

Nonlinear extension of the Kolosov-Muskhelishvili stress function formalism

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The method of stress function in elasticity theory is a powerful analytical tool with applications to a wide range of physical systems, including defective crystals, fluctuating membranes, and more. A complex coordinates formulation of stress function, known as the Kolosov-Muskhelishvili formalism, enabled the analysis of elastic problems with singular domains, particularly cracks, forming the basis for fracture mechanics. A shortcoming of this method is its limitation to linear elasticity, which assumes Hookean energy and linear strain measure. Under finite loads, the linearized strain fails to describe the deformation field adequately, reflecting the onset of geometric nonlinearity. The latter is common in materials experiencing large rotations, such as regions close to the crack tip or elastic metamaterials. While a nonlinear stress function formalism exists, the Kolosov-Muskhelishvili complex representation had not been generalized and remained limited to linear elasticity. This paper develops a Kolosov-Muskhelishvili formalism for the nonlinear stress function. Our formalism allows us to port methods from complex analysis to nonlinear elasticity and to solve nonlinear problems in singular domains. Upon implementing the method to the crack problem, we discover that nonlinear solutions strongly depend on the applied remote loads, excluding a universal form of the solution close to the crack tip and questioning the validity of previous studies of nonlinear crack analysis.

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I. INTRODUCTION

Elasticity theory is the main pillar in studying and analyzing complex physical phenomena in solids, including defects mediated plasticity in crystals [1], fracture [2], wrinkling, and growth of living matter [3–5]. A central motif in these examples is the need to find the stressed state of a system prior to a more detailed analysis that accounts for additional mechanisms. Under the requirement of mechanical equilibrium, finding the Cauchy stress field σ reduces to solving a bulk force balance constitutive equation

$$\text{div } \sigma = 0, \quad (1)$$

accompanied by conditions on the traction forces or displacement on the boundaries [6]. For example, in the case of crack propagation, upon solving Eq. (1), balancing relaxation of elastic energy with fracture energy at the vicinity of the crack tip determines crack trajectory [2].

One particularly useful method to solve Eq. (1) in two-dimensional (2D) elasticity is by representing the stress in terms of a single scalar elastic potential χ , formally denoted as [7–9]

$$\sigma = \text{curl curl } \chi, \quad (2)$$

where “curl” stands for the two-dimensional curl operator defined $\epsilon^{\mu\nu} \nabla_\nu$ ($\mu, \nu = 1, 2$) with ϵ being the Levi-Civita symbol. The potential χ , also known as the Airy stress function,

is determined by a geometric compatibility equation. Within linear elasticity, this condition reduces to a differential relation of the strain tensor, leading to the famous biharmonic equation [6]

$$\nabla^2 \nabla^2 \chi = 0. \quad (3)$$

Airy’s stress function approach is one of the main tools for solving problems in linear elasticity, with a wide range of applications, including statistical physics of fluctuating membranes [10,11], assemblies of structural defects as screening elastic fields [12], wrinkling patterns in thin sheets [13], and granular matter [14].

For highly symmetric domains and loadings, the biharmonic equation is analytically solvable. In contrast, in problems with singular boundaries, e.g., the crack problem, or problems with low symmetry, solving the biharmonic equation is a daunting task. An elegant and advantageous mathematical method for analyzing Eq. (3) is the use of complex coordinates, known as the Kolosov-Muskhelishvili formalism [2,15,16]. For example, the crack problem is canonically solved within this formalism, resulting in the celebrated universal $1/\sqrt{r}$ stress singularity at the crack tip, and forming the basis for linear elastic fracture mechanics [2].

The elastic energy functional for finite strains is generically nonlinear. Therefore, analytical methods for solving linear elastic problems must be generalized accordingly. Earlier attempts to develop analytical methods for finite elasticity, including perturbative and complex analytical methods, focused on special cases, e.g., incompressible neo-Hookean solids [17–19]. However, despite the success in developing

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new methods within finite elasticity, the case of non-Euclidean solids, that is, solids with geometrically incompatible reference states, is still lacking.

Note that the stress function formalism, particularly the Kolosov-Muskhelishvili formalism, is limited to linear elasticity, which assumes *two* distinct linearizations: (i) Hookean elasticity with small strains and (ii) strain that is linear in deformation gradients. However, in the case of fracture, the elastic solution for the stresses diverges at the crack tip, where the fracture process occurs, and this questions the validity of the linear elastic solution in this region.

Indeed, experiments directly identified deviations from linear elasticity close to the crack tip and therefore called for nonlinear analysis of the elastic crack problem [20]. Nonlinearity, in this case, is rooted in the appearance of large rotations, for which a nonlinear measure of strain is needed, and is geometric in nature. Furthermore, a series of publications reported nonlinear perturbative analysis of the asymptotic solution close to the crack tip [20–22]. Since asymptotic analysis lacks the necessary and sufficient boundary conditions (e.g., stress at infinity), a new type of boundary condition was invoked to select the nonlinear solution [22]. This condition, the requirement of divergence-free stress on the crack tip, determines a stress singularity of $1/r$, stronger than the singularity predicted by the linear theory. We claim that the need for additional boundary conditions reflects the absence of a systematic methodology for solving the complete problem that accounts for the remote boundary conditions.

In this paper, we build on a previously suggested nonlinear generalization of Airy's stress function approach [23–25] and reformulate it using a complex function representation. Specifically, we show that the nonlinear stress function written as the sum of i th-order corrections $\chi = \sum_i \chi^{(i)}$ with each order satisfying an equation that depends on lower-order terms is of the form of the linear equation

$$\Delta \Delta \chi^{(i)} = F_i(\chi^{(i-1)}, \dots). \quad (4)$$

Thus, our method forms a nonlinear generalization of the Kolosov-Muskhelishvili formalism, allowing us to calculate nonlinear corrections to classical linear results. To demonstrate our theory, we revisit the prototypical problem of a finite crack in an elastic domain subject to remote stresses and show that one can solve the complete problem to an arbitrary level of accuracy. Furthermore, we explicitly show that our solution, which is uniquely determined by remote boundary conditions and free stresses on the crack lips, does not satisfy the boundary conditions proposed in [21]. Furthermore, we also discover that contrary to the linear case, the asymptotic solution is not universal; different remote loads may result in different stress singularities at the crack tip. We conclude by discussing the nonlinear crack solution and its relevance in certain recognized fracture mechanisms.

II. LINEAR ELASTICITY AND THE KOLOSOV-MUSKHELISHVILI FORMALISM

A. Linear elasticity and the Airy potential

In elasticity, a deformation is quantified by the displacement field $\mathbf{d}(\mathbf{x})$, which maps a point in an undeformed state to its new position. Importantly, note that in our formulation \mathbf{x}

labels material elements in a *Lagrangian* coordinate system, which coincides with the position in the undeformed state. Upon deforming the system, an element originally located at \mathbf{x} is shifted to $\mathbf{x}' = \mathbf{x} + \mathbf{d}(\mathbf{x})$.

The change in the distance between neighboring material elements is $d\ell'^2 - d\ell^2 = 2u_{\mu\nu}dx^\mu dx^\nu$, where $d\ell$ and $d\ell'$ are the distances between two infinitesimally close points before and after the deformation, and $u_{\mu\nu}$ is the strain tensor given by

$$u_{\mu\nu} = \frac{1}{2}(\partial_\mu d_\nu + \partial_\nu d_\mu + \partial_\mu d_\lambda \partial_\nu d^\lambda). \quad (5)$$

In what follows we are considering homogeneous and isotropic Hookean elastic solids. This framework assumes small deformations, meaning that stresses are linearly proportional to strains:

$$\sigma^{\mu\nu} = \mathcal{A}^{\mu\nu\rho\sigma} u_{\rho\sigma}, \quad (6)$$

and the elastic tensor \mathcal{A} , which encodes material properties and symmetries, depends only on two parameters, e.g., Poisson's ratio ν and Young's modulus Y .

Furthermore, linear elasticity theory assumes small displacement gradients and thus omits terms quadratic in ∂d :

$$u_{\mu\nu}^{(1)} \approx \frac{1}{2}(\partial_\mu d_\nu + \partial_\nu d_\mu). \quad (7)$$

The superscript (i) stands for i th order. In equilibrium, the net force on each material element vanishes, manifested by σ being divergence free as imposed by Eq. (1), which in coordinates, and assuming Eq. (7), takes the form $\partial_\mu \sigma^{\mu\nu} = 0$. According to Airy's formalism, the solution to Eq. (1) is represented in terms of a single stress function $\chi^{(1)}$ [7,26]:

$$\sigma^{\alpha\beta(1)} = \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} \partial_{\mu\nu} \chi^{(1)}, \quad (8)$$

where ε is the antisymmetric Levi-Civita symbol [27]. The stress function $\chi^{(1)}$ is determined by enforcing a geometric compatibility condition reflecting the differential relation between the strain and displacement fields:

$$\varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} \partial_{\alpha\beta} u_{\mu\nu}^{(1)} = 0. \quad (9)$$

Upon extracting the strain from the constitutive relation Eq. (6) and expressing the stress using Eq. (8), the compatibility condition reduces to the biharmonic equation for the stress function as in Eq. (3).

