_\zeta S_{\zeta'}}=0$ for $\zeta+\zeta'$ is odd. Also, $\hat{\Sigma}_{P0}=0$, $\hat{\Sigma}_{PQ}=\hat{\Sigma}_{QP}$, $\overline{S_\zeta}=0$, and $\overline{S_\zeta S_{\zeta'}} = \overline{S_{\zeta'} S_\zeta}$.

B. Covariances $\overline{S_\zeta S_{\zeta'}}$

Using Eq. (24) successively with $P+Q$ increasing from two, it is easy to see that the covariances $\overline{S_\zeta S_{\zeta'}}$ can be written in terms of the moments $\hat{\Sigma}_{PQ}$. Formulas for the moments can be derived for low values of $P+Q$ and, as presented in Sec. IV, at present we can go up to $P+Q=8$ (there are some restrictions for $P+Q=6$ and 8). With this, $\overline{S_\zeta\, S_{\zeta'}}$ for $\zeta+\zeta'\leqslant 8$ are

$$\begin{aligned}
\overline{S_1\, S_1} &= \hat{\Sigma}_{11}, \\
\overline{S_3\, S_1} &= \hat{\Sigma}_{31} - C_{13}\, \hat{\Sigma}_{11}, \\
\overline{S_2\, S_2} &= \hat{\Sigma}_{22}, \\
\overline{S_5\, S_1} &= \hat{\Sigma}_{51} - C_{15}\, \overline{S_3\, S_1} - C_{25}\, \overline{S_1\, S_1}, \\
\overline{S_4\, S_2} &= \hat{\Sigma}_{42} - C_{14}\, \overline{S_2\, S_2}, \\
\overline{S_3\, S_3} &= \hat{\Sigma}_{33} - C_{13}^2\, \overline{S_1\, S_1} - 2C_{13}\, \overline{S_1\, S_3}, \\
\overline{S_7\, S_1} &= \hat{\Sigma}_{71} - C_{17}\, \overline{S_5\, S_1} - C_{27}\, \overline{S_3\, S_1} - C_{37}\, \overline{S_1\, S_1}, \\
\overline{S_6\, S_2} &= \hat{\Sigma}_{62} - C_{16}\, \overline{S_4\, S_2} - C_{26}\, \overline{S_2\, S_2}, \\
\overline{S_5\, S_3} &= \hat{\Sigma}_{53} - C_{13}\, \overline{S_5\, S_1} - C_{15}\, \overline{S_3\, S_3} \\
&\quad - [C_{25} + C_{15}C_{13}]\, \overline{S_1\, S_3} - C_{25}C_{13}\, \overline{S_1\, S_1}, \\
\overline{S_4\, S_4} &= \hat{\Sigma}_{44} - 2C_{14}\, \overline{S_2\, S_4} - C_{14}^2\, \overline{S_2\, S_2}.
\end{aligned} \tag{25}$$

In the above, we have dropped q in $C_{r,n}(q)$ for brevity. In order to apply Eq. (25), Eq. (23) for $C_{r,n}(q)$ is simplified for $r=1$, 2, and 3 (note that $n\geqslant 2r+1$). First, $C_{0P}(q)=1$ for any P. Similarly, the formula for $r=1$ is simple:

$$C_{1,P}(q) = \sum_{\kappa=2}^{P} (\kappa-1) q^{P-\kappa}. \tag{26}$$

Then, for example $C_{1,2}(q)=1$, $C_{1,3}=q+2$, $C_{1,4}=q^2+2q+3$, $C_{1,5}=q^3+2q^2+3q+4$, and so on. Besides this, we need the formulas for C_{25}, C_{26}, C_{27}, and C_{37} for applying Eq. (25). These are

$$\begin{aligned}
C_{2,5}(q) &= [q^3+3q^2+6q+5], \\
C_{2,6}(q) &= [q^5+3q^4+7q^3+12q^2+13q+9], \\
C_{2,7}(q) &= [q^7+3q^6+7q^5+13q^4 \\
&\quad +21q^3+24q^2+22q+14], \\
C_{3,7}(q) &= [q^6+4q^5+10q^4+20q^3+28q^2+28q+14].
\end{aligned} \tag{27}$$

It is important to note that $\overline{S_i S_j} = \overline{\langle He_i(H)\rangle^m\, \langle He_j(H)\rangle^m}$, and this can be used to verify Eq. (25). Now we will derive formulas for the bivariate moments Σ_{PQ}, with finite N corrections, so that we can obtain lower order covariances of the S_ζ's.

IV. FORMULAS FOR LOWER ORDER BIVARIATE MOMENTS OF TWO-POINT CORRELATION FUNCTION

In this section we will derive formulas for the moments Σ_{PQ} (thereby, for $\hat{\Sigma}_{PQ}$) of the two-point correlation function by restricting to EGUE(k) for a system of m fermions in N single particle states. As established in [10], these will follow from the Wigner-Racah algebra of $U(N)$. For EGUE(k) Hamiltonians, all the m-fermion states belong to the totally antisymmetric irreducible representation (irrep) $f_m=\{1^m\}$ of $U(N)$ (note that we are using Young tableaux notation for irreps; see Appendix A). Then, the conjugate irrep is $\overline{f_m}=\{1^{N-m}\}$. Given a k-body H, it will decompose into $U(N)$ tensors $B^\nu(k)$ with the irreps $\nu=\{2^\nu 1^{N-2\nu}\}$; note that $\nu=\overline{\nu}$ (the "bar" used here for denoting conjugate irrep should not be confused with the "bar" used for ensemble averages). As $SU(N)$ instead of $U(N)$ is used in the derivations, $\nu=0$ corresponds to $\{1^N\}=\{0\}$ irrep. With m particle states denoted by $|f_m,\alpha\rangle$, we need the $SU(N)$ Clesh-Gordan (CG) coefficients $\langle f_m\alpha_1\;\; \overline{f_m}\,\overline{\alpha_2}|\nu\;\omega_\nu\rangle$ where α's and ω_ν are additional labels needed for complete specification of various states. In the following we will often use the short hand notation $C^{\nu,\omega_\nu}_{\alpha_1\;\overline{\alpha_2}}$ by dropping the f_m label; as always we will deal with m-particle states. Some important properties of the CG coefficients $C^{f_{ab}v_{ab}}_{f_a v_a\; f_b v_b} = \langle f_a v_a\;\; f_b v_b|f_{ab}v_{ab}\rangle$ are [10,40,41]

$$\begin{aligned}
&C^{0,0}_{\alpha_1\;\overline{\alpha_1}} = \frac{1}{\sqrt{d(f_m)}}, \quad C^{\nu,\omega_\nu}_{\alpha_2\;\overline{\alpha_1}} = \left\{C^{\nu,\omega_\nu}_{\alpha_1\;\overline{\alpha_2}}\right\}^*, \\
&\sum_{\alpha_1,\alpha_2} C^{\nu_1,\omega_{\nu_1}}_{\alpha_1\;\overline{\alpha_2}} \left\{C^{\nu_2,\omega_{\nu_2}}_{\alpha_1\;\overline{\alpha_2}}\right\}^* = \delta_{\nu_1\nu_2}\,\delta_{\omega_{\nu_1}\omega_{\nu_2}}, \\
&\sum_{\alpha_1} C^{\nu,\omega_\nu}_{\alpha_1\;\overline{\alpha_1}} \left\{C^{0,0}_{\alpha_1\;\overline{\alpha_1}}\right\}^* = \delta_{\nu,0}, \\
&C^{f_{ab}v_{ab}}_{f_a v_a\; f_b v_b} = (-1)^{\phi(f_a,f_b,f_{ab})}\, C^{f_{ab}v_{ab}}_{f_b v_b\; f_a v_a}, \\
&C^{f_{ab}v_{ab}}_{f_a v_a\; f_b v_b} = C^{\overline{f_{ab}}\,\overline{v_{ab}}}_{\overline{f_a}\,\overline{v_a}\; \overline{f_b}\,\overline{v_b}}, \\
&C^{f_{ab}v_{ab}}_{f_a v_a\; f_b v_b} = (-1)^{\phi(f_a,f_b,f_{ab})} \sqrt{\frac{d(f_{ab})}{d(f_a)}}\, C^{f_a v_a}_{f_{ab}v_{ab}\; \overline{f_b}\,\overline{v_b}}.
\end{aligned} \tag{28}$$

Here $d(f)$ is the $U(N)$ dimension of the irrep $\{f\}$ and the formula for this is well known [42,43]. We have, for example, $d(f_m)=\binom{N}{m}$. Also, $\phi(f_a,f_b,f_{ab})=\Theta(f_a)+\Theta(f_b)+\Theta(f_{ab})$, and in the present work we do not need the explicit form of the function Θ. Just as the Wigner or CG coefficients, one can define the Racah coefficients for $SU(N)$ [40,41]. The Wigner and Racah (or $U-$) coefficients and their various properties will allow one to derive the following important results for the ensemble average of the product any two m-particle matrix elements $\langle f_m\alpha_1|H|f_m\alpha_2\rangle$ of H. As proved in [9,10] we have

$$\begin{aligned}
\overline{H_{\alpha_1\alpha_2} H_{\alpha_3\alpha_4}} &= \overline{\langle f_m\alpha_1|H|f_m\alpha_2\rangle\; \langle f_m\alpha_3|H|f_m\alpha_4\rangle} \\
&= \sum_{\nu=0,1,\ldots,k;\omega_\nu} \Lambda^\nu(N,m,m-k)\, C^{\nu,\omega_\nu}_{\alpha_1\;\overline{\alpha_2}}\, C^{\nu,\omega_\nu}_{\alpha_3\;\overline{\alpha_4}},
\end{aligned} \tag{29}$$

and

$$\overline{H_{\alpha_1\alpha_2} H_{\alpha_3\alpha_4}} = \sum_{\mu=0,1,\ldots,m-k;\omega_\mu} \Lambda^\mu(N,m,k)\, C^{\mu,\omega_\mu}_{\alpha_1\;\overline{\alpha_4}}\, C^{\mu,\omega_\mu}_{\alpha_3\;\overline{\alpha_2}} \tag{30}$$

with

$$\Lambda^\nu(N,m,r) = \binom{m-\nu}{r}\binom{N-m+r-\nu}{r}. \tag{31}$$

These equations are important as we use "binary correlation approximation." In this approximation, in the ensemble averages involving sums of product of many particle matrix elements of the H operator (similarly any other operator) only terms with pair wise correlated parts will dominate [6,9–11]. Equations (28), (29), (30), and (31) along with the "binary correlation approximation" are used to derive formulas for

Σ_{PQ} and, hence, for $\hat{\Sigma}_{PQ}$. Now, we will present the results for $\hat{\Sigma}_{PQ}$ with $P+Q=2$, 4, 6, and 8.

