Leftward, rightward, and complete exit-time distributions of jump processes

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First-passage properties of continuous stochastic processes confined in a one-dimensional interval are well described. However, for jump processes (discrete random walks), the characterization of the corresponding observables remains elusive, despite their relevance in various contexts. Here we derive exact asymptotic expressions for the leftward, rightward, and complete exit-time distributions from the interval $[0, x]$ for symmetric jump processes starting from $x_0 = 0$, in the large x and large time limit. We show that both the leftward probability $F_{\underline{0},x}(n)$ to exit through 0 at step n and rightward probability $F_{0,\underline{x}}(n)$ to exit through x at step n exhibit a universal behavior dictated by the large-distance decay of the jump distribution parametrized by the Levy exponent μ. In particular, we exhaustively describe the $n \ll (x/a_\mu)^\mu$ and $n \gg (x/a_\mu)^\mu$ limits and obtain explicit results in both regimes. Our results finally provide exact asymptotics for exit-time distributions of jump processes in regimes where continuous limits do not apply.

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I. INTRODUCTION

In many physical systems, exit-time distributions, which quantify the time taken by a random process to exit a given confining region, play a key role in understanding the relevant time scales driving the system [1–4]. Although the geometrical constraints can be defined in any dimension, the escape of random processes from the one-dimensional interval $[0, x]$ appears as a highly recurrent and instructive physical model in a variety of fields, from chemical reaction kinetics [2,4] to foraging animals [5] to financial asset modeling [6,7]. A classic example of application is the Wright-Fisher evolutionary model [8], describing the dynamics of a population of two alleles A and B. The first time n at which one of the alleles completely disappears from the population is schematically described by the first exit-time distribution of a random process in the interval [0,1], with initial position x_0 describing the initial fraction of, say, allele A. In fact, in this representative example of a stochastic process with two alternative outcomes, not only the exit time but also the exit side matters. The fixation or extinction time distributions of the allele A are indeed given, respectively, by the rightward or leftward exit-time distributions of the corresponding process.

While these observables are well documented for one-dimensional continuous stochastic processes [1,4], their discrete time counterparts, namely, for jump processes, remain elusive; this is in essence because the integral equations satisfied by exit-time distributions are notoriously difficult to analyze in bounded domains [9]. Jump processes are, however, relevant to a variety of situations [10], and have been the subject of multiple recent works in the context of self-propelled particles, such as active colloids, or larger-scale animals [11–15]. In addition, experimental data of typical tracking experiments (be it of single molecules, animals, or asset prices) are discrete in time by nature, because of a finite sampling rate. Thus, no matter the microscopic nature of the underlying process, experimental trajectories of a tracer particle invariably take the form of discrete time series of successive positions of the tracer. They therefore constitute intrinsic realizations of jump processes, which are for this reason of clear interest in this context. In what follows, we focus on leftward, rightward, and complete exit-time distributions of general jump processes. These observables are of particular interest at a theoretical level, and also relevant to analyze experimental data involving transport processes with competitive outcomes.

The one-dimensional jump processes considered hereafter are defined as follows: starting from $0 \leqslant x_0 \leqslant x$, the random walker successively performs jumps drawn from a symmetric continuous distribution $p(\ell)$, with Fourier transform $\tilde{p}(k) = \int_{-\infty}^{\infty} e^{ik\ell} p(\ell) d\ell$, until it strictly exits the interval $[0, x]$ by crossing either 0 or x. The corresponding first exit-time probability (FETP) at step n is denoted by $F_{\underline{0},\underline{x}}(n|x_0)$. Importantly, because the random walk is defined in discrete time, the FETP is nonvanishing for $x_0 = 0$. For example, the one-step FETP, given by $F_{\underline{0},\underline{x}}(n = 1|0) = 1 - \int_0^x p(y)\mathrm{d}y$, is strictly positive. As a result, the FETP cannot be determined by simply taking the continuous limit of the process, which would invariably lead to a vanishing value for a process starting at $x_0 = 0$. In addition, the determination of first exit observables, and in particular the FETP for $x_0 = 0$, is key in understanding experimental data [16]. As an example, it was recently shown in the context of photon and neutron scattering [16–20] that the transmission probability through a slab of width x was given by the splitting probability $\pi_{0,\underline{x}}$ to reach x before 0 starting from 0. The latter was determined asymptotically in Ref. [21] as

$$\pi_{0,\underline{x}} \underset{x\to\infty}{\sim} \frac{2^{\mu-1}\Gamma\left(\frac{1+\mu}{2}\right)}{\sqrt{\pi}} \left[\frac{a_\mu}{x}\right]^{\frac{\mu}{2}}, \tag{1}$$

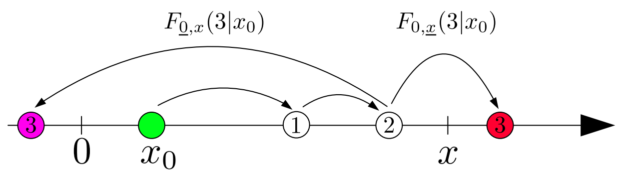

FIG. 1. Rightward and leftward FETPs. In this specific realization, after taking two steps inside the interval, the jump process escapes either through x or through 0 on its third step, with respective probabilities $F_{0,\underline{x}}(3|x_0)$ (red rightward dot) or $F_{\underline{0},x}(3|x_0)$ (purple leftward dot).

