Nonstationary but quasisteady states in self-organized criticality

S. S. Manna
Satyendra Nath Bose National Centre for Basic Sciences, Block-JD, Sector-III, Salt Lake, Kolkata-700106, India

(Received 24 January 2023; accepted 31 March 2023; published 19 April 2023)

The notion of self-organized criticality (SOC) was conceived to interpret the spontaneous emergence of long-range correlations in nature. Since then many different models have been introduced to study SOC. All of them have a few common features: externally driven dynamical systems self-organize themselves to nonequilibrium stationary states exhibiting fluctuations of all length scales as the signatures of criticality. In contrast, we have studied here in the framework of the sandpile model a system that has mass inflow but no outflow. There is no boundary, and particles cannot escape from the system by any means. Therefore, there is no current balance, and consequently it is not expected that the system would arrive at a stationary state. In spite of that, it is observed that the bulk of the system self-organizes to a quasisteady state where the grain density is maintained at a nearly constant value. Power law distributed fluctuations of all lengths and time scales have been observed, which are the signatures of criticality. Our detailed computer simulation study gives the set of critical exponents whose values are very close to their counterparts in the original sandpile model. This study indicates that (i) a physical boundary and (ii) the stationary state, though sufficient, may not be the necessary criteria for achieving SOC.

DOI: 10.1103/PhysRevE.107.044113

I. INTRODUCTION

An externally driven, nonlinear system with open boundaries is key in the prescription of a self-organized critical system [1]. The driving instrument adds intermittent mass (or energy) to the system in the form of tiny particles. The dynamics is nonlinear since the rule does not allow the local accumulation of particles indefinitely [2–5]. This is incorporated using a cutoff in the particle number, beyond which the particles get distributed. This way the system responds to the external drive to minimize the effect of the drive that creates inhomogeneity in particle density. The mass distribution takes place in a "domino" process and creates a series of activity in the form of an avalanche. Eventually, all these activities subside due to spreading of particles through the self-organizing diffusion process and also by flushing out of particles from the system across the boundary. The system continues to be driven ever after, repeatedly [6–9].

Thus, in their original prescription [1] Bak *et al.* designed such a nonequilibrium system with a steady inflow of mass through the driving process and outflow through the boundary. As a result, a stationary state sets in when these two currents balance each other. In this state the avalanches in the system are observed to be of all lengths and time scales, which are considered to be the signatures of the long-range spatiotemporal correlations and appearance of the critical state in the system [10,11].

It was claimed that the steady flow of particle current through the system and the settling of the system in a stationary state are the necessary conditions to achieve the self-organized criticality (SOC) state. After a careful look, however, one realizes that since the ratio of the numbers of boundary to bulk sites becomes very small in the limit of asymptotically large systems, there may be little effect of the boundary in this problem. It had been observed that indeed an increasing number of avalanches remain confined to the bulk as the system sizes become larger which are not touched by the boundary. This is because the probability distribution of linear extent (diameter) of the avalanches is also observed to decay as a power law [12].

This observation leads us to argue that the presence of a physical boundary and establishing a stationary state under the external drive may not be the absolutely necessary criteria to achieve self-organized criticality. In particular, no current balance to attain the stationary state is really required. In the following we describe that even on an infinite system without a boundary the system can in fact self-organize to a nearly steady state. We devise a model system using the frameworks of the well-known sandpile models of SOC [13–16], where such a nonstationary but quasisteady state in the bulk is produced.

II. THE MODEL

We construct a growing sandpile on an infinite square lattice which we consider as the x-y plane. Sand particles are dropped one by one only at the origin of the coordinate system. When a particle is dropped, some activity is generated in the system through the hard core collision process following the dynamical rule of the non-Abelian stochastic sandpile model [11]. A collision is said to take place if more than one particle share a lattice site at the same time when each particle selects one neighbor site randomly with uniform probability and moves there. As time evolves, the number of collisions initially grows, reaches a maximum, and then decreases, and eventually this activity dies after some time. Such a state is referred to as the "passive state" when no particle moves. The next particle is then dropped again at the origin. Therefore, the addition of a single sand particle takes the system from one passive state to another passive state through a sequence

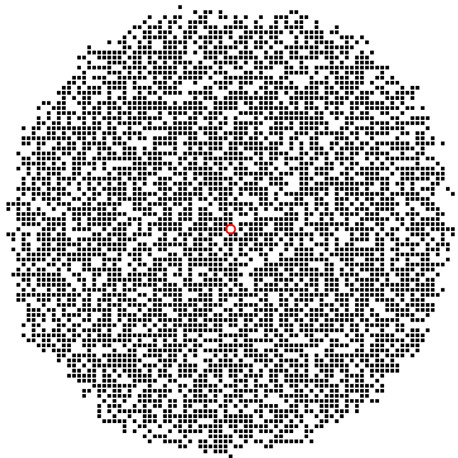

FIG. 1. At the origin (marked by the red circle) of the infinite square lattice $N = 4096$ particles have been dropped one by one. The distribution of particles (black square symbols) in the final passive state has been shown. Apart from a thin outer interface, the bulk of the system has a nearly constant density of particles.