A. Formulas for $\hat{\Sigma}_{PQ}$ with $P+Q=2$

Formulas for $\Sigma_{2,0}=\Sigma_{0,2}$ and $\Sigma_{1,1}$ are already presented in [9,10] and they are briefly discussed here for completeness. First, the variance $\Sigma_{2,0}$ is simply

$$\begin{aligned}\Sigma_{2,0} = \overline{\langle H^2\rangle^m} &= \frac{1}{d(f_m)}\sum_{\alpha_1,\alpha_2}\overline{H_{\alpha_1\alpha_2}H_{\alpha_2\alpha_1}} \\ &= \frac{1}{d(f_m)}\sum_{\mu=0,1,\ldots,m-k;\omega_\mu}\sum_{\alpha_1,\alpha_2}\Lambda^\mu(N,m,k)\,C^{\mu,\omega_\mu}_{\alpha_1\,\overline{\alpha_1}}\,C^{\mu,\omega_\mu}_{\alpha_2\,\overline{\alpha_2}} \\ &= \Lambda^0(N,m,k). \end{aligned} \tag{32}$$

Here, in the second step, we have used Eq. (30) and in the last step the fact that $C^{0,0}_{\alpha,\overline{\alpha}}=1/\sqrt{d(f_m)}$ and the sum rules given in Eq. (28). Similarly, the $\Sigma_{1,1}$ or the covariance in the eigenvalue centroids is

$$\begin{aligned}\Sigma_{1,1} = \overline{\langle H\rangle^m\langle H\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2}\overline{H_{\alpha_1\alpha_1}H_{\alpha_2\alpha_2}} \\ &= \frac{1}{[d(f_m)]^2}\Lambda^0(N,m,m-k)\sum_{\alpha_1,\alpha_2}C^{0,0}_{\alpha_1\,\overline{\alpha_1}}\,C^{0,0}_{\alpha_2\,\overline{\alpha_2}} \\ &= \frac{1}{d(f_m)}\,\Lambda^0(N,m,m-k). \end{aligned} \tag{33}$$

Here, in the second step we have used the result that only a $SU(N)$ scalar ($\nu=0$) part of H contributes to the eigenvalue centroids, and also Eq. (29). In the last step used is the result $C^{0,0}_{\alpha,\overline{\alpha}}=1/\sqrt{d(f_m)}$ and the sum over α's will give $[d(f_m)]^2$. Combining Eqs. (32) and (33) will give the formula for $\hat{\Sigma}_{11}$,

$$\hat{\Sigma}_{11} = \frac{\Lambda^0(N,m,m-k)}{d(f_m)\,\Lambda^0(N,m,k)}. \tag{34}$$

B. Formulas for $\hat{\Sigma}_{PQ}$ with $P+Q=4$

With $P+Q=4$, we have $\Sigma_{4,0}=\Sigma_{0,4}$, $\Sigma_{3,1}=\Sigma_{1,3}$, and $\Sigma_{2,2}$. For $\Sigma_{4,0}$,

$$\Sigma_{4,0} = \overline{\langle H^4\rangle^m} = \frac{1}{d(f_m)}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\overline{H_{\alpha_1\alpha_2}H_{\alpha_2\alpha_3}H_{\alpha_3\alpha_4}H_{\alpha_4\alpha_1}} \tag{35}$$

using binary correlation approximation, there will be two binary correlated terms. Denoting the correlated pairs as A, B, etc., and applying the cyclic invariance of m-particle averages, the two terms are $2\overline{\langle AABB\rangle^m}=2[\overline{\langle AA\rangle^m}]^2$ and $\overline{\langle ABAB\rangle^m}$. Then, Eq. (35) simplifies to

$$\begin{aligned}\Sigma_{4,0} = \overline{\langle H^4\rangle^m} &= 2[\Sigma_{2,0}]^2 + \frac{1}{d(f_m)} \\ &\times \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\overline{H_{\alpha_1\alpha_2}H_{\alpha_3\alpha_4}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_4\alpha_1}}. \end{aligned} \tag{36}$$

Simplifying the last binary correlated term $\overline{\langle ABAB\rangle^m}$ using Eqs. (28), (29), and (30) and properties of $SU(N)$ Racah coefficients, we have [10] (see also [9]):

$$\begin{aligned}\overline{\langle ABAB\rangle^m} &= \frac{1}{d(f_m)}\sum_{\nu=0}^{min\{k,m-k\}}\Lambda^\nu(N,m,m-k) \\ &\times \Lambda^\nu(N,m,k)\,d(\nu), \end{aligned} \tag{37}$$

where $d(\nu)=\binom{N}{\nu}^2-\binom{N}{\nu-1}^2$. Then,

$$\Sigma_{4,0} = \overline{\langle H^4\rangle^m} = 2[\Lambda^0(N,m,k)]^2 + \overline{\langle ABAB\rangle^m} \tag{38}$$

with the last term given by Eq. (37). Note that Eq. (37) also gives the formula for the q parameter for EGUE(k) [30],

$$q = \frac{\sum_{\nu=0}^{min\{k,m-k\}}\Lambda^\nu(N,m,m-k)\,\Lambda^\nu(N,m,k)\,d(\nu)}{d(f_m)\,[\Lambda^0(N,m,k)]^2}. \tag{39}$$

Turning to $\Sigma_{3,1}$,

$$\Sigma_{3,1} = \Sigma_{1,3} = \overline{\langle H\rangle^m\langle H^3\rangle^m}, \tag{40}$$

clearly the H matrix element in $\langle H\rangle^m$ has to correlate with one of the H matrix elements in $\langle H^3\rangle^m$ in the binary correlation approximation. Denoting the correlated terms again as A, B, etc., we have the three terms $\overline{\langle A\rangle^m\langle ABB\rangle^m}$, $\overline{\langle A\rangle^m\langle BAB\rangle^m}$, and $\overline{\langle A\rangle^m\langle BBA\rangle^m}$, and these three are same due to the cyclic invariance of m-particle averages. Then, we have the simple result

$$\begin{aligned}\Sigma_{3,1} = \Sigma_{1,3} &= 3\overline{\langle H\rangle^m\langle H\rangle^m}\;\;\overline{\langle H^2\rangle^m} \\ &= \frac{3}{d(f_m)}\,\Lambda^0(N,m,k)\,\Lambda^0(N,m,m-k) \end{aligned} \tag{41}$$

and the ensemble averages here follow from Eqs. (32) and (33). With this, $\hat{\Sigma}_{31}$ is

$$\hat{\Sigma}_{31} = 3\,\hat{\Sigma}_{11}. \tag{42}$$

Finally, let us consider $\Sigma_{2,2}$,

$$\Sigma_{2,2} = \overline{\langle H^2\rangle^m\langle H^2\rangle^m}. \tag{43}$$

Here, again, there will be three correlated terms $\overline{\langle AA\rangle^m\,\langle BB\rangle^m}$, $\overline{\langle AB\rangle^m\,\langle AB\rangle^m}$, and $\overline{\langle AB\rangle^m\,\langle BA\rangle^m}$ with the later two equal due to the cyclic invariance of m-particle averages. Simplifying these will easily give [9,10]

$$\begin{aligned}\Sigma_{2,2} &= \overline{\langle H^2\rangle^m\langle H^2\rangle^m} \\ &= [\overline{\langle H^2\rangle^m}]^2 + 2\overline{\langle AB\rangle^m\langle AB\rangle^m} \\ \overline{\langle AB\rangle^m\langle AB\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_a,\alpha_b}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_1}H_{\alpha_b\alpha_a}} \\ &= \frac{1}{[d(f_m)]^2}\sum_{\nu=0,1,\ldots,k}[\Lambda^\nu(N,m,m-k)]^2 d(\nu). \end{aligned} \tag{44}$$

Note that the formula for $\overline{\langle H^2\rangle^m}$ is given by Eq. (32). Using this, for $\hat{\Sigma}_{22}$ we have

$$\hat{\Sigma}_{22} = \frac{2\,\sum_{\nu=0}^{k}[\Lambda^\nu(N,m,m-k)]^2 d(\nu)}{[d(f_m)\,\Lambda^0(N,m,k)]^2}. \tag{45}$$

C. Formulas for $\hat{\Sigma}_{PQ}$ with $P+Q=6$

With $P+Q=6$, we have $\Sigma_{6,0}=\Sigma_{0,6}$, $\Sigma_{5,1}=\Sigma_{1,5}$, $\Sigma_{4,2}=\Sigma_{2,4}$, and $\Sigma_{3,3}$. For $\Sigma_{6,0}$ with

$$\Sigma_{6,0}=\Sigma_{0,6}=\overline{\langle H^6\rangle^m}, \tag{46}$$

there will be four different binary correlated terms:

$$\Sigma_{6,0}=5\overline{\langle AABBCC\rangle^m}+6\overline{\langle AABCBC\rangle^m}+3\overline{\langle ABACBC\rangle^m}+\overline{\langle ABCABC\rangle^m} \tag{47}$$

and the first two terms follow from Eq. (32) giving

$$\begin{aligned}5\overline{\langle AABBCC\rangle^m}&=5[\Lambda^0(N,m,k)]^3,\\ 6\overline{\langle CCABAB\rangle^m}&=6\Lambda^0(N,m,k)\ \overline{\langle ABAB\rangle^m}\end{aligned} \tag{48}$$

with $\overline{\langle ABAB\rangle}$ given by Eq. (37). Now let us consider the third term,

$$\overline{\langle ABACBC\rangle^m}=\frac{1}{d(f_m)}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6}\overline{H_{\alpha_1\alpha_2}H_{\alpha_3\alpha_4}}\ \overline{H_{\alpha_2\alpha_3}H_{\alpha_5\alpha_6}}\ \overline{H_{\alpha_4\alpha_5}H_{\alpha_6\alpha_1}}. \tag{49}$$

Applying Eq. (30) to the first and third ensemble averages in Eq. (49) and Eq. (29) to the second term will give

$$\begin{aligned}\overline{\langle ABACBC\rangle^m}=&\frac{1}{d(f_m)}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_6}\ \sum_{\mu_1=0,1,\dots,m-k;\omega_{\mu_1}}\Lambda^{\mu_1}(N,m,k)\,C^{\mu_1,\omega_{\mu_1}}_{\alpha_1\,\overline{\alpha_4}}\ C^{\mu_1,\omega_{\mu_1}}_{\alpha_3\,\overline{\alpha_2}}\\ &\times\sum_{\mu_2=0,1,\dots,m-k;\omega_{\mu_2}}\Lambda^{\mu_2}(N,m,k)\,C^{\mu_2,\omega_{\mu_2}}_{\alpha_4\,\overline{\alpha_1}}\ C^{\mu_2,\omega_{\mu_2}}_{\alpha_6\,\overline{\alpha_5}}\sum_{\nu_1=0,1,\dots,k;\omega_{\nu_1}}\Lambda^{\nu_1}(N,m,m-k)\,C^{\nu_1,\omega_{\nu_1}}_{\alpha_2\,\overline{\alpha_3}}\ C^{\nu_1,\omega_{\nu_1}}_{\alpha_5\,\overline{\alpha_6}}.\end{aligned} \tag{50}$$

Now, applying the sum rules for the CG coefficients using Eq. (28), the final result is obtained:

$$\overline{\langle ABACBC\rangle^m}=\frac{1}{d(f_m)}\sum_{\nu=0}^{min(k,m-k)}\Lambda^{\nu}(N,m,m-k)\,[\Lambda^{\nu}(N,m,k)]^2\,d(\nu). \tag{51}$$

We are now left with the term $\overline{\langle ABCABC\rangle^m}$ and this can be written as

$$\overline{\langle ABCABC\rangle^m}=\frac{1}{d(f_m)}\sum_{\alpha_i,\alpha_j,\alpha_k,\alpha_\ell,\alpha_P,\alpha_Q}\overline{H_{\alpha_i\alpha_j}H_{\alpha_k\alpha_\ell}}\ \overline{H_{\alpha_j\alpha_P}H_{\alpha_\ell\alpha_Q}}\ \overline{H_{\alpha_P\alpha_k}H_{\alpha_Q\alpha_i}}. \tag{52}$$

It is easy to see that Eq. (52) is the same as the S_3 term in [44]; see Eq. (32) in this paper. Then, its simplification involves $SU(N)$ Racah (or $U-$) coefficients. The final result follows from Eq. (36) of [44] with $t=k$ giving

$$\begin{aligned}\overline{\langle ABCABC\rangle^m}=&\frac{1}{[d(f_m)]^2}\sum_{\mu_1,\mu_2=0}^{k}\ \sum_{\nu=0}^{min(2k,m-k)}d(\mu_1)\,d(\mu_2)\ |U(f_m\mu_1f_m\mu_2\,;\,f_m\nu)|^2\\ &\times\Lambda^{\mu_1}(N,m,m-k)\,\Lambda^{\mu_2}(N,m,m-k)\,\Lambda^{\nu}(N,m,k).\end{aligned} \tag{53}$$

In Eq. (53), for simplicity we are not showing the multiplicities that appear in the U-coefficient. See [40,41] for $SU(N)$ Racah coefficients and for some of their properties. Equation (48) of [44] gives the formula in the asymptotic limit for the U coefficient appearing above and this is used in Appendix C. Equations (47), (48), (51), and (53) together will give the formula for $\Sigma_{6,0}=\Sigma_{0,6}$,

$$\begin{aligned}\Sigma_{6,0}=&\,5\,[\Lambda^0(N,m,k)]^3+\frac{6}{d(f_m)}\ \Lambda^0(N,m,k)\sum_{\nu=0}^{min(k,m-k)}\Lambda^{\nu}(N,m,m-k)\ \Lambda^{\nu}(N,m,k)\ d(\nu)\\ &+\frac{3}{d(f_m)}\sum_{\nu=0}^{min(k,m-k)}\Lambda^{\nu}(N,m,m-k)\,[\Lambda^{\nu}(N,m,k)]^2\,d(\nu)+\frac{1}{[d(f_m)]^2}\sum_{\mu_1,\mu_2=0}^{k}\ \sum_{\nu=0}^{min(2k,m-k)}d(\mu_1)\,d(\mu_2)\\ &\times|U(f_m\mu_1f_m\mu_2\,;\,f_m\nu)|^2\Lambda^{\mu_1}(N,m,m-k)\,\Lambda^{\mu_2}(N,m,m-k)\,\Lambda^{\nu}(N,m,k).\end{aligned} \tag{54}$$

Though $\hat{\Sigma}_{6,0}=0$, we need the formula for $\Sigma_{6,0}$ when we consider $\hat{\Sigma}_{PQ}$ with $P+Q\geqslant 8$; see Sec. D.

