# Parametric amplification of electromagnetic plasma waves in resonance with a dispersive background gravitational wave

Swadesh M. Mahajan[1,*] and Felipe A. Asenjo[2,†]
[1]*Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA*
[2]*Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago 7941169, Chile*

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It is shown that a subluminal electromagnetic plasma wave, propagating in phase with a background subluminal gravitational wave in a dispersive medium, can undergo parametric amplification. For these phenomena to occur, the dispersive characteristics of the two waves must properly match. The response frequencies of the two waves (medium dependent) must lie within a definite and restrictive range. The combined dynamics is represented by a Whitaker-Hill equation, the quintessential model for parametric instabilities. The exponential growth of the electromagnetic wave is displayed at the resonance; the plasma wave grows at the expense of the background gravitational wave. Different physical scenarios, where the phenomenon can be possible, are discussed.

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## I. INTRODUCTION

Resonant interactions between two distinct waves, controlled by the same wave operators, are expected to be especially strong. In a series of recent papers, it was shown that hyperbolic waves, in particular the dispersive electromagnetic and the gravitational waves, could very efficiently transfer energy to relativistic electrons described by a Klein-Gordon wave [1–3]. These semiclassical calculations (Klein-Gordon equation representing a quantum relativistic spinless particle [4]) demonstrated that both electromagnetic and gravitational waves, through a wave-wave interaction, could resonantly accelerate relativistic electrons to high energies. The resonance occurs when the energy propagation speed of the Klein-Gordon wave and the classical (electromagnetic or gravitational) wave are equal. This is possible only when the classical wave is propagating in a dispersive medium and its group velocity is close to but less than unity.

This paper is an inquiry into what happens when such a mildly subluminal electromagnetic wave propagates in a plasma medium embedded in curved spacetime, specifically when the spacetime curvature is due to a similarly subluminal background dispersive gravitational wave. We show that the resonant interaction rises to an altogether different level of interest; the phenomenon of parametric amplification of the electromagnetic wave is observed. That is, when the dispersive properties of the two waves (in the absence of the other) bear specific quantitative relations to one another, even a very small-amplitude gravitational wave can drive an electromagnetic plasma wave to large amplitudes. An important and necessary requirement for this phenomenon to occur is that the background gravitational wave must be dispersive (possible in a massive medium background) [5–16]. Only then does the gravitational wave become subluminal, and thus couples to the subluminal electromagnetic field in a resonant manner in a plasma.

In a general context, parametric instabilities occur in a variety of physical systems with periodic potentials. Perhaps the most familiar manifestation is the existence of the band structure (Brillouin zones) in metals, derived by solving the Schrödinger equation for an electron moving in the periodic ionic background [17]. Parametric resonance is an ubiquitous effect in every branch of physics, such as, for example, in plasmas [18–20], lasers [21–27], cosmology [28,29], astrophysics and particle physics [30–32], etc. Though the wave-wave interaction and energy exchange processes permeate all physics, the phenomenon reported here is rarer. It is possible when the two waves are resonant and simultaneously satisfy certain rather "rigid" relationships in their dispersion characteristics, especially when the amplitude of the driving gravitational wave is low. These limiting features of the parametric process will be later discussed when the coupled equations are analyzed.

We must emphasize that the interaction of different waves under diverse conditions is one of the most studied subjects. After discussing some representative references, we will point out how our work is different and adds other thinking to the field. On the different previous studies on this subject, one may find, for example, how gravity modifies electromagnetic fields in Refs. [33–38], or how gravitational waves may be excited by light waves propagating in vacuum in Ref. [39], with constant magnetic fields in Ref. [40], and in plasmas in Ref. [41]. The topic of electromagnetic waves driven by gravitational waves is highly discussed: in vacuum [42–49], and in pointlike charge media [50,51]. The effects of a plasma medium on the modes of the gravitational waves have also been calculated [52–54]. Our focus, however, is on the phenomenon resulting from the resonant coupling

*mahajan@mail.utexas.edu
†felipe.asenjo@uai.cl

 

between gravitational and electromagnetic plasma waves (which becomes possible due to their dispersive properties); the possibility of parametric amplification is an exciting consequence of this physics.

