# Dual ε-closed-loop Nash equilibrium method to study pandemic by numerical analysis

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In this paper, an approach to the disease transmission dynamic of a coronavirus pandemic is presented. Compared to models commonly known from the literature, new classes that describe this dynamic to our model were added, which are a class representing costs of the pandemic and a class of the individuals being vaccinated but without antibodies. Parameters which at most of them depend on time were used. Sufficient conditions for a dual ε-closed-loop Nash equilibrium in the form of the verification theorem are formulated. A numerical algorithm and numerical example are constructed.

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## I. INTRODUCTION

Over the years, the world has struggled with various global diseases. A new pandemic, known by the name coronavirus pandemic (COVID-19), broke out in 2019 in Wuhan, China. This situation prompted scientists to create papers, in which various models of the coronavirus transmission dynamic are developed (see, e.g., [1–10] and literature therein). These compartmental mathematical models differ slightly, especially in terms of the number of classes (which can be distinguished as, e.g., susceptible, exposed, infectious, superspreaders, hospitalized, or fatality). Some authors include in their models a vaccinated class (see, e.g., [1,2,4,11–14] and literature therein). In our paper we have also included a vaccinated class (and susceptible, infectious, hospitalized, and fatality). In addition we take into account new classes, including part of the population being vaccinated but having not antibodies ($R$) and pandemic costs ($C$) (this class was first introduced in [15]). These new groups generate costs which governments of the countries incur. Further costs related to the pandemic concern lockdown of the community, quarantine, isolation, hospitalization or surveillance, and serology testing.

We try in this article to find a relation between the number of individuals having antibodies (which should be maximized), basic reproduction number $R_0$ (which should be minimized), and pandemic costs. The factor $R_0$ is very important in endemicity or pandemics because it gives us information about the expected number of the secondary cases created in a susceptible population by a "typical" infective individual. In other words, it can be understood as the average number of cases in which one infected individual infects healthy individuals. If $R_0 < 1$, it means that one infected individual infects less than one new individual during his infectious period and as a consequence the pandemic dies out. On the other hand, if $R_0 > 1$, then one infected individual infects more than one new individual during his infectious period, which means that the pandemic expands.

The mathematical models presented in the papers cited above vary in parameters which are constant, independent of time, and chosen suitably according to the case study. Generally, they are calculated for a given local pandemic situation (type of virus). We change in our model the role of these parameters because they can change in time (taking into account in this way mutations of the virus) and can differ for different countries and nations. Therefore, parameters in our paper are functions (controls) which depend on time.

This paper presents an approach to the coronavirus dynamic transmission which is based on a game theory. We construct a noncooperative differential game with two main players, instead of optimization with respect to all parameters. The first player tries to maximize (using their own own optimal strategies) the number of the population having antibodies, while the second player tries to minimize the basic reproduction number $R_0$ and the costs of the pandemic by choosing their own (optimal) strategies. Both players use their own suitable defined sets of controls (strategies) which in previous mathematical models were constants.

Nash equilibrium is one of the most useful and most fundamental concepts used in noncooperative game theory. It conceptualizes the behavior between players and allows decisions to be made by the opponents as if they are making them at the same time. Each player's strategy is optimal, taking the choice of his opponents as fixed. Nash equilibrium is used, e.g., in economy to illustrate that decision-making is a system of strategic actions based on the operations of the players.

We can distinguish, e.g., open-loop or closed-loop Nash equilibria which are usually not equivalent (but if there exists a unique Nash equilibrium in every subgame, then open-loop equilibrium and closed-loop equilibrium coincide). The open loop means that the players do not see the game play of their opponents. On the other hand, closed loop means that the players have knowledge about all previous steps of the game before the next step.

Epidemiological models can be described as evolutionary games (see, e.g., [16–19] and literature therein). But in these papers strategies do not depend on time, and solutions which are a kind of Nash equilibrium were found by discussing the parameters. Our game theoretical approach is definitely different because we use functions (strategies) as controls and therefore we develop a dual dynamic programming method-

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ology. Using this method we formulate sufficient conditions for a dual $\varepsilon$-closed-loop Nash equilibrium. Dual dynamic programming was first introduced in [20] (see also, e.g., [14,15,21,22]).

We construct a verification theorem which provides the background to build a numerical algorithm and find an approximate solution for a dual $\varepsilon$-closed-loop Nash equilibrium, which is really the Nash equilibrium, accordingly to our mathematical model. This approach has at least one undeniable advantage: it helps us to verify obtained results. In our case, using the verification theorem, we easily check if the final results are good enough. If not, we must repeat the whole procedure. We also do not need to check the convergence of our numerical algorithm, because if we have a candidate to be a solution of our problem, we use the verification theorem and simply check the obtained results.

## II. MATHEMATICAL MODEL

We use in our compartmental model of the disease transmission dynamic the following states and strategies.

The states used are $V(t)$, a part of the population having antibodies; $R(t)$, vaccinated but having not antibodies; $S(t)$, susceptible class; $I(t)$, symptomatic and infectious class; $H(t)$, hospitalized class; $F(t)$, fatality class; and $C(t)$, costs of the pandemic. The strategies used are $v(t)$, control of the population having antibodies; $\kappa(t)$, control of the vaccinated patients not having the antibodies or requiring medical care; $r_1(t)$, control of the vaccinated without antibodies entering the infectious class; $r_2(t)$, control of the vaccinated without antibodies entering the hospitalized class; $r_3(t)$, control of the vaccinated without antibodies entering the fatality class; $s_1(t)$, control of susceptible individuals entering the symptomatic and infectious class; $\gamma_a(t)$, control of the average rate at which symptomatic individuals become hospitalized; $\gamma_i(t)$, recovery control without being hospitalized; $\gamma_r(t)$, recovery control of hospitalized; $\varphi(t)$, control of patients losing antibodies; $c(t)$, costs of lockdown; $c_1(t)$, costs of maintaining social distancing and using face masks in public; $c_2(t)$, costs of quarantine and isolation of the confirmed cases; $c_3(t)$, hospitalization costs of the confirmed cases; and $c_4(t)$, costs of the vaccination; where $t \in [0, T]$, $T > 0$ and all strategies have positive values.

Let us take a closer look at the following classes:

(a) $V$ contains the part of the population having antibodies which means vaccinated (and acquisition of immunity) and obtaining immunity by being infected and then recovered.

(b) $R$ contains the part of the population being vaccinated but without of the immunity or having side effects. This is not a large group. It is about 4–6 % of vaccinated people. It depends on the type of vaccine. However, those people often require medical care or go to the hospital because of complications after vaccinations. That situation generates costs and we want to take into account all pandemic costs in our model.

(c) $C$ describes costs incurred by the government, such as lockdown of the community, quarantine, isolation, hospitalization, or surveillance and serology testing.

One of the most used epidemiological models is the so-called susceptible-exposed-infected-recovered (SEIR) model belonging to the class of compartmental models (see [23]; compare [24]). It assumes that the total population can be divided into four classes of individuals: susceptible, $S$; exposed, $E$; infected, $I$; and recovered or dead, $R$ (assumed to be not susceptible to reinfection). The model is based on the following assumptions:

(1) The total population does not vary in time.

(2) Susceptible individuals become infected and then can only recover or die.

(3) Exposed individuals have encountered an infected person but are not yet infectious themselves.

(4) Recovered or dead individuals are forever immune.

However, the longevity of the antibody response is still unknown, but it is known that antibodies wane over time. Assumption 3 given above is not well recognized when we apply vaccination of the population. Assumption 2 is too strong because we can observe that many infected people suffer long-time consequences of infection and require medical care. It is difficult to expect that during the period of one-half or one year the total population does not vary; the population is not living in a hermetic box. The SEIR model is presented as

$$\frac{dS}{dt} = -\lambda S(t)I(t),$$
$$\frac{dE}{dt} = -\lambda S(t)I(t) - \alpha E(t),$$
$$\frac{dI}{dt} = \alpha E(t) - \gamma I(t),$$
$$\frac{dR}{dt} = \gamma I(t).$$

To deal with uncertainties in long-term extrapolations and with the time dependency of control parameters, the authors of [24] introduce a stochastic approach into modeling of the epidemic, making parameters depend on time and adding three more equations.

The main goal of this paper is not continuing to get one more extension of the existing epidemic models. Our aim is first of all to concentrate on the costs of the pandemic depending on parameters changing in time. We want to be still deterministic; however, to take into account different situations and uncertainties changing in time, which generate costs, we introduce parameters (controls) that should control the costs in a better way. This is one of the reasons why we do not consider the exposed compartment, but we introduce three new classes: having antibodies ($V$), vaccinated but without immunity or having side effects ($R$), and a new variable describing costs ($C$). It is clear that with our approach we cannot require the above assumptions for the SEIR model because they are not satisfied.

Looking carefully at Figs. 1 and 2, we see that our compartmental model of the pandemic dynamic has seven classes. Six of them ($S$, $V$, $R$, $I$, $H$, and $F$) concern flow of the population between them during the pandemic. It is obvious that each person may be located only in one class in the moment. We can treat classes $S$ and $R$ as "main" classes because each person was either susceptible or was vaccinated but does not have antibodies. We do not assume that part of the population being susceptible can be transformed into vaccinated but without antibodies because our model can be adapted during the pandemic (and of course when vaccines

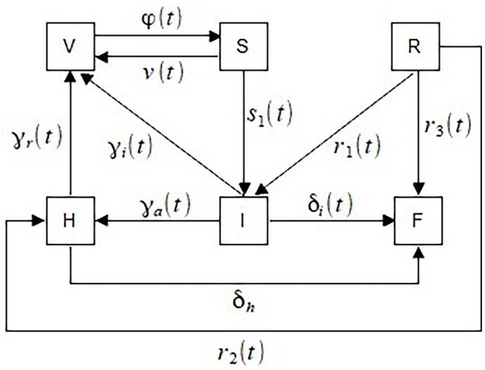


FIG. 1. Flow diagram of the model described by Eqs. (1)–(6).

are available) and not at the start, when the pandemic begins. People who are in classes $S$ or $R$ can pass to other classes during the pandemic. Susceptible people can be infected and go to class $I$. Some models contain a superspreader class (see, e.g., [6]). We assume in our model that all infected people are grouped in class $I$. People who are infected can be hospitalized and go to class $H$. Other people who were slightly sick and do not go to the hospital but can acquire antibodies go to class $V$. We have in this class people having antibodies, meaning vaccinated or people who are recovered.

Let us see that people lose immunity over time. This time is different and depends on many factors, e.g., the type of vaccine. Therefore, people go back in our model to the susceptible class. Our model describes a real situation when people who are recovered can be infectious again or get antibodies after receiving the next dose of the vaccine. We added to our model a new class which does not exist in the literature to the best of the authors' knowledge. This class represents people who were vaccinated but do not have antibodies. As we mention above, this class is not large, but as the research shows, there exist cases for which people do not have immunity after vaccination (see, e.g., [25]). People who have side effects after vaccination can also be in this class. Because part of the population being in class $R$ does not have antibodies, they can be infected easily and can spread the virus. Spreading virus is of course not equivalent to hospitalization. Milder illness may not be noticed by the individuals. More severe illness may cause hospitalization or death.

Parameters $\delta_h$ and $\delta_i(t)$ visible in Fig. 1 are not strategies. Both of them describe disease-induced death rates, the first one due to hospitalization and the second one due to being infected.

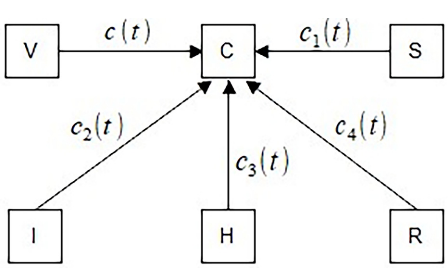


FIG. 2. Flow diagram of the model described by Eq. (7).

Separating class $R$ is important because people who are vaccinated and do not have antibodies can also generate costs. The same applies to vaccinated people being in class $V$. Costs are involved with purchase and distribution of the vaccines. People who are infected also generate costs because of, e.g., isolation of the confirmed cases, and surveillance and serology testing. Costs are generated by hospitalized people, who are under medical care. Pandemic costs are generated by all classes except class $F$ (see Fig. 2) given below. As we see, pandemic costs depend on time because they are not constant and change during the pandemic. The same situation concerns parameters which can also change in time.

Taking into account Figs. 1 and 2 and the above considerations, our mathematical model of COVID-19 disease consists of the seven following differential equations:

$$\frac{dV(t)}{dt} = v(t)S(t) + \gamma_i(t)I(t) + \gamma_r(t)H(t) - \varphi(t)V(t),\ \ t \in [0, T], \tag{1}$$

$$\frac{dR(t)}{dt} = \kappa(t)R(t) - (r_1(t) + r_2(t) + r_3(t))R(t),\ \ t \in [0, T], \tag{2}$$

$$\frac{dS(t)}{dt} = \varphi(t)V(t) - s_1(t)S(t)I(t) - (v(t) + s_1(t))S(t),\ \ t \in [0, T], \tag{3}$$

$$\frac{dI(t)}{dt} = s_1(t)S(t) + r_1(t)R(t) + s_1(t)S(t)I(t) - (\gamma_i(t) + \gamma_a(t) + \delta_i(t))I(t),\ \ t \in [0, T], \tag{4}$$

$$\frac{dH(t)}{dt} = \gamma_a(t)I(t) + r_2(t)R(t) - (\gamma_r(t) + \delta_h)H(t),\ \ t \in [0, T], \tag{5}$$

$$\frac{dF(t)}{dt} = \delta_i(t)I(t) + \delta_h H(t) + r_3(t)R(t),\ \ t \in [0, T], \tag{6}$$

$$\frac{dC(t)}{dt} = c(t)V(t) + c_1(t)S(t) + c_2(t)I(t) + c_3(t)H(t) + c_4(t)R(t),\ \ t \in [0, T], \tag{7}$$

where $\delta_i(t), t \in [0, T]$, is the disease-induced death rate due to the infected individals and $\delta_h$ is the diseaseinduced death rate due to the hospitalized individuals. Simulations made with different parameters (functions) (see Table III) acknowledge that all mentioned strategies have an influence on the behavior of all states. Hence the choice of large numbers of strategies as well as the proposed system of equations describe more exactly the behavior of the pandemic. However, simulations are not sufficient to study them. We need mathematical tools which help us to infer more correct corollaries. To this effect we develop a game-theoretic methodology in Secs. III and IV.