The formula for $\Sigma_{5,1}$ is simple and this follows from the same arguments that gave Eq. (41). Then,

$$\Sigma_{5,1} = \Sigma_{1,5} = \overline{\langle H^5\rangle^m \langle H\rangle^m} = 5\,\overline{\langle H\rangle^m \langle H\rangle^m}\;\overline{\langle H^4\rangle^m} \tag{55}$$

with the first factor given by Eq. (33) and the second factor by Eq. (38). Then,

$$\hat{\Sigma}_{5,1} = 10\,\hat{\Sigma}_{1,1} + \frac{5\,\hat{\Sigma}_{1,1}\,\sum_{\nu=0}^{min\{k,m-k\}}\;\Lambda^\nu(N,m,m-k)\;\Lambda^\nu(N,m,k)\;d(\nu)}{d(f_m)\,[\Lambda^0(N,m,k)]^2}. \tag{56}$$

Coming to $\Sigma_{4,2}$, it is easy to see that there are three different binary correlation terms giving

$$\begin{aligned}\Sigma_{4,2} = \Sigma_{2,4} &= \overline{\langle H^4\rangle^m\,\langle H^2\rangle^m}\\ &= \overline{\langle H^2\rangle^m}\;\overline{\langle H^4\rangle^m} + 8\overline{\langle ABCC\rangle^m\langle AB\rangle^m} + 4\overline{\langle ABCB\rangle^m\langle AC\rangle^m}\\ &= \overline{\langle H^2\rangle^m}\;\overline{\langle H^4\rangle^m} + 8\overline{\langle H^2\rangle^m}\;\overline{\langle AB\rangle^m\langle AB\rangle^m} + 4\overline{\langle ABCB\rangle^m\langle AC\rangle^m}.\end{aligned} \tag{57}$$

Here the first two terms follow from Eqs. (32), (38), and (44) and the third term is simplified as follows. First as in Eq. (52),

$$\begin{aligned}\overline{\langle ABCB\rangle^m\langle AC\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_4\alpha_1}}\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_b\alpha_a}}\\ &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b}\;\sum_{\mu_1=0,1,\ldots,k;\omega_{\mu_1}}\Lambda^{\mu_1}(N,m,m-k)\,C^{\mu_1,\omega_{\mu_1}}_{\alpha_1\,\overline{\alpha_2}}\;C^{\mu_1,\omega_{\mu_1}}_{\alpha_a\,\overline{\alpha_b}}\\ &\times\sum_{\mu_2=0,1,\ldots,k;\omega_{\mu_2}}\Lambda^{\mu_2}(N,m,m-k)\,C^{\mu_2,\omega_{\mu_2}}_{\alpha_3\,\overline{\alpha_4}}\;C^{\mu_2,\omega_{\mu_2}}_{\alpha_b\,\overline{\alpha_a}}\sum_{\nu_1=0,1,\ldots,m-k;\omega_{\nu_1}}\Lambda^{\nu_1}(N,m,k)\,C^{\nu_1,\omega_{\nu_1}}_{\alpha_2\,\overline{\alpha_1}}\;C^{\nu_1,\omega_{\nu_1}}_{\alpha_4\,\overline{\alpha_3}}.\end{aligned} \tag{58}$$

Now the sum rules for the CG coefficients, as given by Eq. (28), allow us to carry out the sum over all the α's giving $\mu_1 = \mu_2 = \nu_1$, and similarly $\omega_{\mu_1} = \omega_{\mu_2} = \omega_{\nu_1}$. With these, Eq. (58) simplifies to

$$\overline{\langle ABCB\rangle^m\langle AC\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\nu=0}^{min(k,m-k)}\Lambda^\nu(N,m,k)\,[\Lambda^\nu(N,m,m-k)]^2\,d(\nu). \tag{59}$$

Combining Eqs. (59) and (57) will give the formula for $\Sigma_{4,2} = \Sigma_{2,4}$. Now, the formula for $\hat{\Sigma}_{4,2}$ is

$$\hat{\Sigma}_{4,2} = 4\,\hat{\Sigma}_{2,2} + \frac{4\,\sum_{\nu=0}^{min(k,m-k)}\Lambda^\nu(N,m,k)\,[\Lambda^\nu(N,m,m-k)]^2\,d(\nu)}{[d(f_m)]^2\,[\Lambda^0(N,m,k)]^3}. \tag{60}$$

Finally, for $\Sigma_{3,3}$ defined by

$$\Sigma_{3,3} = \overline{\langle H^3\rangle^m\langle H^3\rangle^m}, \tag{61}$$

there will be three binary correlated terms:

$$\Sigma_{3,3} = 9\overline{\langle ABB\rangle^m\langle ACC\rangle^m} + 3\overline{\langle ABC\rangle^m\langle ACB\rangle^m} + 3\overline{\langle ABC\rangle^m\langle ABC\rangle^m}. \tag{62}$$

Note that by definition, for EGUE(k), $\overline{\langle H^3\rangle^m} = 0$. The first term in Eq. (62) is simple,

$$\overline{\langle ABB\rangle^m\langle ACC\rangle^m} = [\overline{\langle H^2\rangle^m}]^2\;\overline{\langle H\rangle^m\langle H\rangle^m}. \tag{63}$$

The second term in Eq. (62) has a structure quite similar to the one in Eq. (58),

$$\overline{\langle ABC\rangle^m\langle ACB\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_a,\alpha_b,\alpha_c}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_c\alpha_a}}\;\overline{H_{\alpha_3\alpha_1}H_{\alpha_b\alpha_c}}. \tag{64}$$

Simplifying just as in Eqs. (58) and (59) will give the final formula:

$$\overline{\langle ABC\rangle^m\langle ACB\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\mu=0,1,\ldots,m-k}[\Lambda^\mu(N,m,k)]^3\;d(\mu). \tag{65}$$

The third term $\overline{\langle ABC\rangle^m\langle ABC\rangle^m}$ has a structure quite similar to $\overline{\langle ABCABC\rangle^m}$. Following the same steps that led to Eq. (53) will give the formula involving $SU(N)$ U coefficients. First,

$$\overline{\langle ABC\rangle^m\langle ABC\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_a,\alpha_b,\alpha_c}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_b\alpha_c}}\;\overline{H_{\alpha_3\alpha_1}H_{\alpha_c\alpha_a}}. \tag{66}$$

Now, simplifying this using the same procedure as in Eqs. (32)–(36) of [44] will generate the following formula:

$$\overline{\langle ABC\rangle^m\,\langle ABC\rangle^m} = \frac{1}{[d(f_m)]^3}\sum_{\nu,\nu_1,\nu_2=0}^{k} d(\nu_1)\,d(\nu_2)\,|U(f_m\nu_1 f_m\nu_2\,;\,f_m\nu)|^2 \times \Lambda^{\nu_1}(N,m,m-k)\,\Lambda^{\nu_2}(N,m,m-k)\,\Lambda^{\nu}(N,m,m-k). \quad (67)$$

As in Eq. (53), again in Eq. (67), for simplicity, we are not showing the multiplicities that appear in the U coefficient. Equations (62), (63), (65), and (67) will give the formula for $\Sigma_{3,3}$. With these, the formula for $\hat{\Sigma}_{3,3}$ is

$$\hat{\Sigma}_{3,3} = 9\,\hat{\Sigma}_{1,1} + \frac{3\,\sum_{\mu=0}^{m-k}[\Lambda^{\mu}(N,m,k)]^3\,d(\mu)}{[d(f_m)]^2[\Lambda^0(N,m,k)]^3} + \frac{3}{[d(f_m)\,\Lambda^0(N,m,k)]^3}\sum_{\nu,\nu_1,\nu_2=0}^{k} d(\nu_1)\,d(\nu_2)\,|U(f_m\nu_1 f_m\nu_2\,;\,f_m\nu)|^2 \times \Lambda^{\nu_1}(N,m,m-k)\,\Lambda^{\nu_2}(N,m,m-k)\,\Lambda^{\nu}(N,m,m-k). \quad (68)$$

Thus, we have simple finite-N formulas for all $\hat{\Sigma}_{P,Q}$ with $P+Q=6$, except for the U coefficient in Eq. (68).

D. Formulas for $\hat{\Sigma}_{PQ}$ with $P+Q=8$

With $P+Q=8$, we need to derive formulas for $\hat{\Sigma}_{7,1}=\hat{\Sigma}_{1,7}$, $\hat{\Sigma}_{6,2}=\hat{\Sigma}_{2,6}$, $\hat{\Sigma}_{5,3}=\hat{\Sigma}_{3,5}$, and $\hat{\Sigma}_{4,4}$. First, $\hat{\Sigma}_{7,1}$ is simple,

$$\hat{\Sigma}_{7,1} = \hat{\Sigma}_{1,7} = \frac{\overline{\langle H^7\rangle^m\,\langle H\rangle^m}}{[\Sigma_{2,0}]^4} = 7\,\hat{\Sigma}_{11}\,\frac{\Sigma_{6,0}}{[\Lambda^0(N,m,k)]^3} \quad (69)$$

and the formula for $\Sigma_{6,0}$ is given by Eq. (54).

The formula for $\hat{\Sigma}_{6,2}$ is more complicated and $\Sigma_{6,2}$ contains four different terms:

$$\Sigma_{6,2} = \Sigma_{2,6} = \overline{\langle H^6\rangle^m\,\langle H^2\rangle^m} = \overline{\langle H^2\rangle^m}\;\;\overline{\langle H^6\rangle^m} + 12\,\overline{\langle ABH^4\rangle^m\langle AB\rangle^m} + 12\,\overline{\langle ABCDEF\rangle^m\langle AC\rangle^m} + 6\,\overline{\langle ABCDEF\rangle^m\langle AD\rangle^m}. \quad (70)$$

The second term here is simple, giving

$$\overline{\langle ABH^4\rangle^m\langle AB\rangle^m} = \overline{\langle AB\rangle^m\langle AB\rangle^m}\;\;\overline{\langle H^4\rangle^m}, \quad (71)$$

and the formulas for the two terms on the R.H.S. are given by Eqs. (44), (38), and (37). The third term in Eq. (70) is

$$\overline{\langle ABCDEF\rangle^m\langle AC\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_a,\alpha_b} X_1\,[X_2+X_3+X_4];$$

$$X_1 = \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_b\alpha_a}},\quad X_2 = \overline{H_{\alpha_2\alpha_3}H_{\alpha_4\alpha_5}}\;\;\overline{H_{\alpha_5\alpha_6}H_{\alpha_6\alpha_1}},$$

$$X_3 = \overline{H_{\alpha_2\alpha_3}H_{\alpha_5\alpha_6}}\;\;\overline{H_{\alpha_4\alpha_5}H_{\alpha_6\alpha_1}},\quad X_4 = \overline{H_{\alpha_2\alpha_3}H_{\alpha_6\alpha_1}}\;\;\overline{H_{\alpha_4\alpha_5}H_{\alpha_5\alpha_6}}. \quad (72)$$