where μ and a_μ characterize the small k behavior of $\tilde{p}(k)$:

$$\tilde{p}(k) \underset{k\to 0}{=} 1 - (a_\mu |k|)^\mu + o(k^\mu). \tag{2}$$

Of note, the splitting probability does not contain any information on the exit time. To go further and quantify the time at which exit events occur, one needs the leftward, rightward, and complete FETPs. The leftward FETP $F_{\underline{0},x}(n)$ is defined as the probability for the walker starting from 0 to exit through 0 at time n exactly without having crossed x before, and $F_{0,\underline{x}}(n)$ is its rightward counterpart (see Fig. 1). The complete FETP $F_{\underline{0},\underline{x}}(n)$ is then given by

$$F_{\underline{0},\underline{x}}(n) = F_{\underline{0},x}(n) + F_{0,\underline{x}}(n). \tag{3}$$

A natural strategy to compute these FETPs is to consider the continuous limit of the problem, defined here as the limit $a_\mu \ll x_0$, which implies that typical exit times satisfy $n \gg 1$. Two limit behaviors then arise depending on the value of μ [22,23]: for $0 < \mu < 2$ the process converges to an α-stable Levy process of parameters μ and a_μ, while for $\mu = 2$ the limit distribution is that of a Brownian motion with diffusion coefficient $D = a_2^2$. In this continuous limit, the complete first exit-time distribution has been the focus of several works and is given by [24,25]

$$\begin{aligned} F^{(c)}_{\underline{0},\underline{x}}(n|x_0) &= \sum_{k=1}^{\infty} C_k(x_0)\lambda_k 2^\mu \left[\frac{a_\mu}{x}\right]^\mu e^{-\lambda_k 2^\mu [\frac{a_\mu}{x}]^\mu n} \\ C_k(x_0) &= \left[\int_0^2 \psi_k(u)\mathrm{d}u\right]\psi_k\left(\frac{2}{x}x_0\right), \end{aligned} \tag{4}$$

where λ_k and ψ_k are, respectively, the eigenvalues and eigenfunctions of the fractional diffusion equation of order μ on the interval [0,2] with absorbing boundary conditions [26]. Of note, only approximates of ψ_k and λ_k have been obtained so far for $0 < \mu < 2$ [25]. For illustration we provide $\lambda_1 \simeq [\frac{\pi}{2} - \frac{(2-\mu)\pi}{8}]^\mu$; see also Appendix A.

Although the continuous limit Eq. (4) describes the regime $a_\mu \ll x_0$ for $F_{\underline{0},\underline{x}}(n|x_0)$, it fails to capture the regime $x_0 \lesssim a_\mu$, which depends on the microscopic details of the process. In particular, taking $x_0 \to 0$ in Eq. (4) would yield $F_{\underline{0},\underline{x}}(n|0) = 0$, which is clearly incorrect for a discrete time jump process. The quantitative understanding of the regime $x_0 \lesssim a_\mu$ for leftward, rightward, and complete FETPs for general jump processes, which is key to analyzing experimentally relevant situations and, in particular, the transmission properties stated above, thus calls for a new approach, which is the objective of this paper. For the sake of simplicity, we focus here on the $x_0 = 0$ case (see Appendix D for the full regime $0 \leqslant x_0 \ll a_\mu$).

II. SUMMARY OF RESULTS

In this article, we derive exact asymptotics for both $F_{0,\underline{x}}(n)$ and $F_{\underline{0},x}(n)$ in the $n \to \infty$ and $x \to \infty$ limit. More precisely, we show that the rightward FETP displays the following universal asymptotic behavior:

$$F_{0,\underline{x}}(n) \underset{\substack{n\to\infty \\ x\to\infty \\ \tau\ \text{fixed}}}{\sim} \pi_{0,\underline{x}}\, h_\mu(\tau)\, n^{-1} \tag{5}$$

where $\pi_{0,\underline{x}}$ is the splitting probability defined above, h_μ is a universal μ-dependent function, and $\tau = [\frac{a_\mu}{x}]^\mu n$ is the rescaled number of steps, with respect to the typical number of steps needed to escape the interval, $n_{\text{typ}} = (x/a_\mu)^\mu$. For $\mu = 2$, we find

$$h_2(\tau) = 2\tau\pi^2 \sum_{k=1}^{\infty} k^2(-1)^{k+1} e^{-k^2\pi^2\tau}, \tag{6}$$

while for $0 < \mu < 2$ we obtain the following asymptotic behaviors:

$$h_\mu(\tau) \underset{\tau\ll 1}{\sim} \Gamma(\mu/2)\sin(\pi\mu/2)\pi^{-\frac{3}{2}}\sqrt{\tau} \tag{7a}$$

$$h_\mu(\tau) \underset{\tau\gg 1}{\sim} \frac{C\mu\Gamma^2(\frac{\mu}{2})}{\Gamma(\mu)} 2^{\frac{\mu}{2}-2}[\lambda_1 2^\mu \tau] e^{-\lambda_1 2^\mu \tau} \tag{7b}$$

where λ_1 is defined above and C is a constant which reads

$$C = \lim_{x_0\to 0} \frac{\psi_1(x_0)}{x_0^{\frac{\mu}{2}}}\left[\int_0^2 \psi_1(u)du\right]. \tag{8}$$