B. Kolosov-Muskhelishvili formalism

In complex coordinates, where $z = x + iy$, the Laplace operator takes the form $\nabla^2 = \partial_{z\bar{z}}$. Therefore, real harmonic functions can be represented as the real part of an analytic function, and similarly, biharmonic functions, which are solutions of Eq. (3), are represented as [16]

$$\begin{aligned} \chi^{(1)} &= \text{Re}\{\bar{z}\phi^{(1)}(z) + \eta^{(1)}(z)\} \\ &= \frac{1}{2}[\bar{z}\phi^{(1)}(z) + z\overline{\phi^{(1)}(z)} + \eta^{(1)}(z) + \overline{\eta^{(1)}(z)}] \end{aligned} \quad (10)$$

where both $\phi^{(1)}(z)$ and $\eta^{(1)}(z)$ are analytic functions, and the overbar denotes the complex conjugate operator. We see that in this formalism, often referred to as the Kolosov-Muskhelishvili formalism, the force balance equation, as well as the biharmonic equation, are identically satisfied. Therefore, it remains to determine $\phi^{(1)}(z)$ and $\eta^{(1)}(z)$

by requiring appropriate boundary conditions. Upon defining $\psi^{(1)}(z) = \eta^{(1)}(z)$, Kolosov-Muskhelishvili formulas for the stress components are [16,28]

$$\begin{aligned} \sigma^{xx(1)} + \sigma^{yy(1)} &= 2[\phi^{(1)}(z) + \overline{\phi^{(1)}(z)}] \\ &= 4\text{Re}\{\phi^{(1)}(z)\}, \end{aligned} \quad (11a)$$

$$\sigma^{yy(1)} - \sigma^{xx(1)} + 2i\sigma^{xy(1)} = 2[\bar{z}\phi''^{(1)}(z) + \psi'^{(1)}(z)]. \quad (11b)$$

To complete the solution in a domain Ω one should enforce boundary conditions on $\partial\Omega$, which in a stress controlled setup reads

$$\boldsymbol{\sigma} \cdot \hat{n}|_{\partial\Omega} = \bar{\boldsymbol{\tau}} \quad (12)$$

with \hat{n} being a normal unit vector and $\bar{\boldsymbol{\tau}}$ representing traction forces on $\partial\Omega$. Kolosov-Muskhelishvili equations form the basis for solving multiple problems in elasticity, characterized by complicated geometry, singular behavior (such as cracks), and multiply connected domains by implementing methods from complex analysis [29]. This is similar to the use of complex analysis, and in particular conformal maps, in solving the Laplace equation in the complex domain.

III. EXTENSION TO NONLINEAR ELASTICITY

A nonlinear extension of Airy's stress function approach, given with details in this section, is transparently derived within a geometric formulation of elasticity developed in [30–33]. In this formalism, the strain tensor is defined by $u = \frac{1}{2}(\mathbf{g} - \mathbf{g}^0)$ where the reference and actual metrics \mathbf{g}^0 and \mathbf{g} represent the rest and actual distances between material elements. While \mathbf{g} is defined by $\mathbf{g} = (\nabla f)^T \cdot (\nabla f)$ where f is the embedding representing the configuration, \mathbf{g}^0 is a material property and is fixed for purely elastic materials, which are at the focus of the current paper. If the reference metric is Euclidean, that is, a stress-free configuration exists, then the definition of strain reduces to its classical nonlinear definition Eq. (5). The nonlinear terms of the strain are extremely important when the displacement gradients are larger than 1, even if the strain itself is small. It occurs in systems that experience large rotations, such as cracks [21] and metamaterials [25]. Within this approach, the force balance equation that generalizes Eq. (1) is nonlinear and implicit, with the differential operator being dependent on the yet unknown actual configuration [32] (see Appendix A for a complete derivation)

$$\nabla_\mu \sigma^{\mu\nu} + (\Gamma^{0\lambda}_{\lambda\mu} - \Gamma^{\lambda}_{\lambda\mu}) \sigma^{\mu\nu} = 0, \quad (13)$$

where Γ and Γ^0 are the Christoffel symbols associated with \mathbf{g} and \mathbf{g}^0 , and ∇ is the covariant derivative with respect to \mathbf{g} , that is [34],

$$\nabla_\mu \sigma^{\mu\nu} = \partial_\mu \sigma^{\mu\nu} + \Gamma^{\mu}_{\mu\beta} \sigma^{\beta\nu} + \Gamma^{\nu}_{\mu\beta} \sigma^{\beta\mu}. \quad (14)$$

A representation of the stress in terms of a scalar function, including nonlinear corrections, takes the following form [23–25]:

$$\sigma^{\mu\nu} = \frac{1}{\sqrt{\det \mathbf{g} \det \mathbf{g}^0}} \varepsilon^{\mu\rho} \varepsilon^{\nu\sigma} \nabla_{\rho\sigma} \chi, \quad (15)$$

where ε is the Levi-Civita symbol and $\frac{1}{\sqrt{\mathbf{g}}}\varepsilon$ and $\frac{1}{\sqrt{\mathbf{g}^0}}\varepsilon$ are Levi-Civita tensors with respect to the different metrics. Note

that this representation is proved to be complete only in multiply connected domains [26]. Nevertheless, while a mathematical proof is still lacking, the representation in terms of χ is practically sufficient as it allows finding solutions that satisfy the boundary conditions and topological constraints, as in problems of defects, cracks, and more [1,16]. Equation (15) is implicit since it is expanded in terms of the unknown metric \mathbf{g} , reflecting the implicit form of Eq. (13). Similar to Airy's stress function, the nonlinear stress function χ is determined by imposing a geometric compatibility condition. Since the actual metric \mathbf{g} describes distances in the Euclidean plane, which is flat, the compatibility condition is the vanishing of the Gaussian curvature

$$K(\mathbf{g}) = 0 \quad (16)$$

where the Gaussian curvature is obtained from the metric [34] by

$$K(\mathbf{g}) = \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\beta\gamma\delta} \quad (17)$$

and $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \frac{1}{2} (\partial_{\beta\gamma} g_{\alpha\delta} + \partial_{\alpha\delta} g_{\beta\gamma} - \partial_{\alpha\gamma} g_{\beta\delta} - \partial_{\beta\delta} g_{\alpha\gamma}) \\ &+ g_{\mu\nu} (\Gamma^{\mu}_{\beta\gamma} \Gamma^{\nu}_{\alpha\delta} - \Gamma^{\mu}_{\beta\delta} \Gamma^{\nu}_{\alpha\gamma}). \end{aligned} \quad (18)$$

In the case of small deformations relative to a Euclidean reference metric \mathbf{g}^0 , this condition coincides with Eq. (9). The implicit form of Eq. (15) prevents a direct implementation of the condition $K(\mathbf{g}) = 0$, hence an exact analog of the biharmonic equation for the fully nonlinear problem is still lacking. To avoid this difficulty, a perturbative approach is invoked under the assumption that the nonlinear stress function can be expanded in powers of a formal small parameter δ : $\chi = \chi^{(1)} + \chi^{(2)} + \dots$ with the small parameter encoded in the n th-order stress function $\chi^{(n)} \propto \delta^n$. In the same way, the stress can be written as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)} + \boldsymbol{\sigma}^{(3)} + \dots \quad (19)$$

In order to determine the importance of n th-order corrections of the stress, one must examine the magnitude of $\sigma^{(n)}$.

Generally, one can determine the small parameter retrospectively, but in the case of an imposed remote stress σ , the small parameter is $\delta = \sigma/Y$ where Y is Young's modulus. If the reference metric \mathbf{g}^0 is flat (in any desired coordinates system), enforcing the geometric compatibility condition, namely, $K(\mathbf{g}) = 0$, order by order yields

$$\begin{aligned} \nabla_{\mathbf{g}^0}^2 \nabla_{\mathbf{g}^0}^2 \chi^{(1)} &= 0, \\ \nabla_{\mathbf{g}^0}^2 \nabla_{\mathbf{g}^0}^2 \chi^{(2)} &= F_2(\chi^{(1)}), \\ \nabla_{\mathbf{g}^0}^2 \nabla_{\mathbf{g}^0}^2 \chi^{(3)} &= F_3(\chi^{(1)}, \chi^{(2)}), \\ &\vdots \end{aligned} \quad (20)$$

Here $\nabla_{\mathbf{g}^0}^2 = \frac{1}{\sqrt{\det \mathbf{g}^0}} \partial_\mu (\sqrt{\det \mathbf{g}^0} \mathbf{g}^{0\mu\nu} \partial_\nu)$ is the Laplace operator with respect to the reference metric \mathbf{g}^0 , and F_n are known (nonlinear) functions of stress functions of orders lower than n . All the equations are explicit from this point, thanks to the perturbative approach, enabling us to advance analytically.

Equation (20) provides a prescription for solving the nonlinear elastic problem to an arbitrary order of accuracy.

The solution for $\chi^{(n)}$ consists of a (homogeneous) generic biharmonic function and a particular solution solving the non-homogeneous equation. Again, the homogeneous parts are determined by boundary conditions, as in the linear case. The main difficulty arises from the nonlinearity of F_n , which prevents analytical progress. The following section shows how the complex formulation of this nonlinear formalism enables significant analytical progress.

IV. NONLINEAR EXTENSION OF THE KOLOSOV-MUSKHELISHVILI FORMALISM

In this section, we generalize the Kolosov-Muskhelishvili complex formulation of Airy's stress function approach. For that purpose, we reformulate Eq. (20) together with its accompanying boundary conditions in complex coordinates. As before, the solution of the first-order stress function is the most general real biharmonic function given by Eq. (10).