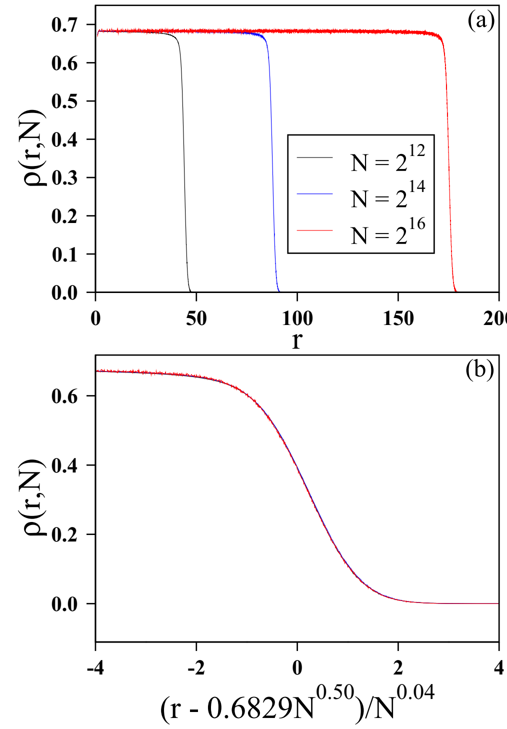

FIG. 2. (a) The particle densities $\rho(r, N)$ in the passive state at a distance r from the origin have been plotted after dropping N particles one by one at the origin. (b) The same data of (a) have been plotted again after scaling the x axis by $(r - 0.6829N^{0.50})/N^{0.04}$. The three plots collapse nicely on top of one another.

of activities. This entire set of activities together is called an "avalanche." Different avalanches create different impacts to the system, and their strengths are measured by the size of the avalanches. Most commonly, the size s of an avalanche is measured by the total number of collisions that take place in the entire avalanche.

III. RESULTS OF THE STOCHASTIC SANDPILE

In the passive state, a lattice site is either occupied by one particle or it is vacant. Typically occupied sites are randomly distributed on the lattice (Fig. 1). Sometimes the origin may also be occupied by one particle. Therefore, when a particle is dropped at the origin, it is likely that a collision takes place, which then triggers an avalanche. If some of the neighboring sites are also occupied, there would be further collisions at these sites and a cascade of collisions results.

We first characterize the passive state by the variation of particle density after dropping N particles one at a time at the origin. The density $\rho(r, N)$ is the average number of particles at a site located at a distance r from the origin. When we plot $\rho(r, N)$ against r in Fig. 2(a) for the three different values of the total number $N = 2^{12}$, 2^{14}, and 2^{16} of particles dropped, we observe a flat bulk region for all N. In this region, the particle densities are nearly the same, though there is a very small but systematic N dependence. We find the average bulk density $\rho(N) = \rho(\infty) - AN^{-x_1}$, where $\rho(\infty) = 0.6835$ and $x_1 = 0.484$ are found. For this reason we say the bulk of the system has reached the quasi-steady state.

As the distance r from the origin increases, the bulk region is followed by an interface where the particle density gradually decreases and finally vanishes. This interface shifts to larger r values as N increases. The steepness of the fall of density profile increases on increasing N. We define a half radius $r_{1/2}(N)$ where the density is half of its average bulk value $\rho(N)$. On plotting (not shown) $r_{1/2}(N)$ against N on a double logarithmic scale, we find $r_{1/2}(N) = 0.6829N^{0.50}$ on the average. We use it in Fig. 2(b) for a scale transformation $r - r_{1/2}(N)$. It makes all three curves pass through nearly the same point, but their slopes at this point are different. To make them collapse on one another, we have to scale the x axis by $N^{-0.04}$. Therefore, we finally plot again $\rho(r, N)$ against $(r - 0.6829N^{0.50})/N^{0.04}$ to obtain a nice data collapse.

The next question we ask is how the particle density in the bulk is maintained as the system evolves. Other than a constant inflow of particles at the origin, and since no particle goes out of the system by evaporation or by other means, it is a fully conservative system. These particles only get themselves distributed to the larger space through the diffusive collision process, but they maintain the bulk density, and consequently the interface of the particle system moves outwards.

To see this point in more detail we consider a circle $\mathcal{C}$ of radius R centered at the origin, situated well inside the bulk region created by dropping N particles. Now we continue to drop ΔN particles at the origin, one at a time again, and observe the flux of the particle current through the circle. There are collisions at sites both inside and outside that are adjacent to the circle. Therefore, for each particle addition at the origin, some particles cross the circle from inside to outside, where as other particles come to the inside from outside. In an actual simulation, we mark all sites inside the circle and keep the outer sites as unmarked. Corresponding to each avalanche, we count how many particles jumped from marked to unmarked sites, which constitutes the flux of outflow current $f(t, \text{out})$.