To calculate the basic reproduction number we must first construct some matrices, using a next generation matrix concept from [26]. We must take into account differential equations (1), (2), (4), and (5) because these equations indicate COVID-19 transmissibility. Vector $\mathcal{Z}$, which is given below, concerns classes $V(t)$, $R(t)$, $I(t)$, and $H(t)$ only, which occur in the right-hand sides of Eqs. (1), (2), (4), and (5). This vector is associated to the rate of appearance of new infections. On the other hand, vector $\mathcal{F}$ takes into account class $S(t)$, which is associated to the net rate out of the corresponding compartments. Hence we have the following vectors:

$$\mathcal{F} = \begin{bmatrix} 0 \\ 0 \\ s_1SI \\ 0 \end{bmatrix} \text{ and } \mathcal{Z} = \begin{bmatrix} \varphi V - \gamma_i I - \gamma_r H \\ (r_1 + r_2 + r_3 - \kappa)R \\ -r_1R + (\gamma_a + \gamma_i + \delta_i)I \\ -r_2R - \gamma_a I + (\gamma_r + \delta_h)H \end{bmatrix}.$$

From above we get the following matrices:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s_1SI & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} \varphi V & 0 & -\gamma_i I & -\gamma_r H \\ 0 & (r_1 + r_2 + r_3 - \kappa)R & 0 & 0 \\ 0 & -r_1R & (\gamma_a + \gamma_i + \delta_i)I & 0 \\ 0 & -r_2R & -\gamma_a I & (\gamma_r + \delta_h)H \end{bmatrix}.$$

As a consequence we have the following Jacobian matrices:

$$JF = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } JZ = \begin{bmatrix} \varphi & 0 & -\gamma_i & -\gamma_r \\ 0 & r_1 + r_2 + r_3 - \kappa & 0 & 0 \\ 0 & -r_1 & \gamma_a + \gamma_i + \delta_i & 0 \\ 0 & -r_2 & -\gamma_a & \gamma_r + \delta_h \end{bmatrix}.$$

The basic reproduction number $R_0$ is computed as the spectral radius of $F Z^{-1}$ and is given by

$$R_0(t) = \frac{s_1(t)}{\gamma_a(t) + \gamma_i(t) + \delta_i(t)}. \tag{8}$$

We see from Eq. (8) that the strategies $s_1$, $\gamma_a$, and $\gamma_i$ and parameter $\delta_i$ have influence on the basic reproduction number. The factor $R_0$ will be smaller, while $s_1$ is smaller and $\gamma_a$, $\gamma_i$, and $\delta_i$ are greater (see Fig. 3).

Denote $x = (V, R, S, I, H, F, C)$ and $u = (v, \kappa, r_1, r_2, r_3, s_1, \gamma_a, \gamma_i, \gamma_r, \varphi, c, c_1, c_2, c_3, c_4)$. Let us write the right-hand sides of Eqs. (1)–(7) in the following form:

$$\begin{aligned} f_1(t, V, S, I, H, v, \gamma_i, \gamma_r, \varphi) &= vS + \gamma_i I + \gamma_r H - \varphi V, \\ f_2(t, R, \kappa, r_1, r_2, r_3) &= (\kappa - r_1 - r_2 - r_3)R, \\ f_3(t, V, S, I, \varphi, s_1, v) &= \varphi V - s_1SI - (v + s_1)S, \\ f_4(t, R, S, I, s_1, r_1, \gamma_i, \gamma_a) &= s_1S + r_1R + s_1SI \\ &\quad - (\gamma_i + \gamma_a + \delta_i)I, \end{aligned}$$

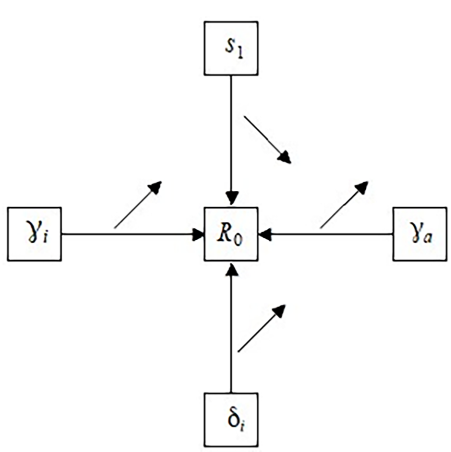


FIG. 3. Impact of the strategies on a basic reproduction number.

$$\begin{aligned} f_5(t, R, I, H, \gamma_a, r_2, \gamma_r) &= \gamma_a I + r_2R - (\gamma_r + \delta_h)H, \\ f_6(t, R, I, H, r_3) &= \delta_i I + \delta_h H + r_3R, \\ f_7(t, V, R, S, I, H, c, c_1, c_2, c_3, c_4) &= cV + c_1S + c_2I + c_3H \\ &\quad + c_4R. \end{aligned}$$

Put

$$\begin{aligned} f(t, x, u) = (&f_1(t, V, S, I, H, v, \gamma_i, \gamma_r, \varphi), \\ &f_2(t, R, \kappa, r_1, r_2, r_3), \\ &f_3(t, V, S, I, \varphi, s_1, v), \\ &f_4(t, R, S, I, s_1, r_1, \gamma_i, \gamma_a), \\ &f_5(t, R, I, H, \gamma_a, r_2, \gamma_r), \quad f_6(t, R, I, H, r_3), \\ &f_7(t, V, R, S, I, H, c, c_1, c_2, c_3, c_4)). \end{aligned}$$

Taking into account the above considerations we can write system of the differential equations (1)–(7) as

$$\frac{dx}{dt} = f(t, x, u). \tag{9}$$

## III. GAME APPROACH TO COVID-19

We have at our disposal 15 strategies $u = (v, \kappa, r_1, r_2, r_3, s_1, \gamma_a, \gamma_i, \gamma_r, \varphi, c, c_1, c_2, c_3, c_4)$. In order to construct a game, we need to divide those strategies for at least two groups and, having these groups of strategies, we should distinguish two players who want to cooperate or not. We decided that our potential players will not cooperate because we do not believe that the virus wants to cooperate with people. Thus the next step in building a game is to define a suitable functional depending on those strategies and suitable states. It is rather obvious that the functional should contain $R_0$ which we want to minimize, and the costs of the pandemic, i.e., the state $C(t)$ and the state $V(t)$, which we want to maximize. In order to control the consumption (costs) of all controls we add to the functional the norm of $u(t)$, i.e.,

$\|u(t)\|$. To justify that $R_0$ should be minimized note that for each fixed $t \in [0, T]$, $R_0(t)$ defines the status of the pandemic: it persists ($R_0(t) > 1$) or it is going to die out ($R_0(t) < 1$). If we look at matrices $F$ and $Z$ from the former section, we see that $R_0$ is computed as the spectral radius of $FZ^{-1}$. That implies that if $V(t)$ is greater, then $R_0(t)$ is smaller. Hence we assume that our cost functional has the form

$$J(x, u) = \int_0^T (V(t) + C(t) + \|u(t)\|)dt + R_0(T). \qquad (10)$$

As we mentioned above, we want to maximize the number of people having antibodies at each step of time and sum them up (the first term in functional (10)) and to minimize the basic reproduction number $R_0$ as well the consumption of the strategies. This approach suggests that classical optimal control theory is not applicable here and that a better way is to apply a kind of noncooperative game theory.

*Remark 1.* Let us see that we can guarantee that increasing $R_0$ and $C$ does not imply maximizing the values of functional (10) and, on the other hand, that decreasing $V$ does not cause this functional to be minimized. It happens because the first player (with fixed opponents' strategies) can always choose his own strategies, which will minimize (10) even when strategies fixed by the second player (one set) cause increasing $R_0$ and $C$. On the other hand, the second player (with fixed opponents' strategies also) can always choose his own strategies which will maximize (10) even when strategies fixed by the first player (one set) cause decreasing $V$.

Functional (10) is subject to z seven-dimensional dynamical system (9) for states

$$x = (V, R, S, I, H, F, C),$$

controlled by a profile of 15 strategies

$$u = (v, \kappa, r_1, r_2, r_3, s_1, \gamma_a, \gamma_i, \gamma_r, \varphi, c, c_1, c_2, c_3, c_4) \qquad (11)$$

over a finite time interval $[0, T]$.

We assume that each strategy $v$, $\kappa$, $r_1$, $r_2$, $r_3$, $s_1$, $\gamma_a$, $\gamma_i$, $\gamma_r$, $\varphi_1$, $c$, $c_1$, $c_2$, $c_3$, $c_4$ satisfies $v(t) \in U_1, \dots, c_4(t) \in U_{15}$, $U_i \subset \mathbb{R}_+$, $i = 1, \dots, 15$, $U = U_1 \times \dots \times U_{15}$. The strategies are measurable functions on $[0, T]$, but additionally we assume that $s_1(t)$, $\gamma_a(t)$, and $\gamma_i(t)$ are continuous in $[0, T]$ and a state equation is formulated as

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), u(t)), \quad t \in [0, T] \\ &x(0) = x_0 \\ &u(t) \in U, \ t \in [0, T], \end{aligned} \qquad (12)$$

with $f : [0, T] \times \mathbb{R}^7 \times \mathbb{R}^{15} \to \mathbb{R}^7$ a given Carathéodory function and the initial state is denoted by the vector $x_0 \in \mathbb{R}^{7+}$. We are searching for solutions to (12) in the space $H^1(0, T; \mathbb{R}^7)$, i.e., absolutely continuous functions with square integrable $\dot{x}(t)$.

The game approach to issues of epidemiology has many examples in the literature. For example, we get to know them in [19] as evolutionary games. We emphasize that the behavior of the people where it concerns vaccinations against disease has produced interest in game approaches (evolutionary games) (see, e.g., [1,11,12,17,18]). The issues of quarantine and isolation policy or a risk of infection also drew some attention (see, e.g., [16,18]). The evolutionary game approach arose from game theory by applying the basic concept of Darwinism. The idea was to compensate for time evolution, which in the original game theory has appeared (as it mainly deals with equilibrium). In that approach game players behave more intelligently and realistically; however, then theory predicts that game players should act defectively. In game theory, a noncooperative game is a game with a competition between individual players. A noncooperative game tries to predict players' individual strategies and payoffs and to find Nash equilibria. It is also more general than cooperative games because cooperative games can be analyzed using the terms of noncooperative game theory. Then it is enough to state sufficient assumptions to encompass all the possible strategies which players may adapt, in relation to the arbitration. We want to consider a noncooperative game for the problem (10)–(12) in which strategies are evaluated in time (are functions) and the cost functional is defined on the set of suitable strategies. That approach is opposite to the papers mentioned earlier, except partially in [18]. It implies that our approach to the game is more general than those presented in [1,11,12,16–18].

To construct a noncooperative game, we divide 15 strategies from (11) across two players, taking into account the influence of them on the behavior of functional (10). The "player" means here only a set of the strategies. The first player $\nu$ should choose those strategies among the 15 which maximize the first term in the functional $J$, i.e., $V$, and can help to minimize the last term, i.e., $R_0$, by making the denominator in (8) larger. It is done by $v$, $\gamma_a$, and $\gamma_i$. Thus we assume that player $\nu$ has at its disposal three strategies $(v, \gamma_a, \gamma_i)$ and hence we denote $\nu = (v, \gamma_a, \gamma_i)$. The second player $\sigma$ should use the strategies from among those remaining from the 15 which can help to minimize (10). Looking carefully at (10) and (11), as well as (8), i.e., the basic reproduction number $R_0(t)$, each of the strategies $\kappa$, $r_1$, $r_2$, $r_3$, $s_1$, $\gamma_r$, $\varphi$, $c$, $c_1$, $c_2$, $c_3$, and $c_4$ has a greater or lesser influence on minimizing functional (10). Therefore, we put $\sigma = (\kappa, r_1, r_2, r_3, s_1, \gamma_r, \varphi, c, c_1, c_2, c_3, c_4)$. Player $\nu$ wants to use the strategies $v$, $\gamma_a$, and $\gamma_i$ to maximize (10) and player $\sigma$ uses the strategies $\kappa$, $r_1$, $r_2$, $r_3$, $s_1$, $\gamma_r$, $\varphi$, $c$, $c_1$, $c_2$, $c_3$, and $c_4$ to minimize (10). For the given player $\nu$, on the opponent $\sigma$'s profile of strategies we write in game terminology

$$u^{\sim\sigma} = u_\nu(v, \gamma_a, \gamma_i) \qquad (13)$$

and for the given player $\sigma$

$$u^{\sim\nu} = u_\sigma(\kappa, r_1, r_2, r_3, s_1, \gamma_r, \varphi, c, c_1, c_2, c_3, c_4). \qquad (14)$$

For a state satisfying (12) with player $\nu$ we put $x^{u^{\sim\nu}}$ for the given opponents' strategy $u^{\sim\nu}$; i.e., $x^{u^{\sim\nu}}$ satisfies

$$\begin{aligned} &\dot{x}^{u^{\sim\nu}}(t) = f(t, x^{u^{\sim\nu}}(t), (u_\nu(t), u^{\sim\nu}(t))), \quad t \in [0, T], \\ &x^{u^{\sim\nu}}(0) = x_0, \quad u_\nu(t) \in U_\nu = U_1 \times U_7 \times U_8, \ t \in [0, T], \\ &u^{\sim\nu}(t) \in U_\sigma = U_2 \times U_3 \times U_4 \times U_5 \times U_6 \times U_9 \times U_{10} \\ &\qquad \times U_{11} \times U_{12} \times U_{13} \times U_{14} \times U_{15}, t \in [0, T]. \end{aligned} \qquad (15)$$