For $n > 1$, the solution of the n th-order stress function consists of the sum of a biharmonic solution and a particular solution to Eq. (20) denoted by \tilde{F}_n :

$$\chi^{(n)}(z, \bar{z}) = \text{Re}\{\bar{z}\phi^{(n)}(z) + \eta^{(n)}(z)\} + \tilde{F}_n(\chi^{(1)}, \dots, \chi^{(n-1)}). \quad (21)$$

The analytic functions $\phi^{(n)}, \eta^{(n)}$ can be determined once boundary conditions are specified, hence expressions for the stresses in terms of $\chi^{(n)}$ are required.

We derive the expressions for the stresses and generalize the Kolosov-Muskhelishvili equations (11) to the nonlinear setting:

$$\begin{aligned} \sigma^{xx} + \sigma^{yy} &= \sum_{n=1} [4\text{Re}\{\phi^{(n)}(z)\} \\ &\quad + \Sigma_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)})], \quad (22a) \\ \frac{1}{2}(\sigma^{yy} - \sigma^{xx}) + i\sigma^{xy} &= \sum_{n=1} [\bar{z}\phi^{(n)}(z) + \psi^{(n)}(z) \\ &\quad + \tilde{\Sigma}_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)})] \quad (22b) \end{aligned}$$

where $\Sigma_{(n)}$ and $\tilde{\Sigma}_{(n)}$ are known functions that represent the contribution of stress functions of order smaller than n to the stress field at order n . Therefore, $\Sigma_{(1)}, \tilde{\Sigma}_{(1)} = 0$. Although our method can be applied to any order in the expansion, to demonstrate this rather abstract formalism we focus on the second-order solution:

$$\chi^{(2)}(z, \bar{z}) = \text{Re}\{\bar{z}\phi^{(2)}(z) + \eta^{(2)}(z)\} + \tilde{F}_2(\chi^{(1)}). \quad (23)$$

The particular solution for the nonlinear and nonhomogeneous correction of Eq. (20) is analytically solvable in closed form and reads

$$\tilde{F}_2(\chi^{(1)}) = c_1|\phi^{(1)}(z)|^2 + c_2|\psi^{(1)}(z) + \bar{z}\phi^{(1)'}(z)|^2, \quad (24)$$

where $c_{1,2}$ are numerical factors that depend on material properties and are given in Appendix B. The particular solution \tilde{F}_2 contributes to the right-hand side of Eq. (22) via the right-hand side of Eq. (21), and thus induces effective traction forces that determine the second-order functions $\phi^{(2)}(z)$ and $\eta^{(2)}(z)$.

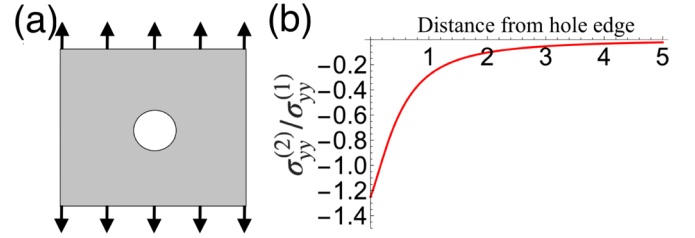


FIG. 1. Nonlinear stress at the vicinity of a circular hole subjected to uniaxial tension. (a) A two-dimensional elastic medium containing a circular hole and subjected to remote uniaxial stress. (b) The second-order stress correction $\sigma_{yy}^{(2)}$ evaluated along the positive x axis, normalized by the linear stress $\sigma_{yy}^{(1)}$ (for zero Poisson's ratio). The horizontal axis is the distance from the hole boundary, with the origin corresponding to the hole edge.

V. EXAMPLES

In this section we solve two prototypical problems in 2D elasticity using the nonlinear approach, namely, a circular hole and a finite crack subjected to constant remote stresses. To emphasize the generality of the formalism in each example we implement the nonlinear generalization of Kolosov-Muskhelishvili equations (22) using a different technique. While the circular hole problem is presented mostly for illustrating the method, the crack case is of physical importance as it questions the validity of previously reported analysis of a similar problem [21].

A. Circular hole under stress

Consider a circular hole of radius $R = 1$ in a 2D elastic material subjected to remote uniaxial stress $\sigma_{\infty}^{yy} = \sigma_{\infty}$ [see Fig. 1(a)], a problem that was first solved by Kirsch [35], and later by Kolosov [28] and Muskhelishvili [16] within the linear approximation. We express the normal traction force on the hole boundary as $\tau = \tau_x + i\tau_y$. Upon expressing τ in terms of ϕ and η via Eqs. (11) and (12) the linearized boundary condition reads [2]

$$\tau_x^{(1)} + i\tau_y^{(1)} = -i\frac{d}{d\theta}[\phi^{(1)}(z) + z\overline{\phi^{(1)'}}(\bar{z}) + \overline{\psi^{(1)}}(\bar{z})] \quad (25)$$

where $z = \exp(i\theta)$ and θ is the angle along the circular boundary. In the case of traction free boundary conditions on the hole, Eq. (25) reduces to

$$\phi^{(1)}(z) + z\overline{\phi^{(1)'}}(\bar{z}) + \overline{\psi^{(1)}}(\bar{z}) = \text{const}. \quad (26)$$

The solution that satisfies both zero normal stress on the hole boundary Eq. (26), and the uniform stresses at infinity, is given by

$$\begin{aligned} \phi^{(1)}(z) &= \frac{\sigma_{\infty}}{2}\left(\frac{z}{2} - \frac{1}{z}\right), \\ \psi^{(1)}(z) &= \frac{\sigma_{\infty}}{2}\left(z - \frac{1}{z} - \frac{1}{z^3}\right). \end{aligned} \quad (27)$$

The stresses derived from this solution according to Eq. (11) are given in Appendix C1.

To solve higher-order stress functions it is required to extend the boundary conditions Eq. (25) to the nonlinear

setup. Indeed, in analogy to deriving Eq. (25) from the linear expression for the stress field Eq. (11), we express the nonlinear stress field (that is, the nonlinear corrections to the stress emerging from nonlinear strains) in terms of complex variables, using Eq. (22):

$$\begin{aligned} \tau_x + i\tau_y = & -i \frac{d}{d\theta} \sum_n [\phi^{(n)}(z) + z\overline{\phi^{(n)}(z)} + \overline{\psi^{(n)}(z)}] \\ & + \sum_n \tilde{\tau}^{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}). \end{aligned} \quad (28)$$

Since the first-order stress function satisfies the boundary conditions, higher-order terms obey zero traction. Yet, we note that the stress functions of order $< n$ contribute to an effective traction on the boundaries, to be balanced by solution at order $n \geq 2$, that is,

$$i \frac{d}{d\theta} [\phi^{(n)}(z) + z\overline{\phi^{(n)}(z)} + \overline{\psi^{(n)}(z)}] = \tilde{\tau}^{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}). \quad (29)$$

The general form of $\tilde{\tau}^{(2)}$ in the case of a circular boundary (with arbitrary boundary conditions) is given in Mathematica notebook [36]. We apply this formalism to the second-order solution for the stress function, explicitly given by Eqs. (23) and (24). In the following, we will assume for simplicity that $\nu = 0$, while a solution for the general Poisson's ratio appears in Appendix C2. We use the fact that on the hole boundary $z = e^{i\theta}$ and therefore $\bar{z} = z^{-1}$, the second-order boundary condition reads

$$\begin{aligned} i \frac{d}{d\theta} [\phi^{(2)}(z) + z\overline{\phi^{(2)}(z)} + \overline{\psi^{(2)}(z)}] \\ = \frac{\sigma_\infty^2 (-5z^8 + 42z^6 - 22z^4 + 14z^2 + 3)}{16z^4 Y}, \end{aligned} \quad (30)$$

with the solution

$$\begin{aligned} \phi^{(2)}(z) &= \frac{\sigma_\infty^2}{16Y} \left[z + \frac{5}{z} + \frac{1}{z^3} \right], \\ \psi^{(2)}(z) &= \frac{9\sigma_\infty^2}{16Y} \left[z + \frac{20}{9z} - \frac{1}{z^3} + \frac{4}{9z^5} \right]. \end{aligned} \quad (31)$$

This form of the solution recovers our formal small parameter to be $\delta = \sigma_\infty/Y$ upon comparison with the first-order expression. This is in accord with our small-strains assumption. Moreover, when compared with the linear solution $\phi^{(1)}$ and $\eta^{(1)}$, the nonlinear corrections $\phi^{(2)}$ and $\eta^{(2)}$ contain a higher-order term in $1/z$. Consequently, the nonlinear stress field contains higher-order angular modes, and higher powers of $1/r$. Specifically, the ratio between the largest value of the stress on the hole boundary and the remote stress, also known as the stress concentration factor $K_t = \sigma_{\max}/\sigma_\infty$, is corrected at the second order, as obtained by estimating σ^{yy} on the hole boundary at $\theta = 0$:

$$K_t = 3 \left(1 - \frac{5\sigma_\infty}{4Y} \right). \quad (32)$$

Figure 1(b) shows the ratio between the second-order correction of σ^{yy} and the first-order solution, i.e., $\delta\sigma^{yy(2)}$ along the $\theta = 0$ line. We emphasize that the solutions Eqs. (31) and (32)

are valid for $\nu = 0$, and expressions for the arbitrary Poisson's ratio are given in Appendix C2.