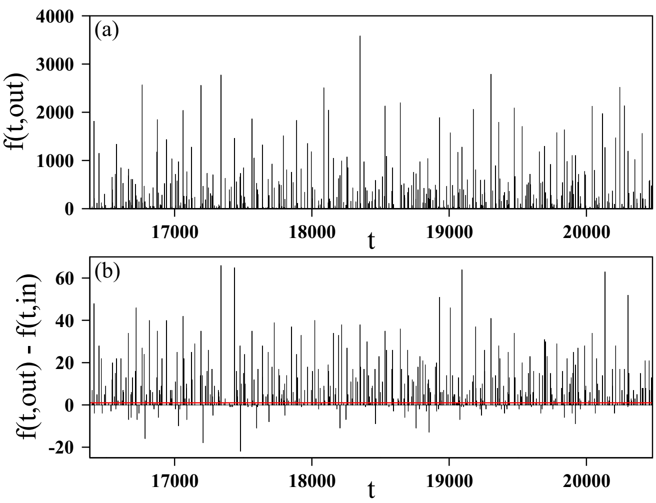

FIG. 3. (a) The out flux $f(t,\mathrm{out})$ through a circle of radius $R=32$ centered at the origin per particle addition has been plotted against time t between N and $N+\Delta N$, with $N=2^{14}$ and $\Delta N=2^{12}$. For each avalanche there is an influx $f(t,\mathrm{in})$ in general which is found to be of the same order as $f(t,\mathrm{out})$ and therefore it has not been plotted. (b) The net outward flux $f(t,\mathrm{out})-f(t,\mathrm{in})$ has been plotted. It is observed that most of the time it is positive, i.e., outward, whereas less frequently it is inward as well. The red line shows the average net flux on this interval, which is very close to unity due to unit rate of particle addition.

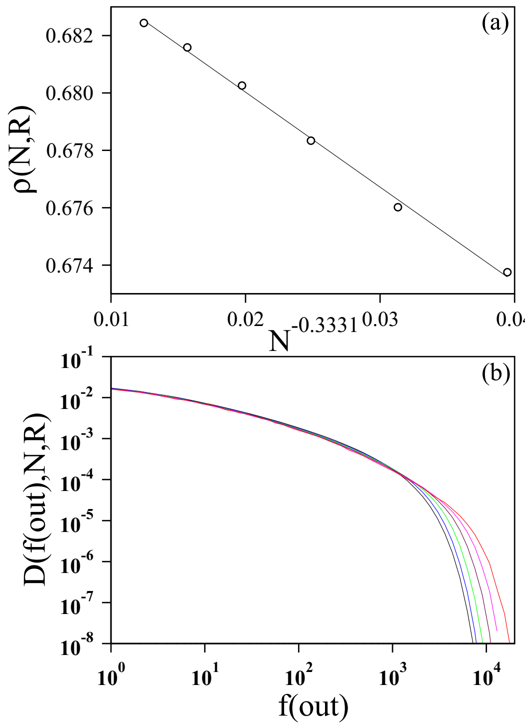

FIG. 4. (a) The average particle density $\rho(N,R)$ within the circle $\mathcal{C}$ of radius R measured within the time interval between N and $N+\Delta N$ has been plotted against $N^{-0.3331}$ and is extrapolated to $\rho(\infty,R)=0.6866$. (b) Probability distribution of $D[f(\mathrm{out}),N,R]$ of the outward flux $f(\mathrm{out})$ across the same circle for every particle addition at the origin within the time interval between N and $N+\Delta N$ has been plotted against $f(\mathrm{out})$. When N is increased from left to right, the distribution is enlarged and larger values of the outward fluxes become more probable. In both plots $N=2^{14}, 2^{15}, ..., 2^{19}$ and $\Delta N=2^{15}$.

Similarly, the number of particles that jumped from unmarked to marked sites constitute flux of the inflow current $f(t,\mathrm{in})$.

In Fig. 3(a) we have shown the variation of $f(t,\mathrm{out})$ for $\Delta N=4096$ time units after dropping $N=2^{14}$ particles. In almost all avalanches $f(t,\mathrm{in})$ is smaller than $f(t,\mathrm{out})$ but of the same order; occasionally, however, $f(t,\mathrm{in})$ is larger. Therefore, we do not plot the variation of $f(t,\mathrm{in})$, which almost looks the same, but plot the net flux $f(t,\mathrm{out})-f(t,\mathrm{in})$ against time in Fig. 3(b), which is mostly positive, but sometimes negative too. The average net flux $\langle f(t,\mathrm{out})-f(t,\mathrm{in})\rangle$ over the entire interval is very close to its exact value unity and has been marked using the red line.

Similarly, for any finite volume within the bulk region like the circle $\mathcal{C}$ the flux of outflow and inflow currents must balance on the average. No particle leaves the system on an infinite lattice, the system only self-organizes itself and particles spread out through the collision process. Within the circle $\mathcal{C}$ the system tries to achieve the steady state of constant density, but it never succeeds in finite time, it only approaches its asymptotic value as N increases. In Fig. 4(a) we have plotted $\rho(N,R)$, which is the average particle density within $\mathcal{C}$ during the time interval N to $N+\Delta N$. On extrapolation it gives the density 0.6866 in the asymptotic limit of $N\to\infty$. This shows that even the core of the bulk region has not become completely steady but it has attained a quasi-steady state and slowly approaches its asymptotic state.