First, the "X_1" term is simplified using Eq. (29). Similarly, the second part of X_2, i.e., $\overline{H_{\alpha_5\alpha_6}H_{\alpha_6\alpha_1}}$, gives $\Lambda^0(N,m,k)\,\delta_{\alpha_1\alpha_5}$. Then, X_1X_2 with sum over all the α's, after applying Eq. (29), is given by

$$\sum_{\alpha' s} X_1X_2 = \sum_{\nu=0}^{k}[\Lambda^{\nu}(N,m,m-k)]^2\Lambda^0(N,m,k)\sum_{\mu=0,1,\ldots,m-k}\Lambda^{\mu}(N,m,k)\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\omega_\nu,\omega_\mu} C^{\nu,\omega_\nu}_{\alpha_1\overline{\alpha_2}}\,C^{\nu,\omega_\nu}_{\alpha_3\overline{\alpha_4}}\,C^{\mu,\omega_\mu}_{\alpha_2\overline{\alpha_1}}\,C^{\mu,\omega_\mu}_{\alpha_4\overline{\alpha_3}}$$

$$= \Lambda^0(N,m,k)\sum_{\nu=0}^{min(k,m-k)}[\Lambda^{\nu}(N,m,m-k)]^2\Lambda^{\nu}(N,m,k)\,d(\nu). \quad (73)$$

In the last step here we have used the sum rules for the CG coefficients as given in Eq. (28). Going further it is easy to see that the X_1X_4 with sum over all the α's is the same as X_1X_2 with sum over all the α's. Then we are left with X_1X_3. Applying Eqs. (29) and (30) will give

$$\sum_{\alpha' s} X_1X_3 = \sum_{\nu=0}^{k}[\Lambda^{\nu}(N,m,m-k)]^2\;\sum_{\mu=0}^{k}\Lambda^{\mu}(N,m,m-k)\sum_{\mu'=0,1,\ldots,m-k}\Lambda^{\mu'}(N,m,k) \times \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\omega_\nu,\omega_\mu,\omega_{\mu'}} C^{\nu,\omega_\nu}_{\alpha_1\overline{\alpha_2}}\,C^{\nu,\omega_\nu}_{\alpha_3\overline{\alpha_4}}\,C^{\mu,\omega_\mu}_{\alpha_2\overline{\alpha_3}}\,C^{\mu,\omega_\mu}_{\alpha_5\overline{\alpha_6}}\,C^{\mu',\omega_{\mu'}}_{\alpha_4\overline{\alpha_1}}\,C^{\mu',\omega_{\mu'}}_{\alpha_6\overline{\alpha_5}}. \quad (74)$$

This is simplified using Eq. (28) and the transformation of product of two CG coefficients as given by Eq. (21) of [10]. Applying these will give the formula:

$$\sum_{\alpha' s} X_1 X_3 = \sum_{\nu=0}^{k} \sum_{\mu=0}^{min(k,m-k)} [\Lambda^\nu(N,m,m-k)]^2 \Lambda^\mu(N,m,m-k)\Lambda^\mu(N,m,k)\sqrt{d(\nu)d(\mu)}\, U(f_m\overline{f_m}f_mf_m\,;\,\nu\mu). \tag{75}$$

Combining Eqs. (73) and (75) will give the formula for $\overline{\langle ABCDEF\rangle^m \langle AC\rangle^m}$. Turning to the fourth term in $\Sigma_{6,2}$, first we have

$$\begin{aligned}
\overline{\langle ABCDEF\rangle^m \langle AD\rangle^m} &= \frac{1}{[d(f_m)]^2} \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_a,\alpha_b} Y_1\,[Y_2+Y_3+Y_4];\\
Y_1 &= \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\;\overline{H_{\alpha_4\alpha_5}H_{\alpha_b\alpha_a}},\quad Y_2 = \overline{H_{\alpha_2\alpha_3}H_{\alpha_3\alpha_4}}\;\;\overline{H_{\alpha_5\alpha_6}H_{\alpha_6\alpha_1}},\\
Y_3 &= \overline{H_{\alpha_2\alpha_3}H_{\alpha_5\alpha_6}}\;\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_6\alpha_1}},\quad Y_4 = \overline{H_{\alpha_2\alpha_3}H_{\alpha_6\alpha_1}}\;\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_5\alpha_6}}.
\end{aligned} \tag{76}$$

The Y_1Y_2 term is simplified easily using Eqs. (28)–(30) and similarly, the Y_1Y_4 term giving

$$\begin{aligned}
\sum_{\alpha' s} Y_1Y_2 &= \sum_{\nu=0}^{k} [\Lambda^\nu(N,m,m-k)]^2[\Lambda^0(N,m,k)]^2\, d(\nu),\\
\sum_{\alpha' s} Y_1Y_4 &= \sum_{\nu=0}^{min(k,m-k)} [\Lambda^\nu(N,m,m-k)\;\Lambda^\nu(N,m,k)]^2\, d(\nu).
\end{aligned} \tag{77}$$

Simplification of the Y_1Y_3 term needs not only Eqs. (28)–(30) but also Eq. (21) of [10] for Y_1 and and Eq. (35) of [44] for Y_3. With these we have

$$\begin{aligned}
\sum_{\alpha' s} Y_1Y_3 = \sum_{\mu=0}^{2k} \sum_{\mu_1,\mu_2,\nu=0}^{k} &[\Lambda^\nu(N,m,m-k)]^2\;\Lambda^{\mu_1}(N,m,m-k)\;\Lambda^{\mu_2}(N,m,m-k)\\
&\times \frac{d(\mu_1)d(\mu_2)\sqrt{d(\nu)}}{d(f_m)\sqrt{d(\mu)}}\;U(f_m\overline{f_m}f_mf_m;\nu\mu)|U(f_m\mu_1f_m\mu_2;f_m\mu)|^2.
\end{aligned} \tag{78}$$

Combining Eqs. (77) and (78) will give the formula for $\overline{\langle ABCDEF\rangle^m \langle AD\rangle^m}$. With all these, the formula for $\hat{\Sigma}_{6,2}$ is

$$\begin{aligned}
\hat{\Sigma}_{6,2} &= \hat{\Sigma}_{2,6} = [\Lambda^0(N,m,k)]^{-4}\;\{A1+A2+A3\};\\
A1 &= \frac{12}{[d(f_m)]^2}\sum_{\nu=0}^{k}\{[\Lambda^\nu(N,m,m-k)]^2 d(\nu)\}\left\{2[\Lambda^0(N,m,k)]^2 + \frac{1}{d(f_m)}\sum_{\nu=0}^{min(k,m-k)}\Lambda^\nu(N,m,m-k)\;\Lambda^\nu(N,m,k)\;d(\nu)\right\},\\
A2 &= \frac{24}{[d(f_m)]^2}\;\Lambda^0(N,m,k)\sum_{\nu=0}^{min(k,m-k)}[\Lambda^\nu(N,m,m-k)]^2\Lambda^\nu(N,m,k)\;d(\nu) + \frac{12}{[d(f_m)]^2}\\
&\quad\times\left\{\sum_{\nu=0}^{k}\sum_{\mu=0}^{min(k,m-k)}[\Lambda^\nu(N,m,m-k)]^2\Lambda^\mu(N,m,m-k)\Lambda^\mu(N,m,k)\sqrt{d(\nu)d(\mu)}\,U(f_m\overline{f_m}f_mf_m\,;\,\nu\mu)\right\},\\
A3 &= \frac{6}{[d(f_m)]^2}\;[\Lambda^0(N,m,k)]^2\sum_{\nu=0}^{k}[\Lambda^\nu(N,m,m-k)]^2\;d(\nu) + \frac{6}{[d(f_m)]^2}\sum_{\nu=0}^{min(k,m-k)}[\Lambda^\nu(N,m,m-k)]^2\;[\Lambda^\nu(N,m,k)]^2\;d(\nu)\\
&\quad+\frac{6}{[d(f_m)]^2}\left\{\sum_{\mu=0}^{2k}\sum_{\mu_1,\mu_2,\nu=0}^{k}[\Lambda^\nu(N,m,m-k)]^2\;\Lambda^{\mu_1}(N,m,m-k)\;\Lambda^{\mu_2}(N,m,m-k)\right.\\
&\quad\left.\times\frac{d(\mu_1)d(\mu_2)\sqrt{d(\nu)}}{d(f_m)\sqrt{d(\mu)}}\;U(f_m\overline{f_m}f_mf_m;\nu\mu)|U(f_m\mu_1f_m\mu_2;f_m\mu)|^2\right\}.
\end{aligned} \tag{79}$$

Now, we will consider $\hat{\Sigma}_{5,3}$ and $\hat{\Sigma}_{4,4}$.

TABLE I. Covariances $\overline{S_i S_j}$ for EGUE(k). Results are shown for $(N, m) = (12, 6)$ with $k = 2$ and 3, and for $(N, m) = (30, 10)$, $k = 2$, 3, and 4. Numbers in the brackets are from asymptotic limit formulas given by Eq. (82). The q values obtained using Eq. (39) are also given.

$\overline{S_iS_j}$	$N = 12,\ m = 6$ $k = 2$	$k = 3$	
	$q = 0.287$	$q = 0.006$	
$\overline{S_1S_1}$	8.117×10^{-3} (3.444×10^{-3})	1.082×10^{-3} (4.132×10^{-4})	
$\overline{S_3S_1}$	5.787×10^{-3} (3.444×10^{-3})	1.021×10^{-3} (4.132×10^{-4})	
$\overline{S_2S_2}$	1.160×10^{-3} (4.591×10^{-4})	1.227×10^{-4} (4.132×10^{-5})	
$\overline{S_5S_1}$	6.835×10^{-3} (4.845×10^{-3})	1.075×10^{-3} (3.968×10^{-4})	
$\overline{S_4S_2}$	1.533×10^{-3} (5.251×10^{-4})	1.497×10^{-4} (4.064×10^{-5})	
$\overline{S_3S_3}$	3.062×10^{-2} (1.303×10^{-2})	5.176×10^{-3} (1.974×10^{-3})	
$\overline{S_7S_1}$	1.249×10^{-2} (1.063×10^{-2})	1.282×10^{-3} (3.993×10^{-4})	
$\overline{S_6S_2}$	(3.676×10^{-4})	(4.276×10^{-5})	
$\overline{S_5S_3}$	(1.539×10^{-2})	(2.08×10^{-3})	
$\overline{S_4S_4}$	(3.907×10^{-4})	(4.126×10^{-5})	
$\overline{S_iS_j}$	$N = 30,\ m = 10$ $k = 2$	$k = 3$	$k = 4$
	$q = 0.524$	$q = 0.208$	$q = 0.005$
$\overline{S_1S_1}$	4.478×10^{-4} (2.378×10^{-4})	1.669×10^{-5} (7.28×10^{-6})	7.211×10^{-7} (2.796×10^{-7})
$\overline{S_3S_1}$	2.13×10^{-4} (2.378×10^{-4})	1.322×10^{-5} (7.28×10^{-6})	6.884×10^{-7} (2.796×10^{-7})
$\overline{S_2S_2}$	1.718×10^{-5} (1.057×10^{-5})	2.577×10^{-7} (1.213×10^{-7})	6.719×10^{-9} (2.663×10^{-9})
$\overline{S_5S_1}$	2.354×10^{-4} (2.413×10^{-4})	1.528×10^{-5} (9.725×10^{-6})	7.182×10^{-7} (3.151×10^{-7})
$\overline{S_4S_2}$	1.559×10^{-5} (9.731×10^{-6})	3.025×10^{-7} (1.365×10^{-7})	7.532×10^{-9} (2.797×10^{-9})
$\overline{S_3S_3}$	1.177×10^{-3} (6.252×10^{-4})	6.887×10^{-5} (3.004×10^{-5})	3.473×10^{-6} (1.345×10^{-6})
$\overline{S_7S_1}$	5.852×10^{-4} (6.316×10^{-4})	2.016×10^{-5} (1.826×10^{-5})	7.659×10^{-7} (3.885×10^{-7})
$\overline{S_6S_2}$	(1.703×10^{-6})	(1.141×10^{-7})	(2.746×10^{-9})
$\overline{S_5S_3}$	(6.91×10^{-4})	(3.472×10^{-5})	(1.405×10^{-6})
$\overline{S_4S_4}$	(5.566×10^{-6})	(1.111×10^{-7})	(2.652×10^{-9})