B. Finite straight cracks

In this section we consider a finite straight crack embedded in an infinite medium, and subjected to uniform remote stresses. Specifically, we solve a nonlinear mode I problem, and give the nonlinear mode II solution in Appendix D2. The crack problem is characterized by its geometry $C = \{(x, y) : -1 < x < 1 \wedge y = 0\}$, by the traction forces acting at infinity σ_∞ , and by the traction free boundary conditions on the crack faces. We note that the displacements may be discontinuous across the crack. Correspondingly, it is necessary to enforce boundary conditions on the top and bottom crack faces separately. At the linear level we follow the Westergaard method [2,16]. Using Eq. (11) and defining $\Psi = \psi'$, $\Phi = \phi'$, and $\Omega = \Phi + \Psi + z\Phi'$, we find

$$\sigma^{yy(1)} + i\sigma^{xy(1)} = \Omega^{(1)}(z) + \overline{\Phi^{(1)}(z)} + (\bar{z} - z)\Phi'^{(1)}(z). \quad (33)$$

Upon approaching the crack from above or below, we denote $z = x \pm i\epsilon$, and the normal stress in complex form reads

$$(\sigma^{yy(1)} + i\sigma^{xy(1)})|_C = \Omega^{(1)}(x \pm i\epsilon) + \overline{\Phi^{(1)}(x \mp i\epsilon)}, \quad (34)$$

where the third term in Eq. (33) vanishes as $\epsilon \rightarrow 0$.

Adding and subtracting the upper and lower limits of Eq. (34), and defining $\Omega^{(1)}(z) = \frac{1}{2}[I^{(1)}(z) + J^{(1)}(z)]$ and $\overline{\Phi^{(1)}(z)} = \frac{1}{2}[I^{(1)}(z) - J^{(1)}(z)]$, yields

$$\begin{aligned} 0 &= I^{(1)}(x + i\epsilon) + I^{(1)}(x - i\epsilon), \\ 0 &= J^{(1)}(x + i\epsilon) - J^{(1)}(x - i\epsilon), \end{aligned} \quad (35)$$

where the left-hand side vanished due to traction free boundary conditions. These equations for $I^{(1)}(z)$ and $J^{(1)}(z)$ replace the equations for $\Omega^{(1)}(z)$ and $\Phi^{(1)}(z)$, and correspondingly for $\psi^{(1)}(z)$ and $\phi^{(1)}(z)$. Once the equations for $I^{(1)}(z)$ and $J^{(1)}(z)$ are solved, we can directly derive the solution for the stress function and the stress fields.

Soon we will see that the form of the equation for $J^{(1)}(z)$ in Eq. (35) is analytically solvable. Therefore, we would prefer to rewrite the equation for $I^{(1)}(z)$ in the same form as that for $J^{(1)}(z)$. For that, we define $L^{(1)}(z) = G(z)I^{(1)}(z)$ with $G(z)$ an analytic function that, on the crack boundary, satisfies $G(x + i\epsilon) = -G(x - i\epsilon)$. Indeed, we note that the specific choice $G(z) = \sqrt{(z+1)(z-1)}$ satisfies this condition, and a direct substitution in Eq. (35) yields

$$\begin{aligned} L^{(1)}(x + i\epsilon) - L^{(1)}(x - i\epsilon) &= 0, \\ J^{(1)}(x + i\epsilon) - J^{(1)}(x - i\epsilon) &= 0. \end{aligned} \quad (36)$$

Since $L^{(1)}(z)$ and $J^{(1)}(z)$ have the same value on both sides of the branch cut, they are analytic everywhere.

From the relation $L^{(1)}(z) = G(z)I^{(1)}(z)$ and from the relation between $I^{(1)}(z)$ and $J^{(1)}(z)$ and $\Phi^{(1)}(z)$ and $\Omega^{(1)}(z)$

we find

$$\begin{aligned} \Omega^{(1)}(z) &= \frac{1}{2} \left[\frac{L^{(1)}(z)}{\sqrt{z^2 - 1}} + J^{(1)}(z) \right], \\ \overline{\Phi^{(1)}}(z) &= \frac{1}{2} \left[\frac{L^{(1)}(z)}{\sqrt{z^2 - 1}} - J^{(1)}(z) \right]. \end{aligned} \quad (37)$$

Since $L^{(1)}(z)$ and $J^{(1)}(z)$ are analytic functions, the celebrated $1/\sqrt{r}$ singularity at the crack tip can already be seen by direct substitution in Eq. (33). The analytic functions $L^{(1)}(z)$ and $J^{(1)}(z)$ are determined by enforcing the boundary conditions on the remote stresses at infinity, as well as the continuity of the elastic fields at $y = 0$ out of the crack domain. In the case of mode I crack, where the stress at infinity is

$$\sigma_\infty = \begin{pmatrix} \sigma_\infty^{xx} & 0 \\ 0 & \sigma_\infty^{yy} \end{pmatrix}, \quad (38)$$

the solution for $L^{(1)}(z)$ and $J^{(1)}(z)$ is given by

$$\begin{aligned} L^{(1)}(z) &= \sigma_\infty^{yy} z, \\ J^{(1)}(z) &= \frac{\sigma_\infty^{yy} - \sigma_\infty^{xx}}{2}. \end{aligned} \quad (39)$$

To solve the second-order stress function we first generalize Eq. (33) to the nonlinear setup. For that we derive a nonlinear generalization of Eq. (33) using the nonlinear Kolosov-Muskhelishvili equations (22):

$$\begin{aligned} &\sigma^{yy} + i\sigma^{xy} \\ &= \sigma^{yy(1)} + i\sigma^{xy(1)} + \sum_{n=2} [\Omega^{(n)}(z) + \overline{\Phi^{(n)}}(z)] \\ &\quad + (\bar{z} - z)\Phi^{(n)}(z) + \sum_{n=2} \left[\frac{1}{2} \Sigma_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}) \right. \\ &\quad \left. + \tilde{\Sigma}_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}) \right]. \end{aligned} \quad (40)$$

We denote

$$\begin{aligned} \Delta\sigma^{yy(n)} &= \text{Re} \frac{1}{2} \Sigma_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}) \\ &\quad + \tilde{\Sigma}_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}), \\ \Delta\sigma^{xy(n)} &= \text{Im} \frac{1}{2} \Sigma_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}) \\ &\quad + \tilde{\Sigma}_{(n)}(\chi^{(1)}, \dots, \chi^{(n-1)}) \end{aligned} \quad (41)$$

and interpret them as the n th-order tensile and shear stresses induced by lower-order solutions.

As in the circular hole case, the first-order solution for the stress function balances the imposed boundary conditions. Therefore, higher-order stresses should identically vanish on the boundaries. Specifically, upon requiring zero traction forces on the crack faces we find at all orders

$$\begin{aligned} &\Omega^{(n)}(x \pm i\epsilon) + \overline{\Phi^{(n)}}(x \mp i\epsilon) \\ &= -\Delta\sigma^{yy(n)}(x \pm i\epsilon) - i\Delta\sigma^{xy(n)}(x \pm i\epsilon). \end{aligned} \quad (42)$$

We express the boundary conditions at second order by a direct substitution of the first-order solution in Eq. (41) with

$n = 2$ on the crack faces and find

$$\begin{aligned} \Delta\sigma^{yy(2)}(x \pm i\epsilon) &= \xi_1(\nu) \frac{(\sigma_\infty^{xx})^2 - 2\sigma_\infty^{xx}\sigma_\infty^{yy} - 3(\sigma_\infty^{yy})^2}{Y}, \\ i\Delta\sigma^{xy(2)}(x \pm i\epsilon) &= \pm \frac{\sigma_\infty^{yy}(\sigma_\infty^{xx} - \sigma_\infty^{yy})}{Y(x^2 - 1)^{3/2}} [\xi_2(\nu)x + \xi_3(\nu)x^3], \end{aligned} \quad (43)$$

where ξ_i are numerical factors that depend on Poisson's ratio (see Appendix D1). As before, our goal is to solve for $\Omega^{(2)}(z)$ and $\Phi^{(2)}(z)$. For that we repeat the technique from the first-order analysis. Upon adding and subtracting the upper and lower limits of Eq. (42), and defining $\Omega^{(2)}(z) = \frac{1}{2}[I^{(2)}(z) + J^{(2)}(z)]$ and $\overline{\Phi^{(2)}}(z) = \frac{1}{2}[I^{(2)}(z) - J^{(2)}(z)]$, we find

$$\begin{aligned} I^{(2)}(x + i\epsilon) + I^{(2)}(x - i\epsilon) &= -2\Delta\sigma^{yy(2)}(x + i\epsilon), \\ J^{(2)}(x + i\epsilon) - J^{(2)}(x - i\epsilon) &= -i[\Delta\sigma^{xy(2)}(x + i\epsilon) \\ &\quad - \Delta\sigma^{xy(2)}(x - i\epsilon)]. \end{aligned} \quad (44)$$

At this stage, upon defining $L^{(2)}(z) = G(z)I^{(2)}(z)$, we can transform the first equation to the form of the second one, similar to the transition from Eq. (35) to Eq. (36). Instead, here we perform a stronger transformation with which the equations become homogeneous:

$$\begin{aligned} G_2(z) &= \pm \frac{\sigma_\infty^{yy}(\sigma_\infty^{xx} - \sigma_\infty^{yy})}{Y(z^2 - 1)^{3/2}} (\xi_2 z + \xi_3 z^3), \\ L^{(2)}(z) &= G(z)[I^{(2)}(z) + \Delta\sigma^{yy(2)}], \\ M^{(2)}(z) &= G_2(z) + J^{(2)}(z), \end{aligned} \quad (45)$$

where $G_2(z)$ is a nonhomogeneous solution for $J^{(2)}(z)$ as reflected from the complex continuation of $\Delta\sigma^{xy(2)}$ in Eq. (43). Indeed, we identify $i\Delta\sigma^{xy(2)}(x \pm i\epsilon) = G_2(x \pm i\epsilon)$, with which Eqs. (44) now read

$$\begin{aligned} L^{(2)}(x + i\epsilon) - L^{(2)}(x - i\epsilon) &= 0, \\ M^{(2)}(x + i\epsilon) - M^{(2)}(x - i\epsilon) &= 0. \end{aligned} \quad (46)$$

The choice of $L^{(2)}(z)$ and $M^{(2)}(z)$ transformed Eq. (44) into a set of homogeneous equations.