A similar conclusion can also be drawn by looking at the probability distribution $D[f(\mathrm{out}),N,R]$ of the outward fluxes $f(\mathrm{out})$ from the same circle $\mathcal{C}$ calculated within the time interval ΔN after first skipping an initial time N [Fig. 4(b)]. It has been observed that on increasing N the distribution shifts to the larger out flux regime and there is no trace of the distribution reaching a steady time-independent form. This study shows that the bulk of the system does not reach a true steady state in finite time but slowly approaches its quasisteady form.

Now we would like to explore whether this self-organized state is critical or not. For that we have to check whether there are fluctuations of all length scales present in the system. Accordingly, we have defined the avalanche sizes s and life times T in the following way. When a particle is dropped at the origin, it creates a sequence of activities in the system. The lattice sites where particle collisions take place are updated synchronously. Let the intraavalanche time be denoted by T. The list of collision sites at time T are updated to create the same list at time $T+1$. The avalanche is finished when the length of the list shrinks to zero. The total number of time steps T is the lifetime of the avalanche and the total number of collisions s is the avalanche size. Initially, their magnitudes are very small, but they gradually grow and soon become quite large. In Fig. 5(a) we have shown the variation of the size $s(t)$ of the avalanche created by dropping the tth particle at the origin. The time series is for a single run when a total of $N=2^{15}$ particles have been dropped. Next we average the avalanche size over many different runs and plot the average avalanche size $\langle s(t)\rangle$ against t on a double logarithmic scale

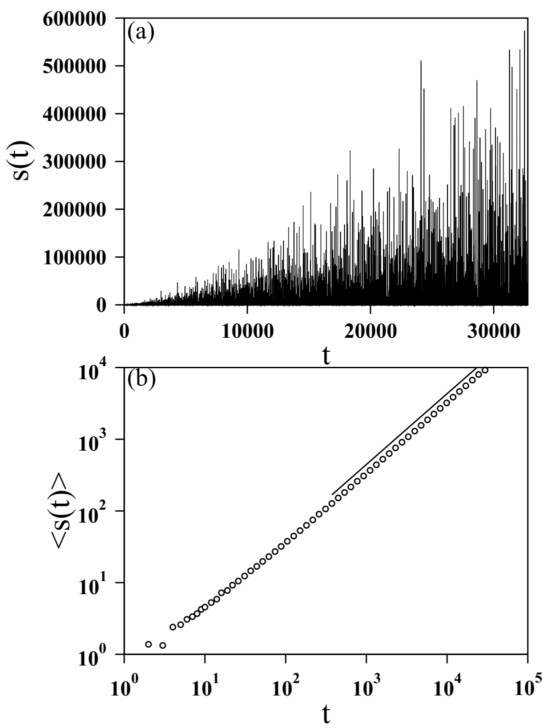

FIG. 5. (a) Plot of size $s(t)$ of the avalanche created by dropping the tth particle at the origin for a single run of $N = 2^{15}$ particles. (b) The average value of the avalanche size $\langle s(t)\rangle$ has been plotted against time t on a log - log scale, and the slope of the straight line fitting the curve is 0.988.

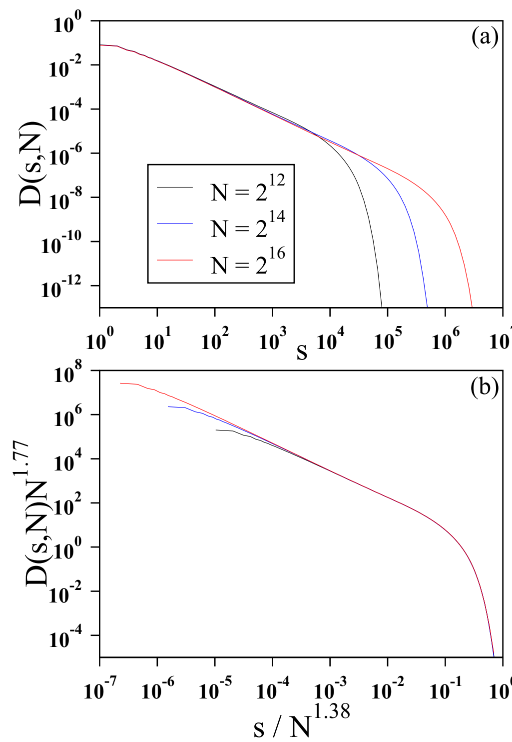

FIG. 6. (a) The probability distributions $D(s, N)$ of the sizes of avalanches have been plotted against the avalanche size s for N particles dropped one by one. The data have been collected over one million runs in each case. (b) The same data of (a) have been plotted again after scaling $D(s, N)N^{1.77}$ against $s/N^{1.38}$, yielding the value of the avalanche size exponent $\tau = 1.77/1.38 \approx 1.28$.

in Fig. 5(b). The slope of the curve for large time is found to be 0.988, which indicates that the avalanche size possibly increases linearly with time.