It is easy to see that $\hat{\Sigma}_{5,3}$ and $\hat{\Sigma}_{4,4}$ will involve a much larger number of terms than $\Sigma_{6,2}$ and they will also involve several $SU(N)$ U coefficients. Details of various terms in $\hat{\Sigma}_{5,3}$ and $\hat{\Sigma}_{4,4}$ are given in Appendix B. Following this, the formula for $\hat{\Sigma}_{5,3}$ is

$$\hat{\Sigma}_{5,3} = 15(2+q)\,\hat{\Sigma}_{1,1} + 5(1+q)[\hat{\Sigma}_{3,3} - 9\hat{\Sigma}_{1,1}] + X_{53}. \tag{80}$$

Here, Eqs. (B2), (B3), (65), (67), (68), and (B5) are used with X_{53} defined by Eq. (B5). Similarly, the formula for $\hat{\Sigma}_{4,4}$ is

$$\begin{aligned}\hat{\Sigma}_{4,4} = {} & 16\hat{\Sigma}_{2,2} + 8[\hat{\Sigma}_{4,2} - 4\hat{\Sigma}_{2,2}] + \frac{8}{[d(f_m)]^2} \\ & \times \sum_{\nu=0}^{min(k,m-k)} \frac{[\Lambda^{\nu}(N,m,m-k)\,\Lambda^{\nu}(N,m,k)]^2\,d(\nu)}{[\Lambda^0(N,m,k)]^4} \\ & + \frac{4}{[d(f_m)]^2}\sum_{\nu=0}^{m-k}\frac{[\Lambda^{\nu}(N,m,k)]^4\,d(\nu)}{[\Lambda^0(N,m,k)]^4} + X_{44}. \end{aligned} \tag{81}$$

Here, Eqs. (B8), (32), (44), (59, (B10), and (B14) are used with X_{44} defined by Eqs. (B11) and (B12). Note that X_{53} in Eq. (80) and X_{44} in Eq. (81) involve $SU(N)$ U coefficients for which formulas are not available. However, both X_{53} and X_{44} can be neglected in the asymptotic limit (see Appendix c).

Formulas derived in this section along with Eq. (25) will allow us to calculate $\overline{S_iS_j}$ numerically for $i + j \leqslant 8$, as well as allow to examine their asymptotic structure. We will turn to these in the following section.

V. ASYMPTOTIC LIMIT RESULTS FOR THE COVARIANCES AND EXPANSION FOR THE NUMBER VARIANCE

In the previous section we have derived formulas for $\hat{\Sigma}_{P,Q}$ with $P + Q = 2 - 8$. In particular, the formula for $(P, Q) = (1, 1)$ is given by Eq. (34); for (3,1) by Eq. (42); for (2,2) by Eq. (45); for (5,1) by Eq. (56); for (4,2) by Eq. (60); for (3,3) by Eq. (68); for (7,1) by Eqs. (69) and (54); for (6,2) by Eq. (79); for (5,3) by Eq. (80): and finally, for (4,4) by Eq. (81). Also, note that the formula for the q parameter is given by Eq. (39) and $\Lambda^{\nu}(N, m, r)$ is given by Eq. (31). In addition, the dimensions $d(f_m) = \binom{N}{m}$ and $d(\nu) = \binom{N}{\nu}^2 - \binom{N}{\nu-1}^2$. Using all these equations along with Eq. (25), the covariances $\overline{S_iS_j}$ for $(i, j) = (1, 1)$, (3,1), (2,2), (5,1), (4,2), (3,3), and (7,1) are calculated and the results are shown in Table I. For (3,3), the last term in Eq. (68) is not included as the U coefficients needed here are not available. Finite N results for (6,2) are not shown as formulas, for the two U coefficients appearing in Eq. (79) are not available. Similarly, the finite N results for (5,3) and (4,4) are not shown in the table. It is seen from the table that in general, the covariances are small and they are of the same order of magnitude as in the SYK model (for Majorana fermions) reported earlier in [22].

A. Asymptotic limit formulas for $\overline{S_iS_j}$

For further insight into the structure of $\overline{S_iS_j}$, the formulas in Sec. IV are used to derive asymptotic limit formulas for $\hat{\Sigma}_{PQ}$ and these are given in Appendix C. Now, using the formulas in Eq. (C3) and Eqs. (25)–(27), the following asymptotic limit formulas are obtained for $\overline{S_iS_j}$ with $i+j \leqslant 8$,

$$
\begin{aligned}
\overline{S_1S_1} &= \frac{\binom{m}{k}}{\binom{N}{k}^2},\\
\overline{S_3S_1} &= (1-q)\,\overline{S_1S_1},\\
\overline{S_2S_2} &= \frac{1}{\binom{N}{k}^2},\\
\overline{S_5S_1} &= (1-q^2)^2\,\overline{S_1S_1},\\
\overline{S_4S_2} &= (1-q^2)\,\overline{S_2S_2},\\
\overline{S_3S_3} &= 3(1-q)\,\overline{S_1S_1} + \frac{3}{\binom{m}{k}\binom{N}{k}^2} + O\left(\frac{1}{\binom{N}{k}^4}\right),\\
\overline{S_7S_1} &= (1-q)(1-q^2)^2[1+2q+3q^2+2q^3+q^4]\,\overline{S_1S_1},\\
\overline{S_6S_2} &= (q^6+q^5-q^4+4q^3-7q^2+q+1)\,\overline{S_2S_2}\\
&\quad + O\left(\frac{1}{\binom{N}{k}^4}\right),\\
\overline{S_5S_3} &= (1-q)^2\left[q^3+7q^2+11q+5\right]\overline{S_1S_1}\\
&\quad + \frac{3(1-q)(q^2+3q+1)}{\binom{m}{k}\binom{N}{k}^2} + O\left(\frac{1}{\binom{N}{k}^4}\right),\\
\overline{S_4S_4} &= (1-q^2)^2\,\overline{S_2S_2} + \frac{4}{\binom{m}{k}^2\binom{N}{k}^2} + O\left(\frac{1}{\binom{N}{k}^4}\right);\\
q &= \frac{\binom{m-k}{k}}{\binom{m}{k}}.
\end{aligned} \tag{82}
$$

All these formulas agree with the GUE ($k=m$ giving $q=0$) results given in [7,34]. They also agree with results for EGUE(k) in the $k/m \to 1$ limit as given in [6,7] and this corresponds to $q \to 1$ in Eq. (82). Thus, the formulas in Eq. (82) cover the two extreme limits and therefore expected to apply to all k values (validity of this needs further testing of the approximations used in Appendix C). In addition, the results in Sec. IV give finite N corrections to the formulas in Eq. (82) for all k values.

Going further, numerical results given by Eq. (82) are shown in brackets in Table I. These results are not too far from the finite N results. The correlations, as seen from the asymptotic limit formulas in Eq. (82), are of the order of $1/[\binom{N}{k}]^2$. With the $1/[\binom{N}{k}]^2$ scaling, correlations $\overline{S_iS_j}$ shown in Table I are no longer small. More strikingly, for $q \to 1$ (i.e., $k/m \to 0$) the $\overline{S_iS_j} = 0$ for $i \neq j$ and $\overline{S_i^2} = \binom{m}{k}^{2-i}\binom{N}{k}^{-2}$. Similarly, for $q=0$ (i.e., $k=m$) the structure of $\overline{S_iS_j}$ is simple and $\overline{S_iS_j} \neq 0$ both for $i=j$ and $i \neq j$. However, for intermediate k values (between $k << m$ and $k=m$), $\overline{S_iS_j}$ are a combination of q, $\overline{S_1S_1}$, $\overline{S_2S_2}$, and $\binom{m}{k}^r\binom{N}{k}^{-2}$ with $r \leqslant 1$. Besides these, the case $k/m \to 0$ (i.e., $q \to 1$), the so-called dilute limit, seems to imply an uncorrelated spectra, since $\overline{S_iS_j} = 0$ for $i \neq j$. This seems to agree with a conclusion obtained first in [46] (see also the discussion in Sec. I and [9,15]).

Numerical evaluation of $\Sigma_{6,0}$, $\Sigma_{3,3}$, and $\Sigma_{6,2}$ (also, for $\Sigma_{5,3}$ and $\Sigma_{4,4}$, see Appendix B) requires formulas for $SU(N)$ U coefficients of the type $U(f_m\overline{f_m}f_mf_m;\nu\mu)$ and $U(f_m\nu_1f_m\nu_2;f_m\nu)$. The situation here is similar to the U coefficients needed even for the fourth moment for EGUE's with spin and spin-isospin $SU(4)$ symmetries [47,48] as encountered before. Thus, much of the progress in analytical approach to EGUE(k)'s will depend on extending our knowledge on $SU(N)$ U coefficients. One approach is to further develop the so-called pattern calculus introduced by Louck and Biedenharn many years back for $SU(N)$ Wigner-Racah algebra [49–51]. Another is to derive asymptotic expansions for the $SU(N)$ Racah coefficients as attempted in the past by French [52].

B. Expansion for number variance $\Sigma^2(\overline{n})$

Before concluding the paper, as an example it is instructive to consider the expansion for the number variance $\Sigma^2(\overline{n})$ in terms of $\overline{S_iS_j}$, and this follows from the expansion for the two-point function. The definition given by Eq. (5) together with Eqs. (4) and (19) will give the expansion,

$$
\begin{aligned}
\Sigma^2(\overline{n}) &= d^2 \sum_{\zeta,\zeta'=1}^{\infty} \overline{S_\zeta\, S_{\zeta'}}\,[R_\zeta(x|q) - R_\zeta(y|q)]\\
&\quad \times [R_{\zeta'}(x|q) - R_{\zeta'}(y|q)];\\
R_\zeta(x|q) &= \int_{-\frac{2}{\sqrt{1-q}}}^{x} f_{qN}(z|q)\frac{He_\zeta(z|q)}{[\zeta]_q!}\,dz.
\end{aligned} \tag{83}
$$

Note that we have used the property $\overline{S_\zeta} = 0$. In Eq. (83), with $\Sigma^2(\overline{n})$ defined over $x_0 \pm (\overline{n}\overline{D})/2$, $x = x_0 - (\overline{n}\overline{D})/2$, and $y = x_0 + (\overline{n}\overline{D})/2$. Note that $\overline{D}$ is the average mean spacing (in σ units) and x_0 is the eigenvalue around which $\Sigma^2(\overline{n})$ is evaluated. It is expected that $\Sigma^2(\overline{n})$ is to be independent of x_0, except perhaps near the spectrum ends. With formulas for $\overline{S_iS_j}$ and for $i+j \leqslant 8$ available as given by the equations in Sec. IV along with Eq. (25), the series given by Eq. (83) can be evaluated up to $\zeta + \zeta' \leqslant 8$ terms. Alternatively, one can use the asymptotic limit formulas given by Eq. (82). Note that at present the function $R_\zeta(x|q)$ need to be evaluated numerically as no analytical formula for the integral defining $R_\zeta(x|q)$ is available except for $q=1$ and 0.

Finally, direct derivation of asymptotic limit formulas for many other $\hat{\Sigma}_{PQ}$ for higher $P+Q$ values (i.e., $P+Q > 8$) may prove to be useful in the future as they will provide systematics for $\hat{\Sigma}_{PQ}$ and hence for $\overline{S_iS_j}$. With this, it may be possible to carry out the sum in Eq. (21) [or the sum in Eq. (83)] and obtain the two-point function (or the number variance) for EGUE(k) just as it was carried out using the moment method for GOE and GUE in the past [7,18,34]. This work is left for the future.