Needless to say, there are different studies on linear and nonlinear resonant energy exchange processes in which plasma waves in curved spacetime (Alfvén waves, cyclotron waves, etc.) grow at the cost of gravitational waves [5,55–57], or particles can be accelerated by gravitational waves [58–62]. Also, a direct resonant amplification of plasma waves due to a lightlike (nondispersive) gravitational wave was studied in Ref. [56]. However, the proposition of a similar process for a subluminal gravitational wave for plasmas is lacking. It is this property that allows the occurrence of parametric resonance when the relation between the dispersive properties of the systems is very specific. This produces a unique amplification process for electromagnetic waves, even in unmagnetized plasmas, contrasting with previous studied effects under lightlike propagation and pointlike charge media conditions [50], or in magnetized plasmas [54]. Therefore, this parametric resonant process opens the possibility that this effect could be even relevant for diluted plasmas under low-amplitude gravitational waves, as it is discussed below.

Although the essential physics behind parametric amplification will be accessible in a rather simple equation, we will begin by deriving in Sec. II the exact equations obeyed by transverse electromagnetic plasma waves propagating in a plasma immersed in a curved spacetime background. In Sec. III, we explore the solutions for those electromagnetic waves when the curvature is due to a gravitational wave background. We will then go on to investigate the rather simplified equation (pertaining to small-amplitude gravitational waves). We will first extract the precise conditions when parametric amplification is possible, and then calculate the characteristics of the enhanced electromagnetic plasma wave. In Sec. IV, we will discuss the implications of this work.

## II. ELECTROMAGNETIC PLASMAS WAVES IN GENERAL CURVED SPACETIME BACKGROUND

Our model system consists of a fluid plasma, immersed in a gravitational field, that has one dynamic charged component (electrons) moving in a neutralizing background (provided by ions, for example). Such an ideal general relativistic one-component plasma fluid (with charge $q$, mass $m$, and density $n$) can be described in a unified form as [63,64]

$$qU_\nu M^{\mu\nu} = T\nabla^\mu\sigma, \tag{1}$$

where $\nabla_\mu$ is a covariant derivative for a metric $g_{\mu\nu}$ with signature $(-,+,+,+)$, and with $\mu,\nu = 0,1,2,3$. Here, $U^\mu$ is the plasma fluid four-velocity, $T$ is the plasma temperature, $\sigma$ is its entropy, and

$$M^{\mu\nu} = F^{\mu\nu} + \frac{m}{q}S^{\mu\nu} \tag{2}$$

is the magnetofluid tensor unifying the electromagnetic field, described by the tensor $F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$ (with the electromagnetic four-potential $A^\mu$), and the fluid vorticity antisymmetric tensor $S^{\mu\nu} = \nabla^\mu(fU^\nu) - \nabla^\nu(fU^\mu)$, with $f = h/mn$, where $h$ is the plasma enthalpy density. Here, $f$ contains the thermal-inertial effects of the plasma, and it can be calculated to be $f = K_3[(mc^2)/(k_BT)]/K_2[(mc^2)/(k_BT)]$, where $K_2$ and $K_3$ are the modified Bessel functions of the second kind of orders 2 and 3, respectively, $c$ is the speed of light, and $k_B$ is the Boltzmann constant [63,65]. The above description is valid for an isentropic plasma, as $U_\nu\nabla^\nu\sigma = 0$. This plasma dynamics will provide the four-current $qnU^\mu$ needed to close the system through Maxwell equations

$$\nabla_\nu F^{\mu\nu} = qnU^\mu. \tag{3}$$

For a homentropic plasma fluid $\nabla^\mu\sigma = \partial^\mu\sigma = 0$, and electromagnetic plasma waves are a straightforward solution for a plasma in any curved spacetime. In fact, the dynamics is reduced to $M^{\mu\nu} = 0$ that leads to a simple relationship between the transverse components ($\mu = 1,2$) of the current and the vector potential,

$$A^\mu + \frac{mf}{q}U^\mu = 0. \tag{4}$$

Substituting (4) into (3), we arrive at the wave equation

$$\nabla_\nu F^{\mu\nu} + \Omega_p^2 A^\mu = 0, \tag{5}$$

where $\Omega_p = \omega_p/\sqrt{f}$ is a constant, with the plasma frequency $\omega_p = \sqrt{nq^2/m}$. This equation describes electromagnetic plasma waves in any spacetime.