We have calculated the probability distributions $D(s, N)$ and $D(T, N)$ of the sizes and lifetimes of the avalanches, respectively, when a total of N particles are dropped. In Fig. 6(a) we have plotted $D(s, N)$ against s for three different N values, namely, 2^{12}, 2^{14}, and 2^{16}, and the avalanche size data have been collected over many independent runs. The plots exhibited the characteristics of power law distributions measured in finite systems. On the double logarithmic scale they have the linear region in the middle leading to a bending and sharp fall at some high-value cutoff size $s_c(N)$. The linear regions have slopes of ∼1.200, 1.226, and 1.231, respectively, for the three N values. The cutoff size increases with N by approximately the same amount on the log-log scale when N is increased by the same factor. This implies $s_c(N) \sim N^{\alpha}$, where α is the scaling exponent to be determined. Further, we have done a finite-size scaling analysis in Fig. 6(b). We observe that the distribution $D(s, N)$ scales nicely using suitable powers of N, like

$$D(s, N)N^{\beta} \propto \mathcal{G}(s/N^{\alpha}), \quad (1)$$

where $\mathcal{G}(x)$ is the scaling function such that $\mathcal{G}(x) \to x^{-\tau}$ for $x << 1$ and $\mathcal{G}(x) \to$ constant for $x \gg 1$. The limiting distribution $D(s) = \lim_{N\to\infty} D(s, N) \propto s^{-\tau}$ must be independent of N, which leads to $\tau = \beta/\alpha$. To try this finite size scaling analysis we have scaled the x axis by $s/N^{1.38}$ and the y axis as $D(s, N)N^{1.77}$. We have tuned the scaling exponents and selected these values for the best fit. Therefore, this scaling analysis gives $\tau = 1.77/1.38 \approx 1.28$. A similar analysis for the lifetime distribution yields

$$D(T, N)N^{\beta_T} \propto \mathcal{G}_T(T/N^{\alpha_T}), \quad (2)$$

where $\alpha_T = 0.77$, $\beta_T = 1.15$, and $\tau_T = 1.494$. The average values of avalanche size and lifetimes are found to grow like $\langle s(N)\rangle \sim N$ and $\langle T(N)\rangle \sim N^{0.41}$.

To check if these results are consistent with the model of an ordinary finite size sandpile we have studied the same system having a fixed open boundary. On an $L \times L$ system, the particles have been dropped one by one only at the center of the lattice. In this model, the collisions which take place on the boundary may throw particles outside the system if these directions are randomly selected. Consequently, the stationary state corresponds to the balance of inflow and outflow currents of sand particles. The avalanche sizes have been measured for different system sizes L, namely, 65, 129, and 257. Again, a data collapse analysis has turned out to be very successful when $D(s, L)L^{3.46}$ have been plotted against $s/L^{2.72}$ (figure not shown). This implies the exponent τ is $3.46/2.72 \approx 1.272$, which matches very well with the value 1.28 of the same exponent in the infinite system.

In the original sandpile model [11] the steady state is robust with respect to the choice of the initial state to start with, which is the signature of the self-organizing dynamical process. Consequently, the particle density in the steady state

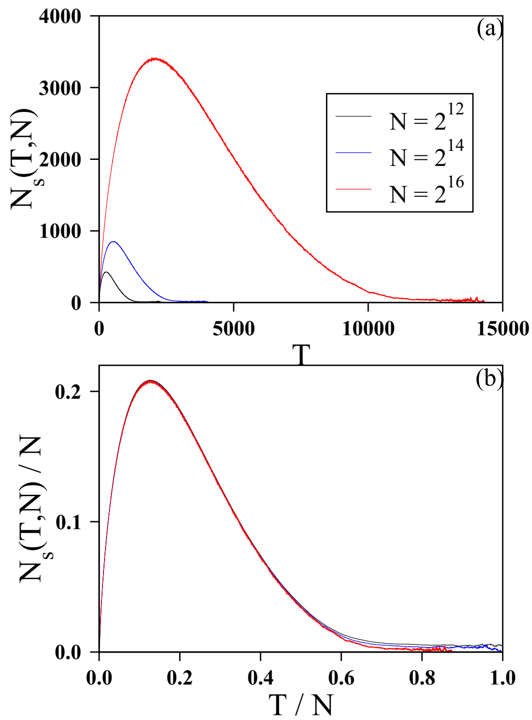

FIG. 7. Relaxation of N particles dropped simultaneously at the origin at time $T = 0$. (a) The number $N_s(T, N)$ of active sites at the intraavalanche time T first sharply increases to a maximum and then gradually decreases to zero. (b) The same data of (a) have been plotted again after scaling the axes by N to obtain a nice data collapse. The scaled curve fits to the generalized gamma distribution.