VI. CONCLUSIONS AND FUTURE OUTLOOK

Two-point correlation function in eigenvalues of embedded random matrix ensembles with k-body interactions is not yet available though these ensembles are applied to many different quantum systems in the last 50 years (see [18,53–62] and references therein for the previous and more recent applications of EE). With the recent recognition that the one-point function for these ensembles follows q-normal form, it is possible to seek an expansion of the eigenvalue density of the members of the ensemble in terms of q-Hermite polynomials. Covariances $\overline{S_\zeta S_{\zeta'}}$ of the expansion coefficients S_ζ with $\zeta \geqslant 1$ determine the two-point function here. As the covariances are a linear combination of the bivariate moments Σ_{PQ} of the two-point function (see Sec. III), in this paper, in Sec. IV, formulas are derived for the bivariate moments Σ_{PQ} with $P+Q \leqslant 8$ for the embedded Gaussian unitary ensembles with k-body interactions [EGUE(k)], as appropriate for systems with m fermions in N single particle states. The Wigner-Racah algebra for $SU(N)$ plays a central role in deriving the formulas with finite N corrections [10,44]. However, the Σ_{PQ} with $P+Q=6$ and 8 need extension of the available knowledge in calculating $SU(N)$ U coefficients; see Sec. IV. Using the finite N formulas, in Sec. V asymptotic limit ($N \to \infty$, $m \to \infty$, $m/N \to 0$ with k finite) formulas for $\overline{S_\zeta S_{\zeta'}}$ with $\zeta + \zeta' \leqslant 8$ are derived. In summary, the present work extends to all k values, the results known in the past in the two extreme limits with $k/m \to 0$ (same as $q \to 1$) and $k = m$ (same as $q = 0$). Also, the case $k/m \to 0$ (i.e., $q \to 1$), seems to imply an uncorrelated spectra as here $\overline{S_i S_j} = 0$ for $i \neq j$. This seems to agree with a conclusion obtained in [46].

In the future, expecting the availability of new methods for evaluating general $SU(N)$ U coefficients, it may be possible to get systematics of Σ_{PQ} and $\overline{S_\zeta S_{\zeta'}}$, and with these it may be possible to derive the two-point correlation function for EGUE(k) ensemble [perhaps also for EGOE(k) and EGSE(k)]. Once the two-point function is available, this may also open the possibility of studying ergodicity and stationarity properties of EGUE(k); see [6,7,63] for some past attempts in this direction.

ACKNOWLEDGMENT

Thanks are due to N. D. Chavda and Manan Vyas for some useful correspondence.

APPENDIX A

Reduction of the Kronecker product of the irreps ν_1 and ν_2, giving irreps ν_3, is symbolically denoted by

$$\nu_1 \times \nu_2 = \sum_{\nu_3} \Gamma_{\nu_1\nu_2\nu_3}\ \nu_3, \tag{A1}$$

where $\times$ denotes the Kronecker product and Γ gives the multiplicity, i.e., the number of times ν_3 appears in the Kronecker product. If $\Gamma_{\nu_1\nu_2\nu_3} = 0$, it implies that the irrep ν_3 will not appear in the Kronecker product. In our applications, the irreps ν correspond to the Young tableaux $\{2^\nu 1^{N-2\nu}\}$ of $U(N)$. Then, Eq. (A1) changes to

$$\{2^{\nu_1} 1^{N-2\nu_1}\} \times \{2^{\nu_2} 1^{N-2\nu_2}\} = \sum_{\nu_3} \Gamma_{\nu_1\nu_2\nu_3} \{2^{\nu_3} 1^{N-2\nu_3}\}. \tag{A2}$$

Though the methods to obtain the reduction given by Eq. (A2) are well known [42,43], a simpler approach is to first evaluate the Kronecker product of the transpose of the irreps and then take the transpose of the final irreps. By taking transpose, the two column irreps $\{2^\nu 1^{N-2\nu}\}$ change to two rowed irreps $\{N-\nu, \nu\}$ giving

$$\{N-\nu_1, \nu_1\} \times \{N-\nu_2, \nu_2\} = \sum_{\nu_3} \Gamma_{\nu_1\nu_2\nu_3} \{N-\nu_3, \nu_3\}. \tag{A3}$$

The Kronecker product here is easy to evaluate using the identity

$$\begin{aligned} &\{N-\nu_1, \nu_1\} \times \{N-\nu_2, \nu_2\} \\ &\quad = \{N-\nu_1, \nu_1\} \times [\{N-\nu_2\} \times \{\nu_2\} \\ &\qquad - \{N-\nu_2+1\} \times \{\nu_2 - 1\}]. \end{aligned} \tag{A4}$$

Now the product $\{n_1, n_2\} \times \{n_3\}$ is simply the sum of the irreps $\{n_1+n_a, n_2+n_b, n_c\}$ with $n_a \geqslant 0$, $n_b \leqslant n_1 - n_2$, $n_c \leqslant n_2$, and $n_a + n_b + n_c = n_3$. Similarly, for the product $\{n_1, n_2, n_3\} \times \{n_4\}$; see [42,43] and Eq. (B9) in [45]. Applying this to Eq. (A4) gives, in general, two, three, and four rowed irreps, however, we need only two rowed irreps. Regularization of the three and four rowed irreps is done using the rules: (i) four rowed irreps $\{n_1, n_2, n_3, n_4\} = 0$ if $n_1 \neq N$ and $n_2 \neq N$. As $n_1+n_2+n_3+n_4 = 2N$, the allowed irrep is just $\{N, N, 0, 0\}$; (ii) three rowed irreps $\{n_1, n_2, n_3\} = \{n_2, n_3\}$ if $n_1 = N$ and zero otherwise. Also, note that $\nu = 0$ corresponds to $\{1^N\}$ for $U(N)$ and $\{0\}$ for $SU(N)$. Using all these, we find the following results for $N \gg \nu$ and N large:

$$\begin{aligned} \nu \times 1 &= (\nu \pm 1)^1, (\nu)^2, \\ \nu \times 2 &= (\nu \pm 2)^1, (\nu \pm 1)^2, (\nu)^3, \\ \nu \times 3 &= (\nu \pm 3)^1, (\nu \pm 2)^2, (\nu \pm 1)^3, (\nu)^4, \\ \nu \times 4 &= (\nu \pm 4)^1, (\nu \pm 3)^2, (\nu \pm 2)^3, (\nu \pm 1)^4, (\nu)^5. \end{aligned} \tag{A5}$$

In the above, r in $(\mu)^r$ denotes multiplicity of the irrep μ. Continuing the above for $\nu \times 5$, $\nu \times 6$, etc., it is easy to see that $\nu \times \nu$ always gives the irrep ν but with multiplicity.

APPENDIX B

Let us consider $\Sigma_{5,3}$,

$$\Sigma_{5,3} = \Sigma_{3,5} = \overline{\langle H^5 \rangle^m \langle H^3 \rangle^m}. \tag{B1}$$

First, $\overline{\langle H^3 \rangle^m} = 0$ and $\overline{\langle H^5 \rangle^m} = 0$ for EGUE(k). Therefore,

$$\hat{\Sigma}_{5,3} = \hat{\Sigma}_{3,5} = [\overline{\langle H^2 \rangle^m}]^{-4}\, \Sigma_{5,3}. \tag{B2}$$

In the binary correlation approximation, for $\Sigma_{5,3}$ there are two possibilities: (i) one H in $\langle H^3 \rangle^m$ correlates with one of the H's in $\langle H^5 \rangle^m$; (ii) the three H's in $\langle H^3 \rangle^m$ correlate pairwise with three of the H's in $\langle H^5 \rangle^m$. These will give five binary correlated terms:

$$\begin{aligned}\Sigma_{5,3} &= 15\overline{\langle AH^4\rangle^m\ \langle AH^2\rangle^m} + 15\overline{\langle ABCDD\rangle^m\ \langle ABC\rangle^m} + 15\overline{\langle ABCDD\rangle^m\ \langle ACB\rangle^m} \\ &\quad + 15\overline{\langle ABDCD\rangle^m\ \langle ABC\rangle^m} + 15\overline{\langle ABDCD\rangle^m\ \langle ACB\rangle^m} \\ &= 15\overline{\langle H\rangle^m\ \langle H\rangle^m}\ \ \overline{\langle H^4\rangle^m}\ \ \overline{\langle H^2\rangle^m} + 15\overline{\langle H^2\rangle^m}\ [\overline{\langle ABC\rangle^m\langle ABC\rangle^m} + \overline{\langle ABC\rangle^m\langle ACB\rangle^m}] \\ &\quad + 15\overline{\langle ABDCD\rangle^m\ \langle ABC\rangle^m} + 15\overline{\langle ABDCD\rangle^m\ \langle ACB\rangle^m}. \end{aligned} \tag{B3}$$

Except for the last two terms, formulas for rest of the terms in Eq. (B3) are already given in Sec. IV; see Eqs. (65) and (67). The last two terms are

$$\begin{aligned}\overline{\langle ABDCD\rangle^m\langle ABC\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_a,\alpha_b,\alpha_c} \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\ \ \overline{H_{\alpha_2\alpha_3}H_{\alpha_b\alpha_c}}\ \ \overline{H_{\alpha_3\alpha_4}H_{\alpha_5\alpha_1}}\ \ \overline{H_{\alpha_4\alpha_5}H_{\alpha_c\alpha_a}}, \\ \overline{\langle ABDCD\rangle^m\langle ACB\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_a,\alpha_b,\alpha_c} \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\ \ \overline{H_{\alpha_2\alpha_3}H_{\alpha_c\alpha_a}}\ \ \overline{H_{\alpha_3\alpha_4}H_{\alpha_5\alpha_1}}\ \ \overline{H_{\alpha_4\alpha_5}H_{\alpha_b\alpha_c}}. \end{aligned} \tag{B4}$$

Further simplification of these follow from the $SU(N)$ algebra given in [10,40,41,44]. Clearly, these will involve several $SU(N)U$ coefficients. However, assuming $N \to \infty$ and (m, k) are finite, it is possible to use the approximation $\langle ABDCD\rangle^m \sim [\binom{m}{k}]^{-1}\binom{m-k}{k}\ \langle ABCDD\rangle^m$; see [6,18]. Also, $q \sim [\binom{m}{k}]^{-1}\binom{m-k}{k}$. With these, the last two terms in Eq. (B3) can be written as

$$15\overline{\langle ABDCD\rangle^m\langle ABC\rangle^m} + 15\overline{\langle ABDCD\rangle^m\langle ACB\rangle^m} = 15\,q\ \overline{\langle H^2\rangle^m}[\overline{\langle ABC\rangle^m\langle ABC\rangle^m} + \overline{\langle ABC\rangle^m\langle ACB\rangle^m}] + X_{53}. \tag{B5}$$

Here, X_{53} is the correction to the approximation given by the first term and this is expected to be of the order of $[\binom{N}{k}]^{-4}$. With this, $\Sigma_{5,3}$ is

$$\Sigma_{5,3} = \quad 15\ \overline{\langle H\rangle^m\ \langle H\rangle^m}\ \ \overline{\langle H^4\rangle^m}\ \ \overline{\langle H^2\rangle^m} + 15(1+q)\ \overline{\langle H^2\rangle^m}\ [\overline{\langle ABC\rangle^m\langle ABC\rangle^m} + \overline{\langle ABC\rangle^m\langle ACB\rangle^m}] + X_{53}. \tag{B6}$$

Turning to $\Sigma_{4,4}$,

$$\Sigma_{4,4} = \overline{\langle H^4\rangle^m\ \langle H^4\rangle^m}, \tag{B7}$$

in the binary correlation approximation, there are three possibilities: (i) the two $\langle H^4\rangle^m$'s are independent; (ii) two of H's in one $\langle H^4\rangle^m$ correlate with two H's in the other $\langle H^4\rangle^m$; (iii) the four H's in $\langle H^4\rangle^m$ correlate pairwise with the four H's in the other $\langle H^4\rangle^m$. These will give one, three, and six binary correlated terms, respectively,

$$\begin{aligned}\Sigma_{4,4} &= \overline{\langle H^4\rangle^m}\ \ \overline{\langle H^4\rangle^m} + 32\overline{\langle ABCC\rangle^m\ \langle ABDD\rangle^m} + 32\overline{\langle ABCC\rangle^m\ \langle ADBD\rangle^m} + 8\overline{\langle ACBC\rangle^m\ \langle ADBD\rangle^m} \\ &\quad + 4\overline{\langle ABCD\rangle^m\ \langle ABCD\rangle^m} + 4\overline{\langle ABCD\rangle^m\ \langle ABDC\rangle^m} + 4\overline{\langle ABCD\rangle^m\ \langle ACBD\rangle^m} \\ &\quad + 4\overline{\langle ABCD\rangle^m\ \langle ACDB\rangle^m} + 4\overline{\langle ABCD\rangle^m\ \langle ADBC\rangle^m} + 4\overline{\langle ABCD\rangle^m\ \langle ADCB\rangle^m}. \end{aligned} \tag{B8}$$