In curved spacetime, gravity will enter Eq. (5) through the tensor derivatives. The propagation characteristics of the electromagnetic waves then will depend on the background curved spacetime in addition to the dielectric properties of the medium that enter through the plasma frequency (and make the propagation subluminous). In the following section, we will show that, under appropriate conditions, the background curved spacetime provided by a dispersive gravitational wave is able to trigger an instability driving the electromagnetic wave to high amplitudes.

## III. PARAMETRIC RESONANCE DUE TO DISPERSIVE GRAVITATIONAL WAVE BACKGROUND

A gravitational wave propagating in a massive medium is dispersive and subluminal; the medium is endowed with a refractive index [5–16]. An alternative interpretation is that the graviton acquires an effective mass in a massive medium just as the photon does in a plasma.

Let us model a dispersive gravitational wave as a a perturbation on the flat spacetime. Without loss of generality, let us assume that it is propagating in the $z$ direction. The spacetime interval $ds^2 = g_{\mu\nu}dx^\mu dx^\mu$ $(\mu,\nu = 0,1,2,3)$, described by the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, is split into $\eta_{\mu\nu} = (-1,1,1,1)$, the flat spacetime metric, and $h_{\mu\nu}$ the perturbation ($h_{\mu\nu} \ll \eta_{\mu\nu}$). Let us further restrict to a simple gravitational wave with two nonzero components $h_{11} = -h_{22} = h(\chi)$ that are functions only of the wave phase, $\chi = \omega t - kz$, where $\omega$ and $k$ are, respectively, the frequency and wave number [5–16].

The simplest example of a dispersive gravitational wave has a dispersion relation of the form $\omega^2 - k^2 \equiv \omega_G^2 \neq 0$ (in close analogy with the electromagnetic wave in a plasma) where $\omega_G$ (determined by the properties of the medium) forces its group velocity $(d\omega/dk)$ to fall below unity. In general, it is expected that the response frequency be quite small compared to the frequency of the wave, $\omega_G \ll \omega, k$; the gravitational wave will travel with group velocities near (but below) the speed of light.

We are now ready to go back to seek more explicit solutions of Eq. (5). Spelling out the tensor derivatives (notice that the amplitude of the electromagnetic wave has to be small enough that it does not affect gravity), we have

$$\frac{1}{\sqrt{-g}}\partial_\nu[\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)] + \Omega_p^2 g^{\mu\alpha}A_\alpha = 0. \quad (6)$$

Fixing $z$ to be the direction of propagation, we let the transverse components of the electromagnetic wave potential to be $A_1 = A_+$ and $A_2 = A_-$. Using the metric for the gravitational wave (specified above), these potentials obey

$$\frac{\partial}{\partial t}\left(f_\pm \frac{\partial A_\pm}{\partial t}\right) - \frac{\partial}{\partial z}\left(f_\pm \frac{\partial A_\pm}{\partial z}\right) + \Omega_p^2 f_\pm A_\pm = 0, \quad (7)$$

where $f_+ = \sqrt{-g}g^{11} \approx 1 - h$, and $f_- = \sqrt{-g}g^{22} \approx 1 + h$.

Since we are out to explore the interaction of resonant electromagnetic plasma waves and gravitational waves, we demand that we seek solutions in which the electromagnetic potentials are functions of exactly the same phase as gravitational waves, i.e., $A_\pm = A_\pm(\chi)$. In this form, the phase velocity of the electromagnetic plasma waves coincide with the phase velocity of the gravitational wave background. The wave Eq. (7) then becomes an ordinary differential equation

$$\frac{d^2A_\pm}{d\chi^2} \mp \frac{dh}{d\chi}\frac{dA_\pm}{d\chi} + \left(\frac{\Omega_p}{\omega_G}\right)^2 A_\pm = 0, \quad (8)$$

representing a driven homogeneous oscillator, the driver being the gravitational field $h$.