In like manner we follow with $x^{u^{\sim\sigma}}$ for the state satisfying (12) for player $\sigma$ with the given opponents' strategy $u^{\sim\sigma}$; thus $x^{u^{\sim\sigma}}$

satisfies

$$\dot{x}^{u^{\sim\sigma}}(t) = f(t, x^{u^{\sim\sigma}}(t), (u_\sigma(t), u^{\sim\sigma}(t))), \quad t \in [0,T],$$
$$x^{u^{\sim\sigma}}(0) = x_0, \quad u^{\sim\sigma}(t) \in U_\nu = U_1 \times U_7 \times U_8, \ t \in [0,T],$$
$$u_\sigma(t) \in U_\sigma = U_2 \times U_3 \times U_4 \times U_5 \times U_6 \times U_9 \times U_{10} \times U_{11} \times U_{12} \times U_{13} \times U_{14} \times U_{15}, t \in [0,T]. \tag{16}$$

A pair $(u_\nu, x^{u^{\sim\nu}})$ with the strategies of player $\nu$ and the given opponent $\sigma$ we call an admissible process for the game (10) subject to (12). The set of all such pairs we denote by

$$Ad_\nu(u^{\sim\nu}) = \{(u_\nu, x^{u^{\sim\nu}}) : (u_\nu, x^{u^{\sim\nu}}) \text{ satisfies (15)}\}.$$

A pair $(u_\sigma, x^{u^{\sim\sigma}})$ with the strategies of player $\sigma$ and the given opponent $\nu$ we call an admissible process for the game (10) subject to (12). The set of all such pairs we denote by

$$Ad_\sigma(u^{\sim\sigma}) = \{(u_\sigma, x^{u^{\sim\sigma}}) : (u_\sigma, x^{u^{\sim\sigma}}) \text{ satisfies (16)}\}.$$

For the given opponents' strategies $u^{\sim\nu}$ we constitute a differential game with cost functional (10) in the form

$$J(x^{u^{\sim\nu}}, u_\nu) = \int_0^T (V^{u^{\sim\nu}}(t) + C^{u^{\sim\nu}}(t) + \|(u_\nu(t), u^{\sim\nu}(t))\|)dt + \frac{s_1^{\sim\nu}(T)}{\gamma_{a\nu}(T) + \gamma_{i\nu}(T) + \delta_i(T)}, \tag{17}$$

which we maximize in the set $Ad_\nu(u^{\sim\nu})$ and for the given opponents' strategies $u^{\sim\sigma}$ with cost functional (10) in the form

$$J(x^{u^{\sim\sigma}}, u_\sigma) = \int_0^T (V^{u^{\sim\sigma}}(t) + C^{u^{\sim\sigma}}(t) + \|(u_\sigma(t), u^{\sim\sigma}(t))\|)dt + \frac{s_{1\sigma}(T)}{\gamma_a^{\sim\sigma}(T) + \gamma_i^{\sim\sigma}(T) + \delta_i(T)}, \tag{18}$$

which we minimize in the set $Ad_\sigma(u^{\sim\sigma})$.

## IV. A DUAL GAME, $\varepsilon$-CLOSED-LOOP STRATEGIES

The dual game means that we do not consider our objects in the space $[0,T] \times \mathbb{R}^7$, but in a dual set $P$, and our objects satisfy dual dynamic inequalities. Even more, we do not study value functions, but instead we define auxiliary functions satisfying dual dynamic inequalities. Then with the help of auxiliary functions we derive a kind of verification conditions for primal value functions.

Thus, we define dual sets: $P_\nu, P_\sigma \subset \mathbb{R}^7$ of the variables $(t, p)$, $p \in \mathbb{R}^7$, $t \in [0,T]$ and their projections on the space of the variable $p$, $\boldsymbol{P}_\nu$, $\boldsymbol{P}_\sigma$. The last sets are chosen by us. By $W^1(P_\nu)$ and $W^1(P_\sigma)$ we denote the special Sobolev spaces of functions of the variables $(t, p)$ with the following properties: there exists the first order weak derivative with respect to $t$ which is continuous with respect to the variable $p$. We define in $P_\nu$ and $P_\sigma$ dual strategies $\boldsymbol{u}_\nu(t,p) = (\boldsymbol{v}(t,p), \boldsymbol{\gamma}_a(t,p), \boldsymbol{\gamma}_i(t,p))$ and $\boldsymbol{u}_\sigma(t,p) = (\boldsymbol{\kappa}(t,p), \boldsymbol{r_1}(t,p), \boldsymbol{r_2}(t,p), \boldsymbol{r_3}(t,p), \boldsymbol{s}_1(t,p), \boldsymbol{\gamma_r}(t,p), \boldsymbol{\varphi_1}(t,p), \boldsymbol{c}(t,p), \boldsymbol{c_1}(t,p), \boldsymbol{c_2}(t,p), \boldsymbol{c_3}(t,p), \boldsymbol{c_4}(t,p))$ with values in $U_\nu$, $U_\sigma$, respectively. We also use opponents' strategies $\boldsymbol{u}^{\sim\nu}(t, p^{\sim\nu})$ and $\boldsymbol{u}^{\sim\sigma}(t, p^{\sim\sigma})$, defined in $P_\sigma$, $P_\nu$, respectively. We assume that the strategies $\boldsymbol{u}_\nu(t,p)$ and $\boldsymbol{u}_\sigma(t,p)$ are Borel measurable and the states corresponding to them $z_\nu^{\boldsymbol{u}^{\sim\nu}}(\cdot, p)$, $p \in \boldsymbol{P}_\nu$, $p^{\sim\nu} \in \boldsymbol{P}_\sigma$ and $z_\sigma^{\boldsymbol{u}^{\sim\sigma}}(\cdot, p)$, $p \in \boldsymbol{P}_\sigma$, $p^{\sim\sigma} \in \boldsymbol{P}_\nu$ satisfy

$$\dot{z}_\nu^{\boldsymbol{u}^{\sim\nu}}(t,p) = f\big(t, z_\nu^{\boldsymbol{u}^{\sim\nu}}(t,p), (\boldsymbol{u}_\nu(t,p), \boldsymbol{u}^{\sim\nu}(t, p^{\sim\nu}))\big), \quad t \in [0,T],$$
$$z^{\boldsymbol{u}^{\sim\nu}}\big(0, p_0^\nu\big) = x_0, \ \text{for some fixed } p_0^\nu \in \boldsymbol{P}_\nu,$$
$$\boldsymbol{u}_\nu(t,p) \in U_\nu, \ t \in [0,T],$$
$$\boldsymbol{u}^{\sim\nu}(t,p) \in U_\sigma, \ t \in [0,T], \tag{19}$$

$$\dot{z}_\sigma^{\boldsymbol{u}^{\sim\sigma}}(t,p) = f\big(t, z_\sigma^{\boldsymbol{u}^{\sim\sigma}}(t,p), (\boldsymbol{u}_\sigma(t,p), \boldsymbol{u}^{\sim\sigma}(t, p^{\sim\sigma}))\big), \quad t \in [0,T],$$
$$z^{\boldsymbol{u}^{\sim\sigma}}\big(0, p_0^\sigma\big) = x_0, \ \text{for some fixed } p_0^\sigma \in \boldsymbol{P}_\sigma,$$
$$\boldsymbol{u}^{\sim\sigma}(t,p) \in U_\nu, \ t \in [0,T],$$
$$\boldsymbol{u}_\sigma(t,p) \in U_\sigma, \ t \in [0,T], \tag{20}$$

respectively. Having opponents' strategies $\boldsymbol{u}^{\sim\nu}$ and $\boldsymbol{u}^{\sim\sigma}$ we define for them dual $\varepsilon$-closed-loop strategies

$$\mathcal{U}_\nu = \big\{\boldsymbol{u}_\nu : \text{exists } z_\nu^{\boldsymbol{u}^{\sim\nu}}(\cdot, p), \ p \in \boldsymbol{P}_\nu \text{ satisfies (19)}\big\},$$
$$\mathcal{U}_\sigma = \big\{\boldsymbol{u}_\sigma : \text{exists } z_\sigma^{\boldsymbol{u}^{\sim\sigma}}(\cdot, p), \ p \in \boldsymbol{P}_\sigma \text{ satisfies (20)}\big\}.$$

The corresponding trajectories $z_\nu^{\boldsymbol{u}^{\sim\nu}}$ and $z_\sigma^{\boldsymbol{u}^{\sim\sigma}}$ we call the dual trajectories and a set of all of them we denote by $X_d$. By

$$\mathcal{P}_\nu(\boldsymbol{u}^{\sim\nu}) = \big\{p : [0,T] \to \boldsymbol{P}_\nu : \ p(0) = p_0^\nu, \text{exists } \big(\boldsymbol{u}_\nu, z_\nu^{\boldsymbol{u}^{\sim\nu}}\big) \in \mathcal{U}_\nu \times X_d, \ \boldsymbol{u}_\nu(t, p(t)) = u_\nu(t), \ p(\cdot) \text{ is continuous}, z_\nu^{\boldsymbol{u}^{\sim\nu}}(t, p(t)) = x^{u^{\sim\nu}}(t), \ (u_\nu(\cdot), x^{u^{\sim\nu}}(\cdot)) \in Ad_\nu(u^{\sim\nu})\big\}$$

and

$$\mathcal{P}_\sigma(\boldsymbol{u}^{\sim\sigma}) = \big\{p : [0,T] \to \boldsymbol{P}_\sigma : \ p(0) = p_0^\sigma, \text{exists } \big(\boldsymbol{u}_\sigma, z_\sigma^{\boldsymbol{u}^{\sim\sigma}}\big) \in \mathcal{U}_\sigma \times X_d, \ \boldsymbol{u}_\sigma(t, p(t)) = u_\sigma(t), \ p(\cdot) \text{ is continuous}, z_\sigma^{u^{\sim\sigma}}(t, p(t)) = x^{u^{\sim\sigma}}(t), \ (u_\sigma(\cdot), x^{u^{\sim\sigma}}(\cdot)) \in Ad_\sigma(u^{\sim\sigma})\big\}$$

we set the families of auxiliaries trajectories.

Rewrite functionals (17) and (18) in terms of the dual notions

$$J_{dn}(z^{\boldsymbol{u}^{\sim\nu}}, \boldsymbol{u}_\nu) = \int_0^T (\boldsymbol{V}^{\boldsymbol{u}^{\sim\nu}}(t, p(t)) + \boldsymbol{C}^{\boldsymbol{u}^{\sim\nu}}(t, p(t)) + \|(\boldsymbol{u}_\nu(t, p(t)), \boldsymbol{u}^{\sim\nu}(t, p^{\sim\nu}(t)))\|)dt + \frac{\boldsymbol{s}_1^{\sim\nu}(T, p^{\sim\nu}(T))}{\boldsymbol{\gamma}_{a\nu}(T, p(T)) + \boldsymbol{\gamma}_{i\nu}(T, p(T)) + \delta_i(T)}, \tag{21}$$

$$J_{ds}(z^{\boldsymbol{u}^{\sim\sigma}}, \boldsymbol{u}_\sigma) = \int_0^T (\boldsymbol{V}^{\boldsymbol{u}^{\sim\sigma}}(t, p(t)) + \boldsymbol{C}^{\boldsymbol{u}^{\sim\sigma}}(t, p(t)) + \|(\boldsymbol{u}_\sigma(t, p(t)), \boldsymbol{u}^{\sim\sigma}(t, p^{\sim\sigma}(t)))\|)dt + \frac{\boldsymbol{s}_{1\sigma}(T, p(T))}{\boldsymbol{\gamma}_a^{\sim\sigma}(T, p^{\sim\sigma}(T)) + \boldsymbol{\gamma}_i^{\sim\sigma}(T, p^{\sim\sigma}(T)) + \delta_i(T)}, \tag{22}$$

where $p^{\sim\nu}(\cdot) \in \mathcal{P}_\sigma(\boldsymbol{u}^{\sim\nu})$ and $p^{\sim\sigma}(\cdot) \in \mathcal{P}_\nu(\boldsymbol{u}^{\sim\sigma})$.

*Definition 1.* Let us fix $\varepsilon > 0$. A pair $(\bar{\boldsymbol{u}}_\nu, \bar{\boldsymbol{u}}_\sigma)$ we name a dual $\varepsilon$-closed-loop Nash equilibrium, if there exists an auxiliary trajectory $\bar{p}_\nu \in \mathcal{P}_\nu(\boldsymbol{u}^{\sim\nu})$, such that $\bar{u}_\nu(t) = \bar{\boldsymbol{u}}_\nu(t, \bar{p}_\nu(t))$, $t \in [0, T]$, together with the corresponding trajectory $\bar{x}^{\sim\nu}(t) = \bar{z}_\nu^{\bar{\boldsymbol{u}}^{\sim\nu}}(t, \bar{p}_\nu(t))$ belonging to $Ad_\nu(\bar{u}^{\sim\nu})$ and if there exists an auxiliary trajectory $\bar{p}_\sigma \in \mathcal{P}_\sigma(\boldsymbol{u}^{\sim\sigma})$, such that $\bar{u}_\sigma(t) = \bar{\boldsymbol{u}}_\sigma(t, \bar{p}_\sigma(t))$, $t \in [0, T]$, together with the corresponding trajectory $\bar{x}^{\sim\sigma}(t) = \bar{z}_\sigma^{\bar{\boldsymbol{u}}^{\sim\sigma}}(t, \bar{p}_\sigma(t))$ belonging to $Ad_\sigma(\bar{u}^{\sim\sigma})$ and for any $p(\cdot) \in \mathcal{P}_\nu(\bar{\boldsymbol{u}}^{\sim\nu})$ and $p(\cdot) \in \mathcal{P}_\sigma(\bar{\boldsymbol{u}}^{\sim\sigma})$, the following inequalities hold:

$$J_{dn}\big(\bar{z}_\nu^{\bar{\boldsymbol{u}}^{\sim\nu}}(\cdot, \bar{p}_\nu(\cdot)), \bar{\boldsymbol{u}}_\nu(\cdot, \bar{p}(\cdot))\big) \geqslant J_{dn}\big(z^{\bar{\boldsymbol{u}}^{\sim\nu}}(\cdot, p(\cdot)), \bar{\boldsymbol{u}}_\nu(\cdot, p(\cdot))\big) - T\varepsilon \tag{23}$$

and

$$J_{ds}\big(\bar{z}_\sigma^{\bar{\boldsymbol{u}}^{\sim\sigma}}(\cdot, \bar{p}_\sigma(\cdot)\big), \bar{\boldsymbol{u}}_\sigma(\cdot, \bar{p}_\sigma(\cdot))) \leqslant J_{ds}\big(z^{\bar{\boldsymbol{u}}^{\sim\sigma}}(\cdot, p(\cdot)\big), \bar{\boldsymbol{u}}_\sigma(\cdot, p(\cdot))) + T\varepsilon. \tag{24}$$