It follows from Eq. (46) that the second-order correction to the stress function is

$$\begin{aligned} \Omega^{(2)}(z) &= \frac{1}{2} \left[-G_2(z) + \frac{L^{(2)}(z)}{\sqrt{z^2 - 1}} + M^{(2)}(z) - \Delta\sigma^{yy(2)} \right], \\ \overline{\Phi^{(2)}}(z) &= \frac{1}{2} \left[G_2(z) + \frac{L^{(2)}(z)}{\sqrt{z^2 - 1}} - M^{(2)}(z) - \Delta\sigma^{yy(2)} \right], \end{aligned} \quad (47)$$

where $L^{(2)}(z)$ and $M^{(2)}(z)$ are analytic functions determined by the remote boundary conditions and the displacement continuity outside the crack. As in the linear case, also here we can identify the stress singularity at the crack tip regardless of the exact form of $L^{(2)}(z)$ and $M^{(2)}(z)$. Since $\Omega^{(2)}(z)$ and $\overline{\Phi^{(2)}}(z)$ are linearly dependent on $G_2(z)$, upon expanding it near the crack tip ($z = \pm 1$), its form reveals stronger singularities, which include terms like $1/r$ and $1/r^{3/2}$.

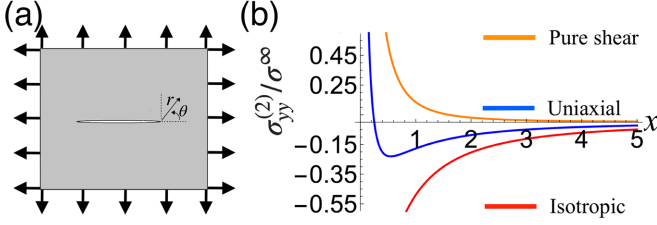


FIG. 2. Nonlinear stress at the vicinity of a mode I crack subjected to biaxial stress. (a) Illustration of a mode I crack under biaxial load. (b) $\sigma_{yy}^{(2)}$, the second-order correction to σ_{yy} normalized by the magnitude of the remote stress σ_∞ , of a mode I crack for three different loadings: isotropic stress $\sigma_\infty^{xx} = \sigma_\infty^{yy}$ (red), uniaxial stress with $\sigma_\infty^{xx} = 0$ (blue), and traceless symmetric stress $\sigma_\infty^{xx} = -\sigma_\infty^{yy}$ corresponding to pure shear (orange). The horizontal axis measure the distance from the hole with the origin corresponding to the hole edge. Close to the crack tip the pure shear dominates over the isotropic stress contribution, reflecting its stronger singularity and emphasizing the lack of a universal solution.

To find the explicit form of $L^{(2)}(z)$ and $M^{(2)}(z)$ we note that the first-order solution contributes to the second-order stresses a uniform stress at infinity:

$$\Delta\sigma_\infty^{(2)} = \begin{pmatrix} s^{xx(2)} & 0 \\ 0 & s^{yy(2)} \end{pmatrix}, \quad (48)$$

where the constants entries are given in Appendix D1. The solution for $L^{(2)}(z)$ and $M^{(2)}(z)$ is

$$\begin{aligned} L^{(2)}(z) &= p_1 z, \\ M^{(2)}(z) &= p_2, \end{aligned} \quad (49)$$

where p_1 and p_2 are constants given by

$$\begin{aligned} p_1 &= (\Delta\sigma_{yy}^{(2)} - s^{yy(2)}), \\ p_2 &= \frac{1}{2}(s^{xx(2)} - s^{yy(2)}) - \frac{(\nu + 1)^2}{8Y}(\sigma_\infty^{xx} - \sigma_\infty^{yy})\sigma_\infty^{yy} \end{aligned} \quad (50)$$

and $\Delta\sigma_{yy}^{(2)}$ is a constant given by Eq. (43).

Interestingly, while the stress singularity at the crack tip in the linear approximation is proportional to $1/\sqrt{r}$ for any externally imposed loads, the singularity at the second order depends on the specific loading. This is seen from the explicit form of $G_2(z)$ given in Eq. (45), which vanishes for isotropic loading $\sigma_\infty^{xx} = \sigma_\infty^{yy}$.

We deduce that the $1/r$ singularity is generic at the second order, while the stronger singularity $1/r^{3/2}$ disappears when the imposed loading is biaxial. The second- and higher-order singularities become important as one approaches the crack tip. Far from the crack tip, the linear solution is clearly valid. In Fig. 2(b), one can see the correction to σ_{yy} , namely, $\sigma_{yy}^{(2)}$, in three cases: isotropic stress, traceless symmetric stress, and a simple superposition of the two where $\sigma_\infty^{xx} = 0$. We see that as we approach the crack tip, the effect of the traceless symmetric stress dominates because of its stronger singularity.

An important consequence of our analysis is the limitation on methods that aim at solving for the asymptotic behavior at the vicinity of the crack tip. In the linear approximation, the asymptotic behavior near the tip of the crack can be found

without imposing the remote stress boundary conditions, although without determining the coefficients, such as the stress intensity factor [37]. Here we find that a second-order solution cannot be found purely asymptotically because the remote stress determines the order of the singularity close to the crack tip. This observation questions the validity of the asymptotic analysis presented in [21,22].

VI. SUMMARY AND DISCUSSIONS

In this paper, based on an earlier geometric extension of the Airy stress function approach, we developed a geometrically nonlinear version of the Kolosov-Muskhelishvili formulas for analyzing 2D elastic problems. Our method allows us to port tools from complex analysis to study nonlinear elasticity of 2D solids, with either regular or singular boundaries. Such problems are essential for understanding natural phenomena such as fractures in solids under large deformations. Motivated by fracture mechanics, we focus on the problem of a straight finite crack and use our generalized Kolosov-Muskhelishvili equations to study the nonlinear elastic solution. While in this paper the calculations performed up to second order, our method is applicable to any desired level of accuracy.

Another exciting consequence that arises from the second-order correction to the mode II crack, known to be unstable compared with mode I crack, is a feature strongly related to the celebrated principle of local symmetry [38]. This phenomenon coheres with our theory, where a prominent distinction between mode I and mode II cracks is observed: Nonlinear corrections to the mode II stress field in the vicinity of the crack tip contain contributions with symmetries characteristic of mode I, consistent with [39]. This is contrary to nonlinear corrections to mode I, which preserve its symmetries. This is shown in Appendix D2, and suggests a future research direction for the derivation of the principle of local symmetry within the framework of nonlinear crack analysis. In particular the relevance of mode I contributions to seemingly pure mode II fracture should be further studied.

It is important to note that the relevance of our work to fracture mechanics is not always obvious. While the nonlinear corrections to the stress field become important only very close to the crack tip, in this region, new physics such as plasticity may dominate. Therefore, judgment regarding the relevance of our results to fracture mechanics reduces to comparing two different length scales: λ_Y , the scale below which fracture occurs, and λ_{NL} , the scale below which nonlinear elastic corrections are non-negligible. If $\lambda_Y < \lambda_{NL}$ then nonlinear corrections are important. A dimensional analysis argument clarifies this comparison of scales. On one hand, the scale λ_Y reflects the balance of linear stresses with the microscopic yield stress $\sigma_Y \approx \sigma^\infty \sqrt{a/\lambda_Y}$ where a is the length of the crack, hence $\lambda_Y \approx a(\sigma^\infty/\sigma_Y)^2$ [40]. On the other hand, the scale where nonlinear correction becomes non-negligible reflects equal contribution from the $1/r$ and $1/\sqrt{r}$ terms $\sigma^\infty/\sqrt{\lambda_{NL}/a} \approx (\sigma^\infty)^2/[Y(\lambda_{NL}/a)]$, hence $\lambda_{NL} \approx a(\sigma^\infty/Y)^2$. The conditions for nonlinear correction to be non-negligible, $\lambda_Y < \lambda_{NL}$, reduce to $\sigma_Y > Y$. This result is reasonable, as it suggests that materials that support large strain before

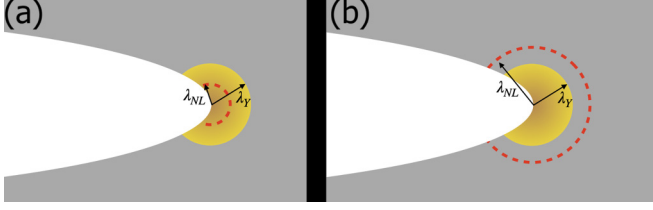


FIG. 3. The importance of nonlinearity at the onset of fracture. Nonlinear contributions to the elastic fields are important in the region bounded by the red dashed line $r < \lambda_{NL}$. Plastic deformations occur inside the yellow region $r < \lambda_Y$. (a) When $\sigma_Y < Y$ the material fails at small strains, thus nonlinear corrections are significant only on scales smaller than the plastic zone size, and therefore irrelevant. (b) When $\sigma_Y > Y$ the material can support large strains prior to failure, hence at the onset of failure strains are sufficiently large, with non-negligible nonlinear corrections on scales larger than the process zone.

failure will require nonlinear corrections. This is summarized in Fig. 3.