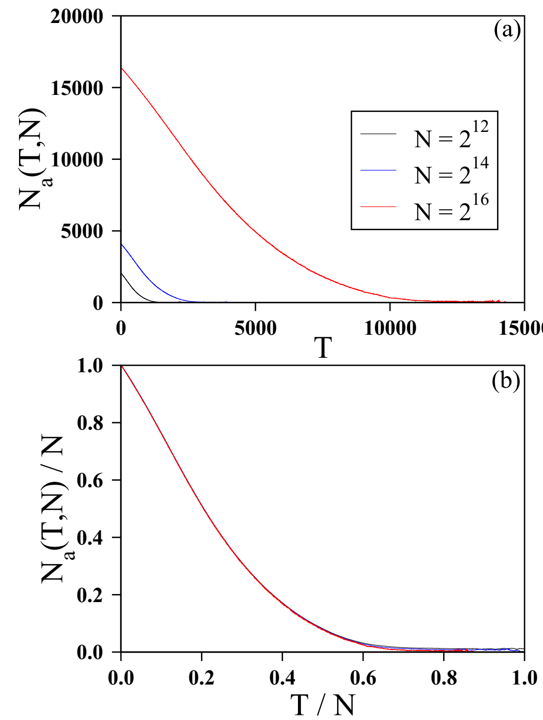

FIG. 8. Relaxation of N particles dropped simultaneously at the origin at time $T = 0$. (a) The number $N_a(T, N)$ of active particles at the intraavalanche time T initially sharply decreases and then slowly vanishes, leading to the passive state. (b) The same data of (a) have been plotted again after scaling both the axes by N to obtain a nice data collapse. The scaled curves fit nicely to the shifted Gaussian [Eq. (4)].

is independent of the density of particles in the initial state. Here also we see the same phenomenon. In another study we add all N particles together at the origin. When such a system evolves to the passive state, we find the density profile indistinguishable from the density profile of the first version when particles were dropped one by one at the origin. Therefore, the nearly same bulk density for all three N values exhibits the signature of self-organization by the dynamical process of this model. These results indicate that even without using a fixed boundary for the mass outflow and current balance, the system can achieve the self-organized state.

Next, we have studied how this system evolves starting from such an initial condition. Specifically, after adding N particles at time $T = 0$ at the origin of the infinite square lattice, we study how the system relaxes as the time T increases. At time $T = 1$, each particle jumps to one of the neighboring sites, selecting it randomly. In the next time step they again jump to their nearest neighbors. In general, collision dynamics is followed, i.e., at any intermediate time if there are more than one particle at a site at a time, then all particles randomly jump to the neighboring sites. As before, this dynamics stops only when there is no active site present in the system, i.e., the system reaches a passive state.

Two quantities are measured. At any arbitrary intermediate time T we count the number $N_s(T, N)$ of active sites, i.e., sites which have more than one particle at that time. It is observed that $N_s(T, N)$ first grows, reaches a maximum, and then decays to a passive state when there is no activity at all. The numbers of such active sites have been averaged over a large number of independent runs to obtain $\langle N_s(T, N)\rangle$. In Fig. 7(a) we have plotted $\langle N_s(T, N)\rangle$ against T for $N = 2^{12}$, 2^{14}, and 2^{16}. As N becomes larger, the height of the peak as well as the duration of the avalanche increases. In Fig. 7(b) we plot again the same data but after scaling the axes. We have plotted $\langle N_s(T, N)\rangle/N$ against T/N and get a nice data collapse. To find its functional form we find that the scaled curve fits best to a generalized gamma distribution function:

$$y = a_0(x/a_1)^{\zeta} \exp[-(x/a_1)^{\eta}], \tag{3}$$

where $y = \langle N_s(T, N)\rangle/N$, $x = T/N$, and the best fitted parameters are $a_0 = 0.456$, $a_1 = 0.234$, $\eta = 0.63$, and $\zeta = 1.484$.

Secondly, we have measured how the number of active particles $N_a(T, N)$ decays with time T and finally vanishes. As time progresses the particles spread to a larger region, where they hardly get other particles to collide and therefore become more and more inactive. In Fig. 8(a) we show the plot of the average number of active particles $\langle N_a(T, N)\rangle$ against T for $N = 2^{12}, 2^{14}$, and 2^{16}. Initially each curve decays fast, but then it slows down and finally vanishes when the passive state is reached. In Fig. 8(b) we scale the axes and plot $\langle N_a(T, N)/N\rangle$ against T/N, which again gave a nice collapse of the data. The best fitted form of this collapsed plot is the shifted Gaussian,