For dual game (21) and (22) we derive existence conditions for a dual $\varepsilon$-closed-loop Nash equilibrium. To this effect we define $\varepsilon$-dual Hamilton-Jacobi inequalities and auxiliary pairs of functions $(y_\nu(t), W_\nu(t, p))$, $t \in [0, T]$, $p \in \boldsymbol{P}_\nu$, $y_\nu \in L^1(0, T)$, $W_\nu \in W^1(P_\nu)$, $(y_\sigma(t), W_\sigma(t, p))$, $t \in [0, T]$, $p \in \boldsymbol{P}_\sigma$, $y_\sigma \in L^1(0, T)$, $W_\sigma \in W^1(P_\sigma)$. First we assume that the opponents' dual $\varepsilon$-closed-loop strategy $\boldsymbol{u}^{\sim\nu}$ is given. We assume that, for given $p^{\sim\nu}$, there exists a pair $(y_\nu(t), W_\nu(t, p))$, $t \in [0, T]$, $p \in \boldsymbol{P}_\nu$ satisfying in $[0, T] \times \boldsymbol{P}_\nu$ the following dual $\varepsilon$-dynamic programming differential inequality

$$\begin{aligned} y_\nu(t) + \varepsilon \geqslant \sup\Big\{ & p(pW_{\nu,t}(t, p)) \\ & + pf(t, -pW_\nu(t, p), u^\nu, \boldsymbol{u}^{\sim\nu}(t, p^{\sim\nu})) \\ & - p_1 W_\nu(t, p) - p_7 W_\nu(t, p) + \|(u^\nu, \boldsymbol{u}^{\sim\nu}(t, p^{\sim\nu})\| \\ & + \frac{1}{T} \cdot \frac{\boldsymbol{s}_1^{\sim\nu}(T, p^{\sim\nu})}{\gamma_a + \gamma_i + \delta_i(T)} : u^\nu \in U_\nu \Big\} \end{aligned} \tag{25}$$

with the initial condition

$$-p_0^\nu W_\nu(0, p_0^\nu) = x_0,\ \ p_0^\nu \in \boldsymbol{P}_\nu \text{ the same as in (19).}$$

We assume that for the given opponents' dual $\varepsilon$-closed-loop strategy $\boldsymbol{u}^{\sim\sigma}$ and $p^{\sim\sigma}$ the pair $(y_\sigma(t), W_\sigma(t, p))$, $t \in [0, T]$, $p \in \boldsymbol{P}_\sigma$ satisfies in $[0, T] \times \boldsymbol{P}_\sigma$ the following $\varepsilon$-dynamic programming differential inequality:

$$\begin{aligned} y_\sigma(t) - \varepsilon \leqslant \inf\Big\{ & p(pW_{\sigma,t}(t, p)) \\ & + pf(-pW_\sigma(t, p), u^\sigma, \boldsymbol{u}^{\sim\sigma}(t, p^{\sim\sigma})) \\ & - p_1 W_\sigma(t, p) - p_7 W_\sigma(t, p) + \|(u^\sigma, \boldsymbol{u}^{\sim\sigma}(t, p^{\sim\sigma})\| \\ & + \frac{1}{T} \cdot \frac{s_1}{\gamma_a^{\sim\sigma}(T, p^{\sim\sigma}) + \gamma_i^{\sim\sigma}(T, p^{\sim\sigma}) + \delta_i(T)} : \\ & \times\ u^\sigma \in U_\sigma \Big\} \end{aligned} \tag{26}$$

with the initial condition

$$-p_0^\sigma W_\sigma\big(0, p_0^\sigma\big) = x_0,\ \ p_0^\sigma \in \boldsymbol{P}_\sigma \text{ is the same as in (20).}$$

## V. VERIFICATION THEOREM FOR DUAL $\varepsilon$-CLOSED-LOOP NASH EQUILIBRIUM

We prove a verification theorem allowing to check whether an approximate dual $\varepsilon$-closed-loop strategy (calculated numerically) is a candidate to be the $\varepsilon$-Nash equilibrium for the game (21) and (22) subject to (19) and (20).

*Theorem 1.* Let us take any dual $\varepsilon$-closed-loop strategies $(\bar{\boldsymbol{u}}_\nu, \bar{\boldsymbol{u}}_\sigma)$ and auxiliary trajectories $\bar{p}_\nu \in \mathcal{P}_\nu(\bar{\boldsymbol{u}}^{\sim\nu})$, $\bar{p}_\sigma \in \mathcal{P}_\sigma(\bar{\boldsymbol{u}}^{\sim\sigma})$, such that $(\bar{\boldsymbol{u}}_\nu(\cdot, \bar{p}_\nu(\cdot)), \bar{z}_\nu^{\bar{\boldsymbol{u}}^{\sim\nu}}(\cdot, \bar{p}_\nu(\cdot))) \in Ad_\nu(\bar{\boldsymbol{u}}^{\sim\nu})$ and $(\bar{\boldsymbol{u}}_\sigma(\cdot, \bar{p}_\sigma(\cdot)), \bar{z}_\sigma^{\bar{\boldsymbol{u}}^{\sim\sigma}}(\cdot, \bar{p}_\sigma(\cdot))) \in Ad_\sigma(\bar{\boldsymbol{u}}^{\sim\sigma})$ (according to (14) and (13), $\bar{\boldsymbol{u}}^{\sim\nu} = \bar{\boldsymbol{u}}_\sigma$ and $\bar{\boldsymbol{u}}^{\sim\sigma} = \bar{\boldsymbol{u}}_\nu$). Assume that there exists a pair $(\bar{y}_\nu(t), \bar{W}_\nu(t, p))$, $t \in [0, T]$, $p \in \boldsymbol{P}_\nu$ satisfying in $[0, T] \times \boldsymbol{P}_\nu$ (25) with "sup" attained at $\bar{p}_\nu(\cdot)$ for $\bar{\boldsymbol{u}}_\nu(\cdot, \bar{p}_\nu(\cdot))$ with opponents' strategy $\bar{\boldsymbol{u}}^{\sim\nu}$ satisfying

$$\begin{aligned} & \bar{p}(t)(\bar{p}(t)\bar{W}_{\nu,t}(t, \bar{p}(t))) \\ & \quad + \bar{p}(t)f(-\bar{p}(t)\bar{W}_\nu(t, \bar{p}(t)), \bar{\boldsymbol{u}}_\nu(t, \bar{p}(t)), \bar{\boldsymbol{u}}^{\sim\nu}(t, \bar{p}^{\sim\nu}(t))) \\ & \quad - \bar{p}_1(t)W_\nu(t, \bar{p}(t)) - \bar{p}_7(t)W_\nu(t, \bar{p}(t)) \\ & \quad + \|(\bar{\boldsymbol{u}}_\nu(t, \bar{p}(t)), \boldsymbol{u}^{\sim\nu}(t, \bar{p}^{\sim\nu}(t))\| \\ & \quad + \frac{1}{T} \cdot \frac{\boldsymbol{s}_1^{\sim\nu}(T, \bar{p}^{\sim\nu}(T))}{\boldsymbol{\gamma}_a(T, \bar{p}(T)) + \boldsymbol{\gamma}_i(T, \bar{p}(T)) + \delta_i(T)} \geqslant \bar{y}_\nu(t). \end{aligned} \tag{27}$$

Assume also that there exists a pair $(\bar{y}_\sigma(t), \bar{W}_\sigma(t, p))$, $t \in [0, T]$, $p \in \boldsymbol{P}_\sigma$ satisfying in $[0, T] \times \boldsymbol{P}_\sigma$ (26) with "inf" attained at $\bar{p}_\sigma(\cdot)$ for $\bar{\boldsymbol{u}}_\sigma(\cdot, \bar{p}_\sigma(\cdot))$ with opponents' strategy $\bar{\boldsymbol{u}}^{\sim\sigma}$ satisfying

$$\begin{aligned} & \bar{p}(t)(\bar{p}(t)\bar{W}_{\sigma,t}(t, \bar{p}(t))) \\ & \quad + \bar{p}(t)f(-\bar{p}(t)\bar{W}_\sigma(t, \bar{p}(t)), \bar{\boldsymbol{u}}_\sigma(t, \bar{p}(t)), \bar{\boldsymbol{u}}^{\sim\sigma}(t, \bar{p}^{\sim\sigma}(t))) \\ & \quad - \bar{p}_1(t)W_\sigma(t, \bar{p}(t)) - \bar{p}_7(t)W_\sigma(t, \bar{p}(t)) \\ & \quad + \|(\bar{\boldsymbol{u}}_\sigma(t, \bar{p}(t)), \boldsymbol{u}^{\sim\sigma}(t, \bar{p}^{\sim\sigma}(t))\| \\ & \quad + \frac{1}{T} \cdot \frac{\boldsymbol{s}_1^{\sim\sigma}(T, \bar{p}^{\sim\sigma}(T))}{\boldsymbol{\gamma}_a(T, \bar{p}(T)) + \boldsymbol{\gamma}_i(T, \bar{p}(T)) + \delta_i(T)} \leqslant \bar{y}_\sigma(t). \end{aligned} \tag{28}$$

Moreover, assume that

$$\bar{z}_\nu^{\bar{\boldsymbol{u}}^{\sim\nu}}(t, p) = -p\bar{W}_\nu(t, p),\ \ t \in [0, T],\ \ p \in \boldsymbol{P}_\nu, \tag{29}$$

$$\bar{z}_\sigma^{\bar{\boldsymbol{u}}^{\sim\sigma}}(t, p) = -p\bar{W}_\sigma(t, p),\ \ t \in [0, T],\ \ p \in \boldsymbol{P}_\sigma. \tag{30}$$

Then, the dual $\varepsilon$-closed-loop strategies $(\bar{\boldsymbol{u}}_\nu, \bar{\boldsymbol{u}}_\sigma)$ are the Nash equilibria for the game (21) and (22) subject to (19) and (20).

*Proof.* We have to show only that for $(\bar{\boldsymbol{u}}_\nu(\cdot, \bar{p}_\nu(\cdot)), \bar{z}_\nu^{\bar{\boldsymbol{u}}^{\sim\nu}}(\cdot, \bar{p}_\nu(\cdot))) \in Ad_\nu(\bar{\boldsymbol{u}}^{\sim\nu})$ and $(\bar{\boldsymbol{u}}_\sigma(\cdot, \bar{p}_\sigma(\cdot)), \bar{z}_\sigma^{\bar{\boldsymbol{u}}^{\sim\sigma}}(\cdot, \bar{p}_\sigma(\cdot))) \in Ad_\sigma(\bar{\boldsymbol{u}}^{\sim\sigma})$ inequalities (23) and (24) hold. We prove inequality (23) because proof of the second one is analogous. Let us take any $p(\cdot) \in \mathcal{P}_\nu(\bar{\boldsymbol{u}}^{\sim\nu})$, put it in (25) in the place of $p$, and assume $p^{\sim\nu} = \bar{p}^{\sim\nu}(t)$. Then, for $t \in [0, T]$, we get

$$\begin{aligned} \bar{y}_\nu(t) + \varepsilon \geqslant\ & p(t)(p(t)\bar{W}_{\nu,t}(t, p(t))) \\ & + p(t)f(-p(t)\bar{W}_\nu(t, p(t)), \bar{\boldsymbol{u}}_\nu(t, p(t)), \bar{\boldsymbol{u}}^{\sim\nu} \\ & \times (t, \bar{p}^{\sim\nu}(t))) - p_1(t)W_\nu(t, p(t)) - p_7(t)W_\nu \\ & \times (t, p(t)) + \|(\bar{\boldsymbol{u}}_\nu(t, p(t)), \boldsymbol{u}^{\sim\nu}(t, p^{\sim\nu}(t))\| \\ & + \frac{1}{T} \cdot \frac{\boldsymbol{s}_1^{\sim\nu}(T, p^{\sim\nu}(T))}{\gamma_a(T) + \gamma_i(T) + \delta_i(T)}. \end{aligned} \tag{31}$$

Assuming (29) along $p(t)$, we can transform (31) to

$$\begin{aligned}\bar{y}_\nu(t)+\varepsilon \geqslant & -p(t)\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,t}(t,p(t))\\ & + p(t)f(\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu}(t,p(t)),\bar{\boldsymbol{u}}_\nu(t,p(t)),\bar{\boldsymbol{u}}^{\sim\nu}(t,\bar{p}^{\sim\nu}(t)))\\ & + \bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,1}(t,p(t)) + \bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,7}(t,p(t))\\ & + \|(\bar{\boldsymbol{u}}_\nu(t,p(t)),\boldsymbol{u}^{\sim\nu}(t,p^{\sim\nu}(t))\|\\ & + \frac{1}{T}\cdot\frac{\boldsymbol{s}_1^{\sim\nu}(T,p^{\sim\nu}(T))}{\gamma_a(T)+\gamma_i(T)+\delta_i(T)},\end{aligned} \tag{32}$$

where $\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,1}$ is the first coordinate of $\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu}$ and $\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,7}$ is the seventh coordinate of $\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu}$. Taking into account that $\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu}(t,p(t))$ satisfies (19) and integrating (32) in the interval $[0,T]$ we get

$$\begin{aligned}\int_0^T \bar{y}_\nu(t)dt+T\varepsilon \geqslant & \int_0^T (\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,1}(t,p(t))+\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,7}(t,p(t))\\ & + \|(\bar{\boldsymbol{u}}_\nu(t,p(t)),\boldsymbol{u}^{\sim\nu}(t,p^{\sim\nu}(t))\|)dt\\ & + \frac{\boldsymbol{s}_1^{\sim\nu}(T,p^{\sim\nu}(T))}{\gamma_a(T)+\gamma_i(T)+\delta_i(T)}.\end{aligned} \tag{33}$$