An important test for any new theory is the comparison with previous results. The most prominent nonlinear analysis of the asymptotic elastic fields in the vicinity of a crack was published in a series of papers [20–22,41–43]. In these works, like in our paper, the authors recovered the $1/r$ and $1/r^{3/2}$ singularities but excluded the latter based on a newly suggested boundary condition at the crack tip, that is requiring the elastic force $\mathbf{f} = \text{div}\sigma$, to vanish on the crack tip. This new boundary condition aimed to replace the remote boundary conditions that are absent from the asymptotic problem. Contrary to the asymptotic analysis, our analysis does not require any additional boundary condition other than the standard normal forces on the boundaries. Our results show that the $1/r^{3/2}$ singularity can survive, depending on the specific remote loading, and therefore show that the problem cannot be solved only asymptotically. This shows the failure of the extra boundary condition of zero force on the crack tip. To settle this disagreement, we emphasize that the bulk equation $\text{div}\sigma = 0$ should hold in the bulk material. A nonzero value on the boundary is in no contradiction with the free boundary conditions that are reflected by $\sigma \cdot \hat{n} = 0$. Indeed, while in the linear approximation at the crack tip, the condition $\text{div}\sigma = 0$ holds, it does not hold along the crack faces, and as mentioned is not in contradiction with the bulk equation and boundary conditions. Our results, which satisfy the boundary conditions and bulk equation to first and second order, question the validity of asymptotic nonlinear crack analysis published in the past and call for revisiting the comparison with experimental results.

The nonlinear geometric approach to elasticity, which is the basis for the nonlinear Kolosov-Muskhelishvili formalism developed in this paper, has many advantages, among which is the ability to calculate nonlinear interactions between sources of stresses [25] and nonlinear residually stressed solids [23] in the presence of large deformation gradients. Similarly, one can use the geometrically nonlinear Kolosov-Muskhelishvili formalism we propose to calculate the nonlinear interaction of cracks with singular sources of stresses, such as topological defects, and with the background stress in residually stressed

solids, such as living growing matter. Another possible extension of this paper may include inelastic deformations. By taking them into account, the strain tensor takes the form $E_e = \frac{1}{2}F_0^{-T}(C - C_0)F_0^{-1}$ where F_0 is the inelastic deformation and C and C_0 are equivalent to g and g_0 , respectively. We leave these promising directions of research for future study.

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APPENDIX A: EQUILIBRIUM EQUATIONS

In this section we derive the nonlinear equilibrium equations Eq. (13).

An elastic solid is modeled as a manifold \mathcal{M} equipped with a reference metric \mathbf{g}^0 . A configuration is an embedding $\phi : \mathcal{M} \rightarrow \mathbb{R}^2$ from which an actual metric, denoted g , is defined on \mathcal{M} as the pullback of the euclidean metric on \mathbb{R}^2 .

The equilibrium equations are derived from an energy variation with respect to the embedding ϕ describing the configuration. The elastic energy to be minimized is

$$U = \int_{\mathcal{M}} \mathcal{W}(\mathbf{g}, \mathbf{g}^0) dS_{\mathbf{g}^0} \quad (\text{A1})$$

where $dS_{\mathbf{g}^0} \equiv \sqrt{\det(\mathbf{g}^0)} dudv$. Upon defining the strain $\epsilon = \frac{1}{2}(\mathbf{g} - \mathbf{g}^0)$ and the stress

$$\sigma^{\alpha\beta} = \frac{\partial \mathcal{W}}{\partial \epsilon_{\alpha\beta}} \quad (\text{A2})$$

we find

$$\delta_{\phi} U = \int_{\mathcal{M}} \frac{1}{2} \sigma^{\alpha\beta} \delta_{\phi} g_{\alpha\beta} dS_{\mathbf{g}^0}. \quad (\text{A3})$$

Writing the metric variation in terms of the configuration and using $\delta_{\phi} g_{\alpha\beta} = (\partial_{\alpha}\phi)(\partial_{\beta}\delta\phi) + (\partial_{\alpha}\delta\phi)(\partial_{\beta}\phi)$ we find

$$\begin{aligned} \delta_{\phi} U &= \int_{\mathcal{M}} \sigma^{\alpha\beta} (\partial_{\alpha}\phi)(\partial_{\beta}\delta\phi) dS_{\mathbf{g}^0} \\ &= \oint_{\partial\mathcal{M}} \sigma^{\alpha\beta} n_{\beta} (\partial_{\alpha}\phi) \delta\phi dl_{\mathbf{g}^0} \\ &\quad - \int_{\mathcal{M}} \frac{1}{\sqrt{\det \mathbf{g}^0}} \partial_{\beta} (\sigma^{\alpha\beta} (\partial_{\alpha}\phi) \sqrt{\det \mathbf{g}^0}) \delta\phi dS_{\mathbf{g}^0}. \end{aligned} \quad (\text{A4})$$

We note that the second integrand can be written as

$$\begin{aligned} \text{div}_{\beta} \sigma^{\alpha\beta} \partial_{\alpha}\phi &\equiv \frac{1}{\sqrt{\det \mathbf{g}^0}} \partial_{\beta} (\sigma^{\alpha\beta} (\partial_{\alpha}\phi) \sqrt{\det \mathbf{g}^0}) \\ &= (\nabla_{\beta} \sigma^{\alpha\beta} + (\Gamma_{\nu\beta}^{0\nu} - \Gamma_{\nu\beta}^{\nu}) \sigma^{\alpha\beta}) \partial_{\alpha}\phi. \end{aligned} \quad (\text{A5})$$

Therefore the variation takes the form

$$\delta_{\phi} U = \oint_{\partial\mathcal{M}} \sigma^{\alpha\beta} n_{\beta} (\partial_{\alpha}\phi) \delta\phi dl_{\mathbf{g}^0} - \int_{\mathcal{M}} \text{div}_{\beta} \sigma^{\alpha\beta} \partial_{\alpha}\phi \delta\phi dS_{\mathbf{g}^0}. \quad (\text{A6})$$

We conclude that the equilibrium equation is

$$\bar{\nabla}_\mu \sigma^{\mu\nu} + (\Gamma_{\alpha\beta}^\nu - \bar{\Gamma}_{\alpha\beta}^\nu) \sigma^{\alpha\beta} = 0, \tag{A7}$$

along with the boundary conditions

$$n_\alpha \sigma^{\alpha\beta} = 0 \tag{A8}$$

and in the presence of external loads

$$n_\alpha \sigma^{\alpha\beta} = t^\beta. \tag{A9}$$

correction is (see Mathematica notebook [36])

$$\begin{aligned} & \frac{Y}{2} \nabla_{\mathbf{g}^0}^2 \nabla_{\mathbf{g}^0}^2 \chi^{(2)}(z, \bar{z}) \\ &= (25 - 22\nu + \nu^2) |\phi^{(1)''}|^2 - (1 + \nu)^2 (2z\phi^{(1)'''} \overline{\phi^{(1)'}} \\ & \quad + 2\bar{z}\phi^{(1)'''} \overline{\phi^{(1)'}} + 2\psi^{(1)'} \overline{\phi^{(1)'}} + 2\overline{\psi^{(1)'}} \phi^{(1)'''} + |\bar{z}\phi^{(1)'''}| \\ & \quad + |\psi^{(1)'}|^2). \end{aligned} \tag{B1}$$

APPENDIX B: SECOND ORDER INCOMPATIBLE STRESS FUNCTION

Our treatment of the nonlinear elastic problem is based on a perturbative solution to the nonlinear stress function, where we expand the stress function in terms of a small parameter. A flat reference metric \mathbf{g}^0 gives a set of nonhomogeneous harmonic equations, in which the nonhomogeneous part depends only on lower-order solutions of the stress function, as shown in Eq. (20). For example, the equation for the second-order

The general solution is

$$\begin{aligned} \chi^{(2)}(z, \bar{z}) &= \frac{1}{2} [\bar{z}\phi^{(2)}(z) + z\overline{\phi^{(2)}(z)} + \eta^{(2)}(z) + \overline{\eta^{(2)}(z)}] \\ & \quad - \frac{1}{8Y} (-29 + 14\nu - 5\nu^2) |\phi^{(1)}(z)|^2 \\ & \quad - \frac{1}{8Y} (1 + \nu)^2 |\bar{z}\phi^{(1)'}(z) + \psi^{(1)}(z)|^2. \end{aligned} \tag{B2}$$

APPENDIX C: RADIAL SYMMETRY

1. Circular hole: First order

Equation (27) shows the linear stress function of a unit circular hole subjected to uniaxial stress σ_∞ along the y axis. The derived stresses are

$$\sigma^{xx(1)} = \sigma_\infty \frac{1}{2r^2} \left[\cos 2\theta + \left(2 - \frac{3}{r^2}\right) \cos 4\theta \right], \tag{C1a}$$

$$\sigma^{xy(1)} = \sigma_\infty \frac{1}{2r^2} \left[-\sin 2\theta + \left(2 - \frac{3}{r^2}\right) \sin 4\theta \right], \tag{C1b}$$

$$\sigma^{yy(1)} = \sigma_\infty \left[1 + \frac{3}{2r^2} \cos 2\theta + \left(\frac{3}{2r^4} - \frac{1}{r^2}\right) \cos 4\theta \right], \tag{C1c}$$

where r is the distance from the hole center and θ is the angle relative to the x axis.