We can consider a gravitational wave described by a plane-wave form $h = h_0 \cos\chi$, with the respective phase $\chi$ and amplitude $h_0 \ll 1$. In this case, Eq. (8) can be written more explicitly as

$$\frac{d^2A_\pm}{d\chi^2} \pm h_0 \sin\chi \frac{dA_\pm}{d\chi} + \left(\frac{\Omega_p}{\omega_G}\right)^2 A_\pm = 0. \quad (9)$$

The oscillator is clearly subject to a periodic potential; this has immensely interesting consequences. Noting that $A_-(\chi) = A_+(\chi - \pi)$, it is enough to solve for only one polarization. For the $A_-$ polarization, a change of variable

$$\mathcal{A}_- = \exp\left(\frac{h_0}{2}\cos\chi\right)A_- \quad (10)$$

converts Eq. (9) into the more standard form of a Whittaker-Hill equation,

$$\frac{d^2\mathcal{A}_-}{d\chi^2} + \left[\left(\frac{\Omega_p}{\omega_G}\right)^2 + \frac{h_0}{2}\cos\chi - \frac{h_0^2}{4}\sin^2\chi\right]\mathcal{A}_- = 0. \quad (11)$$

A similar equation can be found for + polarization. There is vast literature on the exact solutions of equations such as Eq. (11) with periodic coefficients. However, our goal here is to explore the interesting physics and delineate the precise conditions where the parametric resonance is triggered. The key is to recognize that the Whittaker-Hill equation has different instabilities zones for its two-dimensional parameter space $(\Omega_p/\omega_G, h_0)$ [66].

For small enough $|h_0|$, however, most zones of instability shrink, and mostly the only regime where we can observe parametric amplification is the first one. In this limit, neglecting high-order terms $O(h_0^2)$, Eq. (11) reduces to a Mathieu equation, where the first and likely the only relevant unstable region for the dynamics is in the vicinity

$$\left(\frac{\Omega_p}{\omega_G}\right)^2 = \frac{1}{4}, \quad (12)$$

and has a width of order $h_0$ (see, for instance, Ref. [67]). In this part of the parameter space, the electromagnetic plasma wave displays an exponentially growing amplitude with the form

$$|A_-| \propto \exp\left(\frac{h_0}{4}\chi\right). \quad (13)$$

Since the $A_+$ polarization is just phase shifted, it will have the same exponential growth.

This parametric amplification is a manifestation of a combination of two effects: the resonance phase between the electromagnetic plasma wave and the gravitational wave, and their dispersion characteristics having (as remarked earlier) a definitive relationship. Consequently, the electromagnetic plasma wave (of both polarizations) draws energy, very efficiently, from the background gravitational field. In this way, the phenomenon described here has a pure general-relativistic origin.

In order to fully display the parametric amplification for the propagating electromagnetic plasma wave, Eq. (9) [or Eq. (11)] is solved numerically under several conditions. In Fig. 1(a) we plot the solutions for $A_-(\chi)$ in the unstable region (12). We have considered initial conditions $A_-(0) = 1.5 \times 10^{-5}$, with a background gravitational wave amplitude $h_0 = 5 \times 10^{-3}$. The (initial) amplitude of the electromagnetic wave is chosen to be smaller than the amplitude of the gravitational wave in order to maintain the condition of the gravitational wave as a background for the dynamics of the electromagnetic field. The propagating oscillating solution for $A_-$ is displayed as the blue solid line. We also show the exponential grow (dashed black line) of the electromagnetic wave amplitude predicted by Eq. (13). The electromagnetic plasma wave amplitude grows by approximately one order of magnitude by $\chi \approx 1800$. In order to highlight the parametric resonant growth, in the same figure we display the solution for $A_-$ (with the same previous initial conditions) for a departure of the condition (12), by choosing $(\Omega_p/\omega_G)^2 = 3/10$. This solution (magenta line) represents mainly a sinusoidal oscillation with no amplification. Similar behavior can be found for $A_+$.