Because “sup” in (25) is attained at $\bar{p}_\nu(\cdot)$ for $\bar{\boldsymbol{u}}_\nu(\cdot,\bar{p}_\nu(\cdot))$, by (27), we get the inequality

$$\begin{aligned}&\bar{p}(t)(\bar{p}(t)\bar{W}_{\nu,t}(t,\bar{p}(t)))\\ &\quad + \bar{p}(t)f(-\bar{p}(t)\bar{W}_\nu(t,\bar{p}(t)),\bar{\boldsymbol{u}}_\nu(t,\bar{p}(t)),\bar{\boldsymbol{u}}^{\sim\nu}(t,\bar{p}^{\sim\nu}(t)))\\ &\quad - \bar{p}_1(t)W_\nu(t,\bar{p}(t)) - \bar{p}_7(t)W_\nu(t,\bar{p}(t))\\ &\quad + \|(\bar{\boldsymbol{u}}_\nu(t,\bar{p}(t)),\boldsymbol{u}^{\sim\nu}(t,\bar{p}^{\sim\nu}(t))\|\\ &\quad + \frac{1}{T}\cdot\frac{\boldsymbol{s}_1^{\sim\nu}(T,\bar{p}^{\sim\nu}(T))}{\boldsymbol{\gamma}_a(T,\bar{p}(T))+\boldsymbol{\gamma}_i(T,\bar{p}(T))+\delta_i(T)} \geqslant \bar{y}_\nu(t).\end{aligned}$$

Proceeding as above, i.e., using (29), the fact that $\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu}(t,p(t))$ satisfies (19), and integrating, we get the inequality

$$\begin{aligned}&\int_0^T (\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,1}(t,\bar{p}(t))+\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu,7}(t,\bar{p}(t))\\ &\quad + \|(\bar{\boldsymbol{u}}_\nu(t,\bar{p}(t)),\boldsymbol{u}^{\sim\nu}(t,\bar{p}^{\sim\nu}(t))\|)dt\\ &\quad + \frac{\boldsymbol{s}_1^{\sim\nu}(T,\bar{p}^{\sim\nu}(T))}{\boldsymbol{\gamma}_a(T,\bar{p}(T))+\boldsymbol{\gamma}_i(T,\bar{p}(T))+\delta_i(T)} \geqslant \int_0^T \bar{y}_\nu(t)dt.\end{aligned} \tag{34}$$

Comparing (33) and (34) we get

$$\begin{aligned}J_{dn}\big(\bar{z}^{\bar{\boldsymbol{u}}^{\sim\nu}}_{\nu}(\cdot,\bar{p}_\nu(\cdot)),\bar{\boldsymbol{u}}_\nu(\cdot,\bar{p}_\nu(\cdot))\big) \geqslant J_{dn}\big(z^{\bar{\boldsymbol{u}}^{\sim\nu}}(\cdot,p(\cdot)),\bar{\boldsymbol{u}}_\nu(\cdot,p(\cdot))\big)\\ - T\varepsilon,\end{aligned} \tag{35}$$

i.e., the first inequality in the definition of the dual $\varepsilon$-closed-loop Nash equilibrium for the game (23) and (24). The second one is analogous; thus the theorem is proved. ■

## VI. NUMERICAL ALGORITHM

Now we construct a numerical algorithm whose aim is to find suspected strategies (which can be optimal) and verify them using Theorem 1. These strategies are grouped into two players: $\nu$ and $\sigma$. Both of them participate in a noncooperative differential game having $M > 0$ sets of 15 strategies. Based on $M$ sets of three strategies chosen by player $\nu$ and one set of 12 (fixed) strategies of the player $\sigma$, we find $M$ values of the functional (17). At the same time, based on $M$ sets of 12 strategies chosen by player $\sigma$ and one set of three (fixed) strategies of the player $\nu$, we find $M$ values of the functional (18). To do this, we calculate $M$ times the basic reproduction number from (8) and find $M$ values of the population having antibodies and also $M$ values of the pandemic costs, solving $M$ times the system of differential equations (1)–(7) for both players. We find the maximal value of functional (17) among $M$ values of this functional found earlier and also find the minimal value of functional (18) among $M$ values of this functional found earlier. Hence we found 15 strategies for player $\nu$ corresponding to the maximal value of functional (17) and found also 15 strategies for player $\sigma$ corresponding to the minimal value of functional (18). The strategies found in this way are called *suspected optimal*. Having these functions, we build dual $\varepsilon$-closed-loop strategies which are based on the suspected optimal strategies (see steps 6.1–6.2.3 below). Next we find some auxiliary functions. To do this, we solve differential equations (25) and (26), using strategies found previously. In the last step we check inequalities (29) and (30), substituting all functions calculated previously. If these inequalities are satisfied, it means that we found the best strategies which define the dual $\varepsilon$-closed-loop Nash equilibrium.

A Nash equilibrium guarantees that the strategies chosen by the first player are optimal (and with the strategies fixed by the second player) and simultaneously the strategies chosen by the second player (and with the strategies fixed by the first player) are also optimal.

From a medical point of view, the first player chooses their own optimal strategies in order to maximize the part of the population having antibodies and the second player chooses their own optimal strategies in order to minimize the basic reproduction number. Let us see that neither the first player nor the second one wins by changing only one's own strategies. It means that neither maximizing the number of the population having antibodies, $V$, nor minimizing the basic reproduction number $R_0$ is the optimal approach. It is because both players win at the same time (choosing ther own optimal strategies). Hence the most expected situation in order to overcome the pandemic relies on maximizing $V$ and minimizing $R_0$ at the same time.

Below the precise steps of our numerical algorithm are given.

**Algorithm.** Algorithm checking Verification Theorem 1.

**Step 1**
Take time $T > 0$, the number of strategies $M > 0$ and $\varepsilon > 0$.
**Step 2.1**
**for** $j = 1$ to $M$ by 1 **do**
Fix $u_{\nu j} = \{v_j(t), \gamma_{a j}(t), \gamma_{i j}(t)\}$, $t \in [0,T]$, such that $v_j(t) \in U_1$, $\gamma_{a j}(t) \in U_7$, $\gamma_{i j}(t) \in U_8$ and for fixed $u^{\sim\nu} = \{\kappa(t), r_1(t), r_2(t), r_3(t), s_1(t), \gamma_r(t), \varphi(t), c(t), c_1(t), c_2(t), c_3(t), c_4(t)\}$, $t \in [0,T]$, such that $\kappa(t) \in U_2$, $r_1(t) \in U_3$, $r_2(t) \in U_4$, $r_3(t) \in U_5$, $s_1(t) \in U_6$, $\gamma_r(t) \in U_9$, $\varphi(t) \in U_{10}$, $c(t) \in U_{11}$, $c_1(t) \in U_{12}$, $c_2(t) \in U_{13}$, $c_3(t) \in U_{14}$, $c_4(t) \in U_{15}$ solve differential equation (9) which describes system (1)–(7) with initial conditions $x_j(0) = x_{0j}$, where $x_{0j} \in \mathbb{R}^7$, finding $V_j$ and $C_j$.
**end for**

**Algorithm.** *(Continued.)*

**Step 2.2**
**for** $j = 1$ to $M$ by 1 **do**
Fix $u_{\sigma j} = \{\kappa_j(t), r_{1j}(t), r_{2j}(t), r_{3j}(t), s_{1j}(t), \gamma_{rj}(t), \varphi_j(t), c_j(t), c_{1j}(t), c_{2j}(t), c_{3j}(t), c_{4j}(t)\}$, $t \in [0, T]$, such that $\kappa_j(t) \in U_2$, $r_{1j}(t) \in U_3$, $r_{2j}(t) \in U_4$, $r_{3j}(t) \in U_5$, $s_{1j}(t) \in U_6$, $\gamma_{rj}(t) \in U_9$, $\varphi_j(t) \in U_{10}$, $c_j(t) \in U_{11}$, $c_{1j}(t) \in U_{12}$, $c_{2j}(t) \in U_{13}$, $c_{3j}(t) \in U_{14}$, $c_{4j}(t) \in U_{15}$ and for fixed $u^{\sim\sigma} = \{v(t), \gamma_a(t), \gamma_i(t)\}$, $t \in [0, T]$, such that $v(t) \in U_1$, $\gamma_a(t) \in U_7$, $\gamma_i(t) \in U_8$ solve differential equation (9) which describes system (1)–(7) with initial conditions $x_j(0) = x_{0j}$, where $x_{0j} \in \mathbb{R}^7$, finding $V_j$ and $C_j$.
**end for**
**Step 3.1**
**for** $j = 1$ to $M$ by 1 **do**
Calculate $R_{0j}(t)$, $t \in [0, T]$, from (8) for strategies from step 2.1.
**end for**
**Step 3.2**
**for** $j = 1$ to $M$ by 1 **do**
Calculate $R_{0j}(t)$, $t \in [0, T]$, from (8) for strategies from step 2.2.
**end for**
**Step 4.1**
**for** $j = 1$ to $M$ by 1 **do**
For strategies found in step 2.1 and for suitable $R_{0j}(t)$, $t \in [0, T]$, found in step 3.1, calculate $\int_0^T (V_j^{u^{\sim\nu}}(t) + C_j^{u^{\sim\nu}}(t) + \|(u_{\nu j}(t), u^{\sim\nu}(t))\|)$, $t \in [0, T]$, where $V_j^{u^{\sim\nu}}$ and $C_j^{u^{\sim\nu}}$ are solutions of (9) found in step 2.1. For these values calculate values of the functional $J(x_j^{u^{\sim\nu}}, u_{\nu j})$ given in (17), where $x_j^{u^{\sim\nu}}$ is a state variable for player $\nu_j$ for the opponents' strategy $u^{\sim\nu}$.
**end for**
**Step 4.2**
**for** $j = 1$ to $M$ by 1 **do**
For strategies found in step 2.2 and for suitable $R_{0j}(t)$, $t \in [0, T]$, found in step 3.2, calculate $\int_0^T (V_j^{u^{\sim\sigma}}(t) + C_j^{u^{\sim\sigma}}(t) + \|(u_{\sigma j}(t), u^{\sim\sigma}(t))\|)$, $t \in [0, T]$, where $V_j^{u^{\sim\sigma}}$ and $C_j^{u^{\sim\sigma}}$ are solutions of (9) found in step 2.2. For these values calculate values of the functional $J(x_j^{u^{\sim\sigma}}, u_{\sigma j})$ given in (18), where $x_j^{u^{\sim\sigma}}$ is a state variable for player $\sigma_j$ for the opponents' strategy $u^{\sim\sigma}$.
**end for**
**Step 5.1**
**for** $j = 1$ to $M$ by 1 **do**
For strategies found in step 2.1, find the maximal value of the functional $J(x_j^{u^{\sim\nu}}, u_{\nu j})$ among those found in step 4.1.
**end for**
Find strategies corresponding to the maximal value of the functional given above $u_\nu^s = \{v^s(t), \gamma_a^s(t), \gamma_i^s(t)\}$, $t \in [0, T]$, and $u_s^{\sim\nu} = \{\kappa^s(t), r_1^s(t), r_2^s(t), r_3^s(t), s_1^s(t), \gamma_r^s(t), \varphi^s(t), c^s(t), c_1^s(t), c_2^s(t), c_3^s(t), c_4^s(t)\}$, $t \in [0, T]$. We name these strategies *suspected $\varepsilon$-optimal* (see Table I). (see the basic reproduction number $R_0$ in Fig. 4 for the best strategies).
**Step 5.2**
**for** $j = 1$ to $M$ by 1 **do**
For strategies found in step 2.2, find the minimal value of the functional $J(x_j^{u^{\sim\sigma}}, u_{\sigma j})$ among those found in step 4.2.
**end for**
Find strategies corresponding to the minimal value of the functional given above
$u_\sigma^s = \{\kappa^s(t), r_1^s(t), r_2^s(t), r_3^s(t), s_1^s(t), \gamma_r^s(t), \varphi^s(t), c^s(t), c_1^s(t), c_2^s(t), c_3^s(t), c_4^s(t)\}$, $t \in [0, T]$ and $u_s^{\sim\sigma} = \{v^s(t), \gamma_a^s(t), \gamma_i^s(t)\}$, $t \in [0, T]$. We name these strategies *suspected $\varepsilon$-optimal* (see Table II). (see part of the population having antibodies in Fig. 5 for the best strategies).
**Step 6.1**
Choose a set $\boldsymbol{P}_\nu$ and number of vectors $p_j \in \boldsymbol{P}_\nu$ as $K > 0$.
**Step 6.1.1**
**for** $j = 1$ to $K$ by 1 **do**
Fix $p_{0j}^\nu \in \boldsymbol{P}_\nu$ and build a set $\boldsymbol{u}_{\nu j}(t, p_j) = \{\boldsymbol{v}_j(t, p_j), \boldsymbol{\gamma_a}_j(t, p_j), \boldsymbol{\gamma_i}_j(t, p_j)\}$ which consists of the suspected strategies found in step 5.1 in the following way: $\boldsymbol{v}_j(t, p_j) = v^s(t)p_{1j}p_{2j}$, $\boldsymbol{\gamma_a}_j(t, p_j) = \gamma_a^s(t)p_{5j}p_{6j}$, $\boldsymbol{\gamma_i}_j(t, p_j) = \gamma_i^s(t)p_{7j}p_{1j}$, where $p_{1j}, \dots, p_{7j}$ are coordinates of the vectors $p_j \in \boldsymbol{P}_\nu$.
**end for**
**Step 6.1.2**
**for** $j = 1$ to $K$ by 1 **do**
Fix $p_{0j}^\nu \in \boldsymbol{P}_\nu$ and build a set $\boldsymbol{u}_j^{\sim\nu}(t, p_j) = \{\boldsymbol{\kappa}_j(t, p_j), \boldsymbol{r_1}_j(t, p_j), \boldsymbol{r_2}_j(t, p_j), \boldsymbol{r_3}_j(t, p_j), \boldsymbol{s_1}_j(t, p_j), \boldsymbol{\gamma_r}_j(t, p_j), \boldsymbol{\varphi}_j(t, p_j), \boldsymbol{c}_j(t, p_j), \boldsymbol{c_1}_j(t, p_j), \boldsymbol{c_2}_j(t, p_j), \boldsymbol{c_3}_j(t, p_j), \boldsymbol{c_4}_j(t, p_j)\}$ which consists of the suspected strategies by using step 5.2 and according to (14) $u_s^{\sim\nu} = u_\sigma^s$ in the following way:
$\boldsymbol{\kappa}_j(t, p_j) = \kappa^s(t)p_{1j}p_{2j}$, $\boldsymbol{r_1}_j(t, p_j) = r_1^s(t)p_{2j}p_{3j}$,
$\boldsymbol{r_2}_j(t, p_j) = r_2^s(t)p_{3j}p_{4j}$, $\boldsymbol{r_3}_j(t, p_j) = r_3^s(t)p_{4j}p_{5j}$,
$\boldsymbol{s_1}_j(t, p_j) = s_1^s(t)p_{6j}p_{7j}$, $\boldsymbol{\gamma_r}_j(t, p_j) = \gamma_r^s(t)p_{7j}p_{1j}$,
$\boldsymbol{\varphi}_j(t, p_j) = \varphi^s(t)p_{1j}p_{2j}$, $\boldsymbol{c}_j(t, p_j) = c^s(t)p_{2j}p_{3j}$,
$\boldsymbol{c_1}_j(t, p_j) = c_1^s(t)p_{3j}p_{4j}$, $\boldsymbol{c_2}_j(t, p_j) = c_2^s(t)p_{4j}p_{5j}$,
$\boldsymbol{c_3}_j(t, p_j) = c_3^s(t)p_{5j}p_{6j}$, $\boldsymbol{c_4}_j(t, p_j) = c_4^s(t)p_{6j}p_{7j}$, where $p_{1j}, \dots, p_{7j}$ are coordinates of the vectors $p_j \in \boldsymbol{P}_\nu$.
**end for**
**Step 6.1.3**
**for** $j = 1$ to $K$ by 1 **do**
Solve (19) for strategies $\boldsymbol{u}_{\nu j}(t, p_j)$ and $\boldsymbol{u}_j^{\sim\nu}(t, p_j)$ and with fixed initial conditions $z_j^{u^{\sim\nu}}(t, p_j) = x_{0j}$, where $x_{0j}$ are the same vectors as chosen in step 2.1, to find $z_j^{u^{\sim\nu}}(t, p_j)$.
**end for**
**Step 6.2**
Choose a set $\boldsymbol{P}_\sigma$ and number of vectors $p_j \in \boldsymbol{P}_\sigma$ as $K > 0$.
**Step 6.2.1**
**for** $j = 1$ to $K$ by 1 **do**
Fix $p_{0j}^\sigma \in \boldsymbol{P}_\sigma$ and build a set $\boldsymbol{u}_{\sigma j}(t, p_j) = \{\boldsymbol{\kappa}_j(t, p_j), \boldsymbol{r_1}_j(t, p_j), \boldsymbol{r_2}_j(t, p_j), \boldsymbol{r_3}_j(t, p_j), \boldsymbol{s_1}_j(t, p_j), \boldsymbol{\gamma_r}_j(t, p_j), \boldsymbol{\varphi}_j(t, p_j), \boldsymbol{c}_j(t, p_j), \boldsymbol{c_1}_j(t, p_j), \boldsymbol{c_2}_j(t, p_j), \boldsymbol{c_3}_j(t, p_j), \boldsymbol{c_4}_j(t, p_j)\}$ which consists of the suspected strategies found in step 5.2 in the following way:
$\boldsymbol{\kappa}_j(t, p_j) = \kappa^s(t)p_{1j}p_{2j}$, $\boldsymbol{r_1}_j(t, p_j) = r_1^s(t)p_{2j}p_{3j}$,
$\boldsymbol{r_2}_j(t, p_j) = r_2^s(t)p_{3j}p_{4j}$, $\boldsymbol{r_3}_j(t, p_j) = r_3^s(t)p_{4j}p_{5j}$,
$\boldsymbol{s_1}_j(t, p_j) = s_1^s(t)p_{6j}p_{7j}$, $\boldsymbol{\gamma_r}_j(t, p_j) = \gamma_r^s(t)p_{7j}p_{1j}$,
$\boldsymbol{\varphi}_j(t, p_j) = \varphi^s(t)p_{1j}p_{2j}$, $\boldsymbol{c}_j(t, p_j) = c^s(t)p_{2j}p_{3j}$,
$\boldsymbol{c_1}_j(t, p_j) = c_1^s(t)p_{3j}p_{4j}$, $\boldsymbol{c_2}_j(t, p_j) = c_2^s(t)p_{4j}p_{5j}$,
$\boldsymbol{c_3}_j(t, p_j) = c_3^s(t)p_{5j}p_{6j}$, $\boldsymbol{c_4}_j(t, p_j) = c_4^s(t)p_{6j}p_{7j}$, where $p_{1j}, \dots, p_{7j}$ are coordinates of the vectors $p_j \in \boldsymbol{P}_\sigma$.
**end for**