Next, we show that the leftward FETP displays an analogous universal asymptotic behavior:

$$F_{\underline{0},x}(n) \underset{\substack{n\to\infty \\ x\to\infty \\ \tau\ \text{fixed}}}{\sim} F_0(n) g_\mu(\tau), \tag{9}$$

where $F_0(n) \sim (4\pi n^3)^{-\frac{1}{2}}$ is the large n asymptotic first passage time distribution through 0 in the semi-infinite system (starting from 0), obtained from the celebrated Sparre-Andersen theorem, and g_μ is a universal μ-dependant function. For $\mu = 2$, the function g_2 is determined explicitly and reads

$$g_2(\tau) = 4\pi^{\frac{5}{2}}\tau^{\frac{3}{2}} \sum_{k=1}^{\infty} e^{-k^2\pi^2\tau} k^2, \tag{10}$$

while for $0 < \mu < 2$ we obtain the following asymptotic behaviors:

$$g_\mu(\tau) \underset{\tau\ll 1}{\sim} 1 \tag{11a}$$

$$g_\mu(\tau) \underset{\tau\gg 1}{\sim} C\,\Gamma\left(1+\frac{\mu}{2}\right)\sqrt{\pi}\lambda_1^{-\frac{1}{2}}[\lambda_1 2^\mu \tau]^{\frac{3}{2}} e^{-\lambda_1 2^\mu \tau}. \tag{11b}$$

Finally, Eqs. (5)–(11) provide a comprehensive picture of the asymptotic behavior of the rightward and leftward FETPs, which in turn give access to the complete FETP. The obtained universal asymptotic forms capture the dependence on both the system size x and number of steps n.

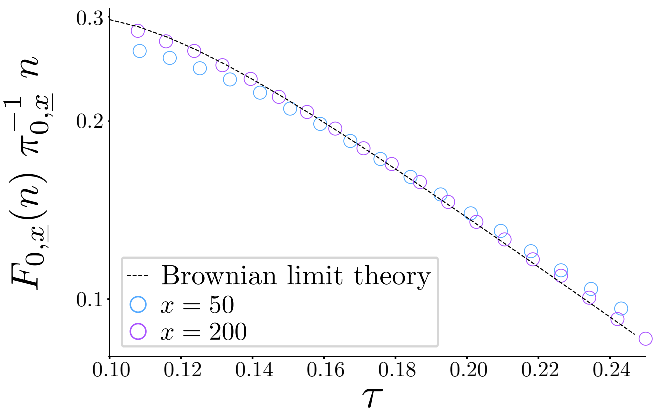

FIG. 2. Rightward FETP for a jump process with $p(\ell) \propto e^{-|l|}$ (yielding $\mu = 2$). Upon rescaling according to Eq. (5), $F_{0,\underline{x}}(n)$ converges to the scaling function $h_2(\tau)$, defined by Eq. (6).

III. RIGHTWARD FETP

We first write the rightward FETP as $F_{0,\underline{x}}(n) = \pi_{0,\underline{x}}\, h(x,n)$, where $h(x,n)$ is the conditional probability to escape through x at step n knowing that the walker reaches x before 0, and $\pi_{0,\underline{x}}$ is the splitting probability defined above. In the large n and x limit, $h(x,n)$ can be written

$$h(x,n) = \frac{F_{0,\underline{x}}(n)}{\pi_{0,\underline{x}}} \underset{\substack{n\to\infty \\ x\to\infty \\ \tau \text{ fixed}}}{\sim} \lim_{x_0\to 0}\left[\frac{F^{(c)}_{0,\underline{x}}(n|x_0)}{\pi^{(c)}_{0,\underline{x}}(x_0)}\right], \tag{12}$$

where $F^{(c)}_{0,\underline{x}}(n|x_0)$ is the rightward FETP of the continuous process and $\pi^{(c)}_{0,\underline{x}}(x_0)$ the corresponding continuous splitting probability [27,28].

Indeed, in the large n and x limit, the typical position X_n of the random walker satisfies $X_n \gg a_\mu$ and the continuous limit can be taken. In turn, since $F^{(c)}_{0,\underline{x}}(n|x_0) \propto \pi^{(c)}_{0,\underline{x}}(x_0)$ for $x_0 \to 0$ (see Appendix B), $h(x,n)$ is a well-defined x_0-independent function. Making use of scale invariance, we then define the μ-dependent universal scaling function $h_\mu(\tau)$, with τ given above, as

$$\lim_{x_0\to 0}\left[\frac{F^{(c)}_{0,\underline{x}}(n|x_0)}{\pi^{(c)}_{0,\underline{x}}(x_0)}\right] \equiv \frac{h_\mu(\tau)}{n}. \tag{13}$$

This yields the result (5). Importantly, the discrete nature of the jump process only enters through $\pi_{0,\underline{x}}$, which yields a nonvanishing rightward FETP as expected. For $\mu = 2$, $h_2(\tau)$ can be derived explicitly from Eq. (12) and leads to (6). This exact asymptotic behavior is confirmed by numerical simulations (see Fig. 2).

For $0 < \mu < 2$, the rightward FETP $F^{(c)}_{0,\underline{x}}(n|x_0)$ of continuous Levy processes is not known, so that h_μ cannot be derived explicitly; its large and small τ asymptotics can, however, be obtained. For $\tau \gg 1$, i.e., $n \gg (x/a_\mu)^\mu$, we remark that the dynamics become independent of the starting point so that $F^{(c)}_{0,\underline{x}}(n|x_0) \sim 2^{-1}F^{(c)}_{0,\underline{x}}(n|x_0)$. Using Eq. (4), this yields the result (7b). Of note, the leading τ behavior of (6) is compatible with Eq. (7b) for $\mu = 2$.