Notice that the growth rate is quite small as compared to the oscillation frequency. It will become commensurately larger for larger $h_0$. To show this, in Fig. 1(b), we plot the numerical solution for $A_-(\chi)$ in the unstable region (12), with initial conditions $A_-(0) = 10^{-5}$ and $h_0 = 10^{-2}$. For this case, by $\chi \approx 1800$, the electromagnetic plasma wave amplitude grows by two orders of magnitude. Again, the dashed line represents the exponential grow (13).

Finally, in Fig. 1(c), we display the solutions for Eq. (9) for the following two unstable regions of Mathieu equation, $(\Omega_p/\omega_G)^2 = 1$ (blue line) and $(\Omega_p/\omega_G)^2 = 9/4$ (magenta

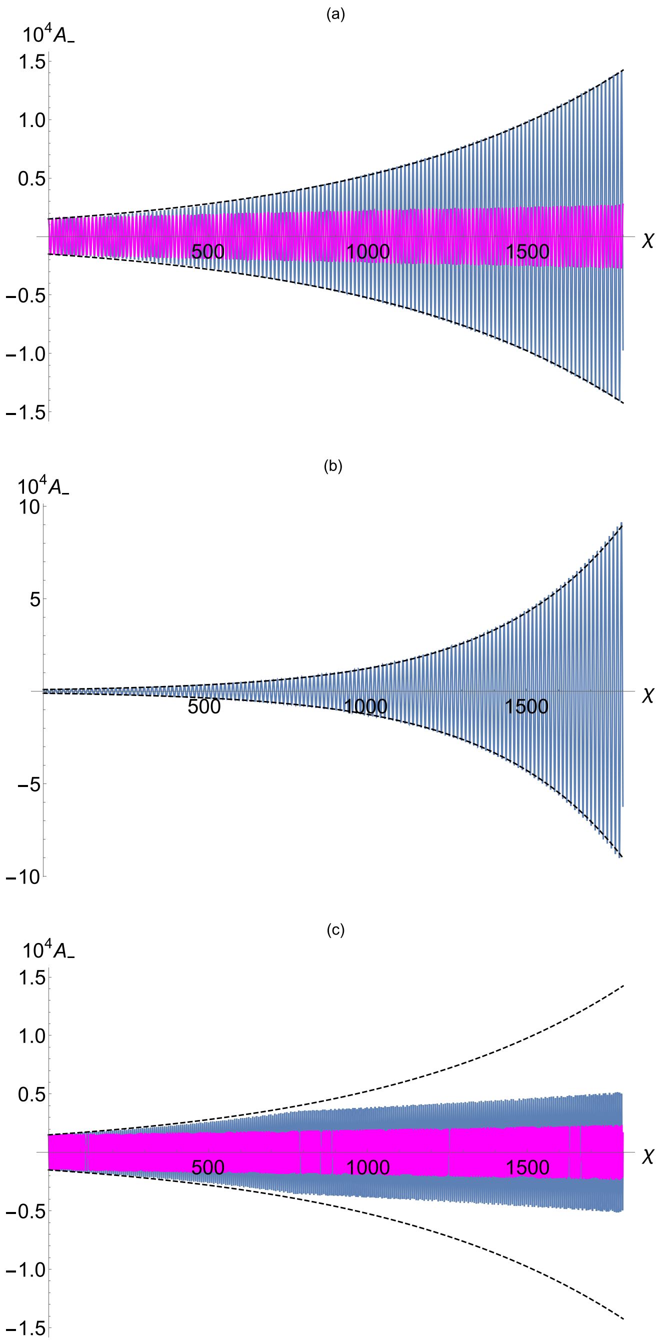


FIG. 1. Numerical solutions for electromagnetic plasma wave $A_-$ from Eq. (9). In all figures, in dashed line is shown the theoretical exponential growing (13) for wave amplitude. (a) Solution in the blue oscillating line for $A_-(0) = 1.5 \times 10^{-5}$, $h_0 = 5 \times 10^{-3}$, and under condition (12). In the magenta line we plot the solution with the same initial conditions but with $(\Omega_p/\omega_G)^2 = 3/10$. (b) Solution in the blue oscillating line for $A_-(0) = 10^{-5}$, $h_0 = 10^{-2}$, and under condition (12). (c) Solutions with initial conditions $A_-(0) = 1.5 \times 10^{-5}$ and $h_0 = 5 \times 10^{-3}$, where we have considered $(\Omega_p/\omega_G)^2 = 1$ (blue line) and $(\Omega_p/\omega_G)^2 = 9/4$ (magenta line).