The formula for the first term is given by Eqs. (38) and (37). Further, the second term reduces to $(\overline{\langle H^2\rangle^m})^2\ \overline{\langle AB\rangle^m\langle AB\rangle^m}$ and the formula for this follows from Eqs. (32) and (44). Similarly, the third term reduces to $\overline{\langle H^2\rangle^m}\ \overline{\langle AB\rangle^m\langle ADBD\rangle^m}$ and the formula for this follows from Eq. (59). The fourth term is explicitly

$$\begin{aligned}\overline{\langle ACBC\rangle^m\langle ADBD\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d} \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\ \ \overline{H_{\alpha_2\alpha_3}H_{\alpha_4\alpha_1}}\ \ \overline{H_{\alpha_3\alpha_4}H_{\alpha_c\alpha_d}}\ \ \overline{H_{\alpha_b\alpha_c}H_{\alpha_d\alpha_a}} \\ &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d}\ \sum_{\nu_1,\nu_3=0}^{k}\ \sum_{\nu_2,\nu_4=0}^{m-k}\ \sum_{\omega_{\nu_1},\omega_{\nu_2},\omega_{\nu_3},\omega_{\nu_4}} \Lambda^{\nu_1}(N,m,m-k)\,C^{\nu_1,\omega_{\nu_1}}_{\alpha_1\overline{\alpha_2}}C^{\nu_1,\omega_{\nu_1}}_{\alpha_a\overline{\alpha_b}}\ \ \Lambda^{\nu_2}(N,m,k) \\ &\quad\times C^{\nu_2,\omega_{\nu_2}}_{\alpha_2\overline{\alpha_1}}C^{\nu_2,\omega_{\nu_2}}_{\alpha_4\overline{\alpha_3}}\,\Lambda^{\nu_3}(N,m,m-k)\,C^{\nu_3,\omega_{\nu_3}}_{\alpha_3\overline{\alpha_4}}C^{\nu_3,\omega_{\nu_3}}_{\alpha_c\overline{\alpha_d}}\ \ \Lambda^{\nu_4}(N,m,k)\,C^{\nu_4,\omega_{\nu_4}}_{\alpha_b\overline{\alpha_a}}\ C^{\nu_4,\omega_{\nu_4}}_{\alpha_d\overline{\alpha_c}}. \end{aligned} \tag{B9}$$

Here we have applied Eqs. (29) and (30). Now, simplifying the CG coefficients will give the formula,

$$\overline{\langle ACBC\rangle^m\langle ADBD\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\nu=0}^{min(k,m-k)} [\Lambda^{\nu}(N,m,m-k)\,\Lambda^{\nu}(N,m,k)]^2\,d(\nu). \tag{B10}$$

For the last six terms in Eq. (B8) we can write formulas similar to the one in Eq. (B9). First, the first five terms are

$$\begin{aligned}\overline{\langle ABCD\rangle^m\langle ABCD\rangle^m} = X_1 &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d} \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\ \ \overline{H_{\alpha_2\alpha_3}H_{\alpha_b\alpha_c}}\ \ \overline{H_{\alpha_3\alpha_4}H_{\alpha_c\alpha_d}}\ \ \overline{H_{\alpha_4\alpha_1}H_{\alpha_d\alpha_a}}, \\ \overline{\langle ABCD\rangle^m\langle ABDC\rangle^m} = X_2 &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d} \overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\ \ \overline{H_{\alpha_2\alpha_3}H_{\alpha_b\alpha_c}}\ \ \overline{H_{\alpha_3\alpha_4}H_{\alpha_d\alpha_a}}\ \ \overline{H_{\alpha_4\alpha_1}H_{\alpha_c\alpha_d}}, \end{aligned}$$

$$\overline{\langle ABCD\rangle^m\langle ACBD\rangle^m} = X_3 = \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_c\alpha_d}}\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_b\alpha_c}}\;\overline{H_{\alpha_4\alpha_1}H_{\alpha_d\alpha_a}},$$

$$\overline{\langle ABCD\rangle^m\langle ACDB\rangle^m} = X_4 = \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_d\alpha_a}}\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_b\alpha_c}}\;\overline{H_{\alpha_4\alpha_1}H_{\alpha_c\alpha_d}},$$

$$\overline{\langle ABCD\rangle^m\langle ADBC\rangle^m} = X_5 = \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_c\alpha_d}}\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_d\alpha_a}}\;\overline{H_{\alpha_4\alpha_1}H_{\alpha_b\alpha_c}}. \tag{B11}$$

Further simplification of these five terms called X_1, X_2, X_3, X_4, and X_5 in Eq. (B9) follow from the $SU(N)$ algebra given in [10,40,41,44] and they involve several $SU(N)U$ coefficients. However, formulas for these U coefficients are not available in literature. For future reference, we call the sum of the five terms as X_{44},

$$X_{44} = X_1 + X_2 + X_3 + X_4 + X_5. \tag{B12}$$

Finally, the sixth term is

$$\begin{aligned}\overline{\langle ABCD\rangle^m\langle ADCB\rangle^m} &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d}\overline{H_{\alpha_1\alpha_2}H_{\alpha_a\alpha_b}}\;\overline{H_{\alpha_2\alpha_3}H_{\alpha_d\alpha_a}}\;\overline{H_{\alpha_3\alpha_4}H_{\alpha_c\alpha_d}}\;\overline{H_{\alpha_4\alpha_1}H_{\alpha_b\alpha_c}}\\ &= \frac{1}{[d(f_m)]^2}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_a,\alpha_b,\alpha_c,\alpha_d}\;\sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{m-k}\;\sum_{\omega_{\nu_1},\omega_{\nu_2},\omega_{\nu_3},\omega_{\nu_4}}\Lambda^{\nu_1}(N,m,k)\,C^{\nu_1,\omega_{\nu_1}}_{\alpha_1\overline{\alpha_b}}C^{\nu_1,\omega_{\nu_1}}_{\alpha_a\overline{\alpha_2}}\\ &\quad\times\Lambda^{\nu_2}(N,m,k)\,C^{\nu_2,\omega_{\nu_2}}_{\alpha_2\overline{\alpha_a}}\;C^{\nu_2,\omega_{\nu_2}}_{\alpha_d\overline{\alpha_3}}\,\Lambda^{\nu_3}(N,m,k)\,C^{\nu_3,\omega_{\nu_3}}_{\alpha_3\overline{\alpha_d}}C^{\nu_3,\omega_{\nu_3}}_{\alpha_c\overline{\alpha_4}}\;\Lambda^{\nu_4}(N,m,k)\,C^{\nu_4,\omega_{\nu_4}}_{\alpha_4\overline{\alpha_c}}\;C^{\nu_4,\omega_{\nu_4}}_{\alpha_b\overline{\alpha_1}}.\end{aligned} \tag{B13}$$

Now, simplifying the CG coefficients will give

$$\overline{\langle ABCD\rangle^m\langle ADCB\rangle^m} = \frac{1}{[d(f_m)]^2}\sum_{\nu=0}^{m-k}[\Lambda^\nu(N,m,k)]^4\,d(\nu). \tag{B14}$$

APPENDIX C

Formulas derived in Sec. IV contain finite N corrections and they can be used to derive asymptotic limit formulas. These will provide a test, as often asymptotic formulas follow from a quite different formulation as given for example, in [6,7,11]. To derive asymptotic formulas we will use the limit $N\to\infty$, $m\to\infty$, $m/N\to 0$ with k finite. Then we have the following approximations:

$$\begin{gathered}\binom{N-p}{r}\overset{p/N\to 0}{\longrightarrow}\frac{N^r}{r!},\quad d(\nu)\overset{\nu/N\to 0}{\longrightarrow}\frac{N^{2\nu}}{(\nu!)^2},\\ \Lambda^0(N,m,k)\to\binom{m}{k}\binom{N}{k},\quad \Lambda^k(N,m,k)\to\binom{m-k}{k}\binom{N}{k},\\ \Lambda^k(N,m,m-k)\to\binom{N}{m-k},\quad \Lambda^0(N,m,m-k)\to\binom{m}{k}\binom{N}{m-k}.\end{gathered} \tag{C1}$$

Using these, first we have

$$\Sigma_{2,0} = \binom{m}{k}\binom{N}{k}. \tag{C2}$$

Using Eqs. (C1) and (C2) and the formulas given in Sec. IV, the following asymptotic limit formulas are obtained for $\hat{\Sigma}_{PQ}$ with $(P,Q) = (1,1)$, (3,1), (2,2), (5,1), (4,2), (3,3), (7,1), (6,2), (5,3), and (4,4). These are

$$\begin{aligned}\hat{\Sigma}_{1,1} &= \frac{\binom{m}{k}}{\binom{N}{k}^2},\\ \hat{\Sigma}_{3,1} &= 3\frac{\binom{m}{k}}{\binom{N}{k}^2} = 3\,\hat{\Sigma}_{1,1},\\ \hat{\Sigma}_{2,2} &= \frac{2}{\binom{N}{k}^2}.\end{aligned}$$

$$
\begin{aligned}
\hat{\Sigma}_{5,1} &= 5\frac{\binom{m}{k}}{\binom{N}{k}^2}\left[2+\frac{\binom{m-k}{k}}{\binom{m}{k}}\right] = (10+5q)\hat{\Sigma}_{1,1},\\
\hat{\Sigma}_{4,2} &= \frac{4}{\binom{N}{k}^2}\left[2+\frac{\binom{m-k}{k}}{\binom{m}{k}}\right] = (4+2q)\hat{\Sigma}_{2,2},\\
\hat{\Sigma}_{3,3} &= 9\frac{\binom{m}{k}}{\binom{N}{k}^2}+\frac{3}{\binom{m}{k}\binom{N}{k}^2}+\frac{3}{\binom{N}{k}^2}|U|^2 = 9\hat{\Sigma}_{1,1}+\frac{3}{\binom{m}{k}\binom{N}{k}^2}+O\left(\frac{1}{\binom{N}{k}^4}\right),\\
\hat{\Sigma}_{7,1} &= 7\hat{\Sigma}_{1,1}\left\{5+6q+3q^2+\frac{\binom{m-2k}{k}}{\binom{m}{k}}(q)\right\} \simeq 7\hat{\Sigma}_{1,1}\,[5+6q+3q^2+q^3],\\
\hat{\Sigma}_{6,2} &= \frac{1}{\binom{N}{k}^2}[30+36q+6q^2+12U_1+6U_2]\\
&= \hat{\Sigma}_{2,2}[15+18q+3q^2+9q^3]+O\left(\frac{1}{\binom{N}{k}^4}\right),\\
\hat{\Sigma}_{5,3} &= 15(2+q)\hat{\Sigma}_{1,1}+5(1+q)[\hat{\Sigma}_{3,3}-9\hat{\Sigma}_{1,1}]+X_{53}\\
&= 15\left\{(2+q)\hat{\Sigma}_{1,1}+\frac{(1+q)}{\binom{m}{k}\binom{N}{k}^2}\right\}+O\left(\frac{1}{\binom{N}{k}^4}\right),\\
\hat{\Sigma}_{4,4} &= 4(q+2)^2\,\hat{\Sigma}_{2,2}+\frac{4}{\binom{m}{k}^2\binom{N}{k}^2}+X_{44}\\
&= 4(q+2)^2\,\hat{\Sigma}_{2,2}+\frac{4}{\binom{m}{k}^2\binom{N}{k}^2}+O\left(\frac{1}{\binom{N}{k}^4}\right);\\
q &= \frac{\binom{m-k}{k}}{\binom{m}{k}}.
\end{aligned}
\tag{C3}
$$

In the above equations, the following approximations (i)–(iv) are adopted. (i) $|U|^2$ in $\hat{\Sigma}_{3,3}$ is the U coefficient appearing in Eq. (68) and it is expected to givea negligible contribution to $\hat{\Sigma}_{3,3}$. More importantly, the GUE formula (i.e., for $m=k$) for $\hat{\Sigma}_{3,3}$ that can be derived easily shows that $|U|^2 \sim \binom{N}{k}^{-2}$ and this gives the final formula in Eq. (C3). (ii) In $\hat{\Sigma}_{7,1}$, for the last term we used the approximation established in [30]. (iii) Going further, U_1 and U_2 in $\hat{\Sigma}_{6,2}$ are the terms with U coefficients in $A2$ and $A3$ in Eq. (79). The GUE formulas and EGUE(k) formulas, assuming $\binom{m-k}{k}/\binom{m}{k}=1$ as given in [7], indicate the plausible result $6U_1+3U_2=9q^3+O(\frac{1}{\binom{N}{k}^2})$. (iv) From the GUE formulas it is plausible that X_{53} and X_{44} introduced in Appendix B will be of the order of $1/\binom{N}{k}^4$ and this is used in Eq. (C3) for $\hat{\Sigma}_{5,3}$ and $\hat{\Sigma}_{4,4}$. Finally, let us add that the diagrammatic method developed in [11] may hopefully give, in the near future, exact asymptotic limit formulas for $\hat{\Sigma}_{6,2}$, $\hat{\Sigma}_{5,3}$, and $\hat{\Sigma}_{4,4}$ and for the last term in $\hat{\Sigma}_{3,3}$.