2. Circular hole: Second-order correction

The homogeneous part of the second-order correction to the stress function with an arbitrary Poisson's ratio is

$$\phi^{(2)}(z) = \frac{\sigma_\infty^2}{16Y} \left[-(v - 1)z + \frac{(5 - 2v + v^2)}{z} + \frac{(1 + v)^2}{z^3} \right], \tag{C2a}$$

$$\psi^{(2)}(z) = \frac{\sigma_\infty^2}{16Y} \left[(v - 3)^2 z + \frac{4(5 - 2v + v^2)}{z} - \frac{(v - 3)^2}{z^3} + \frac{4(1 + v)^2}{z^5} \right] \tag{C2b}$$

and the full derivation is given in Mathematica notebook [36]. From these functions one can derive the second-order correction to the stress, which takes the following form:

$$\sigma^{xx(2)}(r, \theta) = \frac{\sigma_\infty^2}{16Y} [f_1(r) + f_2(r) \cos 2\theta + f_3(r) \cos 4\theta + f_4(r) \cos 6\theta], \tag{C3a}$$

$$\sigma^{xy(2)}(r, \theta) = \frac{\sigma_\infty^2}{16Y} [f_5(r) \cos 2\theta + f_6(r) \cos 4\theta + f_7(r) \cos 6\theta], \tag{C3b}$$

$$\sigma^{yy(2)}(r, \theta) = \frac{\sigma_\infty^2}{16Y} [f_8(r) + f_9(r) \cos 2\theta + f_{10}(r) \cos 4\theta - f_4(r) \cos 6\theta] \tag{C3c}$$

where the functions $f_i(r)$ are

$$f_1(r) = \frac{3(v - 6)v + 11}{256r^4} + \frac{(5v + 9)(v + 1)}{128r^6} - \frac{9(v + 1)^2}{256r^8}, \tag{C4a}$$

$$f_2(r) = \frac{(v - 1)v}{32r^2} + \frac{(14 - 3v)v - 31}{128r^4} + \frac{-9v^2 + 6v + 39}{128r^6} + \frac{3(v - 3)(v + 1)}{64r^8}, \tag{C4b}$$

$$f_3(r) = \frac{\nu(5\nu - 2) - 23}{128r^2} + \frac{(4 - 3\nu)\nu + 9}{32r^4} + \frac{3(\nu - 3)(\nu + 1)}{64r^6}, \tag{C4c}$$

$$f_4(r) = \frac{3(\nu - 3)(\nu + 1)}{128r^2} + \frac{1 - 3(\nu - 2)\nu}{32r^4} + \frac{5(\nu - 1)^2}{64r^6}, \tag{C4d}$$

$$f_5(r) = \frac{\nu^2 - 2\nu + 5}{128r^2} - \frac{3(\nu^2 - 2\nu + 13)}{128r^4} - \frac{3(\nu^2 - 2\nu - 7)}{64r^6} + \frac{3(\nu - 3)(\nu + 1)}{64r^8}, \tag{C4e}$$

$$f_6(r) = \frac{\nu^2 - 2\nu - 19}{128r^2} - \frac{3(\nu^2 - 2\nu - 7)}{64r^4} + \frac{3(\nu - 3)(\nu + 1)}{64r^6}, \tag{C4f}$$

$$f_7(r) = \frac{3(\nu - 3)(\nu + 1)}{128r^2} + \frac{-3\nu^2 + 6\nu + 1}{32r^4} + \frac{5(\nu - 1)^2}{64r^6}, \tag{C4g}$$

$$f_8(r) = -\frac{9(\nu + 1)^2}{256r^8} + \frac{3(\nu - 7)(\nu + 1)}{256r^4} + \frac{3(\nu + 5)(\nu + 1)}{128r^6}, \tag{C4h}$$

$$f_9(r) = -\frac{3(\nu - 3)(\nu + 1)}{64r^8} + \frac{3[(\nu - 6)\nu - 15]}{128r^6} + \frac{-[(\nu - 4)\nu] - 15}{64r^2} + \frac{\nu(3\nu + 2) + 47}{128r^4}, \tag{C4i}$$

$$f_{10}(r) = \frac{-\nu - 6}{16r^4} - \frac{3(\nu - 3)(\nu + 1)}{64r^6} + \frac{\nu(3\nu + 2) + 15}{128r^2}. \tag{C4j}$$

One can observe several features of the second-order solution. First, the angular dependencies of the first order were of 2θ and 4θ . In the second-order solution we get a new angular dependency, of 6θ . Another feature is related to the radial dependency of the solution. In the first-order solution one can find different elements of the stress scales like $\frac{1}{r^2}$ or $\frac{1}{r^4}$, and in the second-order solution one can also find elements of $\frac{1}{r^6}$ and $\frac{1}{r^8}$.

We also get a correction to the stress concentration factor:

$$K_t = 3 \left(1 - \frac{5 - 2\nu + \nu^2}{4} \frac{\sigma_\infty}{Y} \right). \tag{C5}$$

The main characteristic properties of the second-order stress tensor are already reflected from the form Eq. (C3) and the new singularities in f_i . Explicit expressions for the stress tensor are derived in the Mathematica notebook [36].

APPENDIX D: NONLINEAR CRACK

1. Mode I

The linear stress function of the mode I crack is

$$\phi^{(1)}(z) = \frac{\sigma_\infty^{yy}}{4} (-z + 2\sqrt{z^2 - 1}) + \frac{\sigma_\infty^{xx}}{4} z, \tag{D1a}$$

$$\psi^{(1)}(z) = \frac{\sigma_\infty^{yy}}{2} \left(z - \frac{1}{\sqrt{z^2 - 1}} \right) - \frac{\sigma_\infty^{xx}}{2} z \tag{D1b}$$

where σ_∞^{xx} and σ_∞^{yy} are the stresses imposed at infinity.

We want to examine the contribution of the second order to the stress around the tip of the crack. Therefore we expand the elements of the stress around the tip, i.e., around the point

$(1 + r \cos \theta, r \sin \theta)$ in orders of r , and we get that asymptotic stresses take the form

$$\sigma^{xx(1)}(r, \theta) = \frac{\sigma_\infty^{yy}}{4\sqrt{2}r} \left[3 \cos \frac{\theta}{2} + \cos \frac{5\theta}{2} \right], \tag{D2a}$$

$$\sigma^{xy(1)}(r, \theta) = \frac{\sigma_\infty^{yy}}{2\sqrt{2}r} \left[-\sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \right], \tag{D2b}$$

$$\sigma^{yy(1)}(r, \theta) = \frac{\sigma_\infty^{yy}}{4\sqrt{2}r} \left[5 \cos \frac{\theta}{2} - \cos \frac{5\theta}{2} \right]. \tag{D2c}$$

As we showed before, the stresses at each order are affected by the stresses of lower orders. In the case of the mode I crack, the second-order stresses induced on the crack faces by the first-order stresses are

$$\Delta\sigma^{yy(2)}(x \pm i\epsilon) = \frac{(7 - 4\nu + \nu^2)[(\sigma_\infty^{xx})^2 - 2\sigma_\infty^{xx}\sigma_\infty^{yy} - 3(\sigma_\infty^{yy})^2]}{16Y},$$

$$i\Delta\sigma^{xy(2)}(x \pm i\epsilon) = \pm \frac{\sigma_\infty^{yy}(\sigma_\infty^{xx} - \sigma_\infty^{yy})}{8Y(x^2 - 1)^{3/2}} [6(\nu - 1)x - (1 + \nu)^2x^3]. \tag{D3}$$

The second-order stresses at infinity that are induced by the first-order stresses are

$$s^{xx(2)} = \frac{-(\nu^2 - 8\nu + 11)(\sigma_\infty^{xx})^2 + (\nu^2 - 4\nu + 7)(\sigma_\infty^{yy})^2 + 4\nu(\nu + 3)\sigma_\infty^{xx}\sigma_\infty^{yy}}{16Y},$$

$$s^{yy(2)} = \frac{(\nu^2 - 4\nu + 7)(\sigma_\infty^{xx})^2 - (\nu^2 - 8\nu + 11)(\sigma_\infty^{yy})^2 + 4\nu(\nu + 3)\sigma_\infty^{xx}\sigma_\infty^{yy}}{16Y}. \tag{D4}$$

Here we show the asymptotic stresses for $\nu = 0$. Explicit derivation and expressions of the second-order correction to the stress field are given in Mathematica notebook [36].