line) [67]. We have used the initial conditions $A_-(0) = 1.5 \times 10^{-5}$ and $h_0 = 5 \times 10^{-3}$. Both electromagnetic plasma wave solutions do not present a resonant amplification, as it was discussed previously.

In general, the width of the parametric resonance is of order $h_0$; exponentially growing solutions are found only in the range [67]

$$\left(\frac{\Omega_p}{\omega_G}\right)^2 = \frac{1}{4} \pm \frac{h_0}{4}, \tag{14}$$

beyond which pure oscillatory solutions pertain.

## IV. DISCUSSION

Since the basic physics is contained in the rather simple (highly investigated) equation, it is no wonder that numerical and semianalytical treatments give the same results. The most important task, however, is to dwell on precise conditions for the whole dynamic to take place.

The first one is the phase-matching condition in a dispersive realm. Practically, it is insured by demanding that both wave amplitudes are functions of the same propagation phase, $\omega t - kz$. It is this demand that leads to the ordinary differential equation coupling the gravitational wave to the electromagnetic wave. This kind of phase-matching process between the electromagnetic wave and its gravitational wave background can only be possible for subluminal waves; both waves must be dispersive.

The second condition relates the two different (gravitational and electromagnetic) frequency responses of the medium; these must lie in the near neighborhood of $\omega_G = 2\Omega_p$ [see Eqs. (12) and (14)]. When this condition is satisfied, even a low-amplitude gravitational wave (that adds a periodic potential to the equation of the electromagnetic propagation) can trigger a parametric instability.

For a medium with an electron plasma, one can estimate from this condition that $\omega_G = 113,5\sqrt{n/f}$, where $n$ is measured in $\mathrm{m}^{-3}$. We can use this relation to examine what class of media will support this parametric resonance. For an electron density $n \sim 10^{30}\ \mathrm{m}^{-3}$, and relativistic temperatures $T \sim 10^{10}$ K ($f \sim 7$), for instance, $\omega_G \sim 43$ PHz. This is a very high frequency for electromagnetic plasma waves, on the range of ionizing radiation. Therefore, in order for a gravitational wave to trigger the parametric resonance, the frequency $\omega$ of the gravitational wave must be even larger. On the contrary, for a very diluted cold plasma with $n \sim 1\ \mathrm{m}^{-3}$ and $f \sim 1$, it is obtained that the gravitational wave has a frequency response $\omega_G \sim 113.5$ Hz. This is well within the range of future technological capabilities of gravitational wave detectors—frequencies of the order $\omega \sim 300$–$1000$ Hz [68].

On the other hand, under the same conditions, it is very unlikely that a reverse process with the same features takes place, i.e., this kind of parametric amplification of gravitational waves in an electromagnetic background field. The contribution of the electromagnetic wave to the energy momentum tensor is likely to be insignificant to affect the nature of the gravitational wave.

The transfer of energy from the subluminal gravitational to the subluminal electromagnetic wave may be one of the more significant contributors to the presence of electromagnetic energy in the universe. Though this model calculation was done for the low-amplitude gravitational waves, such an exchange is likely to happen even when the gravitational wave is very strong as in some cosmic cataclysmic events. In fact, for low-amplitude waves, one can excite only the first parametric resonance. However, higher unstable bands do become accessible for large-amplitude gravitational drives. In such a case, parametric resonance can occur for much larger ranges of $\omega_G$ and $\Omega_p$. One should expect this energy transfer process between the two waves traveling in the same medium to be ubiquitous.

Since this paper has presented a clear, initial demonstration of what may turn out to be a very efficient source of electromagnetic energy, we plan to investigate the parametric resonance between subluminal electromagnetic and gravitational waves in more depth and detail.


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