For $\tau \ll 1$ (or equivalently $x/a_\mu \gg n^{1/\mu}$), the leading behavior of h_μ cannot be extracted from (4) because there is *a priori* no simple link between h_μ and $F^{(c)}_{0,\underline{x}}(n|x_0)$ in this limit. However, it can conveniently be obtained by making use of the following exact decomposition of $F_{0,\underline{x}}(n)$, which states that during the first $n-1$ steps the walker remains in the interval $[0,x]$, while the nth step takes him beyond x:

$$F_{0,\underline{x}}(n) = \int_0^x G_{0,x}(u,n-1)\left[\int_{x-u}^{\infty} p(l)dl\right]du. \tag{14}$$

Here $G_{0,x}(u,k)$ is defined as the propagator of the jump process in the bounded interval $[0,x]$ after k steps. Next, we note that in the large x limit with n fixed, $G_{0,x}(u,n-1) \sim G_0(u,n-1)$ with G_0 the semi-infinite propagator. This, together with (14), then yields the asymptotic relation

$$F_{0,\underline{x}}(n) \underset{x\to\infty}{\sim} \int_0^x G_0(u,n-1)U(x-u)du, \tag{15}$$

where $U(x) = \int_x^\infty p(l)\mathrm{d}l$ is the cumulative of the jump distribution. Importantly, this shows that the two-target quantity $F_{0,\underline{x}}(n)$ can be expressed asymptotically in terms of the well-characterized one-target quantity $G_0(x,n)$ only. We finally introduce the Laplace transform (in space) of a given function $f(x)$ as $\tilde{f}(p) = \int_0^\infty e^{-px}f(x)dx$ and the generating function (in time) of a given function $g(n)$ as $\hat{g}(\xi) = \sum_{n\geqslant 0} g(n)\xi^n$ and obtain

$$\widehat{\widetilde{F}}_{0,\underline{p}}(\xi) \underset{p\to 0}{\sim} \xi\, \widehat{\widetilde{G}}_0(p,\xi)\, \widetilde{U}(p). \tag{16}$$

Both $\widehat{\widetilde{G}}_0(p,\xi)$ and $\widetilde{U}(p)$ can then be readily analyzed in the $p \to 0$ limit to extract the leading large x behavior of $F_{0,\underline{x}}(n)$. In the case $0 < \mu < 1$ (see Appendix C for $1 \leqslant \mu < 2$), one has [29]

$$\widehat{\widetilde{G}}_0(p,\xi) = \frac{1}{\sqrt{1-\xi}} + o(p^\mu)$$

$$\widetilde{U}(p) = c_\mu\,(a_\mu)^\mu\, p^{\mu-1} + o(p^{\mu-1}), \tag{17}$$

where $c_\mu = \sec(\frac{\pi\mu}{2})/2$. To leading order in $p \to 0$, we obtain $\widehat{\widetilde{F}}_{0,\underline{p}}(\xi) \sim \frac{\xi}{\sqrt{1-\xi}} c_\mu\,(a_\mu)^\mu\, p^{\mu-1}$ and, upon Laplace inversion, we derive the following exact asymptotic form:

$$F_{0,\underline{x}}(n) \underset{x\to\infty}{\sim} q(n-1)\frac{\Gamma(\mu)}{\pi}\sin\left(\frac{\pi\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^\mu \tag{18}$$

where $q(n)$ is the (survival) probability that a symmetric jump process starting from $x = 0$ remains positive up to step n, given by the universal Sparre-Andersen result $q(n) = \binom{2n}{n}2^{-2n}$ [30]. In fact, we show in Appendix C that Eq. (18) holds for all μ such that $0 < \mu < 2$. Last, using $q(n) \sim (\pi n)^{-\frac{1}{2}}$ for large n, identification with Eqs. (5) and (1) yields the announced universal small τ behavior (7a), as displayed in Fig. 3 for different $\mu < 2$.

Of note, both asymptotic behaviors described by Eq. (7) are necessary to recover the large x scaling of the splitting probability $\pi_{0,\underline{x}} = \sum_{n=1}^\infty F_{0,\underline{x}}(n)$ (see Appendix E).

IV. LEFTWARD FETP

As for the rightward FETP, our strategy consists in expressing the two-target quantity $F_{\underline{0},x}(n)$ in terms of a well-characterized one-target quantity—here the first passage time

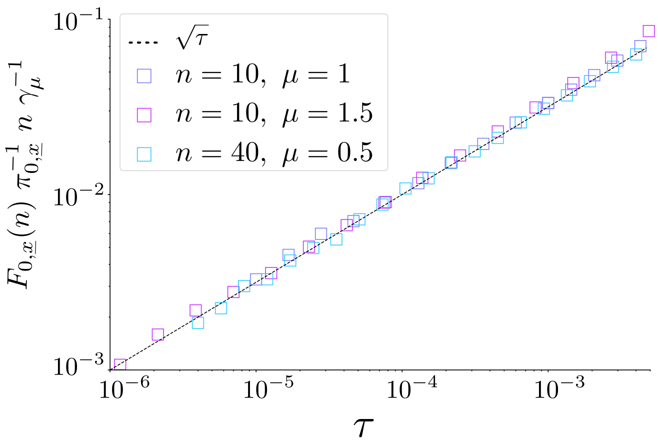

FIG. 3. Small τ behavior of the rightward FETP for various Levy flights with $\mu < 2$. The universal small τ behavior of $F_{0,\underline{x}}(n)$ predicted by Eq. (7a) is displayed, with $\gamma_\mu = \Gamma(\mu/2)\sin(\pi\mu/2)\pi^{-\frac{3}{2}}$.