$$y = a_0 \exp[-a_1(x + a_2)^2], \tag{4}$$

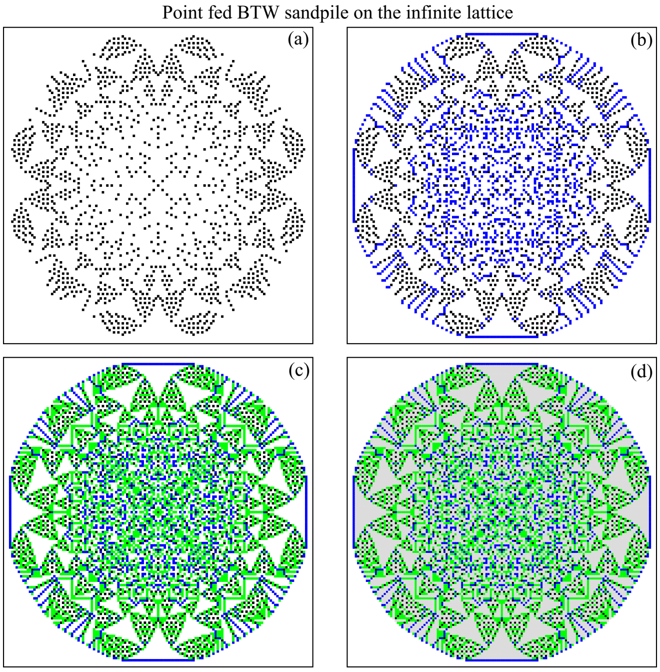

FIG. 9. A total of $N = 2^{15}$ particles have been dropped one by one only at the origin of the infinite square lattice. The final stable state height configuration has been drawn representing sand column heights 0, 1, 2, and 3 by color dots: black (1477), blue (1532), green (4032), and gray (7724), respectively. (a) Sites of height 0 only, (b) sites with heights 0 and 1, (c) sites with heights 0, 1, and 2, and finally (d) sites with heights 0, 1, 2, and 3.

where $y = \langle N_a(T,N)/N\rangle$, $x = T/N$, and the best fitted parameters for $N = 2^{16}$ are $a_0 = 1.21$, which is decreasing towards unity on increasing N; $a_1 = 5.94$, which is increasing; and $a_2 = 0.18$, which is also gradually decreasing to zero.

IV. RESULTS OF THE DETERMINISTIC SANDPILE

Next, we perform a similar study for the deterministic Bak, Tang, and Wiesenfeld (BTW) sandpile model [1,17] on the infinite square lattice, and as before, we drop sand particles one by one only at the origin. As per the rule of the BTW sand pile, the sand column of height $h(i,j)$ becomes unstable only when it exceeds a predefined height $z-1$. An unstable sand column topples and redistributes sand particles as

$$\mathsf{h(i,j)} \to \mathsf{h(i,j)} - \mathsf{z}$$

$$\mathsf{h(i\pm 1, j\pm 1)} \to \mathsf{h(i\pm 1, j\pm 1)} + 1$$

and $z = 4$ is chosen for the square lattice.

We first notice that since the dynamics is entirely deterministic, the underlying symmetries of the square lattice determine the particle distribution patterns. In Fig. 9 the particle distribution patterns have been displayed after dropping $N = 2^{15}$ particles one by one at the origin. The final stable configuration has fourfold symmetry. For clarity we have shown four figures with sites of heights (a) 0 only; (b) 0 and 1; (c) 0, 1, and 2; and (d) 0, 1, 2, and 3.

Since after dropping every four particles the origin becomes active, therefore there are a total of $N/4$ avalanches when N particles are dropped. As more and more particles are dropped, the avalanche sizes become gradually larger. On calculating the probability distribution of the avalanche sizes $D(s,N)$, we find the data to be very much fluctuating. Therefore, we consider the cumulative probability distribution $P(s,N)$, i.e., the probability that a randomly selected avalanche has size s or larger. This integrated distribution is much smoother as displayed in Fig. 10(a) for $N = 2^{12}, 2^{14}$, and 2^{16}. In addition, we execute a finite size scaling here as well. In Fig. 10(b) we have plotted $P(s,N)N^{0.48}$ against $s/N^{1.46}$ and obtain a nice collapse of the data. This implies that the avalanche size exponent $\tau = 1 + 0.48/1.46 \approx 1.33$.