**Algorithm.** *(Continued.)*

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**Step 6.2.2**
**for** $j = 1$ to $K$ by 1 **do**
Fix $p_{0\,j}^{\sigma} \in \boldsymbol{P}_{\sigma}$ and build a set $\boldsymbol{u}_j^{\sim\sigma}(t, p_j) = \{\boldsymbol{v}_j(t, p_j), \boldsymbol{\gamma}_{a\,j}(t, p_j), \boldsymbol{\gamma}_{i\,j}(t, p_j)\}$ which consists of the suspected strategies by using step 5.1 and according to (13) $u_s^{\sim\sigma} = u_v^s$ in the following way: $\boldsymbol{v}_j(t, p_j) = v^s(t) p_{1\,j} p_{2\,j}$, $\boldsymbol{\gamma}_{a\,j}(t, p_j) = \gamma_a^s(t) p_{5\,j} p_{6\,j}$, $\boldsymbol{\gamma}_{i\,j}(t, p_j) = \gamma_i^s(t) p_{7\,j} p_{1\,j}$, where $p_{1\,j}, \ldots, p_{7\,j}$ are coordinates of the vectors $p_j \in \boldsymbol{P}_{\sigma}$.
**end for**
**Step 6.2.3**
**for** $j = 1$ to $K$ by 1 **do**
Solve (20) for strategies $\boldsymbol{u}_{\sigma\,j}(t, p_j)$ and $\boldsymbol{u}_j^{\sim\sigma}(t, p_j)$ and with fixed initial conditions $\boldsymbol{z}_j^{u^{\sim\sigma}}(t, p_j) = x_{0\,j}$, where $x_{0\,j}$ are the same vectors as chosen in step 2.2, to find $z_j^{u^{\sim\sigma}}(t, p_j)$.
**end for**
**Step 7.1**
**for** $j = 1$ to $K$ by 1 **do**
Find a pair of functions $y_{\nu\,j}$ and $W_{\nu\,j}$ solving differential equation (25) for the fixed initial conditions $-p_{0\,j}^{\nu} W_{\nu\,j}(0, p_{0\,j}^{\nu}) = x_{0\,j}$, where $x_{0\,j}$ are the same vectors chosen in step 2.1, $p_{0\,j}^{\nu}$ are the same vectors chosen in step 6.1, for the strategies $\boldsymbol{u}_{\nu\,j}(t, p_j)$ and $\boldsymbol{u}_j^{\sim\nu}(t, p_j)$ found in step 6.1 and for $R_{0\,j}(\boldsymbol{u}_{\nu\,j}(t, p_j), \boldsymbol{u}_j^{\sim\nu}(t, p_j^{\sim\nu}))$.
**end for**
**Step 7.2**
**for** $j = 1$ to $K$ by 1 **do**
Find a pair of functions $y_{\sigma\,j}$ and $W_{\sigma\,j}$ solving differential equation (26) for the fixed initial conditions $-p_{0\,j}^{\sigma} W_{\sigma\,j}(0, p_{0\,j}^{\sigma}) = x_{0\,j}$, where $x_{0\,j}$ are the same vectors chosen in step 2.2, $p_{0\,j}^{\sigma}$ are the same vectors chosen in step 6.2, for the strategies $\boldsymbol{u}_{\sigma\,j}(t, p_j)$ and $\boldsymbol{u}_j^{\sim\sigma}(t, p_j)$ found in step 6.2 and for $R_{0\,j}(\boldsymbol{u}_{\sigma\,j}(t, p_j), \boldsymbol{u}_j^{\sim\sigma}(t, p_j^{\sim\sigma}))$.
**end for**
**Step 8.1**
**for** $j = 1$ to $K$ by 1 **do**
Check equality (29) for $z_j^{u^{\sim\nu}}$ which were found in step 6.1 and for $W_{\nu\,j}$ which were found in step 7.1.
**end for**
**Step 8.2**
**for** $j = 1$ to $K$ by 1 **do**
Check equality (30) for $z_j^{u^{\sim\sigma}}$ which were found in step 6.2 and for $W_{\sigma\,j}$ which were found in step 7.2.
**end for**
**Step 9**
**for** $j = 1$ to $K$ by 1 **do**
**if** equalities in (29) and (30) are satisfied **then**
Finish algorithm because Verification Theorem 1 guarantees that $\bar{\boldsymbol{u}}_{\nu\,j} = \boldsymbol{u}_{\nu\,j}$ and $\bar{\boldsymbol{u}}_{\sigma\,j} = \boldsymbol{u}_{\sigma\,j}$ are $\varepsilon$-optimal strategies.
**else**
Repeat steps 1–9.
**end if**
**end for**

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### A. Practical example realizing numerical algorithm

We use in this numerical example real data from Poland from 8 May 2021 and use them to repeat the steps presented in the numerical algorithm in Section VI: $V(t) = 6\,142\,989$ (which is 16.16% of the whole population) having antibodies, $S(t) = 25\,357\,400$ (which is 66.73% of the whole population) susceptible, $I(t) = 2\,829\,196$ (which is 7.44% of the whole population) symptomatic and infectious, $H(t) = 17\,155$ (which is 0.04% of the whole population) hospitalized, and $F(t) = 69\,866$ (which is 0.18% of the whole population), fatality.

The coefficients $\delta_i$ and $\delta_h$ correspond to the situation in Wuhan (see, e.g., [6]) because we were unable to calculate them.

We take percentages only for the acceleration of the numerical calculations. Hence we have the following initial conditions: $V(0) = 0.1616$, $R(0) = 0.0942$, $S(0) = 0.6673$, $I(0) = 0.0744$, $H(0) = 0.0004$, $F(0) = 0.0018$, and $C(0) = 0$, where $V(0) + R(0) + S(0) + I(0) + H(0) + F(0) = 1$ (which means 100% of the population).

(1) Take $T = 1$ (which means one year), $\varepsilon = 0.1$, $\delta_i = 0.01$, $\delta_h = 0.3$ and fix

$$x_0 = (V(0), R(0), S(0), I(0), H(0), F(0), C(0)).$$

(2.1) Take $M = 5$ of the fixed three strategies $u_{\nu\,j} = \{v_j(t), \gamma_{a\,j}(t), \gamma_{i\,j}(t)\}$, $t \in [0, 1]$, $j = 1, \ldots, 5$ for player $u_\nu$ such that $v_j(t) \in (0, 0.6)$, $\gamma_{a\,j}(t) \in (0, 1.7)$, $\gamma_{i\,j} \in (0, 1.3)$ and for $M = 5$ of the twelve fixed strategies $u^{\sim\nu} = \{\kappa(t), r_1(t), r_2(t), r_3(t), s_1(t), \gamma_r(t), \varphi(t), c(t), c_1(t), c_2(t), c_3(t), c_4(t)\}$, $t \in [0, 1]$, such that $\kappa(t) \in (0, 1.1)$, $r_1(t) \in (0, 1)$, $r_2(t) \in (0, 0.4)$, $r_3(t) \in (0, 0.2)$, $s_1(t) \in (0, 2.8)$, $\gamma_r(t) \in (0, 1)$, $\varphi(t) \in (0, 0.7)$, $c(t) \in (0, 1.6)$, $c_1(t) \in (0, 0.1)$, $c_2(t) \in (0, 0.6)$, $c_3(t) \in (0, 1.5)$,

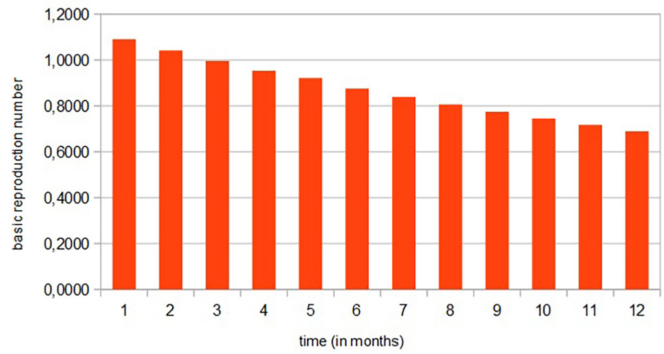


FIG. 4. Basic reproduction number $R_0$ for the best strategies given above.

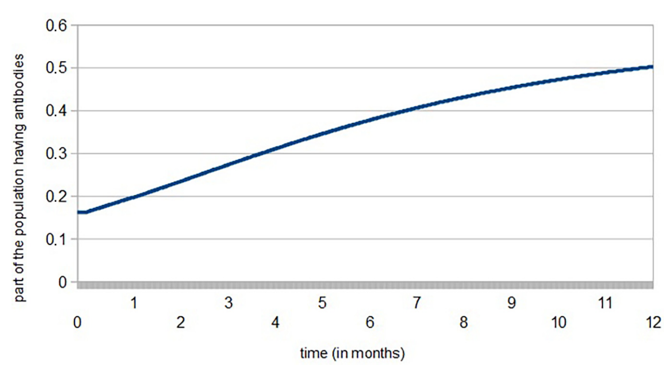


FIG. 5. Part of the population having antibodies for the best strategies given above.