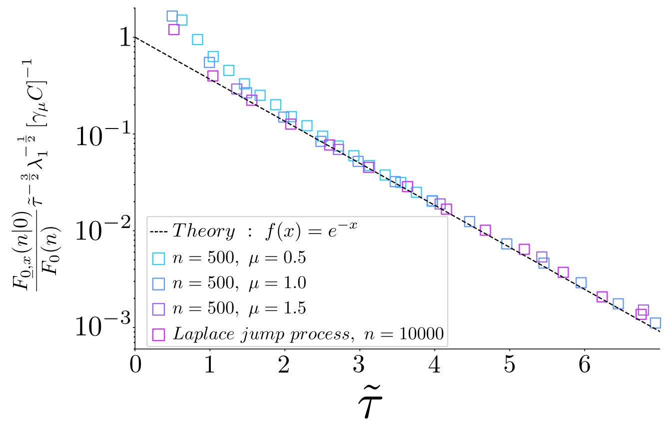

FIG. 4. Leftward FETP for $n \gg (x/a_\mu)^\mu$. Defining $\tilde{\tau} = \lambda_1 2^\mu \tau$ and rescaling $F_{\underline{0},x}(n|0)$ according to (11b), all curves collapse onto a single exponential for various processes with $0 < \mu \leqslant 2$. The Laplace jump process is defined by $p(\ell) \propto e^{-|\ell|}$, corresponding to $\mu = 2$. C is given by Eq. (8), and here $\gamma_\mu = \Gamma(1 + \frac{\mu}{2})\sqrt{\pi}$.

probability through 0 for a jump process starting from 0 in a semi-infinite domain $F_0(n)$. We first recall that for a given jump process, the typical number of steps needed to cover a distance x scales as $n \propto x^\mu$ [22]. We thus argue that, for an interval of typical extension $(x/a_\mu)^\mu \gg n$, $F_{\underline{0},x}(n) \sim F_0(n)$, because trajectories approaching the rightmost target are very unlikely [28,31]. On the other hand, for $n \gg (x/a_\mu)^\mu$, $F_{\underline{0},x}(n)$ vanishes exponentially fast since it is increasingly unlikely for the walker to remain in $[0, x]$. Following the derivation of (12) and (13), we introduce $g(x, n)$ and define its continuous limit $g_\mu(\tau)$ by

$$g(x,n) = \frac{F_{\underline{0},x}(n)}{F_0(n)} \underset{\substack{n\to\infty \\ x\to\infty \\ \tau \text{ fixed}}}{\sim} \lim_{x_0\to 0}\left[\frac{F^{(c)}_{\underline{0},x}(n|x_0)}{F^{(c)}_0(n|x_0)}\right] \equiv g_\mu(\tau), \quad (19)$$

with $F_0^{(c)}$ and $F_{\underline{0},x}^{(c)}$, respectively, the semi-infinite first passage time distribution and leftward FETP of the limit continuous process. It is shown in Appendix B that $F^{(c)}_{\underline{0},x}(n|x_0) \propto F^{(c)}_0(n|x_0)$ for $x_0 \to 0$, which ensures that $g_\mu(\tau)$ is well defined and independent of x_0. Similarly to the rightward FETP, g_2 can be computed explicitly and is given in (10). For $0 < \mu < 2$, only the asymptotic behavior of g_μ for $\tau \ll 1$ and $\tau \gg 1$ can be obtained. For small τ, one has $F_{\underline{0},x}(n) \sim F_0(n)$ (as discussed above), yielding Eq. (11a). Note that this is verified explicitly in the case $\mu = 2$ (see Appendix F). When $\tau \gg 1$, we perform the same analysis as for the rightward FETP. $F_0^{(c)}(n|x_0)$ is known exactly [32]:

$$F_0^{(c)}(n|x_0) \underset{n\to\infty}{\sim} \left[\frac{x_0}{a_\mu}\right]^{\frac{\mu}{2}} \frac{1}{2\sqrt{\pi}\,\Gamma\left(1+\frac{\mu}{2}\right)} \frac{1}{n^{\frac{3}{2}}}, \quad (20)$$

and in the large n limit, $F^{(c)}_{\underline{0},x}(n|x_0) \sim 2^{-1}F^{(c)}_{\underline{0},x}(n|x_0)$. Equation (19) together with Eq. (4) then yields (11b), which is illustrated in Fig. 4 for various $\mu \leqslant 2$.

V. COMPLETE FETP

Finally, the complete FETP can now be obtained from Eq. (3). For $n \ll (x/a_\mu)^\mu$, one finds $F_{\underline{0},\underline{x}}(n) \sim F_0(n)$, which simply reflects the fact that the target at x is never approached by the walker and rightward exit events almost never occur. For $n \gg (x/a_\mu)^\mu$, however, both rightward and leftward FETP contribute and one has $F_{0,\underline{x}}(n) \sim F_{\underline{0},x}(n)$. Indeed, after a large number of steps, the dynamics is independent of the initial condition and exits on both sides are equiprobable. The complete FETP thus reads $F_{\underline{0},\underline{x}}(n) \sim 2F_{\underline{0},x}(n)$.