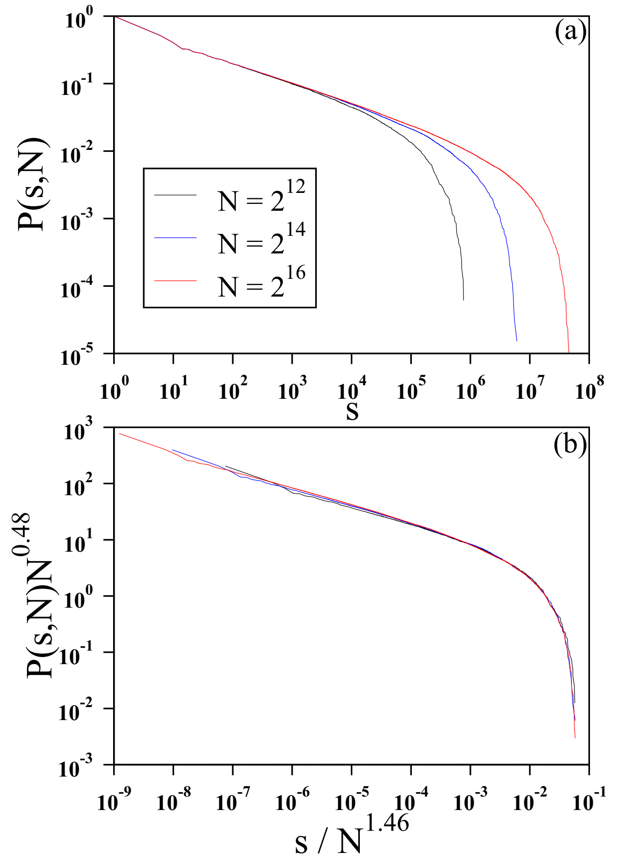

FIG. 10. The cumulative probability distribution $P(s,N)$ of the avalanche sizes of the BTW model where sand particles have been dropped only at the origin. (a) The distribution $P(s,N)$ has been plotted against the avalanche size s for three different values of N. (b) A nice data collapse of the same data is observed when we have plotted $P(s,N)N^{0.48}$ against $s/N^{1.46}$. This analysis implies that the avalanche size exponent $\tau = 1 + 0.48/1.46 \approx 1.328$.

To check if this avalanche size exponent matches with the same system but with a boundary, we have studied the same BTW model on an $L \times L$ square lattice having the center at the origin. As in the ordinary BTW model, particles are dropped outside the boundary, the only difference here being that the system is fed by dropping particles at the origin only. In the stationary state, avalanche sizes are measured for a long time and the cumulative probability distribution $P(s,L)$ has been calculated for $L = 65$, 129, and 257. A finite size scaling plot of $P(s,L)L$ against $s/L^{2.9}$ exhibited a good data collapse (not shown here), yielding $\tau = 1 + 1/2.9 \approx 1.34$. Therefore, the avalanche size distribution exponents for the single-site-fed BTW model on a square lattice with or without boundaries very well match (1.33 against 1.34) each other.

Finally, for the same system we have estimated the probability P_L that an arbitrary avalanche reaches the boundary in the steady state starting from the center of the $L \times L$ lattice. That means P_L is the fraction of avalanches that dropped at least one particle outside the system. Our numerical estimation gives $P_L \sim L^{-0.82}$. We recognize this exponent to be the same as the cumulative probability distribution $P(\xi) \sim \xi^{1-\tau_\xi}$ of the linear extent ξ of the avalanches, and therefore $\tau_\xi \approx 1.82$ [12].

V. THE SUMMARY

To summarize, we have found a way to generate the self-organized critical state without a physical boundary. In the original models of SOC the physical entity, mass or energy for example, can drop out of the system through such a boundary. Here we have studied a growing sandpile where particles have been injected one by one at the origin of the infinite square lattice. The addition of each particle creates an avalanche of activities in the system, which eventually dies down and the system returns to a new passive state. This passive state is not only self-organized but also critical since it exhibits long-range correlations of all length scales. Since there is no boundary, the data are found to be very well behaved.

In contrast to the original prescription of Bak *et al.* [1] we observe that a steady flow of particle current through the system where the average fluxes of global inflow and the global outflow balance each other may not be an absolutely necessary criterion. Instead, only the external drive that injects an inflow current so that the particles only get scattered within the system as per the dynamical rules of the model is sufficient to ensure the self-organized criticality in the system. It is also true that the balance of the outward flux and inward flux of particles through any arbitrarily defined fixed volume within the bulk of the system is always maintained. Because of the particle number conservation and the self-organizing dynamical process, a quasisteady particle density in the bulk is maintained.

A similar study with an Abelian stochastic sandpile where only two particles are transferred in a collision is under progress.

ACKNOWLEDGMENTS

I thank very much one of the referees who suggested the study of the outflux of particles through a box in the bulk of the system. I also acknowledge that a substantial part of the numerical work has been done in S. N. Bose National Centre for Basic Sciences, Kolkata, through a Visiting (Honorary) Fellow position.

[1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett., **59**, 381 (1987).
[2] H. J. Jensen, *Self-Organized Criticality* (Cambridge University, Cambridge, 1998).
[3] S. S. Manna, A. L. Stella, P. Grassberger, and R. Dickman, Self-organized criticality, three decades later, in *Frontiers in Physics*, 2022.
[4] H. Hoffmann and D. W. Payton, Sci. Rep. **8**, 2358 (2018).
[5] A. Garber and H. Kantz, Eur. Phys. J. B **67**, 437 (2009).
[6] D. Dhar, Phys. Rev. Lett. **64**, 2837 (1990).
[7] D. Dhar, Physica A **263**, 4 (1999).
[8] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[9] R. Dickman, T. Tome, and M. J. de Oliveira, Phys. Rev. E **66**, 016111 (2002).
[10] B. Hough, D. C. Jerison, and L. Levine, Commun. Math. Phys. **367**, 33 (2019).
[11] S. S. Manna, J. Phys. A: Math. Gen. **24**, L363 (1991).
[12] S. N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).
[13] D. Dhar, Physica A **186**, 82 (1992).
[14] D. Dhar, Physica A **270**, 69 (1999).
[15] S. S. Manna, Physica A **179**, 249 (1991).
[16] R. Karmakar, S. S. Manna, and A. L. Stella, Phys. Rev. Lett. **94**, 088002 (2005).
[17] K. Wiesenfeld, J. Theiler, and B. McNamara, Phys. Rev. Lett. **65**, 949 (1990).