TABLE I. Suspected $\varepsilon$-optimal strategies, for which functional (17) is maximized.

| | [0, 0.1) | [0.1, 0.2) | [0.2, 0.3) | [0.3, 0.4) | [0.4, 0.5) | [0.5, 0.6) | [0.6, 0.7) | [0.7, 0.8) | [0.8, 0.9) | [0.9, 1) | [1, 1.1) | [1.1, 1.2] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 1.0000 | 1.1000 | 1.2000 | 1.3000 | 1.4000 | 1.5000 | 1.6000 | 1.7000 | 1.8000 | 1.9000 | 2.0000 | 2.1000 |
| $\gamma_a$ | 1.1000 | 1.1500 | 1.2000 | 1.2500 | 1.3000 | 1.3500 | 1.4000 | 1.4500 | 1.5000 | 1.5500 | 1.6000 | 1.6500 |
| $\gamma_i$ | 1.0000 | 1.0250 | 1.0500 | 1.0750 | 1.1000 | 1.1250 | 1.1500 | 1.1750 | 1.2000 | 1.2250 | 1.2500 | 1.2750 |
| $\kappa$ | 0.7500 | 0.7600 | 0.7700 | 0.7800 | 0.7900 | 0.8000 | 0.8100 | 0.8200 | 0.8300 | 0.8400 | 0.8500 | 0.8600 |
| $r_1$ | 0.6000 | 0.6100 | 0.6200 | 0.6300 | 0.6400 | 0.6500 | 0.6600 | 0.6700 | 0.6800 | 0.6900 | 0.7000 | 0.7100 |
| $r_2$ | 0.1000 | 0.1050 | 0.1100 | 0.1150 | 0.1200 | 0.1250 | 0.1300 | 0.1350 | 0.1400 | 0.1450 | 0.1500 | 0.1550 |
| $r_3$ | 0.0500 | 0.0510 | 0.0520 | 0.0530 | 0.0540 | 0.0550 | 0.0560 | 0.0570 | 0.0580 | 0.0590 | 0.0600 | 0.0610 |
| $s_1$ | 2.3000 | 2.2750 | 2.2500 | 2.2250 | 2.2000 | 2.1750 | 2.1500 | 2.1250 | 2.1000 | 2.0750 | 2.0500 | 2.0250 |
| $\gamma_r$ | 0.8000 | 0.8050 | 0.8100 | 0.8150 | 0.8200 | 0.8250 | 0.8300 | 0.8350 | 0.8400 | 0.8450 | 0.8500 | 0.8550 |
| $\varphi$ | 0.5000 | 0.5050 | 0.5100 | 0.5150 | 0.5200 | 0.5250 | 0.5300 | 0.5350 | 0.5400 | 0.5450 | 0.5500 | 0.5550 |
| $c$ | 1.5000 | 1.4750 | 1.4500 | 1.4250 | 1.4000 | 1.3750 | 1.3500 | 1.3250 | 1.3000 | 1.2750 | 1.2500 | 1.2250 |
| $c_1$ | 0.0500 | 0.0475 | 0.0450 | 0.0425 | 0.0400 | 0.0375 | 0.0350 | 0.0325 | 0.0300 | 0.0275 | 0.0250 | 0.0225 |
| $c_2$ | 0.5000 | 0.4800 | 0.4600 | 0.4400 | 0.4200 | 0.4000 | 0.3800 | 0.3600 | 0.3400 | 0.3200 | 0.3000 | 0.2800 |
| $c_3$ | 1.0000 | 0.9750 | 0.9500 | 0.9250 | 0.9000 | 0.8750 | 0.8500 | 0.8250 | 0.8000 | 0.7750 | 0.7500 | 0.7250 |
| $c_4$ | 0.8000 | 0.7500 | 0.7000 | 0.6500 | 0.6000 | 0.5500 | 0.5000 | 0.4500 | 0.4000 | 0.3500 | 0.3000 | 0.2500 |

$c_4(t) \in (0, 0.9)$ (see [27] for all of the tested strategies) solve five times the system of differential equations (1)–(7) with the initial condition fixed in step 1.

(2.2) Take $M = 5$ of the fixed twelve strategies $u_{\sigma j} = \{\kappa_j(t), r_{1j}(t), r_{2j}(t), r_{3j}(t), s_{1j}(t), \gamma_{rj}(t), \varphi_j(t), c_j(t), c_{1j}(t), c_{2j}(t), c_{3j}(t), c_{4j}(t)\}$, $t \in [0, 1]$, $j = 1, \dots, 5$ for player $u_\sigma$ such that $\kappa_j(t) \in (0, 1.1)$, $r_{1j}(t) \in (0, 1)$, $r_{2j}(t) \in (0, 0.4)$, $r_{3j}(t) \in (0, 0.2)$, $s_{1j}(t) \in (0, 2.8)$, $\gamma_{rj}(t) \in (0, 1)$, $\varphi_j(t) \in (0, 0.7)$, $c_j(t) \in (0, 1.6)$, $c_{1j}(t) \in (0, 0.1)$, $c_{2j}(t) \in (0, 0.6)$, $c_{3j}(t) \in (0, 1.5)$, $c_{4j}(t) \in (0, 0.9)$ and for $M = 5$ of the fixed three strategies $u^{\sim\sigma} = \{v(t), \gamma_a(t), \gamma_i(t)\}$, $t \in [0, 1]$, such that $v(t) \in (0, 0.6)$, $\gamma_a(t) \in (0, 1.7)$, $\gamma_i \in (0, 1.3)$ (see [27] for all of the tested strategies) solve five times the system of differential equations (1)–(7) with the initial condition fixed in step 1.

(3.1) For strategies from step 2.1 and from (8) we have $R_{01}(T) = 0.9485$, $R_{02}(T) = 0.8672$, $R_{03}(T) = 0.7988$, $R_{04}(T) = 0.7404$, and $R_{05}(T) = 0.6899$.

(3.2) For strategies from step 2.2 and from (8) we have $R_{01}(T) = 0.6899$, $R_{02}(T) = 0.7240$, $R_{03}(T) = 0.7581$, $R_{04}(T) = 0.7922$, and $R_{05}(T) = 0.8262$.

(4.1) For strategies found in step 2.1, for suitable $R_{0j}(T)$, $j = 1, \dots, 5$, found in step 3.1, calculate $\int_0^T (V_1^{u^{\sim v}}(t) + C_1^{u^{\sim v}}(t) + \|(u_{v1}(t), u^{\sim v}(t))\|) = 4.8652$, $\int_0^T (V_2^{u^{\sim v}}(t) + C_2^{u^{\sim v}}(t) + \|(u_{v2}(t), u^{\sim v}(t))\|) = 4.9772$, $\int_0^T (V_3^{u^{\sim v}}(t) + C_3^{u^{\sim v}}(t) + \|(u_{v3}(t), u^{\sim v}(t))\|) = 5.1013$, $\int_0^T (V_4^{u^{\sim v}}(t) + C_4^{u^{\sim v}}(t) + \|(u_{v4}(t), u^{\sim v}(t))\|) = 5.2279$, $\int_0^T (V_5^{u^{\sim v}}(t) + C_5^{u^{\sim v}}(t) + \|(u_{v5}(t), u^{\sim v}(t))\|) = 5.3602$, where $V_j$ and $C_j$, $j = 1, \dots, 5$, are solutions of (9) found in step 2.1. For these values $J(x_1^{u^{\sim v}}, u_{v1}) = 5.8137$, $J(x_2^{u^{\sim v}}, u_{v2}) = 5.8444$, $J(x_3^{u^{\sim v}}, u_{v3}) = 5.9001$, $J(x_4^{u^{\sim v}}, u_{v4}) = 5.9683$, and $J(x_5^{u^{\sim v}}, u_{v5}) = 6.0501$.

(4.2) For strategies found in step 2.2, for suitable $R_{0j}(T)$, $j = 1, \dots, 5$, found in step 3.2, calculate $\int_0^T (V_1^{u^{\sim\sigma}}(t) + C_1^{u^{\sim\sigma}}(t) + \|(u_{\sigma 1}(t), u^{\sim\sigma}(t))\|) = 5.3602$, $\int_0^T (V_2^{u^{\sim\sigma}}(t) + C_2^{u^{\sim\sigma}}(t) + \|(u_{\sigma 2}(t), u^{\sim\sigma}(t))\|) = 5.7288$, $\int_0^T (V_3^{u^{\sim\sigma}}(t) + C_3^{u^{\sim\sigma}}(t) + \|(u_{\sigma 3}(t), u^{\sim\sigma}(t))\|) = 5.9033$, $\int_0^T (V_4^{u^{\sim\sigma}}(t) + C_4^{u^{\sim\sigma}}(t) + \|(u_{\sigma 4}(t), u^{\sim\sigma}(t))\|) = 6.0707$, $\int_0^T (V_5^{u^{\sim\sigma}}(t) + C_5^{u^{\sim\sigma}}(t) + \|(u_{\sigma 5}(t), u^{\sim\sigma}(t))\|) = 6.2371$, where $V_j$ and $C_j$, $j = 1, \dots, 5$, are solutions of (9) found in step 2.2. For these values $J(x_1^{u^{\sim\sigma}}, u_{\sigma 1}) = 6.0501$, $J(x_2^{u^{\sim\sigma}}, u_{\sigma 2}) = 6.4528$, $J(x_3^{u^{\sim\sigma}}, u_{\sigma 3}) = 6.6614$, $J(x_4^{u^{\sim\sigma}}, u_{\sigma 4}) = 6.8629$, and $J(x_5^{u^{\sim\sigma}}, u_{\sigma 5}) = 7.0633$.

(5.1) For strategies found in step 2.1, the maximal value of the functional among those found in step 4.1 is $J(x_5^{u^{\sim v}}, u_{v5}) = 6.0501$. Strategies which correspond to the maximal value of the functional given above are given in Table I. These strategies are *suspected $\varepsilon$-optimal*.

(5.2) For strategies found in step 2.2, the minimal value of the functional among those found in step 4.2 is $J(x_1^{u^{\sim\sigma}}, u_{\sigma 1}) = 6.0501$. Strategies which correspond to the minimal value of the functional given above are given in Table II. These strategies are *suspected $\varepsilon$-optimal*.

(6.1) Choose a set $\boldsymbol{P_v} = \{p_j \in \mathbb{R}^7 : -0.0001 \leqslant p_j^i \leqslant 0.0001,\ p^i \text{ coordinate of } p_j\}$, take a discreet set $\boldsymbol{P_v^d} = \{p_j^i \in \mathbb{R} : -0.0001 \leqslant p_j^i < 0.0001, j = 1, 2, 3\}$, $i = 1, \dots, 7$, such that $\boldsymbol{P_v^d} \subset \boldsymbol{P_v}$, and take the following vectors:

$$p_1 = (-0.0001, -0.0001, -0.0001, -0.0001, -0.0001, -0.0001, -0.0001),$$

$$p_2 = (-0.00001, -0.00001, -0.00001, -0.00001, -0.00001, -0.00001, -0.00001),$$

$$p_3 = (0.0001, 0.0001, 0.0001, 0.0001, \times\ 0.0001, 0.0001, 0.0001),$$

$p_j \in \boldsymbol{P_v^d}$, $j = 1, 2, 3$.

(6.1.1) Having suspected strategies found in step 5.1 and vectors $p_j$, $j = 1, 2, 3$, we build a set $\boldsymbol{v}_j(t, p_j) = v^s(t)p_{1j}p_{2j}$, $\boldsymbol{\gamma_a}_j(t, p_j) = \gamma_a^s(t)p_{5j}p_{6j}$, $\boldsymbol{\gamma_i}_j(t, p_j) = \gamma_i^s(t)p_{7j}p_{1j}$, where $p_{1j}, \dots, p_{7j}$ are coordinates of the vectors $p_j \in \boldsymbol{P}_v$.

(6.1.2) Having suspected strategies found in step 5.1 and vectors $p_j$, $j = 1, 2, 3$, we build a set $\boldsymbol{\kappa}_j(t, p_j) = \kappa^s(t)p_{1j}p_{2j}$,

TABLE II. Suspected $\varepsilon$-optimal strategies, for which functional (18) is minimized.

| | [0, 0.1) | [0.1, 0.2) | [0.2, 0.3) | [0.3, 0.4) | [0.4, 0.5) | [0.5, 0.6) | [0.6, 0.7) | [0.7, 0.8) | [0.8, 0.9) | [0.9, 1) | [1, 1.1) | [1.1, 1.2] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 1.0000 | 1.1000 | 1.2000 | 1.3000 | 1.4000 | 1.5000 | 1.6000 | 1.7000 | 1.8000 | 1.9000 | 2.0000 | 2.1000 |
| $\gamma_a$ | 1.1000 | 1.1500 | 1.2000 | 1.2500 | 1.3000 | 1.3500 | 1.4000 | 1.4500 | 1.5000 | 1.5500 | 1.6000 | 1.6500 |
| $\gamma_i$ | 1.0000 | 1.0250 | 1.0500 | 1.0750 | 1.1000 | 1.1250 | 1.1500 | 1.1750 | 1.2000 | 1.2250 | 1.2500 | 1.2750 |
| $\kappa$ | 0.7500 | 0.7600 | 0.7700 | 0.7800 | 0.7900 | 0.8000 | 0.8100 | 0.8200 | 0.8300 | 0.8400 | 0.8500 | 0.8600 |
| $r_1$ | 0.6000 | 0.6100 | 0.6200 | 0.6300 | 0.6400 | 0.6500 | 0.6600 | 0.6700 | 0.6800 | 0.6900 | 0.7000 | 0.7100 |
| $r_2$ | 0.1000 | 0.1050 | 0.1100 | 0.1150 | 0.1200 | 0.1250 | 0.1300 | 0.1350 | 0.1400 | 0.1450 | 0.1500 | 0.1550 |
| $r_3$ | 0.0500 | 0.0510 | 0.0520 | 0.0530 | 0.0540 | 0.0550 | 0.0560 | 0.0570 | 0.0580 | 0.0590 | 0.0600 | 0.0610 |
| $s_1$ | 2.3000 | 2.2750 | 2.2500 | 2.2250 | 2.2000 | 2.1750 | 2.1500 | 2.1250 | 2.1000 | 2.0750 | 2.0500 | 2.0250 |
| $\gamma_r$ | 0.8000 | 0.8050 | 0.8100 | 0.8150 | 0.8200 | 0.8250 | 0.8300 | 0.8350 | 0.8400 | 0.8450 | 0.8500 | 0.8550 |
| $\varphi$ | 0.5000 | 0.5050 | 0.5100 | 0.5150 | 0.5200 | 0.5250 | 0.5300 | 0.5350 | 0.5400 | 0.5450 | 0.5500 | 0.5550 |
| $c$ | 1.5000 | 1.4750 | 1.4500 | 1.4250 | 1.4000 | 1.3750 | 1.3500 | 1.3250 | 1.3000 | 1.2750 | 1.2500 | 1.2250 |
| $c_1$ | 0.0500 | 0.0475 | 0.0450 | 0.0425 | 0.0400 | 0.0375 | 0.0350 | 0.0325 | 0.0300 | 0.0275 | 0.0250 | 0.0225 |
| $c_2$ | 0.5000 | 0.4800 | 0.4600 | 0.4400 | 0.4200 | 0.4000 | 0.3800 | 0.3600 | 0.3400 | 0.3200 | 0.3000 | 0.2800 |
| $c_3$ | 1.0000 | 0.9750 | 0.9500 | 0.9250 | 0.9000 | 0.8750 | 0.8500 | 0.8250 | 0.8000 | 0.7750 | 0.7500 | 0.7250 |
| $c_4$ | 0.8000 | 0.7500 | 0.7000 | 0.6500 | 0.6000 | 0.5500 | 0.5000 | 0.4500 | 0.4000 | 0.3500 | 0.3000 | 0.2500 |