VI. CONCLUSION

We have derived asymptotic forms for the rightward, leftward, and complete exit-time probabilities from an interval $[0, x]$ for general jump processes starting from the edge of the domain. While such first-passage properties have been well described for continuous stochastic processes, the case of jump processes has so far remained elusive, despite its relevance in various contexts. In fact, continuous limits provide only vanishing expressions for starting positions close to the edge of the domain, and are thus useless to quantify important observables such as transmission or backscattering-type probabilities. These are key to analyzing experimental data, such as phase delay in neutron-scattering experiments. Our approach fills this gap and provides a comprehensive picture of exit-time probabilities, which yields asymptotically explicit universal forms controlled by the large-distance decay of the jump distribution only.

APPENDIX A: EIGENVALUES AND EIGENFUNCTIONS OF THE FRACTIONAL DIFFUSION EQUATION

In this section, we reproduce results from Ref. [25] regarding the eigenvalues and eigenfunctions of the fractional Laplacian. We emphasize that the fractional Laplacian is an important technical tool to describe Levy-like processes, but that the eigenfunctions $\tilde{\phi}_k$ and eigenvalues λ_k of the operator in a bounded interval are not known analytically. Here we present approximates of these quantities; we closely follow the notations of Ref. [25]. Let $D = (-1, 1)$ be the domain of interest; we aim at finding solutions to the following equation:

$$\left(\frac{d^2}{dx^2}\right)^{\frac{\mu}{2}}\phi(x)=\lambda\phi(x)\ \text{ for } x\in(-1,1), \tag{A1}$$

with $\phi(-1)=\phi(1)=0$. Denoting λ_k the kth eigenvalue, sorted in increasing order, we have

$$\lambda_k=\left(\frac{k\pi}{2}-\frac{(2-\mu)\pi}{8}\right)^{\mu}+O\left(\frac{1}{k}\right). \tag{A2}$$

Approximate expressions ψ_k of the corresponding eigenfunctions can be obtained by combining infinite and semi-infinite eigenfunctions of the fractional Laplacian, which are known explicitly. It is found that

$$\psi_k(x)=q(-x)F_{\mu_k}(1+x)-(-1)^k q(x)F_{\mu_k}(1-x), \tag{A3}$$

with

$$\mu_k=\frac{k\pi}{2}-\frac{(2-\mu)\pi}{8} \tag{A4}$$

and F_λ defined in the following way:

$$F_\lambda(x)=\sin\left(\lambda x+\frac{(2-\mu)\pi}{8}\right)-G(\lambda x). \tag{A5}$$

Here G is the Laplace transform of a positive function $\gamma(s)$:

$$\gamma(s)=\frac{\sqrt{2\mu}\sin\left(\frac{\mu\pi}{2}\right)}{2\pi}\frac{s^\mu}{1+s^{2\mu}-2s^\mu\cos\left(\frac{\mu\pi}{2}\right)}\times\exp\left[\frac{1}{\pi}\int_0^\infty\frac{dr}{1+r^2}\ln\left(\frac{1-r^\mu s^\mu}{1-r^2s^2}\right)\right]. \tag{A6}$$

Finally, q is an interpolating function:

$$q(x)=\begin{cases}0 & \text{for } x\in\left(-\infty,-\frac{1}{3}\right)\\ \frac{9}{2}\left(x+\frac{1}{3}\right)^2 & \text{for } x\in\left(-\frac{1}{3},0\right)\\ 1-\frac{9}{2}\left(x-\frac{1}{3}\right)^2 & \text{for } x\in\left(0,\frac{1}{3}\right)\\ 1 & \text{for } x\in\left(\frac{1}{3},\infty\right).\end{cases} \tag{A7}$$

From these combined expressions, one can numerically evaluate the approximates ψ_k; this is used to determine numerically C in Eq. (8) of the main text.

APPENDIX B: SMALL x_0 BEHAVIOR OF THE CONTINUOUS QUANTITIES

In this section, we show that all continuous quantities considered in the main text, namely, $F_0^{(c)}(n|x_0)$, $F_{0,\underline{x}}^{(c)}(n|x_0)$, $F_{\underline{0},x}^{(c)}(n|x_0)$, $\pi_{\underline{0},x}^{(c)}(x_0)$, vanish similarly to $x_0\to 0$. Denoting here ψ_k the eigenfunctions of the fractional Laplacian operator on $[0,x]$ and ϕ_k the eigenfunctions of the fractional Laplacian operator on $[0,+\infty]$, Kwasnicki shows that for $x_0\to 0$:

$$\psi_k(x_0)\propto\phi_k(x_0)\propto\left[\sqrt{\frac{\mu}{2}}\Gamma\left(\frac{\mu}{2}\right)\right]^{-1}[\mu_k x_0]^{\frac{\mu}{2}}. \tag{B1}$$

Since all of the $F_0^{(c)}(n|x_0)$, $F_{0,\underline{x}}^{(c)}(n|x_0)$, $F_{\underline{0},x}^{(c)}(n|x_0)$, $\pi_{\underline{0},x}^{(c)}(x_0)$ can be projected either on the ψ_k or ϕ_k basis, we obtain that they all vanish as $x_0^{\frac{\mu}{2}}$ as $x_0\to 0$.

APPENDIX C: DETAILS ON THE RIGHTWARD EXIT-TIME PROBABILITY FOR THE CASE $1\leqslant\mu<2$

In this section, we focus on the asymptotic behavior of $F_{0,\underline{x}}(n|0)$ in the case $1\leqslant\mu<2$, by performing the same asymptotic expansion as in the main text for the $\mu<1$ case.