The asymptotic stress field takes the form

$$\sigma^{xx(2)}(r, \theta) = \frac{\sigma_{\infty}^{yy}}{Y} \left[\frac{g_1^I(\theta)}{\sqrt{r}} + \frac{g_2^I(\theta)}{r} + \frac{g_3^I(\theta)}{r^{3/2}} \right], \tag{D5a}$$

$$\sigma^{xy(2)}(r, \theta) = \frac{\sigma_{\infty}^{yy}}{Y} \left[\frac{g_4^I(\theta)}{\sqrt{r}} + \frac{g_5^I(\theta)}{r} + \frac{g_6^I(\theta)}{r^{3/2}} \right], \tag{D5b}$$

$$\sigma^{yy(2)}(r, \theta) = \frac{\sigma_{\infty}^{yy}}{Y} \left[\frac{g_7^I(\theta)}{\sqrt{r}} + \frac{g_8^I(\theta)}{r} + \frac{g_9^I(\theta)}{r^{3/2}} \right] \tag{D5c}$$

where $r = 0$ denotes the crack tip and

$$g_1^I(\theta) = \frac{9 \cos \frac{\theta}{2} (61\sigma_{\infty}^{yy} - 125\sigma_{\infty}^{xx}) + \cos \frac{5\theta}{2} (95\sigma_{\infty}^{yy} - 287\sigma_{\infty}^{xx}) + 72 \cos \frac{9\theta}{2} (\sigma_{\infty}^{xx} - \sigma_{\infty}^{yy})}{512\sqrt{2}}, \tag{D6a}$$

$$g_2^I(\theta) = -\frac{1}{128} (96 \cos \theta - 32 \cos 2\theta + 64 \cos 3\theta + 9 \cos 4\theta - 57) \sigma_{\infty}^{yy}, \tag{D6b}$$

$$g_3^I(\theta) = \frac{15(\cos \frac{3\theta}{2} + 3 \cos \frac{7\theta}{2})(\sigma_{\infty}^{yy} - \sigma_{\infty}^{xx})}{128\sqrt{2}}, \tag{D6c}$$

$$g_4^I(\theta) = \frac{3 \sin \theta [\cos \frac{3\theta}{2} (29\sigma_{\infty}^{yy} - 93\sigma_{\infty}^{xx}) + 24 \cos \frac{7\theta}{2} (\sigma_{\infty}^{xx} - \sigma_{\infty}^{yy})]}{256\sqrt{2}}, \tag{D6d}$$

$$g_5^I(\theta) = -\frac{(-42 \sin 2\theta + 64 \sin 3\theta + 9 \sin 4\theta) \sigma_{\infty}^{yy}}{128}, \tag{D6e}$$

$$g_6^I(\theta) = \frac{45 \sin \theta \cos(\frac{5\theta}{2})(\sigma_{\infty}^{yy} - \sigma_{\infty}^{xx})}{64\sqrt{2}}, \tag{D6f}$$

$$g_7^I(\theta) = \frac{-\cos \frac{\theta}{2} (403\sigma_{\infty}^{xx} + 557\sigma_{\infty}^{yy}) + \cos \frac{5\theta}{2} (415\sigma_{\infty}^{xx} - 223\sigma_{\infty}^{yy}) - 72 \cos \frac{9\theta}{2} (\sigma_{\infty}^{xx} - \sigma_{\infty}^{yy})}{512\sqrt{2}}, \tag{D6g}$$

$$g_8^I(\theta) = \frac{\cos^2(\frac{\theta}{2})(-153 \cos \theta + 46 \cos 2\theta + 9 \cos 3\theta + 82) \sigma_{\infty}^{yy}}{32}, \tag{D6h}$$

$$g_9^I(\theta) = \frac{15(7 \cos \frac{3\theta}{2} - 3 \cos \frac{7\theta}{2})(\sigma_{\infty}^{yy} - \sigma_{\infty}^{xx})}{128\sqrt{2}}. \tag{D6i}$$

2. Mode II

The linear stress function of the mode II crack is

$$\phi^{(1)}(z) = -\frac{i\sigma_{\infty}}{2} \sqrt{z^2 - 1}, \tag{D7a}$$

$$\psi^{(1)}(z) = \frac{i\sigma_{\infty}}{2} \left(\frac{2z^2 - 1}{\sqrt{z^2 - 1}} \right), \tag{D7b}$$

which gives the following asymptotic behavior around the crack tip:

$$\sigma^{xx(1)}(r, \theta) = -\frac{\sigma_{\infty}}{4\sqrt{2r}} \left[7 \sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \right], \tag{D8a}$$

$$\sigma^{xy(1)}(r, \theta) = \frac{\sigma_{\infty}}{2\sqrt{2r}} \left[3 \cos \frac{\theta}{2} + \cos \frac{5\theta}{2} \right], \tag{D8b}$$

$$\sigma^{yy(1)}(r, \theta) = \frac{\sigma_{\infty}}{4\sqrt{2r}} \left[-\sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \right]. \tag{D8c}$$

Analogous to mode I, the second-order stresses induced on the crack faces and at infinity by the first-order stresses are

$$\Delta \sigma^{yy(2)}(x \pm i\epsilon) = -\frac{\sigma_{\infty}^2}{4Y} (7 - 4\nu + \nu^2), \tag{D9a}$$

$$i\Delta \sigma^{xy(2)}(x \pm i\epsilon) = 0, \tag{D9b}$$

$$s^{xx(2)} = -\frac{\sigma_{\infty}^2}{4Y} (-5 + 8\nu + \nu^2), \tag{D9c}$$

$$s^{yy(2)} = \frac{\sigma_{\infty}^2}{4Y} (7 - 4\nu + \nu^2). \tag{D9d}$$

The second-order correction to the stress function of the mode II crack is given by

$$\Omega^{(2)}(z) = \frac{1}{2} \left[\frac{L^{(2)}(z)}{\sqrt{z^2 - 1}} + M^{(2)}(z) - \Delta \sigma^{yy(2)} \right], \tag{D10a}$$

$$\Phi^{(2)}(z) = \frac{1}{2} \left[\frac{L^{(2)}(z)}{\sqrt{z^2 - 1}} - M^{(2)}(z) - \Delta \sigma^{yy(2)} \right] \tag{D10b}$$

where $\Omega^{(2)}$ and $\Phi^{(2)}$ are defined by $\Psi = \psi'$, $\Phi = \phi'$, and $\Omega = \Phi + \Psi + z\Phi'$. L and M are determined by the boundary

conditions at infinity and by demanding the continuity of the displacement outside of the crack:

$$L^{(2)}(z) = (\Delta\sigma^{yy(2)} - s^{yy(2)})z, \quad (\text{D11a})$$

$$M^{(2)}(z) = \frac{1}{2}(s^{xx(2)} - s^{yy(2)}). \quad (\text{D11b})$$

Here we show the asymptotic stresses for $\nu = 0$. Explicit derivation and expressions of the second-order correction to the stress field are given in Mathematica notebook [36].

The asymptotic stress field takes the form

$$\sigma^{xx(2)}(r, \theta) = \frac{(\sigma_\infty^{yy})^2}{Y} \left[\frac{g_1^{\text{II}}(\theta)}{\sqrt{r}} + \frac{g_2^{\text{II}}(\theta)}{r} \right], \quad (\text{D12a})$$

$$\sigma^{xy(2)}(r, \theta) = \frac{(\sigma_\infty^{yy})^2}{Y} \left[\frac{g_3^{\text{II}}(\theta)}{\sqrt{r}} + \frac{g_4^{\text{II}}(\theta)}{r} \right], \quad (\text{D12b})$$

$$\sigma^{yy(2)}(r, \theta) = \frac{(\sigma_\infty^{yy})^2}{Y} \left[\frac{g_5^{\text{II}}(\theta)}{\sqrt{r}} + \frac{g_6^{\text{II}}(\theta)}{r} \right] \quad (\text{D12c})$$

where $r = 0$ denotes the crack tip and

$$g_1^{\text{II}}(\theta) = -\frac{(5 + 52\pi)(3 \cos \frac{\theta}{2} + \cos \frac{5\theta}{2})}{64\sqrt{2}\pi}, \quad (\text{D13a})$$

$$g_2^{\text{II}}(\theta) = \frac{1}{128}(96 \cos \theta + 96 \cos 2\theta + 64 \cos 3\theta + 27 \cos 4\theta - 43), \quad (\text{D13b})$$

$$g_3^{\text{II}}(\theta) = -\frac{(5 + 52\pi) \sin \theta \cos \frac{3\theta}{2}}{32\sqrt{2}\pi}, \quad (\text{D13c})$$

$$g_4^{\text{II}}(\theta) = \frac{1}{128}(66 \sin 2\theta + 64 \sin 3\theta + 27 \sin 4\theta) \quad (\text{D13d})$$

$$g_5^{\text{II}}(\theta) = \frac{(5 + 52\pi) \cos^3(\frac{\theta}{2})(2 \cos \theta - 3)}{16\sqrt{2}\pi}, \quad (\text{D13e})$$

$$g_6^{\text{II}}(\theta) = \frac{1}{32} \cos^2\left(\frac{\theta}{2}\right)(11 \cos \theta - 10 \cos 2\theta - 27 \cos 3\theta + 42). \quad (\text{D13f})$$

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