$\boldsymbol{r_1}_j(t, p_j) = r_1^s(t)p_{2j}p_{3j}$, $\boldsymbol{r_2}_j(t, p_j) = r_2^s(t)p_{3j}p_{4j}$, $\boldsymbol{r_3}_j(t, p_j) = r_3^s(t)p_{4j}p_{5j}$, $\boldsymbol{s_1}_j(t, p_j) = s_1^s(t)p_{6j}p_{7j}$, $\boldsymbol{\gamma_r}_j(t, p_j) = \gamma_r^s(t)p_{7j}p_{1j}$, $\boldsymbol{\varphi}_j(t, p_j) = \varphi^s(t)p_{1j}p_{2j}$, $\boldsymbol{c}_j(t, p_j) = c^s(t)p_{2j}p_{3j}$, $\boldsymbol{c_1}_j(t, p_j) = c_1^s(t)p_{3j}p_{4j}$, $\boldsymbol{c_2}_j(t, p_j) = c_2^s(t)p_{4j}p_{5j}$, $\boldsymbol{c_3}_j(t, p_j) = c_3^s(t)p_{5j}p_{6j}$, $\boldsymbol{c_4}_j(t, p_j) = c_4^s(t)p_{6j}p_{7j}$ which consists of the suspected strategies and according to (14) $u_s^{\sim v} = u_\sigma^s$.

(6.1.3) Fix initial conditions $z_j^{u\sim v} = x_{0j}$, $j = 1, 2, 3$, and solve differential equation (19) for the 15 strategies found above.

(6.2) Choose a set $\boldsymbol{P_\sigma} = \{p_j \in \mathbb{R}^7 : -0.0001 \leqslant p_j^i \leqslant 0.0001,\ p^i \text{ coordinate of } p_j\}$, take a discreet set $\boldsymbol{P_\sigma^d} = \{p_j^i \in \mathbb{R} : -0.0001 \leqslant p_j^i < 0.0001, j = 1, 2, 3\}$, $i = 1, \ldots, 7$, such that $\boldsymbol{P_\sigma^d} \subset \boldsymbol{P_\sigma}$, and take the following vectors:

$$p_1 = (-0.0001, -0.0001, -0.0001, -0.0001, -0.0001, -0.0001, -0.0001),$$

$$p_2 = (-0.00001, -0.00001, -0.00001, -0.00001, -0.00001, -0.00001, -0.00001),$$

$$p_3 = (0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001),$$

$$p_j \in \boldsymbol{P_\sigma^d},\ j = 1, 2, 3.$$

(6.2.1) Having suspected strategies found in step 5.2 and vectors $p_j$, $j = 1, 2, 3$, we build a set $\boldsymbol{\kappa}_j(t, p_j) = \kappa^s(t)p_{1j}p_{2j}$, $\boldsymbol{r_1}_j(t, p_j) = r_1^s(t)p_{2j}p_{3j}$, $\boldsymbol{r_2}_j(t, p_j) = r_2^s(t)p_{3j}p_{4j}$, $\boldsymbol{r_3}_j(t, p_j) = r_3^s(t)p_{4j}p_{5j}$, $\boldsymbol{s_1}_j(t, p_j) = s_1^s(t)p_{6j}p_{7j}$, $\boldsymbol{\gamma_r}_j(t, p_j) = \gamma_r^s(t)p_{7j}p_{1j}$, $\boldsymbol{\varphi}_j(t, p_j) = \varphi^s(t)p_{1j}p_{2j}$, $\boldsymbol{c}_j(t, p_j) = c^s(t)p_{2j}p_{3j}$, $\boldsymbol{c_1}_j(t, p_j) = c_1^s(t)p_{3j}p_{4j}$, $\boldsymbol{c_2}_j(t, p_j) = c_2^s(t)p_{4j}p_{5j}$, $\boldsymbol{c_3}_j(t, p_j) = c_3^s(t)p_{5j}p_{6j}$, $\boldsymbol{c_4}_j(t, p_j) = c_4^s(t)p_{6j}p_{7j}$, where $p_{1j}, \ldots, p_{7j}$ are coordinates of the vectors $p_j \in \boldsymbol{P}_\sigma$.

(6.2.2) Having suspected strategies found in step 5.2 and vectors $p_j$, $j = 1, 2, 3$, we build a set $\boldsymbol{v}_j(t, p_j) = v^s(t)p_{1j}p_{2j}$, $\boldsymbol{\gamma_a}_j(t, p_j) = \gamma_a^s(t)p_{5j}p_{6j}$, $\boldsymbol{\gamma_i}_j(t, p_j) = \gamma_i^s(t)p_{7j}p_{1j}$, which consists of the suspected strategies and according to (13) $u_s^{\sim\sigma} = u_v^s$.

(6.2.3) Fix initial conditions $z_j^{u\sim\sigma} = x_{0j}$, $j = 1, 2, 3$, and solve differential equation (20) for 15 strategies found above.

(7.1) For fixed initial conditions $-p_{0j}^v W_{vj}(0, p_{0j}^v) = x_{0j}$, $j = 1, 2, 3$, where $x_{0j}$ are the same vectors chosen in step 2.1, and $p_{0j}^v$ are the same vectors chosen in step 6.1, for the strategies $\boldsymbol{u}_{vj}(t, p_j)$, $\boldsymbol{u}_j^{\sim v}(t, p_j)$ found in step 6.1 and $y_v = 1$, solve differential equation (25).

(7.2) For fixed initial conditions $-p_{0j}^\sigma W_{\sigma j}(0, p_{0j}^\sigma) = x_{0j}$, $j = 1, 2, 3$, where $x_{0j}$ are the same vectors chosen in step 2.2, and $p_{0j}^\sigma$ are the same vectors chosen in step 6.2, for the strategies $\boldsymbol{u}_{\sigma j}(t, p_j)$, $\boldsymbol{u}_j^{\sim\sigma}(t, p_j)$ found in step 6.2 and $y_\sigma = 1$, solve differential equation (26).

(8.1) Having solutions $z_j^{u\sim v}$ found in step 6.1 and for $W_{vj}$, $j = 1, 2, 3$, from step 7.1, check inequality (29) (see [27]). For vectors $p_1$, $p_2$, and $p_3$ given above we have $z_{vj} + p_j W_{vj} < 0.67$, $j = 1, 2, 3$, which shows that equality (29) holds with accuracy 0.67.

(8.2) Having solutions $z_j^{u\sim\sigma}$ found in step 6.2 and for $W_{\sigma j}$, $j = 1, 2, 3$, from step 7.2, check inequality (30) (see [27]). For

TABLE III. Interpretation of the results.

| | $v$ | $\gamma_a$ | $\gamma_i$ | $\kappa$ | $r_1$ | $r_2$ | $r_3$ | $s_1$ | $\gamma_r$ | $\varphi$ | $c$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $R_0$ ↘ | × | ↗ | ↗ | × | × | × | × | ↘ | × | × | × | × | × | × | × |
| $V$ ↗ | ↗ | × | ↗ | ↗ | × | ↗ | × | ↗ | ↗ | ↘ | ↗ | ↗ | ↗ | × | ↗ |
| $H$ ↘ | × | ↘ | ↘ | ↘ | × | ↘ | × | ↘ | ↗ | × | × | × | × | × | × |
| $F$ ↘ | × | ↘ | × | ↘ | × | ↘ | ↘ | × | × | × | × | × | × | × | × |
| $C$ ↘ | ↘ | ↘ | ↘ | ↘ | × | ↘ | × | ↘ | × | ↗ | ↘ | ↘ | ↘ | ↘ | ↘ |

vectors $p_1$, $p_2$, and $p_3$ given above we have $z_{\sigma j} + p_j W_{\sigma j} < 0.67$, $j = 1, 2, 3$, which shows that equality (30) holds with accuracy 0.67.

(9) Because equalities (29) and (30) are satisfied, we finish this algorithm.

## VII. INTERPRETATION OF THE RESULTS

It is worth asking how to choose the strategies to minimize the basic reproduction number, the part of the population being hospitalized, fatality cases, and costs of the pandemic and how to maximize part of the population having antibodies.

Based on multiple tests done in MATLAB, we see that decreasing strategy $s_1(t)$ or increasing strategies $\gamma_a(t)$ or $\gamma_i(t)$ causes a decrease in the basic reproduction number $R_0(t)$. The other strategies have no effect on $R_0(t)$, because these strategies are not included in (8) which allows us to compute $R_0(t)$.

To increase the part of the population having antibodies, we should increase strategies $v(t)$, $\gamma_i(t)$, $\kappa(t)$, $r_2(t)$, $s_1(t)$, $\gamma_r(t)$, $c(t)$, $c_1(t)$, $c_2(t)$, or $c_4(t)$ or decrease strategy $\varphi(t)$. Changing strategies $\gamma_a(t)$, $r_1(t)$, or $c_3(t)$ has no effect on class $V$.

We can decrease the part of the population being hospitalized by minimizing strategies $\gamma_a(t)$, $\gamma_i(t)$, $\kappa(t)$, $r_2(t)$, or $s_1(t)$ or maximizing strategy $\gamma_r(t)$. Changing the other strategies has no effect on class $H$.

Decreasing strategies $\gamma_a(t)$, $\kappa(t)$, $r_2(t)$, or $r_3(t)$ causes a decrease in fatality cases. The other strategies do not affect class $F$.

We are also interested in minimizing costs of the pandemic. We do this by decreasing strategies $v(t)$, $\gamma_a(t)$, $\gamma_i(t)$, $\kappa(t)$, $r_2(t)$, $s_1(t)$, $c(t)$, $c_1(t)$, $c_2(t)$, $c_3(t)$, or $c_4(t)$ or increasing strategy $\varphi(t)$. Strategies $r_1(t)$, $r_3(t)$, and $\gamma_r(t)$ have no effect on class $C$.

We enclose the above considerations in Table III. As we see, the basic reproduction number $R_0$ decreases, when only strategy $s_1(t)$ decreases (symbol $\searrow$) and when $\gamma_a(t)$ or $\gamma_i(t)$ increase (symbol $\nearrow$). In a similar way we interpret the growth of population $V$ having antibodies and the decreasing fraction of the population being hospitalized, cases, and pandemic costs. The symbol $\times$ means that changing the given strategy has no effect on $R_0$, $V$, $H$, $F$, or $C$.

It is worth noticing that the values of the strategies selected above confirm natural intuition because, e.g., we expect growth of the population having antibodies as the values for strategy $v(t)$ increase. We expect also that pandemic costs are lower when we decrease strategies regarding costs, that is, $c(t)$, $c_1(t)$, $c_2(t)$, $c_3(t)$, or $c_4(t)$.

It may be interesting to compare our results with these from [2,24]. We consider in our paper $M$ sets of the strategies which evolve in time (see the graph in [27], how best strategies change in time). This situation resembles a stochastic approach from the two papers cited above. Therefore, we averaged all tested strategies and using them we studied the behavior part of the population having antibodies and the basic reproduction number. As we see in Fig. 6, using the average strategies gives a smaller number of the population having antibodies (throughout the period considered) compared to using the best strategies.

Similarly, we have worse results for the basic reproduction number when we use averaged strategies. As we see in Fig. 7, $R_0$ is less than one in the ninth month for averaged strategies, while $R_0 < 1$ is definitely faster (in the third month) when we use our best strategies instead of the averaged. The basic reproduction number is smaller in all time periods for the best strategies obtained thanks to our game.

We present [27] five sets of the tested strategies for which functional (17) is maximized and also five sets of strategies for which functional (18) is minimized. We present also results for the verification theorem and a graph of the strategies which change in time.

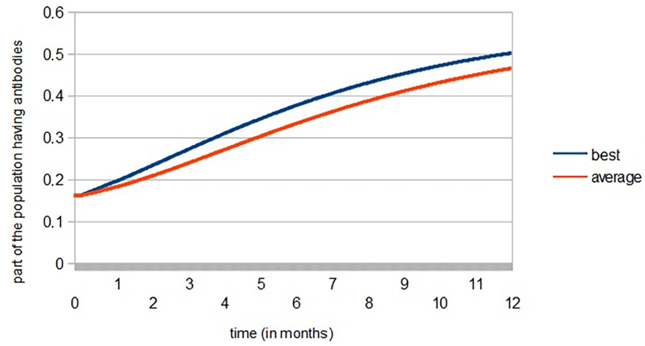


FIG. 6. Comparison values for the part of the population having antibodies $V$ for the best and average strategies.

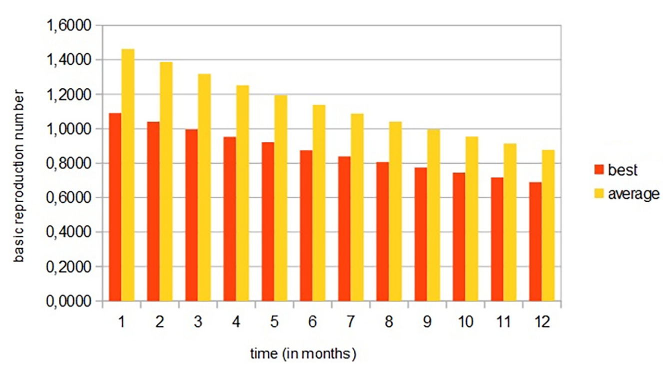


FIG. 7. Comparison values of the basic reproduction number $R_0$ for the best and average strategies.

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