[1] C. E. Porter, *Statistical Theories of Spectra: Fluctuations* (Academic Press, New York, 1965).

[2] M. L. Mehta, *Random Matrices*, 3rd edition (Elsevier B. V., The Netherlands, 2004).

[3] G. Akemann, J. Baik, and P. Di Francesco (eds.), *The Oxford Handbook of Random Matrix Theory* (Oxford University Press, Oxford, 2011).

[4] J. B. French and S. S. M. Wong, Some random-matrix level and spacing distributions for fixed-particle-rank interactions, Phys. Lett. B **35**, 5 (1971).

[5] O. Bohigas and J. Flores, Two-body random Hamiltonian and level density, Phys. Lett. B **34**, 261 (1971).

[6] K. K. Mon and J. B. French, Statistical properties of many-particle spectra, Ann. Phys. (NY) **95**, 90 (1975).

[7] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Random matrix physics: Spectrum and strength fluctuations, Rev. Mod. Phys. **53**, 385 (1981).

[8] F. J. Dyson, Statistical theory of energy levels of complex systems III, J. Math. Phys. **3**, 166 (1962).

[9] L. Benet, T. Rupp, and H. A. Weidenmüller, Spectral properties of the k-body embedded Gaussian ensembles of random matrices, Ann. Phys. (NY) **292**, 67 (2001).

[10] V. K. B. Kota, SU(N) Wigner-Racah algebra for the matrix of second moments of embedded Gaussian unitary

ensemble of random matrices, J. Math. Phys. **46**, 033514 (2005).

[11] R. A. Small and S. Müller, Particle diagrams and statistics of many-body random potentials, Ann. Phys. (NY) **356**, 269 (2015).

[12] F. J. Dyson and M. L. Mehta, Statistical theory of energy levels of complex systems IV, J. Math. Phys. **4**, 701 (1963).

[13] J. B. French, Analysis of distant-neighbour spacing distributions for k-body interaction ensembles, Rev. Mex. de Fis. **22**, 221 (1973).

[14] J. J. M. Verbaarschot and M. R. Zirnbauer, Replica variables, loop expansion, and spectral rigidity of random-matrix ensembles, Ann. Phys. (NY) **158**, 78 (1984).

[15] L. Benet and H. A. Weidenmüller, Review of the k-body embedded ensembles of Gaussian random matrices, J. Phys. A: Math. Gen. **36**, 3569 (2003).

[16] M. Srednicki, Spectral statistics of the k-body random interaction model, Phys. Rev. E **66**, 046138 (2002).

[17] T. Papenbrock and H. A. Weidenmüller, Random matrices and chaos in nuclear spectra, Rev. Mod. Phys. **79**, 997 (2007).

[18] V. K. B. Kota, *Embedded Random Matrix Ensembles in Quantum Physics* (Springer, Heidelberg, 2014).

[19] A. M. García-García and J. J. M. Verbaarschot, Analytical spectral density of the Sachdev-Ye-Kitaev model at finite N, Phys. Rev. D **96**, 066012 (2017).

[20] A. M. García-García, Y. Jia, and J. J. M. Verbaarschot, Universality and Thouless energy in the supersymmetric Sachdev-Ye-Kitaev Model, Phys. Rev. D **97**, 106003 (2018).

[21] A. M. García-García, T. Nosaka, D. Rosa, and J. J. M. Verbaarschot, Quantum chaos transition in a two-site Sachdev-Ye-Kitaev model dual to an eternal traversable wormhole, Phys. Rev. D **100**, 026002 (2019).

[22] Y. Jia and J. J. M. Verbaarschot, Spectral fluctuations in the Sachdev-Ye-Kitaev mode, J. High Energy Phys. 07 (2020) 193.

[23] A. M. García-García, Y. Jia, D. Rosa, and J. J. M. Verbaarschot, Sparse Sachdev-Ye-Kitaev model, quantum chaos, and gravity duals, Phys. Rev. D **103**, 106002 (2021).

[24] L. Sá and A. M. García-García, Q-Laguerre spectral density and quantum chaos in the Wishart-Sachdev-Ye-Kitaev model, Phys. Rev. D **105**, 026005 (2022).

[25] M. Bożejko, K. Burkhard, and R. Speicher, q-Gaussian processes: Non commutative and classical aspects, Commun. Math. Phys. **185**, 129 (1997).

[26] W. Bryc, Stationary random fields with linear regressions, Ann. Probab. **29**, 504 (2001).

[27] P. J. Szablowski, Multidimensional q-Normal and related distributions - Markov case, Electron. J. Probab. **15**, 1296 (2010).

[28] P. J. Szablowski, Moments of q-Normal and conditional q-Normal distribution, Stat. Probab. Lett. **106**, 65 (2015).

[29] M. E. H. Ismail, D. Stanton, and G. Viennot, The combinatorics of q-Hermite polynomials and the Askey-Wilson integral, Eur. J. Comb. **8**, 379 (1987).

[30] M. Vyas and V. K. B. Kota, Quenched many-body quantum dynamics with k-body interactions using q-Hermite Polynomials, J. Stat. Mech.: Theory Exp. (2019) 103103.

[31] M. Vyas and V. K. B. Kota, Bivariate q-normal distribution for transition matrix elements in quantum many-body systems, J. Stat. Mech.: Theory Exp. (2020) 093101.

[32] L. Muñoz, E. Faleiro, R. A. Molina, A. Relaño, and J. Retamosa, Spectral statistics in non-interacting many-particle systems, Phys. Rev. E **73**, 036202 (2006).

[33] R. Prakash and A. Pandey, Saturation of number variance in embedded random-matrix ensembles, Phys. Rev. E **93**, 052225 (2016).

[34] J. B. French, P. A. Mello, and A. Pandey, Statistical properties of many-particle spectra II. Two-point correlations and fluctuations, Ann. Phys. (NY) **113**, 277 (1978).

[35] R. Small, On the unification of random matrix theories, Ph.D. thesis, University of Bristol, Bristol, 2015; arXiv:1503.09121.

[36] G. J. H. Laberge and R. U. Haq, "Universality" of Gaussian orthogonal ensemble fluctuations: The two-body random ensemble and shell model spectra, Can. J. Phys. **68**, 301 (1990).

[37] R. J. Leclair, R. U. Haq, V. K. B. Kota, and N. D. Chavda, Power spectrum analysis of the average-fluctuation density separation in interacting particle systems, Phys. Lett. A **372**, 4373 (2008).

[38] K. Patel, M. S. Desai, V. Potbhare, and V. K. B. Kota, Average-fluctuations separation in energy levels in dense interacting boson systems, Phys. Lett. A **275**, 329 (2000).

[39] N. D. Chavda, Average-fluctuation separation in energy levels in many-particle quantum systems with k-body interactions using q-Hermite polynomials, Pramana-J. Phys. **96**, 223 (2022).

[40] P. H. Butler, Coupling coefficients and tensor operators for chains of groups, Philos. Trans. R. Soc. London, Ser. A **277**, 545 (1975).

[41] P. H. Butler, *Point Group Symmetry Applications: Methods and Tables* (Plenum, New York, 1981).

[42] B. G. Wybourne, *Symmetry Principles and Atomic Spectroscopy* (Wiley, New York, 1970).

[43] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, 2nd edn. (AMS Chelsea Publishing, AMS, Providence, 2006).

[44] V. K. B. Kota and M. Vyas, Random matrix theory for transition strength densities in finite quantum systems: Results from embedded unitary ensembles, Ann. Phys. (NY) **359**, 252 (2015).

[45] V. K. B. Kota, $SU(3)$ *Symmetry in Atomic Nuclei* (Springer Nature, Singapore, 2020).

[46] L. Benet, T. Rupp, and H. A. Weidenmüller, Nonuniversal Behavior of the k-Body Embedded Gaussian Unitary Ensemble of Random Matrices, Phys. Rev. Lett. **87**, 010601 (2001).

[47] V. K. B. Kota, $U(2\Omega) \supset U(\Omega) \otimes SU(2)$ Wigner-Racah algebra for embedded Gaussian unitary ensemble of random matrices with spin, J. Math. Phys. **48**, 053304 (2007).

[48] M. Vyas and V. K. B. Kota, Spectral properties of embedded Gaussian unitary ensemble of random matrices with Wigner's $SU(4)$ symmetry, Ann. Phys. (NY) **325**, 2451 (2010).

[49] J. D. Louck, Recent progress toward a theory of tensor operators in the unitary groups, Am. J. Phys. **38**, 3 (1970).

[50] L. C. Biedenharn, J. D. Louck, E. Chacon, and M. Ciftan, On the structure of the canonical tensor operators in the unitary groups. I. An extension of the pattern calculus rules and the canonical splitting in U(3), J. Math. Phys. **13**, 1957 (1972).

[51] J. D. Louck and L. C. Biedenharn, On the structure of the canonical tensor operators in the unitary groups. III. Further developments of the boson polynomials and their implications, J. Math. Phys. **14**, 1336 (1973).

[52] J. B. French, Special topics in spectral distributions, in *Moment Methods in Many-Fermion Systems*, edited by B. J. Dalton, S. M. Grimes, J. P. Vary, and S. A. Williams (Plenum, New York, 1980), pp. 91–108.

[53] V. K. B. Kota and N. D. Chavda, Embedded random matrix ensembles from nuclear structure and their recent applications, Int. J. Mod. Phys. E **27**, 1830001 (2018).

[54] V. K. B. Kota and N. D. Chavda, Random k-body ensembles for chaos and thermalization in isolated systems, Entropy **20**, 541 (2018).

[55] M. Vyas and T. H. Seligman, Random matrix ensembles for many-body quantum systems, in *Latin-American School of Physics Marcos Moshinsky ELAF2017: Quantum Correlations*, AIP Conf. Proc. 1950 (AIP, Melville, NY, 2018), p. 030009.

[56] A. Ortega, M. Vyas, and L. Benet, Quantum efficiencies in finite disordered networks connected by many-body interactions, Ann. Phys. (Berl.) **527**, 748 (2015).

[57] A. Ortega, T. Stegmann, and L. Benet, Efficient quantum transport in disordered interacting many-body networks, Phys. Rev. E **94**, 042102 (2016).

[58] A. Ortega, T. Stegmann, and L. Benet, Robustness of optimal transport in disordered interacting many-body networks, Phys. Rev. E **98**, 012141 (2018).

[59] E. Carro, L. Benet, and I. P. Castillo, A smooth transition towards a Tracy-Widom distribution for the largest eigenvalue of interacting k-body fermionic Embedded Gaussian Ensembles, J. Stat. Mech. (2023) 043201.

[60] F. Borgonovi and F. M. Izrailev, Emergence of correlations in the process of thermalization of interacting bosons, Phys. Rev. E **99**, 012115 (2019).

[61] F. Borgonovi, F. M. Izrailev, and L. F. Santos, Timescales in the quench dynamics of many-body quantum systems: Participation ratio versus out-of-time ordered correlator, Phys. Rev. E **99**, 052143 (2019).

[62] V. K. B. Kota and Manan Vyas, Statistical Nuclear Spectroscopy with q-normal and bivariate q-normal distributions and q-Hermite polynomials, Ann. Phys. (NY) **446**, 169131 (2022).

[63] J. Flores, M. Horoi, M. Müller, and T. H. Seligman, Spectral statistics of the two-body random ensemble revisited, Phys. Rev. E **63**, 026204 (2001).