1. $\mu=1$ case

Let us list the various small p expansions necessary for the analysis, which can be found in Ref. [33], or easily derived:

$$\begin{aligned}\widehat{\widetilde{G}}_0(p,\xi|0)&=\frac{1}{\sqrt{1-\xi}}\left[1+\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{\mu}}\frac{a_1}{\pi}p\log(p)+O(p)\right]\\ \widetilde{f}(p)&=\frac{1}{2}+\frac{a_1}{\pi}p\log(p)+O(p)\\ \widetilde{F}(p)&=-\frac{a_1}{\pi}\log(p)+O(1),\end{aligned} \tag{C1}$$

To lowest order in p one then obtains

$$\widehat{\widetilde{F}}_{0,\underline{p}}(\xi)=-\frac{\xi}{\sqrt{1-\xi}}\frac{a_1}{\pi}\log(p)+o[\log(p)], \tag{C2}$$

which yields, after inversion,

$$F_{0,\underline{x}}(n|0)\underset{x\to\infty}{\sim}\frac{1}{\pi}q(n-1|0)\left[\frac{a_1}{x}\right], \tag{C3}$$

in agreement with Eq. (18) from the main text.

2. $1<\mu<2$ case

Let us repeat this operation:

$$\begin{aligned}\widehat{\widetilde{G}}_0(p,\xi|0)&=\frac{1}{\sqrt{1-\xi}}\left[1+\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{\mu}}\frac{a_\mu p}{\sin(\pi/\mu)}-\frac{\xi}{1-\xi}c_\mu(a_\mu p)^\mu+o(p^\mu)\right]\\ \widetilde{f}(p)&=\frac{1}{2}-p\langle f\rangle-c_\mu(a_\mu p)^\mu+o(p^\mu)\\ \widetilde{F}(p)&=\langle f\rangle+c_\mu\, a_\mu^\mu\, p^{\mu-1}+o(p^{\mu-1}).\end{aligned} \tag{C4}$$

To lowest order in p one then obtains

$$\widehat{\widetilde{F}}_{0,\underline{p}}(\xi|0)=\frac{\xi}{\sqrt{1-\xi}}\left[\langle f\rangle+c_\mu\, a_\mu^\mu\, p^{\mu-1}\right], \tag{C5}$$

which yields, after inversion,

$$F_{0,\underline{x}}(n|0)\underset{x\to\infty}{\sim}q(n-1|0)\times\left[\langle f\rangle\delta(x)+\frac{1}{\pi}\Gamma(\mu)\sin\left(\frac{\pi\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^\mu\right]. \tag{C6}$$

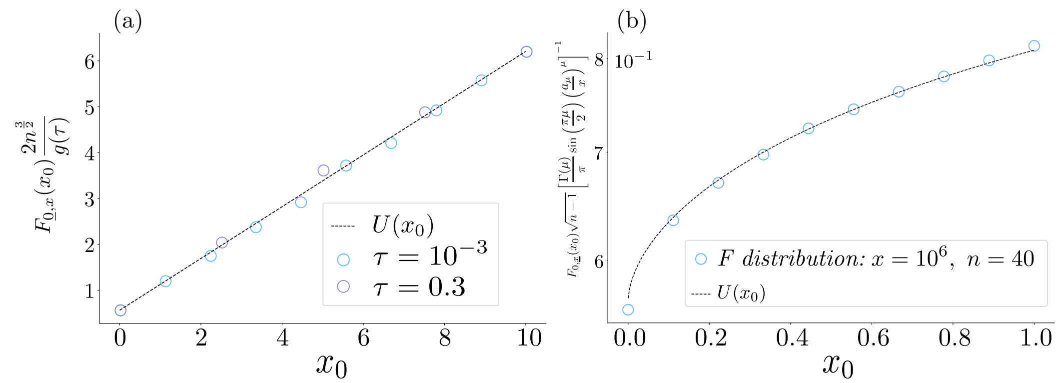

FIG. 5. x_0 dependence of exit-time probabilities. (a) Leftward FETP for a Laplace jump process. After rescaling, the rightward FETP collapses to the $U(x_0) = \frac{1}{\sqrt{\pi}} + V(x_0)$ function, for various τ values. (b) Rescaled rightward FETP for an F-distributed jump process defined by $p(\ell) \propto [\sqrt{|l|}(1+|l|)]^{-1}$. The x_0 dependence is in this case sublinear.

In the large x limit, we finally obtain

$$F_{0,\underline{x}}(n|0) \underset{x\to\infty}{\sim} q(n-1|0)\frac{1}{\pi}\Gamma(\mu)\sin\left(\frac{\pi\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^\mu, \quad \text{(C7)}$$

in agreement with Eq. (18) of the main text.

APPENDIX D: NONZERO INITIAL CONDITIONS

In this section we focus on the derivation of $F_{0,\underline{x}}(n|x_0)$ and $F_{\underline{0},x}(n|x_0)$ for $0 < x_0 \ll a_\mu$. Note that the assumption $x_0 \ll a_\mu$ is important since in the opposite limit $x_0 \gg a_\mu$, the continuous limit is recovered and the rightward and leftward exit-time distributions are known. Our objective here is thus to highlight the peculiar behaviors arising from the discrete time nature of the jump process. Recalling Eqs. (5) and (9) of the main text, we argue that the dependence on the initial position of exit-time probabilities is contained either in the splitting probability (rightward FETP) or in the semi-infinite first passage time probability (leftward FETP):

$$\begin{aligned} F_{0,\underline{x}}(n|x_0) &\underset{\substack{n\to\infty\\ x\to\infty\\ \tau\text{ fixed}}}{\sim} \pi_{0,\underline{x}}(x_0)h_\mu(\tau)n^{-1} \\ F_{\underline{0},x}(n|x_0) &\underset{\substack{n\to\infty\\ x\to\infty\\ \tau\text{ fixed}}}{\sim} F_0(n|x_0)g_\mu(\tau). \end{aligned} \quad \text{(D1)}$$

It was shown in Refs. [33,34] that both splitting and semi-infinite first passage time probabilities exhibit similar asymptotic forms in the large n and x limit. More precisely, for a jump process with jump distribution $p(l)$ and Fourier transform $\tilde{p}(k) = \int_{-\infty}^{\infty} e^{ik\ell}p(\ell)d\ell$ one has

$$F_0(n|x_0) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{4n^3}}\left[\frac{1}{\sqrt{\pi}} + V(x_0)\right]$$

$$\pi_{0,\underline{x}}(x_0) \underset{x\to\infty}{\sim} 2^{\mu-1}\Gamma\left(\frac{1+\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^{\frac{\mu}{2}}\left[\frac{1}{\sqrt{\pi}} + V(x_0)\right], \quad \text{(D2)}$$

where a_μ is the scale of the jump process defined by the small k expansion of $\tilde{p}(k)$:

$$\tilde{p}(k) \underset{k\to 0}{=} 1 - (a_\mu|k|)^\mu + o(k^\mu), \quad \text{(D3)}$$

and $V(x_0)$ is defined by its Laplace transform:

$$\begin{aligned} \mathcal{L}V(\lambda) &= \int_0^\infty V(x_0)e^{-\lambda x_0}dx_0 \\ &= \frac{1}{\lambda\sqrt{\pi}}\left(\exp\left\{-\frac{\lambda}{\pi}\int_0^\infty \frac{dk}{k^2+\lambda^2}\ln[1-\tilde{p}(k)]\right\} - 1\right) \end{aligned} \quad \text{(D4)}$$

Combining Eqs. (D1) and (D2), we thus obtain the small $x_0 \ll a_\mu$ behavior of the rightward and leftward FETPs, along with the complete exit-time distribution $F_{\underline{0},\underline{x}}(n|x_0) = F_{0,\underline{x}}(n|x_0) + F_{\underline{0},x}(n|x_0)$. Agreement with numerical simulations is displayed in Fig. 5.

These expressions constitute the extension of the results derived in the main text to the full regime $0 < x_0 \ll a_\mu$. We stress that the $V(x_0)$ function displays highly nontrivial x_0 behavior, far from the continuous limit $V(x_0) \underset{x_0\gg a_\mu}{\propto} x_0^{\mu/2}$ and clearly identified in Ref. [21].

APPENDIX E: SCALING OF THE SPLITTING PROBABILITY

In this section, we show how to recover the scaling of the splitting probability derived in Ref. [21] from Eqs. (5) and (18). Recall first that the splitting probability starting from $x_0 = 0$ reads

$$\pi_{0,\underline{x}} = \sum_{k=1}^{\infty} F_{0,\underline{x}}(k). \quad \text{(E1)}$$

Splitting this sum into two parts, we obtain the following decomposition:

$$\begin{aligned} \pi_{0,\underline{x}} &= \sum_{k=1}^{x^\mu} q(k-1)\frac{\Gamma(\mu)}{\pi}\sin\left(\frac{\pi\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^\mu + \sum_{k=x^\mu}^{\infty}\pi_{0,\underline{x}}\frac{h_\mu\left(\frac{k}{x^\mu}\right)}{k} \\ &\equiv A_1(x) + A_2(x). \end{aligned} \quad \text{(E2)}$$

Since the survival probability satisfies $q(k) \propto k^{-\frac{1}{2}}$ in the large k limit, one has $A_1(x) \propto x^{-\frac{\mu}{2}}$. In the large x limit, the second sum can be rewritten as an integral:

$$A_2(x) \sim \pi_{0,\underline{x}} \int_1^\infty h_\mu(\tau)\tau^{-1} d\tau. \tag{E3}$$

Recalling the definition of $h_\mu(\tau)$ from Eq. (3) of the main text, we rewrite A_2 as

$$A_2(x) \sim \pi_{0,\underline{x}} \sum_{k=1}^{\infty} P(k) e^{-\lambda_k 2^\mu}, \tag{E4}$$

with $P(k)$ some subexponential function of k that guarantees the convergence of the sum. Summing A_1 and A_2 together, we thus recover the expected scaling of the splitting probability derived in Ref. [21]:

$$A_1(x) + A_2(x) \propto x^{-\frac{\mu}{2}}. \tag{E5}$$

APPENDIX F: SMALL τ BEHAVIOR OF $g_2(\tau)$

In this section we give an alternate expression of $g_2(\tau)$ defined in Eq. (10) of the main text, which is convenient to analyze small τ values. We obtain this expression by using the Poisson sum formula:

$$g_2(\tau) = 4\pi^{\frac{5}{2}} \tau^{\frac{3}{2}} \sum_{k=1}^{\infty} e^{-k^2\pi^2\tau} k^2$$
$$= 1 + \sum_{k=1}^{\infty} e^{-\frac{k^2}{\tau}} \left(2 - \frac{4k^2}{\tau}\right) \tag{F1}$$

In particular, this yields $g_2(\tau) \to 1$ for $\tau \to 0$.

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